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Infinite Lattice Systems of Interacting Diffusion Processes, Existence and Regularity Properties

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Summary. Generalized stochastic gradient systems for infinite lattice models are investigated. The allowed strength of the interaction depends on the dimension of the lattice. The semigroup of transition probabilities is constructed and its regularity properties are also discussed. Some results of Doss and Royer [2] are improved.

0. Introduction

In the last few years several classes of interacting diffusion processes have been proposed as models for the temporal evolution of certain infinite systems of statistical physics, see [2-4, 11, 13, 15, 18] with some further references. Nevertheless, even construction of such processes is as yet poorly understood. Some extra difficulties arise in case of the physically most interesting point systems, because interaction of particles in higher dimensions is so strong that standard methods cease to work. Bearing in mind this problem, the main purpose of this paper is to develop methods for some strongly interacting lattice models; extensions to point systems will be discussed in [6].

We are going to investigate diffusions in an infinite product space $\mathbb{R}^S$, where $\mathbb{R}$ is the real line and S is such a countable subset of $\mathbb{R}^d$ that $|j-k|\geqq 1$ if j and k are distinct points of S; $|\cdot|$ denotes the usual Euclidean norm. Regular lattices are typical examples for S. Elements of $\mathbb{R}^S$ are represented as infinite sequences $x=(x_k)_{k\in S}$; i.e. if $k\in S$ and $x\in\mathbb{R}^S$ then x_k denotes the k-th component of x. Let $\mathbb{R}^S$ be given the product topology and let $\mathscr{R}^V$, $V\subset S$, be the smallest σ-algebra in $\mathbb{R}^S$ such that each projection $x\to x_k$, $k\in V$ is $\mathscr{R}^V$-measurable, then $\mathscr{R}^S$ is just the associated Borel field of $\mathbb{R}^S$. Let $\mathbb{C}(\mathbb{R}^V)$ denote the space of $\mathscr{R}^V$-measurable and continuous real functions and set $\mathbb{C}^1(\mathbb{R}^V)=\{\varphi\in\mathbb{C}(\mathbb{R}^V)$: $\mathsf{D}_k\varphi\in\mathbb{C}(\mathbb{R}^V)$, $k\in S\}$, $\mathbb{C}^2(\mathbb{R}^V)=\{\varphi\in\mathbb{C}^1(\mathbb{R}^V)$: $\mathsf{D}_k\varphi\in\mathbb{C}^1(\mathbb{R}^V)$, $k\in S\}$, where D_k denotes the operator of partial differentiation with respect to the k-th component of the argument. Finally, if $V\subset S$ then $|V|$ denotes cardinality of $|V|$, diam $V=\sup\{|j-k|: j\in V, k\in V\}$ and $S_k(r)=\{j\in S: |j-k|\leqq r\}$.

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Suppose now that for each $k \in S$ we are given some coefficients $c_k \in \mathbb{C}(\mathbb{R}^S)$ and $\sigma_k \in \mathbb{C}(\mathbb{R}^S)$, further $\{w_k\colon k \in S\}$ is a family of independent standard Wiener processes on a probability space $(\mathbb{W}, \mathscr{A}, \mathsf{P})$. We may and do assume that $(\mathbb{W}, \mathscr{A})$ is the space of continuous mappings of $[0, \infty)$ into $\mathbb{R}^S$ with the associated Borel field, thus $w_k = w_k(\cdot)$ is the k-th component of $w \in \mathbb{W}$. Let $\mathscr{A}_t \subset \mathscr{A}$ denote the σ-algebra generated by the family $\{w_k(s)\colon k \in S,\ 0 \leqq s \leqq t\}$ of random variables. We are going to study infinite systems of stochastic differential equations of type

$$X_k(t) = x_k + \int_0^t c_k(X(s))\, ds + \int_0^t \sigma_k(X(s))\, dw_k(s), \qquad k \in S; \tag{0.1}$$

realizations of a solution $X = X(\cdot)$ are supposed to be continuous trajectories in $\mathbb{R}^S$, i.e. they belong to $\mathbb{W}$.

We are not able to discuss the existence problem of (0.1) in full generality. The first restriction we need is locality of the interaction, we assume that it has a finite radius $r \geqq 1$; i.e. $c_k(x)$ and $\sigma_k(x)$ depend on x_j only if $j \in S_k(r)$. If we have smooth and bounded coefficients then global solutions can be constructed for each initial configuration, see [9, 18], the situation, however, is much more complicated otherwise. Pathological behaviour of solutions including explosion and breakdown of uniqueness can be demonstrated in case of very simple linear systems. Indeed, let $S = \{0, 1, \ldots, n, \ldots\}$, $c_k(x) = x_{k+1}$ and $\sigma_k(x) = 0$ for each $k \in S$, then $X_n(t)$ is just the n-th derivative of $X_0(t)$, thus we have a one to one correspondence between solutions and functions $\varphi\colon [0, \infty) \to \mathbb{R}$ possessing derivatives of all orders. Since such a function, $X_0(t)$, is not determined by the sequence of its derivatives at $t = 0$, uniqueness of solutions may hold only in a restricted sense. It is quite natural to remove nonuniqueness in the above example by allowing analytic solutions only, then the phase space of the system will be a proper subset of $\mathbb{R}^S$ characterized by Cauchy's growth criterion. Of course, linear and quasi-linear systems can be treated in a suitably chosen Banach space of sequences indexed by S, in such cases only strongly continuous solutions are considered. Moreover, if the interaction is weak in the sense that $\mathsf{D}_j c_k$ and $\mathsf{D}_j \sigma_k$ are uniformly bounded if $j \neq k$ then a natural stability condition implies similar results, see [2, 17]. If we are given more singular coefficients then a method of Liapunov functions is needed, and concepts of existence and uniqueness of solutions become more sophisticated, cf. [3].

Time dependent models motivated by problems from statistical physics are usually constructed in such a way that Gibbs random fields with a given interaction potential are stationary measures of the process and satisfy the principle of microscopic balance, cf. [2, 8, 10, 11, 18]. In our context an interaction potential is a family $U = \{U_V\colon V \subset S\}$ of real functions such that $U_V \in \mathbb{C}^2(\mathbb{R}^V)$ and $U_V = 0$ if $\operatorname{diam} V > r$; thus it is quite natural to assume that $U_V \geqq 0$ for all $V \subset S$. Interaction energy of site $k \in S$ is defined as

$$H_k(x) = \sum_{V : k \in V} U_V(x), \tag{0.2}$$

and a probability measure μ on $\mathscr{R}^S$ is called a Gibbs random field with interaction potential U if given $\mathscr{R}^{S \setminus \{k\}}$, the conditional distribution of μ is absolutely continuous with respect to Lebesque measure and its density is proportional to $\exp(-H_k(x))$, see [1, 12, 16]. Let G denote the formal generator associated to (0.1), i.e.

$$\mathsf{G}=\sum_{k\in S}(c_k(x)\,\mathsf{D}_k+\tfrac{1}{2}\sigma_k^2(x)\,\mathsf{D}_k^2); \tag{0.3}$$

if μ is a Gibbs random field with interaction potential U then the principle of microscopic balance means that

$$\int\varphi_1(x)\,\mathsf{G}\varphi_2(x)\,\mu(dx)=\int\varphi_2(x)\,\mathsf{G}\varphi_1(x)\,\mu(dx) \tag{0.4}$$

holds for a dense set of smooth functions in $\mathbb{L}^2(\mu)$. An easy integration by parts argument shows that (0.4) is formally equivalent to

$$c_k(x)=\tfrac{1}{2}\exp(H_k(x))\,\mathsf{D}_k(\sigma_k^2(x)\exp(-H_k(x))), \tag{0.5}$$

which makes sense even if there is not any Gibbs random field for U. In some cases (0.4) implies reversibility, i.e. the transition semigroup defined by (0.1) will be self-adjoint in $\mathbb{L}^2(\mu)$. Some useful consequences of this property are discussed in [8].

There are two extreme cases of (0.5). If $\sigma_k=1$, i.e. $c_k(x)=-\frac{1}{2}\,\mathsf{D}_k H_k(x)$, then we obtain the familiar class of stochastic gradient systems, see [11-13, 15, 19]. On the other hand, putting $\sigma_k=\exp(\frac{1}{2}H_k)$ we have $c_k=0$. This second case is very strange from a physical point of view, because in typical situations all solutions explode in a finite time. That is the reason why we can discuss only certain generalizations of stochastic gradient systems; total energy plays the role of Liapunov function in these cases.

