Z. Wahrscheinlichkeitstheorie verw. Gebiete
59, 361–371 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Associated Random Variables and Martingale Inequalities

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Summary. Many of the classical submartingale inequalities, including Doob's maximal inequality and upcrossing inequality, are valid for sequences S_j such that the $(S_{j+1}-S_j)$'s are associated (positive mean) random variables, and for more general "demisubmartingales". The demisubmartingale maximal inequality is used to prove weak convergence to the two-parameter Wiener process of the partial sum processes constructed from a stationary two-parameter sequence of associated random variables $\{X_{ij}\}$ with $\sum_i \sum_j \operatorname{Cov}(X_{00}, X_{ij}) < \infty$.

1. Introduction

A finite collection of random variables, $X_1, \ldots, X_m$, is said to be *associated* if for any two coordinatewise nondecreasing functions f, g on R^m,

$$\operatorname{Cov}(f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geqq 0, \tag{1}$$

whenever the covariance is defined; an infinite collection is associated if every finite subcollection is associated. This definition was introduced in [5] and has found several applications in reliability theory [1]. The basic concept actually appears in [8] in the context of percolation models and it was subsequently applied to the Ising models of statistical mechanics in [6]; in the statistical mechanics literature (see, e.g., [9]), which developed independently of reliability theory, associated random variables are said to satisfy the FKG inequalities.

One of the results originating in statistical mechanics which is of particular probabalistic interest concerns a central limit theorem for certain stationary d-parameter arrays, $\{X_{\vec{j}}: \vec{j} \in \mathbb{Z}^d\}$ of associated random variables [10]. The following theorem is a modified version of that result; it is a direct consequence of [10, Theorem 2] and standard arguments.

* Alfred P. Sloan Research Fellow. Research supported in part by NSF Grants MCS 77-20683 and MCS 80-19384

0044-3719/82/0059/0361/$02.20

Theorem 1. *Let $\{X_{\vec{j}}: \vec{j}=(j_1,\ldots,j_d)\in\mathbb{Z}^d\}$ be a strictly stationary (with respect to translations in $\mathbb{Z}^d$) array of finite variance, mean zero, associated random variables such that*

$$\sigma^2 \equiv \sum_{\vec{j}\in\mathbb{Z}^d} \operatorname{Cov}(X_{\vec{0}}, X_{\vec{j}}) < \infty. \tag{2}$$

For $\vec{t}=(t_1,\ldots,t_d)$ with each $t_j \geqq 0$, define

$$W_n(\vec{t}) = n^{-d/2} \sum_{j_1=1}^{[nt_1]} \cdots \sum_{j_d=1}^{[nt_d]} X_{\vec{j}}, \tag{3}$$

where $[\cdot]$ denotes the usual greatest integer function, and let $W(\vec{t})$ be the d-parameter Wiener process, a mean zero Gaussian process with

$$\operatorname{Cov}(W(\vec{t}), W(\vec{s})) = \sigma^2 \prod_{i=1}^{d} \min(t_i, s_i); \tag{4}$$

then the finite dimensional distributions of W_n converge (in distribution) to those of W.

The question of whether there is weak convergence of W_n to W (in an appropriate function space sense) was left open in [10] but was answered affirmatively in [11] for the case $d=1$. In order to obtain weak convergence, it is necessary to have a sufficiently good estimate concerning the tail of the distribution of $\sup(|W_n(\vec{t})|: 0\leqq t_1 \leqq 1, \ldots, 0 \leqq t_d \leqq 1)$; such an estimate was obtained for $d=1$ as a consequence of certain maximal inequalities (see [11, Theorem 2] and Corollaries 4 and 5 below) which seemed related to some of the standard maximal inequalities for martingales. The results of this paper were motivated first by a desire to obtain weak convergence for $d>1$ and second by the related need to better understand the relation between sums of associated random variables and martingales. The key to that relation is given in the following proposition which is an immediate consequence of the definition of association.

