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On a Problem of Necessary and Sufficient Conditions in the Functional Central Limit Theorem for Local Martingales

R.Sh. Liptser[1] and A.N. Shiryaev[2]

[1] Institute of Control Science, Profsoyuznaya 65, Moscow, USSR
[2] Steclov Mathematical Institute, Vavilova 42, Moscow, USSR

1. It was shown in [8] that under the condition (**sup B**) (see Lemma 1 in Sect. 3.) the weak convergence of a sequence of semimartingales, in particular – local martingales, to a continuous Gaussian martingale holds if and only if the conditions (**A**) and (**C**) hold (Lemma 1).

The aim of the present paper is to show by using the result of [8] under the assumption of uniform integrability of jumps of local martingales (condition (ρ) in Theorem 1), that the same convergence for local martingales holds if and only if a convergence in probability of corresponding quadratic variations (condition (γ) in Theorem 1) takes place.

It should be noted that some particular cases of this result may be found in [1, 3, 11, 12]. For example, Theorem 1 is the direct generalization of a result in [3] (see Corollary 2 in Sect. 6).

The condition (ρ) is not used in Theorem 2 (Sect. 9.), but we give an example (Sect. 8.) which shows that the condition (ρ) cannot be essentially weakened.

2. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a complete probability space, $F^n = (\mathscr{F}_t^n)_{t \geqq 0}$, $n \geqq 1$ and $F = (\mathscr{F}_t)_{t \geqq 0}$ be non-decreasing right continuous family of σ-algebras $\mathscr{F}_t^n \subseteq \mathscr{F}$, $\mathscr{F}_t \subseteq \mathscr{F}$, $n \geqq 1$, $t \geqq 0$ such that the σ-algebras $\mathscr{F}_0^n$ and $\mathscr{F}_0$ contain the P zero sets from $\mathscr{F}$.

Let $M^n = (M_t^n, \mathscr{F}_t^n)_{t \geqq 0}$, $n \geqq 1$ be local martingales with $M_0^n = 0$, $n \geqq 1$ and trajectories in the measurable space $(D, \mathscr{D})$ with Skorokhod topology [2, 13], and $M = (M_t, \mathscr{F}_t)_{t \geqq 0}$ be continuous Gaussian martingale with $M_0 = 0$.

Every local martingale M^n admits a unique decomposition: $M^n = M^{nc} + M^{nd}$, where $M^{nc} = (M_t^{nc}, \mathscr{F}_t^n)_{t \geqq 0}$ is a continuous local martingale and $M^{nd} = (M_t^{nd}, \mathscr{F}_t^n)_{t \geqq 0}$ is a purely discontinuous local martingale. It is well known that M^{nd} admits the representation in the form of the stochastic integral

$$M_t^{nd} = \int_0^t \int_{R_0} x \, d(\mu^n - \nu^n) \tag{1}$$

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by "martingale" measure $(\mu^n - \nu^n)$, where μ^n is the integer-valued random measure of jumps M^n, ν^n is its dual predictable projection (or compensator) with respect to F^n, [4, 6], and $R_0 = R \backslash \{0\}$.

The local martingal M^n has the canonical semimartingale representation:

$$M_t^n = B_t^n + M_t^{nc} + \int_0^t \int_{|x|>1} x\,d\mu^n + \int_0^t \int_{|x|\leqq 1} x\,d(\mu^n - \nu^n) \tag{2}$$

Since (see [6]) for each $t>0$

$$\int_0^t \int_{R_0} x^2(1+|x|)^{-1}\,d\nu^n < \infty \qquad (P\text{-a.s.})$$

then it follows from (1) and (2) that the predictable process $B^n = (B_t^n, \mathscr{F}_t^n)_{t\geqq 0}$ is defined by the formula:

$$B_t^n = -\int_0^t \int_{|x|>1} x\,d\nu^n. \tag{3}$$

In addition, the process B^n admits the decomposition: $B^n = B^{nc} + B^{nd}$ with $B_t^{nd} = -\sum_{0<s\leqq t} \int_{|x|>1} x\nu^n(\{s\}, dx)$. It follows from (3) that the variation $V_T(B^{nd})$ of the function $(B_t^{nd})_{t\geqq 0}$ on the interval $[0, T]$ is defined by the formula:

$$V_T(B^{nd}) = \sum_{0<t\leqq T} \left| \int_{|x|>1} x\nu^n(\{t\}, dx)\right| \tag{4}$$

and the variation of the function $B^n = (B_t^n)_{t\geqq 0}$ has the property

$$V_t(B^n) \leqq \int_0^t \int_{|x|>1} |x|\,d\nu^n.$$

Let $[M^n, M^n] = ([M^n, M^n]_t, \mathscr{F}_t^n)$ be the quadratic variation of the local martingale M^n:

$$[M^n, M^n]_t = \langle M^{nc}\rangle_t + \sum_{0<s\leqq t} (\Delta M_s^n)^2,$$

where $\langle M^{nc}\rangle = (\langle M^{nc}\rangle_t, \mathscr{F}_t^n)_{t\geqq 0}$ is the quadratic characteristic of M^{nc}, i.e. $\langle M^{nc}\rangle$ is a non-decreasing predictable process such that the process $(M^{nc})^2 - \langle M^{nc}\rangle$ is a local martingale, and $\Delta M_s^n = M_s^n - M_{s-}^n$, $s>0$. For a continuous Gaussian martingale M, we have

$$[M, M]_t = \langle M\rangle_t = \mathbf{E}M_t^2.$$

We shall denote $A(t) = \mathbf{E}M_t^2$. We shall denote by the symbol: $M^n \xrightarrow{\mathscr{L}} M$ the fact that the distributions of local martingales M^n, $n \geqq 1$ converge weakly to the distribution of M. The symbol $\xrightarrow{P}$ denotes the convergence in probability.

3. The following result is a particular case of corresponding statements in [8] and [7].

Lemma 1. *a) Let M^n, $n \geqq 1$ be local martingales with $M_0^n = 0$, $n \geqq 1$ and trajectories in $(D, \mathscr{D})$, M be a continuous Gaussian martingale with $M_0 = 0$ and $A(t) = \mathbf{E}M_t^2$.*

Under the condition

$$(\mathbf{sup\,B}):\ \sup_{0 \leqq s \leqq t} \left| \int_0^s \int_{|x|>1} x\, d\nu^n \right| \xrightarrow{p} 0, \qquad t>0,$$

the convergence $M^n \xrightarrow{\mathscr{L}} M$ holds if and only if the following conditions take place: for $t>0$ and $\varepsilon \in (0,1]$

$$(\mathbf{A}):\ \int_0^t \int_{|x|>\varepsilon} d\nu^n \xrightarrow{p} 0,$$

$$(\mathbf{C}):\ \langle M^{nc} \rangle_t + \int_0^t \int_{|x| \leqq \varepsilon} x^2\, d\nu^n - \sum_{0<s \leqq t} \Big(\int_{|x| \leqq \varepsilon} x \nu^n(\{s\}, dx) \Big)^2 \xrightarrow{p} A(t).$$

b) Condition (**A**) *is equivalent to the condition*

$$(\mathbf{A}^*):\ \sup_{0<s \leqq t} |\Delta M_s^n| \xrightarrow{p} 0, \qquad t>0.$$

c) Under condition (**A**), *condition* (**C**) *is equivalent to the condition*

$$(\mathbf{C}^*):\ \langle M^{nc} \rangle_t + \sum_{0<s \leqq t} \Big(\Delta M_s^n - \int_{|x| \leqq 1} x \nu^n(\{s\}, dx) \Big)^2 \xrightarrow{p} A(t), \qquad t>0.$$

4. The main result of the present paper is the following

Theorem 1. *Let M^n, $n \geqq 1$ be local martingales with $M_0^n = 0$, $n \geqq 1$ and trajectories in $(D, \mathscr{D})$, M be a continuous Gaussian martingale with $M_0 = 0$ and $A(t) = \mathbf{E}M_t^2$.*

Let the following condition be satisfied:

(ρ): *for each $t>0$ the family of random variables $\big(\sup_{0<s \leqq t} |\Delta M_s^n|\big)$, $n \geqq 1$ is uniformly integrable.*

Then the convergence $M^n \xrightarrow{\mathscr{L}} M$ holds if and only if the following condition takes place:

$$(\gamma):\ [M^n, M^n]_t \xrightarrow{p} A(t), \qquad t>0.$$

5. For the proof of this theorem, we need the following result.

Lemma 2. *Let $X^n = (X_t^n, \mathscr{F}_t^n)_{t \geqq 0}$, $Y^n = (Y_t^n, \mathscr{F}_t^n)_{t \geqq 0}$ be random processes with $X_0^n = 0$, $Y_0^n = 0$, $n \geqq 1$ and trajectories in $(D, \mathscr{D})$.*

Assume X^n, $n \geqq 1$ are non-negative processes and Y^n, $n \geqq 1$ are non-decreasing processes such that

$$EX_\tau^n \leqq EY_\tau^n \tag{5}$$

for any finite stopping time τ with respect to F^n.