1. Generalized Stochastic Gradient Systems

Suppose that we are given an interaction potential $U=\{U_V\colon V\subset S\}$ such that $U_V\in\mathbb{C}^2(\mathbb{R}^V)$, $U_V\geqq 0$ and $U_V=0$ if diam $V>r$. Then energy per site k is defined as

$$Q_k(x)=\sum_{V:k\in V}\frac{1}{|V|}\,U_V(x); \tag{1.1}$$

$$H_k(x,\rho)=\sum_{V:V\subset S_k(\rho)}(1+U_V(x)) \tag{1.2}$$

is a version of total energy in $S_k(\rho)$. We say that (0.1) is a generalized stochastic gradient system with respect to U if we have such constants $\delta>0$ and $a\geqq 0$ that

$$c_k(x)\,\mathsf{D}_k H_k(x)\leqq-\delta\sigma_k^2(x)\,(\mathsf{D}_k H_k(x))^2+aH_k(x,r) \tag{1.3}$$

holds for each k and x. We are also assuming that

$$|c_k(x)|\leqq a\sigma_k^2(x)\,|\mathsf{D}_k H_k(x)|+a|\sigma_k(x)|\,|H_k(x,r)|^{1/2}, \tag{1.4}$$

$$\sigma_k^2(x)\,(\mathsf{D}_k U_V(x))^2\leqq a(1+U_V(x))\,(H_k(x,r))^\lambda \quad \text{if } |V|\geqq 2, \tag{1.5}$$

where $0 \leqq \lambda \leqq \frac{2}{d}$, but $\lambda < 2$ if $d=1$, further

(1.6) $$\sigma_k^2(x)\, \mathrm{D}_k^2 U_V(x) \leqq a H_k(x, r)$$

for each $V \subset S$, $k \in S$, $x \in \mathbb{R}^S$. Let us remark that (1.4) and (1.5) are needed to control boundary effects, they are easily verified in case of stochastic gradient systems; (1.6) is a natural regularity condition. For applicability of the method of successive approximation we need that

(1.7) $$c_k^2(x) + \sigma_k^2(x) \leqq a u^a \quad \text{whenever } H_k(x, r) \leqq u,$$

and

(1.8) $$\max\{H_k(x, r), H_k(y, r)\} \leqq u \quad \text{implies that}$$
$$(c_k(x) - c_k(y))^2 + (\sigma_k(x) - \sigma_k(y))^2 \leqq a u^a \sum_{j \in S_k(r)} |x_j - y_j|^2.$$

Let us remember that c_k and σ_k are measurable with respect to $\mathscr{R}^{S_k(r)}$. In the first part of the paper the potential U is used as a family of auxiliary functions, later we are going to consider reversible process in particular. Of course, the interaction potential of the underlying Gibbs random field need not be the same as in (1.3)–(1.8).

We shall construct solutions to (0.1) in the following subset, Ω_0, of $\mathbb{R}^S$. Let $g(u) = (1 + \log(1+u))^{1/d}$ for $u \geqq 0$, then

(1.9) $$\bar{H}(x) = \sup_{k \in S} \sup_{\rho \geqq g(|k|)} \rho^{-d} H_k(x, \rho)$$

is the so called logarithmic energy fluctuation, and the set of allowed configurations is defined as $\Omega_0 = \{x \in \mathbb{R}^S \colon \bar{H}(x) < \infty\}$. Since $\bar{H}$ is lower semi-continuous, Ω_0 is an F_σ subset of $\mathbb{R}^S$. It is easy to check that Ω_0 is of full measure with respect to a large class of random fields. For example, if μ is a probability measure on $\mathscr{R}^S$ and we have such constants $a_1 > 0$, $a_2 > 0$ that

(1.10) $$\int \exp(a_1 H_k(x, \rho))\, \mu(dx) \leqq \exp(a_2 \rho^d) \quad \text{if } \rho \geqq g(|k|),$$

then the Borel-Cantelli lemma implies $\mu(\Omega_0) = 1$ directly. This condition can be verified easily for Gibbs random fields with various interaction potentials including U. The very same argument suggests that Ω_0 is essentially the smallest set carrying a sufficiently large class of measures.

(1.11) *Definition.* An Ω_0-valued continuous stochastic process $X = X(t)$, $t \geqq 0$, i.e. a measurable mapping of $(\mathbb{W}, \mathscr{A}, \mathsf{P})$ into $(\mathbb{W}, \mathscr{A})$ is called a tempered solution of (0.1) with initial configuration $x \in \Omega_0$ if $X(t)$ is $\mathscr{A}_t$-measurable, $\mathsf{P}[X(0) = x] = 1$, almost every realization of X satisfies (0.1) for all $t \geqq 0$ and $\mathsf{P}[\sup_{s \leqq t} \bar{H}(X(s)) < \infty] = 1$ for all $t > 0$.

We show that the above conditions are sufficient for existence of a unique tempered solution for each initial configuration $x \in \Omega_0$. Our principal result is the following a priori bound, further results are more or less standard consequences of this estimate.

(1.12) **Proposition.** *Suppose* (1.3)–(1.6), *then for each tempered solution* X *of* (0.1) *there exists an* $\mathscr{A}$*-measurable random variable* $N>0$ *such that* $\mathsf{P}[N>u]\leqq e^{-\delta u}$ *and* $\bar{H}(X(t))\leqq\exp(\exp(qh^2 e^{qt})\log(e+N))$ *holds for all* $t\geqq 0$ *if* $\bar{H}(X(0))\leqq h$; $q=q(d, r, \delta, \lambda, a)$ *is a universal constant.*

Ideas of the proof of this a priori bound go back to [3], where further references are given. However, techniques of [3] work only if $\lambda\leqq 1/d$ in (1.5); let us remark that methods of [2, 17] correspond to the case of $\lambda=0$.

Existence of tempered solutions can be easily proven if $c_k=\sigma_k=0$ for $k\notin S'$ and $|S'|<\infty$. Since (1.12) implies relative compactness of the set of such partial solutions in the weak sense, existence of global solutions to (0.1) follows from (1.6) and (1.7) by an iteration procedure. Consequently, we can construct the semigroup P_t of transition probabilities in $(\Omega_0, \mathscr{R}^S\cap\Omega_0)$. Some regularity properties of P_t are also discussed. We can prove Feller continuity of P_t only in a restricted sense. In some cases existence of a stationary measure follows by similar methods. If μ is a Gibbs random field with interaction potential U, and each c_k is given by (0.5), i.e. (0.1) is formally reversible with respect to μ, then $\mu\mathsf{P}_t=\mu$ and P_t is really self-adjoint in $\mathbb{L}^2(\mu)$. In [5] we are going to discuss conditions under which stationary measures of certain stochastic gradient systems are Gibbs random fields for the associated interaction potential.

2. Proof of (1.12)

As a family of Liapunov functions, the following modification of total energy in balls of rapidly decreasing radius is used, cf. [3]. Consider a positive and non-increasing, twice continuously differentiable function f such that $f(u)=1$ if $u\leqq 1$, $f''(u)\leqq 0$ if $u\leqq 2+r$, $f''(u)\geqq 0$ if $u\geqq 2$ and $f(u)=e^{-bu}$ with some $b>0$ if $u\geqq 3+r$. We may and do assume that

$$-f'(u)\leqq f(u),\quad f(u)\leqq 2f(v)\quad \text{if } |u-v|\leqq r, \tag{2.1}$$

$$|f(u)-f(v)|\leqq -(f'(u)+f'(v))\,|u-v|\quad \text{if } |u-v|\leqq r. \tag{2.2}$$

Let $k\in S$ be fixed, and introduce

$$Q(x, k, \rho)=\sum_{j\in S} f(|j-k|\,\rho^{-1})\,(1+Q_j(x)), \tag{2.3}$$

$$Z(t)=\int_0^t \bar{H}^{\lambda}(X(s))\,ds \tag{2.4}$$

and

$$\rho_n(t, k)=g(|k|)\,[n^2-K g^{\lambda d}(n)\,Z(t)]^{1/2}, \tag{2.5}$$

where $\rho\geqq 1$, n is a positive integer, K is a large constant to be specified later, and $X=X(t)$ is the tempered solution we are interested in. We consider ρ_n only in the maximal interval $[0, T_n)$ such that $\rho_n(t, k)\geqq 1$ if $t\leqq T_n$. Then T_n is a stopping time and $\rho_n(t, k)$ is $\mathscr{A}_t$-measurable, thus $Q e^{-Kt}=e^{-Kt}Q(X(t), k, \rho_n(t, k))$ admits a stochastic differential