Proposition 2. *Suppose $X_1, X_2, \ldots$ are L^1, mean zero, associated random variables and $S_j = X_1 + \ldots + X_j (S_0 = 0)$; then*

$$\begin{aligned} &\text{for } j=1,2,\ldots, E((S_{j+1}-S_j) f(S_1,\ldots,S_j)) \geqq 0 \\ &\text{for all coordinatewise nondecreasing functions } f \end{aligned} \tag{5}$$

such that the expectation is defined.

We note that if f is not required to be increasing, then (5) becomes equivalent (with the obvious choice of σ-fields) to the condition that $S_1, S_2, \ldots$ be a martingale; on the other hand, if f is required to be nonnegative (respectively, nonpositive) rather than nondecreasing, then (5) becomes equivalent to the condition that $S_1, S_2, \ldots$ be a submartingale (respectively, supermartingale). We will call an L^1 sequence satisfying (5) a *demimartingale;* if (5) is modified so that f is required to be nonnegative (resp., nonpositive) *and* nondecreasing, the sequence will be called a *demisubmartingale* (resp., *demisupermartingale*).

The real justification for this terminology is given by the results of Sect. 2 of this paper where we show that both Doob's maximal inequality and the upcrossing inequality are valid for demisubmartingales. The usefulness of demimartingales is demonstrated in Sect. 3 where they play an essential role in obtaining weak convergence for $d=2$. The problem of proving weak convergence for $d \geqq 3$ is presently an open question.

2. Demimartingale Inequalities

The following theorem and proof are based on Garsia's version of Doob's maximal inequality [7]. We extend the inequality from submartingales to demisubmartingales and (at almost no extra cost) from the maximum,

$$S_n^* \equiv \operatorname{Max}(S_1, \ldots, S_n), \tag{6}$$

to more general rank orders $S_{n,j}$, defined by

$$S_{n,j} \equiv \begin{cases} j\text{th largest of } (S_1, \ldots, S_n), & \text{if } j \leqq n, \\ \min(S_1, \ldots, S_n) = S_{n,n}, & \text{if } j > n, \end{cases} \tag{7}$$

so that $S_{n,1} = S_n^*$. In this theorem and this section, we do not explicitly consider the demisupermartingale case since that may be obtained from the demisubmartingale case by replacing all S_i's by their negatives.

Theorem 3. *Suppose $S_1, S_2, \ldots$ is a demimartingale (resp., demisubmartingale) and m is a nondecreasing (resp., non-negative and nondecreasing) function on $(-\infty, \infty)$ with $m(0)=0$; then for any n and j,*

$$E\left(\int_0^{S_{n,j}} u\, dm(u)\right) \leqq E(S_n m(S_{n,j})); \tag{8}$$

and thus for any $\lambda > 0$,

$$\lambda P(S_{n,j} \geqq \lambda) \leqq \int_{\{S_{n,j} \geqq \lambda\}} S_n\, dP. \tag{9}$$

Proof. For fixed n and j, let $Y_k = S_{k,j}$ and $Y_0 = 0$; then

$$S_n m(Y_n) = \sum_{k=0}^{n-1} S_{k+1}(m(Y_{k+1}) - m(Y_k)) + \sum_{k=1}^{n-1} (S_{k+1} - S_k) m(Y_k). \tag{10}$$

Note from the definition of $S_{n,j}$, that

$$\text{for } k<j, \quad \text{either } Y_k = Y_{k+1} \quad \text{or} \quad S_{k+1} = Y_{k+1}, \tag{11a}$$

$$\text{for } k \geqq j, \quad \text{either } Y_k = Y_{k+1} \quad \text{or} \quad S_{k+1} \geqq Y_{k+1}. \tag{11b}$$

Thus for any k,

$$S_{k+1}(m(Y_{k+1}) - m(Y_k)) \geqq Y_{k+1}(m(Y_{k+1}) - m(Y_k)) \geqq \int_{Y_k}^{Y_{k+1}} u\, dm(u), \tag{12}$$

so that (10) yields

$$S_n m(Y_n) \geqq \int_0^{Y_n} u\, dm(u) + \sum_{k=1}^{n-1} ((S_{k+1} - S_k)\, m(Y_k)). \tag{13}$$

Next we note that

$$E((S_{k+1} - S_k)\, m(Y_k)) \geqq 0 \tag{14}$$

by the definition of demimartingale (resp., demisubmartingale) since $m(Y_k)$ is a nondecreasing (resp., non-negative and nondecreasing) function of $S_1, \ldots, S_k$; taking the expectation of (13) and using (14) yields (8) since $Y_n = S_{n,j}$. To obtain (9) from (8), take $m(u)$ to be the indicator function $1_{\{u \geqq \lambda\}}$.