For any finite stopping time T with respect to $\bigcap_{n \geqq 1} F^n$, *let the family of random variables* $(\sup_{0<s\leqq T} \Delta Y_s^n)$, $n \geqq 1$ *be uniformly integrable.*

Then by $n \to \infty$

$$Y_T^n \xrightarrow{P} 0 \;\Rightarrow\; \sup_{0<t\leqq T} X_t^n \xrightarrow{P} 0. \tag{6}$$

Proof. Under the assumption (5), we have the Lenglart-Rebolledo inequality (see [10, 11]) stating that for arbitrary a, $b>0$:

$$\mathbf{P}(\sup_{t\leqq T} X_t^n \geqq a) \leqq 1/a\,\mathbf{E}(Y_T^n \wedge (b + \sup_{t\leqq T} \Delta Y_t^n)) + \mathbf{P}(Y_T^n \geqq b).$$

If $Y_T^n \xrightarrow{P} 0$, then

$$\overline{\lim_n}\, \mathbf{P}(\sup_{t\leqq T} X_t^n \geqq a) \leqq b/a + \overline{\lim_n}\, E \sup_{t\leqq T} \Delta Y_t^n.$$

Since $\sup_{t\leqq T} \Delta Y_t^n \leqq Y_T^n$, we have $\sup_{t\leqq T} \Delta Y_t^n \xrightarrow{P} 0$. By virtue of uniform integrability of the family $(\sup_{0<t\leqq T} \Delta Y_t^n)$, $n \geqq 1$

$$\lim_n \mathbf{E} \sup_{0<t\leqq T} \Delta Y_t^n = 0.$$

Therefore, $\overline{\lim_n}\, \mathbf{P}(\sup_{t\leqq T} X_t^n \geqq a) \leqq b/a$.

The required assertion (6) follows from this inequality by virtue of the arbitrariness of the constant $b>0$.

6. *Proof* of Theorem 1. The sufficiency of the conditions (ρ) and (γ) considered above will follow from Lemma 1 and following implications:

$$\begin{aligned} &(\rho,\gamma) \overset{\{1\}}{\Rightarrow} (\rho,\gamma,\mathbf{A}) \overset{\{2\}}{\Rightarrow} (\rho,\gamma,\mathbf{A},\mathbf{sup\,B}) \\ &\overset{\{3\}}{\Rightarrow} (\rho,\gamma,\mathbf{A},\mathbf{sup\,B},\ \mathbf{C}) \overset{\{4\}}{\Rightarrow} (\rho, M^n \xrightarrow{\mathscr{L}} M) \quad \text{(where Lemma 1 enters in } \{4\}) \end{aligned} \tag{7}$$

and the necessity of the condition (γ) will follow from the implications:

$$\begin{aligned} &(\rho, M^n \xrightarrow{\mathscr{L}} M) \overset{\{5\}}{\Rightarrow} (\rho, M^n \xrightarrow{\mathscr{L}} M, \mathbf{A}) \overset{\{6\}}{\Rightarrow} (\rho, M^n \xrightarrow{\mathscr{L}} M, \mathbf{A}, \mathbf{sup\,B}) \\ &\overset{\{7\}}{\Rightarrow} (\rho, \mathbf{A}, \mathbf{sup\,B}, \mathbf{C}) \overset{\{8\}}{\Rightarrow} (\rho, \mathbf{A}, \mathbf{sup\,B}, \gamma) \overset{\{9\}}{\Rightarrow} (\rho,\gamma). \end{aligned} \tag{8}$$