(2.6) $$dQe^{-Kt} = -Ke^{-Kt}Q\,dt + dI_1 + I_2\,dt - I_3\,dt$$

in $[0, T_n)$, where

$$dI_1 = e^{-Kt}\sum_{j\in S} f_j \sum_{i\in S} \mathsf{D}_i Q_j\,(c_i\,dt + \sigma_i\,dw_i),$$

$$I_2 = \tfrac{1}{2}e^{-Kt}\sum_{j\in S} f_j \sum_{i\in S} \sigma_i^2\,\mathsf{D}_i^2 Q_j,$$

$$I_3 = e^{-Kt}\sum_{j\in S} f_j'\,|j-k|\,\rho_n^{-2}\rho_n'(1+Q_j)$$

$$= -\frac{K}{2}e^{-Kt}\,\bar{H}^{\lambda} g^2(|k|)\,g^{\lambda d}(n)\,\rho_n^{-3}\sum_{j\in S} f_j'|j-k|(1+Q_j);$$

and abbreviations as $f_j = f(|j-k|\rho_n^{-1})$, $f_j' = f'(|j-k|\rho_n^{-1})$, $Q_j = Q_j(X(t))$, $H_i = H_i(X(t))$, $\rho_n' = \frac{d}{dt}\rho_n$, $U_V = U_V(X(t))$, $\bar{H} = \bar{H}(X(t))$, $c_i = c_i(X(t))$, $\sigma_i = \sigma_i(X(t))$ are used and will be used also in what follows. Since X is a tempered solution and f has an exponential tail, infinite sums in the above expressions make sense. First we show that K can be chosen to be so large that the deterministic part of (2.6) turns to be negative. Indeed, as

(2.7) $$\sum_{j\in S} f_j \mathsf{D}_i Q_j = \sum_{j\in S} f_j \sum_{V:\,j\in V} |V|^{-1}\mathsf{D}_i U_V = f_i \mathsf{D}_i H_i + J_i,$$

where

$$J_i = \sum_{V:\,i\in V}\sum_{j\in V}(f_j - f_i)\,|V|^{-1}\mathsf{D}_i U_V;$$

(1.3), (1.4) and the elementary inequality $uv \leqq pu^2/2 + v^2/2p$ if $p>0$ imply that

(2.8) $$\sum_{j\in S} f_j(\mathsf{D}_i Q_j)\,c_i \leqq -\delta f_i \sigma_i^2(\mathsf{D}_i H_i)^2 + a f_i H_i(X,r)$$

$$+\,a\sigma_i^2\,|J_i|\,|\mathsf{D}_i H_i| + a|\sigma_i|\,|J_i|(H_i(X,r))^{1/2} \leqq -\frac{\delta}{2} f_i \sigma_i^2(\mathsf{D}_i H_i)^2$$

$$+\left(\frac{a}{\delta}+\frac{a}{2}\right) f_i^{-1}\sigma_i^2 J_i^2 + \frac{3}{2} a f_i H_i(X,r).$$

Observe that in view of (2.1) we have

(2.9) $$\sum_{i\in S} f_i H_i(X,r) \leqq 2(2r)^d Q(X,k,\rho_n),$$

while (1.6) and a similar argument results in

(2.10) $$I_2 \leqq a(2r)^d e^{-Kt} Q(X,k,\rho_n),$$

thus $f_i^{-1}\sigma_i^2 J_i^2$ is the critical term here; it is enough to show that

(2.11) $$\left(\frac{a}{\delta}+\frac{a}{2}+\frac{\delta}{2}\right) e^{-Kt}\sum_{i\in S} f_i^{-1}\sigma_i^2 J_i^2 \leqq I_3$$

if $K = K(d, r, \delta, \lambda, a)$ is large enough.

(2.12) **Lemma.** *If* $|i-j|\leqq r$ *and* $\max\{|i-k|,|j-k|\}\geqq\rho_n$ *then we have a universal constant* K_1 *such that*

$$\max\{g^{\lambda d}(|i|), g^{\lambda d}(|j|)\}$$
$$\leqq K_1\rho_n^{-1}g^2(|k|)\,g^{\lambda d}(n)\,(1+\min\{|i-k|,|j-k|\}).$$

Proof. Suppose e.g. that $|j-k|\geqq\rho_n$. Since $g^{\lambda d}(u+v)\leqq g^{\lambda d}(u)+g^{\lambda d}(v)$ if $u,v\geqq 1$, we have $g^{\lambda d}(|i|)\leqq g^{\lambda d}(|j|)+g^{\lambda d}(r)$, while $g^{\lambda d}(|j|)\leqq g^{\lambda d}(|k|)+g^{\lambda d}(|j-k|)$. On the other hand, $u^{-1}g^{\lambda d}(u)$ is a decreasing function of $u\geqq 0$ as $g(u)\geqq\lambda d u g'(u)$ reduces to $1+\log(1+u)\geqq\frac{2u}{1+u}$, consequently $g^{\lambda d}(|j-k|)\leqq\rho_n^{-1}\;|j-k|\,g^{\lambda d}(\rho_n)$. Finally, $\rho_n\leqq n g(|k|)$ and $g(ug(v))\leqq g(u)\,g(v)$ for $u,v\geqq 1$, i.e. $g^{\lambda d}(\rho_n)\leqq g^2(|k|)\,g^{\lambda d}(n)$, which completes the proof; K_1 depends only on r. QED

To conclude (2.11) observe that (1.5), (2.2) and (2.1) imply

$$\begin{aligned}(2.13)\qquad & f_i^{-1}\sigma_i^2(f_i-f_j)^2\,(\mathsf{D}_i U_V)^2\\ &\leqq a\bar{H}^\lambda\max\{r^{\lambda d}, g^{\lambda d}(|i|)\}\,f_i^{-1}(f_i-f_j)^2\,(1+U_V)\\ &\leqq -2ar^{2+\lambda d}\bar{H}^\lambda\rho_n^{-2}\,g^{\lambda d}(|i|)\,(f_i'+2f_j')\,(1+U_V)\end{aligned}$$

whenever $|i-j|\leqq r$, and $\max\{|i-k|,|j-k|\}\geqq\rho_n$ may also be assumed because $f_i=f_j=1$ otherwise. Since the number of summands in the expression of J_i is bounded by a constant depending only on r and d, estimating J_i^2 by means of the Cauchy inequality, (2.13) and (2.12) imply (2.11).

Stochastic integrals in dI_1 can be estimated by means of the following maximal inequality for martingales. If each p_i is progressively measurable with respect to $\mathscr{A}_t$ and

$$\sum_{i\in S}\int_0^t p_i^2(s)\,ds<+\infty\qquad\text{a.s.,}$$

then for $u>0$, $z>0$ we have

$$(2.14)\qquad \mathsf{P}\left[\sup_{t\geqq 0}\sum_{i\in S}\int_0^t\left(p_i dw_i-\frac{z}{2}p_i^2\,ds\right)>u\right]\leqq e^{-zu},$$

see [14] for the case of finite sums, whence (2.14) follows by an easy limiting procedure. Now we are in a position to summarize results of the above calculations.

(2.15) **Lemma.** *If* K *is large enough then for each* n *and* k *there exists an* $\mathscr{A}$-*measurable random variable* $N_{n,k}$ *such that*

$$\sup_{t\leqq T_n} e^{-Kt}Q(X(t),k,\rho_n(t,k))\leqq Q(X(0),k,ng(|k|))+N_{n,k}$$

and $\mathsf{P}[N_{n,k}>2v]\leqq 2e^{-\delta v}$.