Inequality (15) of the following corollary extends a result of Pitman [12] from the maximum (or minimum) to general rank orders.

Corollary 4. *If $S_1, S_2, \ldots$ is an L^2 demimartingale, then*

$$E((S_{n,j} - S_n)^2) \leqq E(S_n^2); \tag{15}$$

if $S_1, S_2, \ldots$ is an L^2 demisubmartingale, then

$$E((S_{n,j}^+ - S_n)^2) \leqq E(S_n^2). \tag{16}$$

Proof. In the demimartingale case, take $m(u) = u$ in (8) to obtain $E(S_{n,j}^2/2) \leqq E(S_n S_{n,j})$ which is equivalent to (15); in the demisubmartingale case, take $m(u) = u\, 1_{\{u \geqq 0\}}$.

The next corollary shows how the maximal inequalities derived in [11] to obtain weak convergence for $d = 1$ are related to the Doob inequality (8). Inequality (19) of the corollary is esentially identical to the one usually derived for sums of independent variables (see e.g. [2, Eq. 10.7]).

Corollary 5. *Suppose $S_1, S_2, \ldots$ are as in Proposition 2; then*

$$E(S_{n,j}^2) \leqq E(S_n^2) \equiv s_n^2, \tag{17}$$

and for $\lambda_1 < \lambda_2$,

$$(1 - s_n^2/(\lambda_2 - \lambda_1)^2)\, P(S_n^* \geqq \lambda_2) \leqq P(S_n \geqq \lambda_1), \tag{18}$$

so that for $\alpha_1 < \alpha_2$ with $\alpha_2 - \alpha_1 > 1$,

$$P(\mathrm{Max}(|S_1|, \ldots, |S_n|) \geqq \alpha_2 s_n) \leqq \frac{(\alpha_2 - \alpha_1)^2}{(\alpha_2 - \alpha_1)^2 - 1} P(|S_n| \geqq \alpha_1 s_n). \tag{19}$$

Proof. With $S_k = X_1 + \ldots + X_k$, we define $T_1 = 0$, and

$$T_k = \sum_{i=n-k+2}^{n} X_i, \qquad \text{for } k = 2, 3, \ldots, n+1.$$

$T_1, T_2, \ldots$ is a demimartingale by Proposition 2 and so by (8) with $m(u) = u$, we have (assuming without loss of generality that $j \leqq n$)

$$E(T_{n,n-j+1}^2/2) \leqq E(T_n\, T_{n,n-j+1}) \leqq E(T_{n+1}\, T_{n,n-j+1}) \tag{20}$$

where the second inequality follows from the definition of a demimartingale. Now (20) implies that

$$E((T_{n+1}-T_{n,n-j+1})^2) \leqq E(T_{n+1}^2),$$

and this is the same as (17) since $T_{n+1}=S_n$ and

$$T_{n+1}-T_{n,n-j+1}=j\text{th largest of } (T_{n+1}-T_n, T_{n+1}-T_{n-1}, \ldots, T_{n+1}-T_1)=S_{n,j}.$$

To obtain (18), we first note that

$$\begin{aligned} P(S_n^* \geqq \lambda_2) &= P(S_n^* \geqq \lambda_2, S_n \geqq \lambda_1) + P(S_n^* \geqq \lambda_2, S_n < \lambda_1) \\ &\leqq P(S_n \geqq \lambda_1) + P(S_n^* \geqq \lambda_2, S_n^* - S_n \geqq \lambda_2 - \lambda_1). \end{aligned} \tag{21}$$