To prove (7) and (8), we shall show that

$$(\gamma) \Rightarrow (\mathbf{A}), \tag{9}$$

$$(\mathbf{A},\rho) \Rightarrow (\mathbf{sup\,B}) \tag{10}$$

and under the conditions (**A**) and (ρ)

$$(\gamma) \Leftrightarrow (\mathbf{C}). \tag{11}$$

The validity of (9) follows from the following fact. Since the functions $[M^n, M^n]_t$ and $A(t)$ are non-decreasing and the function $A(t)$ is continuous, then (see Lemma 1 in [9])

$$(\gamma) \Rightarrow \sup_{s \leqq t} |[M^n, M^n]_s - A(s)| \xrightarrow{P} 0, \quad t > 0.$$

Hence, $(\gamma) \Rightarrow \sup_{s \leqq t} (\Delta M_s^n)^2 \xrightarrow{P} 0$, i.e. $(\gamma) \Rightarrow$ (**A***): Now, the implication (9) follows from the equivalence of the conditions (**A**) and (**A***) (see Lemma 1, b)).

In order to prove (10) we shall introduce the following notations:

$$X_t^n = \int_0^t \int_{|x|>1} d\mu^n, \quad \tilde{X}_t^n = \int_0^t \int_{|x|>1} d\nu^n, \quad Z_t^n = \int_0^t \int_{|x|>1} |x| \, d\mu^n.$$

Under condition (**A**) $\tilde{X}_t^n \xrightarrow{P} 0$, $t>0$. Therefore, by Lemma 2 $X_t^n \xrightarrow{P} 0$, $t>0$. Since for any $\delta > 0 (Z_t^n > \delta) \subseteq (X_t^n \geqq 1)$,

$$(\mathbf{A}) \Rightarrow Z_t^n \xrightarrow{P} 0, \quad t > 0. \tag{12}$$

The processes $Z^n = (Z_t^n, \mathscr{F}_t^n)_{t \geqq 0}$ and $V(B^n) = (V_t(B^n), \mathscr{F}_t^n)_{t \geqq 0}$ satisfy the conditions of Lemma 2:

$$\mathbf{E} V_\tau(B^n) \leqq \mathbf{E} \int_0^\tau \int_{|x|>1} |x| \, d\nu^n = \mathbf{E} Z_\tau^n$$

for any finite stopping time τ with respect to F^n and the family $(\sup_{0<s\leqq t} \Delta Z_s^n)$, $n \geqq 1$ is uniformly integrable for each $t > 0$ because $\Delta Z_s^n \leqq |\Delta M_s^n|$ and the condition ($\boldsymbol{\rho}$) takes place. Therefore from (12) and Lemma 2, it follows that

$$(\mathbf{A}, \boldsymbol{\rho}) \Rightarrow V_t(B^n) \xrightarrow{P} 0, \quad t > 0.$$

Now, the inequality $\sup_{0 \leqq s \leqq t} |B_s^n| \leqq V_t(B^n)$ implies the validity of implication (10).

To prove (11) we shall use the fact that under the condition (**A**) the conditions (**C**) and (**C***) are equivalent. Since

$$\langle M^{nc} \rangle_t + \sum_{0<s\leqq t} (\Delta M_s^n - \int_{|x|\leqq 1} x \nu^n(\{s\}, dx))^2 = [M^n, M^n]_t + J_t^n,$$

where

$$J_t^n = \sum_{0<s\leqq t} (\int_{|x|\leqq 1} x \nu^n(\{s\}, dx))^2 - 2 \sum_{0<s\leqq t} \Delta M_s^n \int_{|x|\leqq 1} x \nu^n(\{s\}, dx),$$

to proove (11) it is sufficient to show that

$$(\mathbf{A}, \boldsymbol{\rho}) \Rightarrow J_t^n \xrightarrow{P} 0, \quad t > 0. \tag{13}$$

The processes M^n, $n \geqq 1$ are local martingales. Because of that

$$\int_{R_0} x \nu^n(\{s\}, dx) = 0 \ (P\text{-a.s.}), \quad s > 0$$

(see [4, 6]). By virtue of this equality and (4) we have

$$
\begin{aligned}
|J_t^n| &\leqq \sum_{0<s\leqq t} (\Delta V_s(B^{nd}))^2 + 2 \sup_{0<s\leqq t} |\Delta M_s^n|\, V_t(B^{nd}) \\
&\leqq V_t^2(B^{nd}) + 2 \sup_{0<s\leqq t} |\Delta M_s^n|\, V_t(B^{nd}).
\end{aligned}
\tag{14}
$$

By the implication proved above $(\mathbf{A}, \rho) \Rightarrow V_t(B^{nd}) \xrightarrow{P} 0$, $t>0$ and by virtue of the inequality $V_t(B^{nd}) \leqq V_t(B^n)$ and the equivalence of conditions (**A**) and (**A***), it follows from (14) that $|J_t^n| \xrightarrow{P} 0$, $t>0$.