Proof. Let

$$N_{n,k}=\sup_{t\leqq T_n}\sum_{i\in S}\int_0^t e^{-Ks}\left(\sigma_i J_i dw_i-\frac{\delta}{2}\sigma_i^2 J_i^2 ds\right)$$
$$+\sup_{t\leqq T_n}\sum_{i\in S}\int_0^t e^{-Ks}f_i\left(\sigma_i(\mathsf{D}_i H_i)\,dw_i-\frac{\delta}{2}\sigma_i^2(\mathsf{D}_i H_i)^2\,ds\right).$$

Since $e^{-Kt}\leqq 1$ and $f_i\leqq 1$, (2.14) implies $\mathsf{P}[N_{n,k}>2v]\leqq 2e^{-\delta v}$, thus comparing (2.6)–(2.11) we obtain (2.15). QED

The next step is to transform (2.15) into an integral inequality for $\bar{H}(X(t))$. It is easy to verify that

(2.16) $$\bar{H}(x)\leqq\sup_{m\geqq 1}\sup_{k\in S}2^d(mg(|k|))^{-d}Q(x,k,mg(|k|)),$$

where m varies in the set of positive integers. On the other hand, there exists a constant K_2 depending only on d and r such that

(2.17) $$Q(x,k,\rho)\leqq K_2\bar{H}(x)\,\rho^d \quad \text{if } \rho\geqq g(|k|).$$

Indeed, $f(u)=e^{-bu}$ if $u\geqq 3+r$ with some $b>0$ depending only on r, thus we have

$$\begin{aligned}
Q(x,k,\rho)&\leqq 2\sum_{n=0}^{\infty}f\left(1+\frac{n}{\rho}\right)(H(x,k,\rho+n+1)-H(x,k,\rho+n))\\
&\leqq 2e^{b(3+r)}\sum_{n=0}^{\infty}\exp\left(-b\frac{n}{\rho}\right)(H(x,k,\rho+n+1)-H(x,k,\rho+n))\\
&\leqq K_3\left(1-\exp\left(-\frac{b}{\rho}\right)\right)\sum_{n=0}^{\infty}\exp\left(-b\frac{n}{\rho}\right)H(x,k,\rho+n)\\
&\leqq K_3\bar{H}(x)\left(1-\exp\left(-\frac{b}{\rho}\right)\right)\sum_{n=0}^{\infty}(\rho+n)^d\exp\left(-b\frac{n}{\rho}\right)\\
&\leqq K_3\left(\frac{2d\rho}{b}\right)^d\bar{H}(x)\left(1-\exp\left(-\frac{b}{\rho}\right)\right)\sum_{n=0}^{\infty}\left(1+\frac{bn}{2d\rho}\right)^d\exp\left(-b\frac{n}{\rho}\right)\\
&\leqq K_4\rho^d\bar{H}(x)\left(1-\exp\left(-\frac{b}{\rho}\right)\right)\sum_{n=0}^{\infty}\exp\left(-\frac{bn}{2\rho}\right)\\
&\leqq K_4\rho^d\bar{H}(x)\left(1-\exp\left(-\frac{b}{\rho}\right)\right)\left(1-\exp\left(-\frac{b}{2\rho}\right)\right)^{-1}\leqq 2K_4\rho^d\bar{H}(x).
\end{aligned}$$

Now we can rewrite (2.15) in terms of $\bar{H}$. Let K_5 be so large that

(2.18) $$\sum_{n=1}^{\infty}\sum_{k\in S}\exp(-\delta K_5 n^d g^d(|k|))<\tfrac{1}{2}$$

and introduce

(2.19) $$N=\max\{0,\sup_{n,k}\tfrac{1}{2}(N_{n,k}-2K_5 n^d g^d(|k|))\},$$

then $\mathsf{P}[N>u]<e^{-\delta u}$ and

$$\sup_{t\leqq T_n} e^{-Kt} Q(X(t), k, \rho_n(t,k)) \leqq Q(X(0), k, n g(|k|)) + 2N + K_5 n^d g^d(|k|) \leqq K_2 n^d g^d(|k|)\, \bar{H}(X(0)) + 2N + K_5 n^d g^d(|k|) \tag{2.20}$$

follows by (2.17). Now let $n=n(t,m,k)$ denote the smallest integer $n>0$ for which $\rho_n(t,k)\geqq m g(|k|)$. Since $\lim_{n\to\infty} T_n=\infty$ and $\lim_{n\to\infty}\rho_n(t,k)=\infty$, $n(t,m,k)$ is well defined for all $t>0$, $k\in S$ and positive integer m. Further, $t<T_n$ and $\rho_n(t,k)<m+1)\,g(|k|)$ if $n=n(t,m,k)$. Let us remark that the exceptional set, where $n(t,m,k)$ is not defined is independent of $t>0$. Choosing $n=n(t,m,k)$ in (2.20) we get

$$e^{-Kt} Q(X(t), k, m g(|k|)) \leqq K_2 n^d(t,m,k)\, g^d(|k|)\, \bar{H}(X(0)) + 2N + K_5 n^d(t,m,k)\, g^d(|k|)$$

for all t, m, k, whence

$$\bar{H}(X(t)) \leqq K_6 e^{Kt}(N + h \sup_{k\in S}\sup_{m\geqq 1} m^{-d} n^d(t,m,k)) \tag{2.21}$$

follows by (2.16) with some $K_6=K_6(d,r,\delta,\lambda,a)$ and $h\geqq \bar{H}(X(0))$. On the other hand, if $n=n(t,m,k)$ in the definition (2.5) of ρ_n then

$$n^2 \leqq (m+1)^2 + K g^{\lambda d}(n)\, Z(t) \leqq 4m^2 + 2KnZ(t) \tag{2.22}$$

as $\lambda d\leqq 2$ and $g(u)\leqq 2\sqrt{u}$ if $u\geqq 1$. Since $n\geqq m$ we obtain $n\leqq 4m+2KZ(t)$, i.e.

$$n^2(t,m,k) \leqq K_7 m^2 (1 + Z(t)\, g^{\lambda d}(Z(t))) \tag{2.23}$$

with a new constant K_7, thus (2.21) turns into

$$\bar{H}(X(t)) \leqq M e^{Mt}(N + h(1 + Z(t)\, g^{\lambda d}(Z(t)))^{d/2}), \tag{2.24}$$

where $M\geqq K$ is a universal constant,

$$Z(t)=\int_0^t \bar{H}^{\lambda}(X(s))\, ds.$$

If $\lambda=0$ then (2.24) reduces to

$$\bar{H}(X(t)) \leqq M e^{Mt}(N + h(1+t)^{d/2}), \tag{2.25}$$

while a differential inequality,

$$Z' \leqq M^{\lambda} e^{\lambda M t}(N + h(1 + Z g^{\lambda d}(Z))^{d/2})^{\lambda}, \qquad Z(0)=0, \tag{2.26}$$

follows for Z if $\lambda>0$. Since $\lambda d\leqq 2$ and $\lambda<2$ if $d=1$,

$$\int_0^{\infty} (N + h(1 + z g^{\lambda d}(z))^{d/2})^{-\lambda} dz = +\infty \tag{2.27}$$

for $N \geqq 0$, $h > 0$; consequently (2.26) has a maximal solution which is bounded in finite intervals of time. Indeed, if $\lambda d < 2$ then $g^{\lambda d}(Z)$ can be estimated by a constant multiple of $(1+Z)^{1-\lambda d/2}$, while $\lambda \leqq 1$ if $d > 1$, thus (2.26) implies in both cases that

$$Z' = (Z + e + N^\lambda)' \leqq K_8 e^{\lambda M t}(N^\lambda + h^\lambda(1 + (1 + \log(1+Z))\,Z))$$
$$\leqq 2K_8 e^{\lambda M t} h^\lambda (Z + e + N^\lambda)\log(Z + e + N^\lambda),$$

whence

$$\log(Z(t) + e + N^\lambda) \leqq \log(e + N^\lambda)\exp(2h^\lambda K_8 \int_0^t e^{\lambda M s}\,ds),$$

thus (2.24) implies (1.12) even if $\lambda > 0$. QED

(2.28) **Corollary.** *If* $\lambda d < 2$ *and* $0 < \varepsilon < 1 - \lambda\frac{d}{2}$ *then we have* $\bar{H}(X(t)) \leqq p e^{pt}(N + h^{1/\varepsilon})$ *with some universal* p.

Proof. Since $\lambda d/2 < 1 - \varepsilon$, an elementary calculation shows that $(1 + Z g^{\lambda d}(Z))^{\lambda d/2}$ is bounded by a multiple of $(1+Z)^{1-\varepsilon}$, thus (2.26) yields

$$\frac{d}{dt}\left(1 + \left(\frac{N}{h}\right)^{\lambda/1-\varepsilon} + Z\right) \leqq K_9 h^\lambda e^{\lambda M t}\left(1 + \left(\frac{N}{h}\right)^{\lambda/1-\varepsilon} + Z\right)^{1-\varepsilon},$$

whence we obtain

$$(1 + Z(t))^\varepsilon \leqq \left(1 + \left(\frac{N}{h}\right)^{\lambda/1-\varepsilon}\right)^\varepsilon + \varepsilon K_9 h^\lambda \int_0^t e^{\lambda M s}\,ds,$$

which implies (2.28) by (2.24). QED

(2.29) *Remark.* An important consequence of (1.12) is finiteness of all moments of $\bar{H}$ along a tempered solution with a deterministic initial configuration. If the initial configuration is random, and $\lambda = 0$ in (1.5), then (2.25) yields a time dependent bound for each moment of $\bar{H}(X(t))$; if $\lambda > 0$ then we can not prove such a conservation law of existence of moments of $\bar{H}$. However, if $\lambda d < 2$ and $\bar{H}$ has finite moments of all orders at $t = 0$ then (2.28) implies conservation of this property for all $t > 0$.