We next claim that it follows from the fact that S_n^* and $S_n - S_n^* = T_{n,n}$ are associated, since they are both increasing functions of the X_i's, that

$$P(S_n^* \geqq \lambda_2, S_n^* - S_n \geqq \lambda_2 - \lambda_1) \leqq P(S_n^* \geqq \lambda_2)\, P(S_n^* - S_n \geqq \lambda_2 - \lambda_1); \tag{22}$$

to see that (22) is valid, note that for X, Y associated,

$$-P(X \geqq x, Y \leqq y) + P(X \geqq x)\, P(Y \leqq y) = \operatorname{Cov}(1_{\{X \geqq x\}}, -1_{\{Y \leqq y\}}) \geqq 0.$$

Now by combining (21), (22), Čebyšev's inequality, and (15), we have

$$\begin{aligned} P(S_n^* \geqq \lambda_2) &\leqq P(S_n \geqq \lambda_1) + P(S_n^* \geqq \lambda_2)\, E((S_n^* - S_n)^2)/(\lambda_2 - \lambda_1)^2 \\ &\leqq P(S_n \geqq \lambda_1) + P(S_n^* \geqq \lambda_2)\, E(S_n^2)/(\lambda_2 - \lambda_1)^2, \end{aligned}$$

which immediately yields (18). Finally, we may obtain (19) by taking $\lambda_i = \alpha_i s_n$ and adding to (18) the analogous inequality obtained after replacing all X_i's by their negatives (which are also necessarily associated).

Remark. There seems to be no reason why (22) should be valid for a general demimartingale (or martingale) and thus (18)–(19) are presumably not valid in that generality. The next corollary however gives a similar type maximal inequality (with the tail of S_n appearing on the right hand side) which is valid for demimartingales; it will be used to obtain weak convergence for $d=2$ in Sect. 3 below. We again define $s_n^2 = E(S_n^2)$.

Corollary 6. *If $S_1, S_2, \ldots$ is an L^2 demisubmartingale; then for $0 \leqq \lambda_1 < \lambda_2$,*

$$P(S_n^* \geqq \lambda_2) \leqq (s_n/(\lambda_2 - \lambda_1))(P(S_n \geqq \lambda_1))^{1/2}. \tag{23}$$

If $S_1, S_2, \ldots$ is an L^2 demimartingale, then for $0 \leqq \alpha_1 < \alpha_2$,

$$P(\operatorname{Max}(|S_1|, \ldots, |S_n|) \geqq \alpha_2 s_n) \leqq \sqrt{2}(\alpha_2 - \alpha_1)^{-1}(P(|S_n| \geqq \alpha_1 s_n))^{1/2}. \tag{24}$$

Proof. Starting from (9) with $\lambda = \lambda_2$ and $j=1$, we have

$$\begin{aligned} \lambda_2 P(S_n^* \geqq \lambda_2) &\leqq \int_{\{S_n^* \geqq \lambda_2\}} S_n\, dP \leqq \int_{\{S_n \geqq \lambda_1\}} S_n\, dP + \int_{\{S_n^* \geqq \lambda_2, S_n < \lambda_1\}} S_n\, dP \\ &\leqq \int_{\{S_n \geqq \lambda_1\}} S_n\, dP + \lambda_1 P(S_n^* \geqq \lambda_2), \end{aligned}$$

which immediately yields the basic inequality,

$$P(S_n^* \geqq \lambda_2) \leqq (\lambda_2 - \lambda_1)^{-1} E(S_n 1_{\{S_n \geqq \lambda_1\}}). \tag{25}$$

The Cauchy-Schwarz inequality applied to the right hand side of (25) then yields (23). To obtain (24), we take $\lambda_i = \alpha_i s_n$ and add to (23) the analogous inequality with all S_i's replaced by their negatives (which also form a demimartingale).