Thus, the implication (11) holds.

Now, we can prove the implications (7) and (8).

The implications {1}, {2} and {3} follow from (9), (10) and (11) respectively. The implication {4} follows from Lemma 1. In [8], it was shown (Lemma 1) that

$$M^n \xrightarrow{\mathscr{L}} M \Rightarrow (\mathbf{A}).$$

This fact implies the implication {5}. The implication {6} follows from (10), and the implication {7} follows from Lemma 1. The implication {8} follows from (11), and implication {9} is obvious.

7. Corollary 1 ([7, 10]). *Let M^n, $n \geqq 1$ be continuous local martingales, $M_0^n = 0$, $n \geqq 1$ and M is a continuous Gaussian martingale, $M_0 = 0$.*

Then

$$M^n \xrightarrow{\mathscr{L}} M \Leftrightarrow \langle M^n \rangle_t \xrightarrow{P} \langle M \rangle_t, \qquad t>0.$$

Corollary 2 ([3, 12]). *Let for every $n \geqq 1$ the sequence $\xi^n = (\xi_{kn}, \mathscr{F}_k^n)$, $1 \leqq k \leqq n$ form a martingale-difference*, i.e. $\mathbf{E}|\xi_{kn}| < \infty$, $\mathbf{E}(\xi_{kn} | \mathscr{F}_{k-1}^n) = 0$, $\mathscr{F}_1^n \subseteq \mathscr{F}_2^n \subseteq \ldots \subseteq \mathscr{F}_n^n$ and $\mathscr{F}_0^n = (\phi, \Omega)$.

Put $\xi_{0n} = 0$ and $M_t^n = \sum_{k=0}^{[nt]} \xi_{kn}$, $0 \leqq t \leqq 1$. If the family of the random variables $(\max_{k \leqq n} |\xi_{kn}|)$, $n \geqq 1$ is uniformly integrable then (W be a Wiener process)

$$M^n \xrightarrow{\mathscr{L}} W \Leftrightarrow \sum_{k=0}^{[nt]} \xi_{kn}^2 \xrightarrow{P} t, \qquad 0 \leqq t \leqq 1.$$

8. It is well known that the uniform integrability condition (ρ) is equivalent to the simultaneous realization of the two following conditions:

(ρ_1): for every $t>0$ the family of the random variables $(\sup_{0<s\leqq t} |\Delta M_s^n|)$, $n \geqq 1$ is uniformly bounded, i.e.

$$\sup_n \mathbf{E} \sup_{0<s\leqq t} |\Delta M_s^n| < \infty; \tag{15}$$

(ρ_2): for every $t>0$ the family of the random variables $(\sup_{0<s\leqq t} |\Delta M_s^n|)$, $n \geqq 1$ is uniformly continuous, i.e.

$$\sup_n \mathbf{E} \sup_{0<s\leqq t} |\Delta M_s^n|\, I(A) \to 0, \qquad A \in \mathscr{F}$$

by $\mathbf{P}(A) \to 0$.

It was shown in [5] that under the condition $(\boldsymbol{\rho}_1)$ the implication

$$M^n \xrightarrow{\mathscr{L}} M \Rightarrow (\gamma)$$

takes place.

Now, we shall give an example showing that under the condition $(\boldsymbol{\rho}_1)$ the inverse implication $(\gamma) \Rightarrow M^n \xrightarrow{\mathscr{L}} M$, generally speaking, is not true. Thus in Theorem 1 the condition $(\boldsymbol{\rho})$ cannot be weakened to the condition $(\boldsymbol{\rho}_1)$.

Example. Let (ξ_{kn}), $1 \leqq k \leqq n$, $n \geqq 2$ be an array of independent random variables with $\mathbf{P}(\xi_{kn}=n)=1/n^2$, $\mathbf{P}(\xi_{kn}=-1/n(1-n^{-2}))=1-n^{-2}$.