(2.30) *Remark.* We suspect that just as in case of linear systems, there exist many non-tempered solutions for each initial configuration $x \in \Omega_0$. Apart from some pathological cases we are not able to prove that solutions remaining in Ω_0 for all $t > 0$ are necessarily tempered solutions.

(2.31) *Remark.* Our $\mathscr{A}$-measurable random variable N can be replaced by an $\mathscr{A}_t$-adapted increasing process N_t, and estimating Q more carefully, it is possible to show that $\bar{H}$ is a continuous function of time along tempered solutions.

(2.34) *Remark.* Essential conditions we needed on the shape of our auxiliary function g were (2.27), $\lambda d u g'(u) \leqq g(u)$ and $g(u g(v)) \leqq g(u)\,g(v)$ for $u, v \geqq 1$. Therefore, if $\lambda d<2$ and $0<\varepsilon\leqq\min\left\{\frac{1}{\lambda d}, \frac{2}{\lambda d}-1\right\}$, then g can be replaced by $(1+u)^{\varepsilon}$ and the allowed set of initial configurations gets larger than Ω_0.

(2.35) *Remark.* Results of [2, 17] can be reproduced by choosing $U_V(x)=x_k^2$ if $V=\{k\}$, $U_V=0$ if $|V|>1$; if x_k^2 can be replaced by a bigger power then some more singular interactions get tractable, too. Our stability conditions are optimal in the absence of external field, i.e. when $U_V=0$ for $|V|=1$; cf. the case of point systems. We are not going to discuss stabilizing effects of external fields.

3. Construction of the Transition Semigroup

Here we show that (1.7), (1.12) and our Lipschitz condition (1.8) imply convergence of solutions of finite subsystems of (0.1) to a tempered solution. Conditions (1.3)–(1.8) are assumed without any reference.

(3.1) **Lemma.** *If $\tilde{S}\subset S$ is finite, and $\sigma_k=0$ for $k\notin\tilde{S}$ then* (0.1) *has a unique tempered solution for each initial configuration $x\in\Omega_0$.*

Proof. Since (1.4) implies $c_k=\sigma_k=0$ if $k\notin\tilde{S}$, coefficients of (0.1) are bounded and satisfy a uniform Lipschitz condition in such domains where

$$H_{\tilde{S}}(x)=\sum_{V:\tilde{S}\cap V\neq\emptyset}(1+U_V(x))$$

remains bounded. Therefore we certainly have a unique local solution X defined in a random interval $[0,T)$ such that T is a stopping time and $\lim_{t\to\infty} H_{\tilde{S}}(X(\min\{t,T\}))=+\infty$ if $T<+\infty$. Observe that $\mathsf{D}_k H_{\tilde{S}}=\mathsf{D}_k H_k$, thus using (1.3) and (1.6), an easy calculation shows that

$$\begin{aligned} d(H_{\tilde{S}}e^{-Kt}) &\leqq -\delta e^{-Kt}\sum_{k\in\tilde{S}}\sigma_k^2(\mathsf{D}_k H_k)^2\,dt \\ &\quad + e^{-Kt}\sum_{k\in\tilde{S}}\sigma_k\,\mathsf{D}_k H_k\,dw_k, \end{aligned}$$

where K is a large constant. In view of (2.14) this is possible only if $H_{\tilde{S}}(X(t))\,e^{-Kt}$ remains bounded in finite intervals of time, i.e. $\mathsf{P}[T=\infty]=1$. QED

To prove convergence of these partial solutions as $\tilde{S}\to S$, the following contraction property of (0.1) is needed.

(3.2) **Lemma.** *Let $i\in S$, $n\geqq 0$, $x,y\in\Omega_0$, and consider such Ω_0-valued continuous stochastic processes $X=X(t)$ and $Y=Y(t)$ that $X(t)$, $Y(t)$ are $\mathscr{A}_t$-measurable and satisfy* (0.1) *for $k\in S_i(nr)$ with $X(0)=x$ and $Y(0)=y$, respectively. Set $\varphi(t,u)=1$ if $\sup_{s\leqq t}\max\{\bar{H}(X(s)),\bar{H}(Y(s))\}\leqq u$, $\varphi(t,u)=0$ otherwise, and introduce*

$$d_i(t,m,X,Y,u)=\max_{k\in S_i(mr)} d_k(t,X,Y,u),$$

where $d_k(t, X, Y, u) = \operatorname{arctg}(\mathsf{E}[\sup_{s \le t} \varphi(s, u)|X_k(s) - Y_k(s)|^2])$. *If* $0 \leqq m \leqq n$ *then for all* $t > 0$ *and* $u \geqq 1$ *we have*

$$d_i(t, m, X, Y, u) \leqq 3 d_i(0, m, X, Y, u)$$
$$+ M(1+t)(u g^d(|i| + mr))^a \int_0^t d_i(s, m+1, X, Y, u)\, ds$$

with some $M > 0$ *depending only on* d, r, a.

Proof. Observe that $\varphi(t, u) \leqq \varphi(s, u) = \varphi^2(s, u)$ if $s \leqq t$, thus by the Cauchy inequality we obtain

$$\varphi(t, u)\,|X_k(t) - Y_k(t)|^2 \leqq 3\varphi(t, u)|X_k(0) - Y_k(0)|^2$$
$$+ 3t\,\varphi(t, u) \int_0^t (c_k(X(s)) - c_k(Y(s)))^2\, ds$$
$$+ 3\varphi(t, u) \left[\int_0^t (\sigma_k(X(s)) - \sigma_k(Y(s)))\, dw_k(s)\right]^2$$
$$\leqq 3\varphi(0, u)\,|X_k(0) - Y_k(0)|^2 + 3t \int_0^t (c_k(X(s)) - c_k(Y(s)))^2 \varphi(s, u)\, ds$$
$$+ 3\left[\int_0^t (\sigma_k(X(s)) - \sigma_k(Y(s)))\, \varphi(s, u)\, dw_k(s)\right]^2.$$

On the other hand, $\varphi(s, u)$ is $\mathscr{A}_s$-measurable, thus the maximal inequality

$$\mathsf{E}\left[\sup_{s \leqq t} \left(\int_0^s p_k\, dw_k\right)^2\right] \leqq 4 \int_0^t \mathsf{E}(p_k^2)\, ds$$

implies

$$\mathsf{E}[\sup_{s \leqq t} \varphi(s, u)\,|X_k(s) - Y_k(s)|^2] \leqq 3\varphi(0, u)\,|X_k(0) - Y_k(0)|^2 \tag{3.3}$$
$$+ 3t \int_0^t \mathsf{E}[\varphi(s, u)\,(c_k(X(s)) - c_k(Y(s)))^2]\, ds$$
$$+ 12 \int_0^t \mathsf{E}[\varphi(s, u)\,(\sigma_k(X(s)) - \sigma_k(Y(s)))^2]\, ds.$$

Integrals above can be estimated by means of (1.8) if $1 \geqq \operatorname{tg} d_k(s, 1, X, Y, u)$, and by (1.7) if $\operatorname{tg} d_k(s, 1, X, Y, u) > 1$. Indeed, as $v \leqq 2 \operatorname{arctg} v$ if $0 \leqq v \leqq 1$ and $|S_k(r)| \leqq (2r)^d$, while $2 \operatorname{arctg} v \geqq 1$ if $v > 1$, we obtain that

$$\mathsf{E}[\varphi(s, u)\,((c_k(X(s)) - c_k(Y(s)))^2 + (\sigma_k(X(s)) - \sigma_k(Y(s)))^2)]$$
$$\leqq 2a(2r)^d\,(u \max\{r^d, g^d(|k|)\})^a\, d_k(s, 1, X, Y, u),$$

thus (3.3) turns into

$$\operatorname{tg} d_k(t, X, Y, u) \leqq 3 \operatorname{tg} d_k(0, X, Y, u) \tag{3.4}$$
$$M(1+t)(u g^d(|k|))^a \int_0^t d_k(s, 1, X, Y, u)\, ds.$$