The next theorem extends Doob's upcrossing inequality to demisubmartingales; our proof is a modified version of the standard stopping time argument used for submartingales (see, e.g., [4, Chap. 9]). Given $S_1, S_2, \ldots, S_n$, and $a < b$ we define a sequence of stopping times $J_0 = 0$, $J_1, J_2, \ldots$ as follows (for $k = 1, 2, \ldots$):

$$J_{2k-1} = \begin{cases} n+1, \text{ if } \{j\colon J_{2k-2} < j \leqq n \text{ and } S_j \leqq a\} \text{ is empty} \\ \min\{j\colon J_{2k-2} < j \leqq n \text{ and } S_j \leqq a\}, \quad \text{otherwise,} \end{cases} \tag{26a}$$

$$J_{2k} = \begin{cases} n+1, \text{ if } \{j\colon J_{2k-1} < j \leqq n \text{ and } S_j \geqq b\} \text{ is empty} \\ \min\{j\colon J_{2k-1} < j \leqq n \text{ and } S_j \geqq b\}, \text{ otherwise.} \end{cases} \tag{26b}$$

The number of complete upcrossings of the interval $[a, b]$ by $S_1, \ldots, S_n$ is denoted $U_{a,b}$ and is defined by

$$U_{a,b} = \max\{k\colon J_{2k} < n+1\}. \tag{27}$$

Theorem 7. *If $S_1, S_2, \ldots, S_n$ is a demisubmartingale; then for any $a < b$,*

$$E(U_{a,b}) \leqq \frac{E((S_n - a)^+) - E((S_1 - a)^+)}{b - a}. \tag{28}$$

Proof. We define for $j = 1, \ldots, n-1$,

$$\varepsilon_j = \begin{cases} 1, \text{ if for some } k = 1, 2, \ldots, \quad J_{2k-2} \leqq j < J_{2k-1} \\ 0, \text{ if for some } k = 1, 2, \ldots, \quad J_{2k-1} \leqq j < J_{2k}. \end{cases}$$

so that $1 - \varepsilon_j$ is the indicator function of the event that the "time interval" $[j, j+1)$ is part of an upcrossing (possibly incomplete); an equivalent definition is

$$\varepsilon_j = \begin{cases} 1, \text{ if either } S_i > a \text{ for } i = 1, \ldots, j \text{ or else} \\ \quad \text{for some } i = 1, \ldots, j,\ S_i \geqq b \text{ and } S_k > a \text{ for } k = i+1, \ldots, j \\ 0, \text{ otherwise.} \end{cases} \tag{29}$$

We also define Λ as

$$\Lambda = \{\tilde{J} \equiv J_{2U_{a,b}+1} < n\};$$

Λ is the event that the sequence ends with an incomplete upcrossing. Now

$$(S_n - a)^+ - (S_1 - a)^+ = \sum_{j=1}^{n-1} [(S_{j+1} - a)^+ - (S_j - a)^+] = H_u + H_d, \tag{30}$$

with H_d given by

$$H_d=\sum_{j=1}^{n-1}\varepsilon_j[(S_{j+1}-a)^+-(S_j-a)^+]\geqq\sum_{j=1}^{n-1}\varepsilon_j(S_{j+1}-S_j), \tag{31}$$

where the last inequality is a consequence of the facts that $(S_{j+1}-a)^+\geqq(S_{j+1}-a)$ while $\varepsilon_j(S_j-a)^+=\varepsilon_j(S_j-a)$ since $\varepsilon_j=1$ implies $S_j>a$ (see (29)), and with H_u given by

$$\begin{aligned} H_u&=\sum_{j=1}^{n-1}(1-\varepsilon_j)[(S_{j+1}-a)^+-(S_j-a)^+]\\ &=\sum_{k=1}^{U_{a,b}}\sum_{j=J_{2k-1}}^{J_{2k}-1}[(S_{j+1}-a)^+-(S_j-a)^+]+\sum_{j=\bar{J}}^{n-1}[(S_{j+1}-a)^+-(S_j-a)^+]\\ &=\sum_{k=1}^{U_{a,b}}[(S_{J_{2k}}-a)^+-(S_{J_{2k-1}}-a)^+]+[(S_n-a)^+-(S_{\bar{J}}-a)^+]\,1_A\\ &=\sum_{k=1}^{U_{a,b}}(S_{J_{2k}}-a)^++(S_n-a)^+\,1_A\geqq(b-a)\,U_{a,b}. \end{aligned} \tag{32}$$