Put $\mathscr{F}_k^n=\sigma(\xi_{jn}, j \leqq k)$ and $\mathscr{F}_0^n=(\phi, \Omega)$ and $\mathscr{F}_t^n=\mathscr{F}_{[nt]}^n$, $0 \leqq t \leqq 1$. Let $M_t^n = \sum_{k=0}^{[nt]} \xi_{kn}$ with $\xi_{0n}=0$. It is clear that $M^n=(M_t^n, \mathscr{F}_t^n)$, $0 \leqq t \leqq 1$ is a martingale for every $n \geqq 2$.

Since $\sup_{s \leqq 1} |\Delta M_s^n| \leqq \sum_{k=1}^n |\xi_{kn}|$ and $\sum_{k=1}^n \mathbf{E}\,|\xi_{kn}|=2$, then in the case under consideration the condition $(\boldsymbol{\rho}_1)$ holds. A simple calculation shows that $[M^n, M^n]_1 = \sum_{k=1}^n \xi_{kn}^2 \xrightarrow{P} 0$. Thus, if Theorem 1 holds then $M_t^n \xrightarrow{P} 0$ for every $t \in [0,1]$. However, it is not difficult to show that $M_t^n \xrightarrow{P} -t$.

It should be noted that in this example

$$B_t^n = -\sum_{k=1}^{[nt]} \mathbf{E}\,\xi_{kn} I(|\xi_{kn}|>1) = -[nt]/n \to -t$$

and consequently the condition $(\mathbf{sup\,B})$ does not hold. On the contrary, the conditions $(\mathbf{A})$ and $(\mathbf{C})$ take place. Thus this example shows that the condition $(\mathbf{sup\,B})$ in Lemma 1 is essential.

9. In the proof of Theorem 1 it was shown that $(\rho, \gamma) \Rightarrow V_t(B^n) \xrightarrow{P} 0$, $t>0$. Consequently, we have the following implication:

$$(\rho, \gamma) \Rightarrow \sup_{0 \leqq s \leqq t} |B_s^{nc}| + V_t(B^{nd}) \xrightarrow{P} 0, \qquad t>0.$$

Let us consider the condition

$$(\mathbf{V}): \sup_{0 \leqq s \leqq t} |B_s^{nc}| + V_t(B^{nd}) \xrightarrow{P} 0, \qquad t>0.$$

It is clear that $(\mathbf{V}) \Rightarrow (\mathbf{sup\,B})$ and under conditions $(\mathbf{A})$ and $(\mathbf{V})$ the equivalence (11) holds.

Hence,

$$(\mathbf{V}, \gamma) \Rightarrow (\mathbf{A}, \mathbf{sup\,B}, \mathbf{C}) \Rightarrow M^n \xrightarrow{\mathscr{L}} M.$$

On the other hand,

$$(\mathbf{V}, M^n \xrightarrow{\mathscr{L}} M) \Rightarrow (\mathbf{V}, \mathbf{A}, \mathbf{C}) \Rightarrow (\gamma).$$

Thus, we obtain the following

Theorem 2. *Let M^n, $n \geqq 1$ be local martingales with $M_0^n = 0$, $n \geqq 1$ and trajectories in $(D, \mathscr{D})$ and M is a continuous Gaussian martingale with $M_0 = 0$ and $A(t) = \mathbf{E} M_t^2$.*

Under the condition (**V**)

$$M^n \xrightarrow{\mathscr{L}} M \Leftrightarrow (\gamma).$$

Corollary 3. *If the local martingales M^n, $n \geqq 1$ are quasi-left continuous processes then under condition* (**sup B**)

$$M^n \xrightarrow{\mathscr{L}} M \Leftrightarrow (\gamma).$$

Remark [5]. The implication $M^n \xrightarrow{\mathscr{L}} M \Rightarrow (\gamma)$ takes place under the condition: for every $t > 0$

$$\lim_{b \to \infty} \sup_n P(V_t(B^n) \geqq b) = 0.$$

10. *Remark.* Let M^n, $n \geqq 1$, be local martingales and let the condition (ρ) be satisfied. Then for each $t > 0$

$$\sup_{s \leqq t} |M_s^n| \xrightarrow{P} 0 \Leftrightarrow [M^n, M^n]_t \xrightarrow{P} 0.$$

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