On the other hand, $\operatorname{arctg}(z+v) \leqq \operatorname{arctg} z + \operatorname{arctg} v$ if $z, v \geqq 0$, and $\operatorname{arctg} v \leqq v$ for all $v \geqq 0$, consequently

(3.5)
$$d_k(t, X, Y, u) \leqq 3 d_k(0, X, Y, u) + M(1+t)(u g^d(|k|))^a \int_0^t d_k(s, 1, X, Y, u)\, ds,$$

which implies (3.2) directly. QED

(3.6) **Corollary.** *If n is a positive integer then*

$$d_i(t, X, Y, u) \leqq \frac{2t^{n+1}}{(n+1)!}(M(1+t)(u g^d(|i|+nr))^a)^{n+1} + \sum_{m=0}^{n} d_i(0, m, X, Y, u) \frac{3t^m}{m!}(M(1+t)(u g^d(|i|+mr))^a)^m.$$

Proof. Notice that $d_i(t, X, Y, u) = d_i(t, 0, X, Y, u)$ and $\operatorname{arctg} v < 2$, thus iterating (3.2) n times we obtain (3.6). QED

(3.7) **Proposition.** *If X and Y are tempered solutions of* (0.1) *then*

$$d_k(t, X, Y, u) \leqq \sum_{m=0}^{\infty} d_k(0, m, X, Y, u) \frac{3t^m}{m!}(M(1+t)(u g^d(|k|+mr))^a)^m$$

holds for all $t \geqq 0$, $u \geqq 1$ and $k \in S$.

Proof. Letting n go to infinity in (3.6) we get (3.7). QED

Now we are in a position to summarize some basic properties of solutions of (0.1).

(3.8) **Theorem.** *For each initial configuration $x \in \Omega_0$ there exists a unique tempered solution $X = X(t, x)$ of* (0.1). *Moreover, if $X^{(n)} = X^{(n)}(t, x)$ denotes the partial solution defined in* (3.1) *with $X^{(n)}(0, x) = x$ and $\tilde{S} = S_i(nr)$; $i \in S$ is fixed, then*

$$\mathsf{P}[\lim_{n \to \infty} \sup_{s \leqq t} |X_k^{(n)}(s, x) - X_k(s, x)| = 0] = 1$$

holds for all $t \geqq 0$ and $k \in S$.

Proof. Suppose that Y is another tempered solution with $Y(0) = x$, then letting u go to infinity in (3.7), (1.12) implies $\mathsf{P}[\sup_{s \leqq t} |X_k(s, x) - Y_k(s)| = 0] = 1$ for all $t > 0$ and $k \in S$, which means uniqueness of tempered solutions.

To prove existence of $X(t, x)$, let us apply (3.2) for $X^{(n)}$ and $X^{(n+1)}$ with $u = g^p(n)$, the associated indicator variable will be denoted as $\varphi^{(n)} = \varphi^{(n)}(t, g^p(n))$. An easy calculation shows that (3.6) implies

$$\sum_{n=1}^{\infty} [d_k(t, X^{(n)}, X^{(n+1)}, g^p(n))]^{1/2} < \infty$$

for all $t > 0$, $p > 0$ and $k \in S$, whence

$$\mathsf{E}\left[\sum_{n=1}^{\infty} \sup_{s \leqq t} \varphi^{(n)}(s, g^p(n)) |X_k^{(n)}(s,x) - X_k^{(n+1)}(s,x)|\right] < \infty \tag{3.9}$$

follows by the Cauchy inequality as $\operatorname{arctg} v \geqq \frac{v}{2}$ if $0 \leqq v \leqq 1$. On the other hand, the explicit bound of (1.12) allows us to choose $p = p(t)$ to be so large that

$$\sum_{n=1}^{\infty} \mathsf{P}[\varphi^{(n)}(t, g^{p(t)}(n)) = 0] < \infty \tag{3.10}$$

for each $t > 0$, thus (3.9) and the Borel-Cantelli lemma imply for all $t > 0$ and $k \in S$ that

$$\sum_{n=1}^{\infty} \sup_{s \leqq t} |X_k^{(n)}(s,x) - X_k^{(n+1)}(s,x)| < \infty \tag{3.11}$$

with probability one. This means that our sequence of partial solutions converges in the natural topology of $\mathbb{W}$ to a necessarily continuous and $\mathscr{A}_t$-adapted process $X = X(t,x)$. Since $\bar{H}$ is lower semi-continuous, $\bar{H}(X(t,x))$ remains bounded in finite intervals of time, thus it is easily verified that X is a termpered solution of (0.1) with $X(0,x) = x$. QED

(3.12) **Corollary.** $\mathsf{P}_t(x, A) = \mathsf{P}[X(t,x) \in A]$ *is a semigroup of transition probabilities in the measurable space* $(\Omega_0, \Omega_0 \cap \mathscr{R}^S)$.

Proof. This follows directly from $X = \lim X^{(n)}$. QED

The transition semigroup acts in spaces of $\Omega_0 \cap \mathscr{R}^S$-measurable functions as

$$\mathsf{P}_t \varphi = (\mathsf{P}_t \varphi)(x) = \int \mathsf{P}_t(x, dy)\, \varphi(y), \tag{3.13}$$

regularity properties of P_t are more sophisticated than in the finite dimensional case. Let $\Omega_0^h = \{x \in \Omega_0 : \bar{H}(x) \leqq h\}$, $\|\varphi\|_h = \sup\{|\varphi(x)| : x \in \Omega_0^h\}$,

$$d(x,y) = \sum_{k \in S} 2^{-|k|} (\operatorname{arctg} |x_k - y_k|^2)^{1/2},$$

and notice that $d(x,y)$ is a distance inducing the relative topology of $\Omega_0 \subset \mathbb{R}^S$. Introduce now $\mathbb{C}(\Omega_0)$ as the space of such $\Omega_0 \cap \mathscr{R}^S$-measurable $\varphi : \Omega_0 \to \mathbb{R}$ that $\|\varphi\|_h < \infty$ and the restriction of φ to Ω_0^h is uniformly continuous for each $h > 1$. Further, let $\mathbb{C}_p(\Omega_0)$ be the set of such $\varphi \in \mathbb{C}(\Omega_0)$ that $\|\varphi\|_h$ is bounded by a polynomial of h depending on φ, and let $\mathbb{C}_b(\Omega_0)$ denote the set of bounded $\varphi \in \mathbb{C}(\Omega_0)$.

(3.14) **Theorem.** *Suppose that* $\varphi \in \mathbb{C}_p(\Omega_0)$, *then* $\mathsf{P}_t \varphi$ *exists and belongs to* $\mathbb{C}(\Omega_0)$, *and* $\lim_{t \to \infty} \|\mathsf{P}_t \varphi - \varphi\|_h = 0$ *for each* $h > 1$, *while* $\mathsf{P}_t \varphi \in \mathbb{C}_b(\Omega_0)$ *whenever* $\varphi \in \mathbb{C}_b(\Omega_0)$. *If* $\lambda < 2/d$ *in* (1.5) *then* $\varphi \in \mathbb{C}_p(\Omega_0)$ *implies* $\mathsf{P}_t \varphi \in \mathbb{C}_p(\Omega_0)$.

Proof. It is immediate, bounds for $\|\mathsf{P}_t \varphi\|_h$ follow from (1.12) and (2.28), respectively, conservation of continuity properties is a consequence of (3.7) and (1.12), while norm-continuity of P_t follows from (1.12) and (1.7) for each $h > 1$. QED

Finally we discuss relation of P_t to its formal generator G given by (0.3). The first problem is to give a reasonable definition for G. Let $\mathbb{D}_G$ denote the set of such $\varphi \in \mathbb{C}_p(\Omega_0)$ that $\mathsf{D}_k \varphi$, $\mathsf{D}_k^2 \varphi$ exist and belong to $\mathbb{C}_p(\Omega_0)$ for each $k \in S$, further

$$\sum_{k \in S} (\|c_k \mathsf{D}_k \varphi\|_h + \|\sigma_k^2 \mathsf{D}_k^2 \varphi\|_h) \leqq m h^m$$

holds for $h > 1$ with some $m > 0$ depending on φ. Since $c_k \in \mathbb{C}_p(\Omega_0)$ and $\sigma_k^2 \in \mathbb{C}_p(\Omega_0)$, cf. (1.7), (1.8); $\mathsf{G}\varphi$ is a well defined element of $\mathbb{C}_p(\Omega_0)$ whenever $\varphi \in \mathbb{D}_G$.