Combining (30), (31), and (32) and taking expectations, we obtain

$$E((S_n-a)^+-(S_1-a)^+)\geqq(b-a)\,E(U_{a,b})+\sum_{j=1}^{n-1}E((S_{j+1}-S_j)\,\varepsilon_j). \tag{33}$$

The upcrossing inequality (28) is an immediate consequence of

$$E((S_{j+1}-S_j)\,\varepsilon_j)\geqq 0 \qquad \text{for } j=1,\ldots,n-1, \tag{34}$$

which in turn follows from the definition of demisubmartingale and the fact that ε_j is a non-negative nondecreasing function of $S_1,\ldots,S_j$; the nondecreasing nature of ε_j follows easily from (29). The proof is now complete.

The following convergence theorem is an immediate consequence of Theorem 7 as in the martingale case (see e.g. [4, Chap. 9]).

Corollary 8. *If* $S_1,S_2,\ldots$ *is a demisubmartingale and* $\sup_n E(|S_n|)<\infty$, *then* S_n *converges a.e. to a finite limit.*

3. Inequalities and Weak Convergence for 2-Parameter Arrays

Throughout this section we deal with a 2-parameter array $\{X_{\vec{j}}:\ \vec{j}=(j_1,j_2)\in\mathbb{Z}^2\}$ of mean zero, finite variance, associated random variables, and the related partial sums

$$S_{\vec{j}}=S_{(j_1,j_2)}=\sum_{i=1}^{j_1}\sum_{k=1}^{j_2}X_{(i,k)}.$$

We also define for $m,n\geqq 1$,

$$S^*_{(m,n)}=\operatorname{Max}\{S_{\vec{j}}:\ 1\leqq j_1\leqq m,\ 1\leqq j_2\leqq n\}.$$

In order to strengthen Theorem 1 to obtain weak convergence for $d=2$, we need a maximal inequality which controls the tail of $S^*_{(m,n)}$ in terms of the tail of $S_{(m,n)}$ as was done for $d=1$ by (18) or (23). Our $d=2$ result will in fact be based on (23) and the key step is the following lemma; our approach is modelled after previous work on multiparameter martingale inequalities [3].

Lemma 9. *For fixed m, let*

$$S_j = \operatorname{Max}\{S_{(k,j)}: 1 \leqq k \leqq m\}; \tag{35}$$

then $S_1, S_2, \ldots$ is a demisubmartingale.

Proof. Suppose $Y=f(S_1,\ldots,S_j)$ where f is non-negative and nondecreasing, then by the definition (35)

$$Y=\tilde{f}(\{X_{(i,j)}\}), \quad \tilde{f} \text{ non-negative and nondecreasing;} \tag{36}$$

we must show that

$$E((S_{j+1}-S_j)\,Y) \geqq 0. \tag{37}$$

Let us define K_j by

$$K_j = \min\{k: S_{(k,j)} = \operatorname{Max}(S_{(1,j)},\ldots,S_{(m,j)})\};$$

then

$$\begin{aligned} S_{j+1}-S_j &= S_{j+1}-S_{(K_j,j)} \geqq S_{(K_j,j+1)}-S_{(K_j,j)} \\ &= \sum_{k=1}^{K_j} X_{(k,j+1)} = \sum_{k=1}^{m} X_{(k,j+1)}\, 1_{\{K_j \geqq k\}}, \end{aligned}$$

and so since $Y \geqq 0$ we have

$$\begin{aligned} E((S_{j+1}-S_j)\,Y) &\geqq \sum_{k=1}^{m} E(X_{(k,j+1)}\, 1_{\{K_j \geqq k\}}\, Y) \\ &= \sum_{k=1}^{m} \operatorname{Cov}(X_{(k,j+1)}, 1_{\{K_j \geqq k\}}\, Y) \end{aligned} \tag{38}$$

where we have used the fact that the $X_{(k,j)}$'s have zero mean. It is a simple fact that for any sequence $s_1,\ldots,s_m$, and any $k=1,\ldots,m$, the function,

$$1_{\{\min\{k': s_{k'} = s^*_m\} \geqq k\}}$$

is a nondecreasing function of $\{s_1, s_2-s_1, \ldots, s_m-s_{m-1}\}$; thus $1_{\{K_j \geqq k\}}$ and consequently $1_{\{K_j \geqq k\}}\,Y$ are nondecreasing functions of the $X_{(i,j)}$'s and so the right hand side of (38) is non-negative by the association of the $X_{(i,j)}$'s which yields (37) and completes the proof.