(3.15) **Theorem.** *If $\varphi \in \mathbb{D}_G$ then $\mathsf{P}_t \mathsf{G} \varphi$ is a $\|\cdot\|_h$-continuous function of $t \geqq 0$ for each $h > 1$, and*

$$\mathsf{P}_t \varphi = \varphi + \int_0^t \mathsf{P}_s \mathsf{G} \varphi \, ds.$$

Proof. Continuity of $\mathsf{P}_t \mathsf{G} \varphi$ follows from (3.14). Let $\mathsf{P}_t^{(n)}$ and $\mathsf{G}^{(n)}$ denote the semigroup and the formal generator associated to the partial solution $X^{(n)}$, considered in (3.8). Notice that $\varphi \in \mathbb{D}_G$ is a twice continuously differentiable function of any finite collection of its variables, thus the Ito lemma yields

$$\mathsf{P}_t^{(n)} \varphi = \varphi + \int_0^t \mathsf{P}_s^{(n)} \mathsf{G}^{(n)} \varphi \, ds,$$

whence (3.15) follows directly by (1.12) and (3.8). QED

(3.16) *Remark.* If $\varphi \in \mathbb{D}_G$ then (3.15) implies $\mathsf{G}\varphi = \lim_{t \to 0} \frac{1}{t} (\mathsf{P}_t \varphi - \varphi)$ with respect to each $\|\cdot\|_h$. We do not know such dense $\mathbb{D} \subset \mathbb{D}_G$ that $\mathsf{P}_t \mathbb{D} \subset \mathbb{D}$.

4. Reversible Evolution Laws

In this section we are assuming that c_k are given by (0.5), then the formal generator of our semigroup P_t has a self-adjoint form

(4.1) $$\mathsf{G}\varphi(x) = \tfrac{1}{2} \sum_{k \in S} \exp(H_k(x)) \, \mathsf{D}_k(\sigma_k^2(x) \exp(-H_k(x)) \, \mathsf{D}_k \varphi(x)).$$

In addition to (1.3)–(1.8) we are also assuming that each U_V has continuous third derivatives. Conditions concerning c_k can easily be reformulated in terms of σ_k.

(4.2) **Theorem.** *If μ is a Gibbs random field with interaction potential U and $\mu(\Omega_0) = 1$ then P_t is a semigroup of bounded self-adjoint operators in the Hilbert space $\mathbb{L}^2(\mu)$.*

Proof. Let $\mathbb{C}_b^2(\mathbb{R}^S)$ denote the space of such bounded $\varphi \in \mathbb{C}^2(\mathbb{R}^S)$ that first and second partial derivatives of φ are also bounded. Suppose first that $\tilde{c}_k, \tilde{\sigma}_k \in \mathbb{C}_b^2(\mathbb{R}^S)$ and $\tilde{c}_k = \tilde{\sigma}_k = 0$ if $|k| > nr$, the associated semigroup and formal generator are denoted by $\tilde{\mathsf{P}}_t^{(n)}$ and $\tilde{\mathsf{G}}^{(n)}$, respectively. Observe that $d\mu = \exp(-H_k(x))\,dx_k\,d\mu_k$, where μ_k is a finite measure on $\mathscr{R}^{S\setminus\{k\}}$, thus integrating by parts we obtain that

$$\int \varphi_1 \exp(H_k)\,\mathsf{D}_k(\tilde{\sigma}_k^2 \exp(-H_k)\,\mathsf{D}_k\,\varphi_2)\,d\mu = -\int \tilde{\sigma}_k^2\,\mathsf{D}_k\,\varphi_1\,\mathsf{D}_k\,\varphi_2\,d\mu,$$

provided that $\varphi_1, \varphi_2 \in \mathbb{C}_b^2(R^S)$ vanish at infinity, whence

$$\int \varphi_1\,\tilde{\mathsf{G}}^{(n)}\,\varphi_2\,d\mu = \int \varphi_2\,\tilde{\mathsf{G}}^{(n)}\,\varphi_1\,d\mu \tag{4.3}$$

follows for all $\varphi_1, \varphi_2 \in \mathbb{C}_b^2(\mathbb{R}^S)$ from (4.1) by a simple extension procedure.

On the other hand, $\tilde{\mathsf{P}}_t^{(n)}\varphi \in \mathbb{C}_b^2(\mathbb{R}^S)$ and $\frac{d}{dt}\tilde{\mathsf{P}}_t^{(n)}\varphi = \tilde{\mathsf{G}}^{(n)}\tilde{\mathsf{P}}_t^{(n)}\varphi$ if $\varphi \in \mathbb{C}_b^2(\mathbb{R}^S)$, see e.g. Theorem 1 of Sect. 16 in [7]. Let $\varphi_1, \varphi_2 \in \mathbb{C}_b^2(\mathbb{R}^S)$ and introduce

$$u(t,s) = \int \tilde{\mathsf{P}}_t^{(n)}\varphi_1\,\tilde{\mathsf{P}}_s^{(n)}\varphi_2\,d\mu,$$

then u is differentiable and we obtain from (4.3) that

$$\begin{aligned}\frac{\partial}{\partial s}u(t,s) &= \int \tilde{\mathsf{P}}_t^{(n)}\varphi_1\,\tilde{\mathsf{G}}^{(n)}\,\tilde{\mathsf{P}}_s^{(n)}\varphi_2\,d\mu\\ &= \int \tilde{\mathsf{P}}_s^{(n)}\varphi_2\,\tilde{\mathsf{G}}^{(n)}\,\tilde{\mathsf{P}}_t^{(n)}\varphi_1\,d\mu = \frac{\partial}{\partial t}u(t,s),\end{aligned}$$

i.e. $\frac{d}{dv}u(t-v, s+v) = 0$. This means that $u(0,t) = u(t,0)$, thus

$$\int \varphi_1\,\tilde{\mathsf{P}}_t^{(n)}\varphi_2\,d\mu = \int \varphi_2\,\tilde{\mathsf{P}}_t^{(n)}\varphi_1\,d\mu \tag{4.4}$$

if $\varphi_1, \varphi_2 \in \mathbb{C}_b^2(\mathbb{R}^S)$. Now we are in a position to conclude (4.2). Let $\mathsf{P}_t^{(n)}$ denote the transition probability kernel in the general case. Keeping n fixed we can approximate σ_k by $\tilde{\sigma}_k \in \mathbb{C}_b^2(\mathbb{R}^S)$ for $|k| \leqq nr$ in such a way that $\tilde{\mathsf{P}}_t^{(n)}(x,\cdot)$ converges weakly to $\mathsf{P}_t^{(n)}(x,\cdot)$ for each x, see [7], whence

$$\int \varphi_1\,\mathsf{P}_t^{(n)}\varphi_2\,d\mu = \int \varphi_2\,\mathsf{P}_t^{(n)}\varphi_1\,d\mu \tag{4.5}$$

follows by the dominated convergence theorem. Now letting n go to infinity, (3.8) yields

$$\int \varphi_1\,\mathsf{P}_t\,\varphi_2\,d\mu = \int \varphi_2\,\mathsf{P}_t\,\varphi_1\,d\mu \tag{4.6}$$

for $\varphi_1, \varphi_2 \in \mathbb{C}_b^2(\mathbb{R}^S)$ in the same way, which proves (4.2) because $\mathbb{C}_b^2(\mathbb{R}^S)$ is dense in $\mathbb{L}^2(\mu)$. QED

(4.7) **Corollary.** *If μ is a Gibbs random field with interaction potential U and $\mu(\Omega_0) = 1$ then $\mu\mathsf{P}_t = \mu$ for $t > 0$.*

Proof. If $\varphi_1 = 1$ then (4.6) reduces to (4.7). QED

(4.8) *Remark.* If μ is a probability measure satisfying (0.4) then (0.5) implies that μ is a Gibbs random field with interaction potential U, cf. [2, 8, 11, 18].

5. Weakly Interacting Systems

In this section such situations are considered where $U_V=0$ for $|V|>1$ may be supposed, i.e. $U_{\{k\}}(x)=U_k(x_k)=H_k(x)$ if $V=\{k\}$ for some $k\in S$, $U_V=0$ otherwise. Auxiliary functions of this kind are very suitable for the study of weakly interacting systems; in such cases (1.4) is not needed in the proof of (1.12), while (1.5) turns to be trivial.