The following theorem generalizes (23)–(24) to $d=2$. We define

$$s^2_{m,n} \equiv E(S^2_{(m,n)}).$$

Theorem 10. *The following inequalities apply to the partial sums of a two-parameter array of mean zero associated random variables: For $\lambda_2 > \lambda_1 \geqq 0$,*

$$P(S^*_{(m,n)} \geqq \lambda_2) \leqq 3^{3/2}\, 2^{-1} (s^2_{m,n}/(\lambda_2-\lambda_1)^2)^{3/4} (P(S_{(m,n)} \geqq \lambda_1))^{1/4}; \tag{39}$$

for $0 \leqq \alpha_1 < \alpha_2$,

$$\begin{aligned} &P(\operatorname{Max}\{|S_{(k,j)}|: 1 \leqq k \leqq m, 1 \leqq j \leqq n\} \geqq \alpha_2 s_{m,n}) \\ &\quad \leqq 3^{3/2} \cdot 2^{-1/4} (\alpha_2 - \alpha_1)^{-3/2} (P(|S_{(m,n)}| \geqq \alpha_1 s_{m,n}))^{1/4}. \end{aligned} \tag{40}$$

Proof. It follows immediately from Lemma 9, inequality (23) of Corollary 6, and the fact that with S_j as defined in (35), $S^*_{(m,n)} = S^*_n$, that for $0 \leqq \lambda < \lambda_2$,

$$P(S^*_{(m,n)} \geqq \lambda_2) \leqq (E(S_n^2)/(\lambda_2 - \lambda)^2)^{1/2} P(S_n \geqq \lambda)^{1/2}. \tag{41}$$

If we let $T_k = S_{(k,n)}$, then $T_k = X_1 + \ldots + X_k$ *where* $X_i = X_{(i,1)} + \ldots + X_{(i,n)}$ so that the X_i's are mean zero associated. Since $S_n = T^*_m$ and $S_{(m,n)} = T_m$, it follows from (17) of Corollary 5 that

$$E(S_n^2) = E(T_m^{*2}) \leqq E(S^2_{(m,n)}) \tag{42}$$

and from (23) of Corollary 6 that for $0 \leqq \lambda_1 < \lambda$

$$P(S_n \geqq \lambda) \leqq (E(S^2_{(m,n)})/(\lambda - \lambda_1)^2)^{1/2} (P(S_{(m,n)} \geqq \lambda_1))^{1/2}. \tag{43}$$

Combining (41), (42), and (43), we obtain

$$P(S^*_{(m,n)} \geqq \lambda_2) \leqq \frac{[E(S^2_{(m,n)})]^{3/4}}{(\lambda_2 - \lambda)(\lambda - \lambda_1)^{1/2}} (P(S_{(m,n)} \geqq \lambda_1))^{1/4}; \tag{44}$$

choosing $\lambda = (2\lambda_1 + \lambda_2)/3$ to minimize the righthand side of (44) leads to (39). To obtain (40), we choose $\lambda_i = \alpha_i s_{m,n}$, add to (39) the analogous inequality obtained when all the $X_{(i,j)}$'s are replaced by their negatives, and use the fact that for $u, v \geqq 0$

$$u^{1/4} + v^{1/4} \leqq 2^{3/4} (u + v)^{1/4}.$$

The next theorem gives two-parameter weak convergence as an immediate consequence of Theorems 1 and 10. We choose to consider weak convergence in the sense of [14] for the sake of convenience; Theorem 10 is sufficiently strong to yield other types of weak convergence as well (see e.g. [13]).