(5.1) **Proposition.** *Suppose* (1.3), (1.6) *and* $U_V=0$ *if* $|V|>1$. *Then there exists a universal constant* K, *and for each tempered solution* $X=X(t)$ *of* (0.1) *we have such* $\mathscr{A}$*-measurable random variables,* $N_{k,\rho}$, $k\in S$, $\rho>0$, *that* $\mathsf{P}[N_{k,\rho}>u]\leqq e^{-\delta u}$ *and*

$$\text{(i)} \qquad Q(X(t),k,\rho)\leqq K\,e^{Kt}(Q(X(0),k,\rho)+N_{k,\rho})$$

hold for all $k\in S$, $\rho>1$, *where* Q *is the same as in* (2.3); *further*

$$\text{(ii)} \qquad \bar{H}(X(t))\leqq K\,e^{Kt}(\bar{H}(X(0))+N),$$

where N *is* $\mathscr{A}$*-measurable and* $\mathsf{P}[N>u]\leqq e^{-\delta u}$.

Proof. Since $f(u)\leqq 2f(v)$ if $|u-v|\leqq r$, an easy calculation shows that

$$(5.2) \qquad Q(X(t),k,\rho)\leqq Q(X(0),k,\rho)+K\int_0^t Q(X(s),k,\rho)\,ds$$
$$+\sum_{k\in S} f_k\int_0^t (\sigma_k D_k H_k\,dw_k-\delta\sigma_k^2(\mathsf{D}_k H_k)^2\,dt),$$

which proves (i) if $N_{k,\rho}$ denotes supremum over $t>0$ of the sum on the right hand side of (5.2). (ii) is a direct consequence of (i) and (2.16), (2.17). QED

(5.3) *Remark.* (1.4) was used only in the proof of (1.12), thus all results of the previous two sections hold in this case, too.

(5.4) *Remark.* Results by Doss-Royer [2] can be reproduced by choosing $H_k(x)=x_k^2$. Theorem 4.1. by Shiga-Shimizu [17] also follows in the same way. In [2] and [17] solutions are constructed in $\Omega=\{x\in\mathbb{R}^S\colon \lim_{|k|\to\infty}|x_k|\,e^{-\varepsilon|k|}=0$ for all $\varepsilon>0\}$, but in many cases (i) is sufficient for such a result, too.

Finally we outline a method for proving existence of stationary measures. For convenience we assume that the situation is translation invariant, i.e. S is a d-dimensional additive subgroup of $\mathbb{R}^d$ and $c_k(x)=c_0(\mathsf{T}_k x)$, $\sigma_k(x)=\sigma_0(\mathsf{T}_k x)$ for each $k\in S$, where 0 denotes the neutral element of S, T_k is defined by $(\mathsf{T}_k x)_j=x_{j-k}$. Suppose that we are given such nonnegative $p\colon \mathbb{R}\to\mathbb{R}$ and $q\colon \mathbb{R}\to\mathbb{R}$ that p is twice continuously differentiable, q is continuous and $\lim_{|u|\to\infty} q(u)=\infty$,

$$(5.5) \qquad p'(x_k)\,c_k(x)\leqq -q(x_k)+a+\sum_{j\neq k} b_{k-j}\,q(x_j)+\sum_{V:k\in V} J_{k,V}(x),$$

where a and b_k are constants, $b_k=0$ if $|k|>r$,

$$(5.6) \qquad \sum_{k\in S} b_k=b<1,$$

further $J_{k,V}: \mathbb{R}^S \to \mathbb{R}$ is $\mathscr{R}^V$-measurable, $J_{k,V}=0$ if $k \notin V$ or $\operatorname{diam} V > r$, and

$$\sum_{k \in V} J_{k,V}(x) \leqq a, \tag{5.7}$$

$p'(x_k)$, $q(x_k)$ and $J_{k,V}(x)$ are all bounded by a polynomial of $H_k(x,r)$, finally

$$\sigma_k^2(x)\, p''(x_k) \leqq a. \tag{5.8}$$

These conditions are natural e.g. if $q(u)=p'(u)\,h'(u)$, and

$$c_k(x) = -h'(u) - \sum_{j \in S_k(r)} U'_{k-j}(x_k - x_j),$$

where $U_{k-j}(v)=U_{j-k}(v)=U_{k-j}(-v)$, see [2].

(5.9) **Proposition.** *Under the above conditions there exists a translation invariant probability measure μ on $\mathscr{R}^S$ such that $\int q(x_k)\,\mu(dx) < \infty$, $\int \mathsf{G}\varphi\,\mu(dx)=0$ if $\varphi \in \mathbb{D}_G$ and $\mathsf{G}\varphi$ is bounded, and the restriction of μ to $\Omega_0 \cap \mathscr{R}^S$ is stationary.*

Proof. Let X be the solution (0.1) with $X_k(0)=0$ for all $k \in S$, and consider the following stochastic differential.

$$\begin{aligned} d \sum_{k \in S_0(n)} p(X_k) &= \sum_{k \in S_0(n)} (p'(X_k)\, c_k\, dt + \tfrac{1}{2}\sigma_k^2\, p''(X_k)\, dt) + \sum_{k \in S_0(n)} \sigma_k\, p'(X_k)\, dw_k \\ &\leqq -(1-b) \sum_{k \in S_0(n)} q(X_k)\, dt + \sum_{k \in S_0(n)} \sum_{j \notin S_0(n)} b_{k-j}\, q(X_j)\, dt \\ &\quad + \sum_{k \in S_0(n)} \sum_{V \not\subset S_0(n)} J_{k,V}\, dt + K n^d\, dt + \sum_{k \in S_0(n)} \sigma_k\, p'(X_k)\, dw_k. \end{aligned}$$

Observe now that the distribution of X is also translation invariant, consequently

$$\sum_{k \in S_0(n)} \mathsf{E}[p(X_k(t))] \leqq K_1 n^d (1+t) + K_t n^{d-1} - (1-b) \sum_{k \in S_0(n)} \int_0^t \mathsf{E}[q(X_k(s))]\, ds, \tag{5.10}$$

where K_1 and K_t do not depend on n. Therefore, dividing by n^d and letting n go to infinity we obtain that

$$\frac{1}{t} \int_0^t \int \mathsf{P}_s(\theta, dy)\, q(y_k)\, ds \leqq M, \tag{5.11}$$

where M is a universal constant, and θ denotes the identically zero configuration. This means that we have a probability measure μ_t on $\mathscr{R}^S$ such that

$$\int \varphi(x)\, \mu_t(dx) = \frac{1}{t} \int_0^t \int \mathsf{P}_s(\theta, dy)\, \varphi(y)\, ds \tag{5.12}$$

if $\varphi \in \mathbb{C}(\mathbb{R}^S)$ is bounded, further

$$\int q(x_k)\, \mu_t(dx) \leqq M \tag{5.13}$$

for each $k \in S$. Since $\{u \in \mathbb{R}: q(u) \leqq v\}$ is compact for each v, there exists a probability measure μ on $\mathscr{R}^S$ such that

(5.14) $$\int \varphi(x)\,\mu(dx)=\lim_{n\to\infty}\int \varphi(x)\,\mu_{t_n}(dx)$$

holds for each bounded $\varphi\in\mathbb{C}(\mathbb{R}^S)$, where t_n is an increasing sequence tending to infinity. Thus $\int q(x_k)\,\mu(dx)\leqq M$ follows from (5.13), stationarity of the restriction of μ to $\Omega_0\cap\mathscr{R}^S$ is a direct consequence of (5.12), while $\int \mathsf{G}\varphi\,d\mu=0$ follows from (5.12) by (3.15). QED

(5.15) *Remark.* We cannot prove $\mu(\Omega_0)=1$ in general, but finiteness of $\int q(x_k)\,\mu(dx)$ implies $\mu(\Omega)=1$ by the Borel-Cantelli lemma if $q(u)\geqq\varepsilon|u|^{\varepsilon}-a$ with some $\varepsilon>0$.

(5.16) *Remark.* In a forthcoming paper [5] we investigate conditions implying that a stationary measure is necessarily a Gibbs random field with a given interaction, whence (5.9) implies existence of Gibbs random fields also in such cases when general results do not apply, cf. [1, 12, 16]. We do not claim, of course, that this is the simplest way towards proving existence theorems for Gibbs random fields.

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