Theorem 11. *Let* $W_n(t_1, t_2)$, $W(t_1, t_2)$ *be as in Theorem* 1 *with* $d = 2$ *and with* (t_1, t_2) *restricted to lie in the unit cube,* $[0,1]^2$, *and let* $\bar{P}_n, \bar{P}$ *be the corresponding probability distributions (on the space* (D_2, A) *with* $\bar{P}(C_2) = 1$ *as defined in* [14]); *then* $\bar{P}_n$ *converges weakly in the U-topology (see* [14]) *to* $\bar{P}$.

Proof. By Theorem 1 and [14, Theorem 2], it suffices to show that

$$\forall \varepsilon > 0, \quad \lim_{\delta \to 0} \limsup_{n \to \infty} P(M_n^\delta > \varepsilon) = 0, \tag{45}$$

where

$$M_n^\delta = \sup\{|W_n(\vec{t}) - W_n(\vec{s})| : \vec{s}, \vec{t} \in [0,1]^2, |\vec{s} - \vec{t}| < \delta\} \tag{46}$$

and $|\vec{s} - \vec{t}| = \max(|s_1 - t_1|, |s_2 - t_2|)$. Now simple estimates show that to obtain (45) it suffices to have

$$\forall \varepsilon > 0, \quad \lim_{\delta \to 0} \limsup_{n \to \infty} \delta^{-2} P(\overline{M}_n^\delta \geqq \varepsilon) = 0 \tag{47}$$

where

$$\begin{aligned}\overline{M}_n^{\delta}&=\sup\{|W_n(\vec{t})|:\vec{t}\in[0,\delta]^2\}\\&=n^{-1}\operatorname{Max}\{|S_{(k,j)}|:1\leqq k\leqq n\,\delta,\ 1\leqq j\leqq n\,\delta\}.\end{aligned}\tag{48}$$

Now by Theorem 10, the fact that $E(S^2_{([n\delta],[n\delta])})/n^2\to\sigma^2\,\delta^2$ (see [10]), and Theorem 1 we have that

$$\begin{aligned}\limsup_{n\to\infty}P(\overline{M}_n^{\delta}\geqq\varepsilon)&\leqq C\,\sigma^{3/2}\,\delta^{3/2}\,\varepsilon^{-3/2}\lim_{n\to\infty}[P(S_{([n\delta],[n\delta])}/n\geqq\varepsilon/2)]^{1/4}\\&=C(\sigma^2\,\delta^2/\varepsilon^2)^{3/4}[\lim_{n\to\infty}P(W_n(\delta,\delta)\geqq\varepsilon/2)]^{1/4}\\&=C(\sigma^2\,\delta^2/\varepsilon^2)^{3/4}\left(\int_{\varepsilon/2}^{\infty}(2\pi\,\sigma^2\,\delta^2)^{-1/2}\exp(-u^2/2\,\sigma^2\,\delta^2)\,du\right)^{1/4}\end{aligned}\tag{49}$$

where C is a universal constant. Thus for fixed σ and ε, we have for some constants B and b

$$\lim_{\delta\to0}\limsup_{n\to\infty}\delta^{-2}P(\overline{M}_n^{\delta}\geqq\varepsilon)\leqq\lim_{\delta\to0}B\,\delta^{-1/2}\left(\int_{b\,\delta^{-1}}^{\infty}(2\pi)^{-1/2}\exp(-u^2/2)\,du\right)^{1/4}=0\tag{50}$$

which yields (47) and completes the proof.

As to the question of weak convergence for $d>2$, we conjecture that versions of Doob's inequality for multiparameter martingales (related to the inequalities of [3, Thm. 1]) and/or of Wichura's inequality [14, Eq. (2a)] for sums of independent variables apply to sums of mean zero associated variables; either of these inequalities coupled with Theorem 10 would imply weak convergence.

Acknowledgment. The authors would like to thank J.W. Pitman for discussing his work concerning the inequality (15) prior to publication.

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Received November 29, 1980; in slightly revised form October 22, 1981