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Convergence of Closed Convex Sets and σ-Fields

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Summary. Let $\{C_n\}$ be a sequence of closed convex subsets in a Hilbert space H. We prove that the prediction sequence $\{p(x|C_n)\}$ converges for every $x \in H$ if and only if s-lim C_n exists and is not empty. We further show the relation between the limit of closed convex sets and the one of σ-subfields in probability measure spaces.

0. Introduction

Let (Ω, Σ, μ) be a probability measure space, and $\{\Sigma_n\}$ be a sequence of σ-subfields of Σ. Now we consider the following proposition:

(E) *There exists a σ-subfield Σ_∞ such that every sequence $\{E(f|\Sigma_n)\}$ of conditional expectations of $f \in L^p(\Omega, \Sigma, \mu)$ converges to $E(f|\Sigma_\infty)$ in L^p-norm* $(1 \leqq p < \infty)$;

Doob's martingale convergence theorem says that if $\{\Sigma_n\}$ is monotone increasing (resp. decreasing) with respect to set inclusion, Proposition (E) is true for $\Sigma_\infty = \bigvee_n \Sigma_n$ (resp. $\bigcap_n \Sigma_n$). Let $\liminf \Sigma_n$ be the lower limit and $\limsup \Sigma_n$ be the upper limit of $\{\Sigma_n\}$ with respect to set inclusion. Then the proposition is also true, if $\liminf \Sigma_n = \limsup \Sigma_n = \Sigma_\infty$ (see Dang-Ngoc [4]). On the other hand, Neveu [8, IV.3.2] introduced the notion of strong convergence of σ-subfields. We say Σ_∞ to be the strong limit of $\{\Sigma_n\}$ if every sequence $\{P(A|\Sigma_n)\}$ of conditional probabilities of $A \in \Sigma$ converges to $P(A|\Sigma_\infty)$ in measure. Proposition (E) is satisfied if and only if Σ_∞ is the strong limit of $\{\Sigma_n\}$. Becker [2] pointed out that the proposition is the consequence of weak convergence. Kudō [6] studied the notion of strong convergence of σ-subfields more precisely and applied it to the asymptotic theory of statistics. He defined another lower limit μ-$\liminf \Sigma_n$ and upper limit μ-$\limsup \Sigma_n$ of $\{\Sigma_n\}$. Σ_∞ is the strong limit of $\{\Sigma_n\}$ if and only if μ-$\liminf \Sigma_n = \mu$-$\limsup \Sigma_n = \Sigma_\infty$.

In this paper we investigate analogues theorems in Hilbert spaces. The conditional expectation $E(f|\Sigma')$ of $f \in L^2(\Omega, \Sigma, \mu)$ relative to Σ' is the best ap-

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proximation of f by elements of $L^2(\Omega, \Sigma', \mu)$. Hence Proposition (E) in the case of $p=2$ is generalized as follows. Let H be a Hilbert space, and $\{C_n\}$ be a sequence of, non-empty, closed convex subsets of H. Then we consider the following proposition:

(P) *There exists a closed convex subset* C_∞ *of* H *such that every sequence* $\{p(x|C_n)\}$ *of best approximations of* $x \in H$ *converges to* $p(x|C_\infty)$;

Brunk [3] showed that if $\{C_n\}$ is monotone increasing (resp. decreasing and if $\bigcap_n C_n \neq \emptyset$) with respect to set inclusion, Proposition (P) is true for $C_\infty = \overline{\bigcup_n C_n}$ (resp. $\bigcap_n C_n$), and applied it to the problem of maximum likelihood estimation. We generalize it for neither increasing nor decreasing case. Mosco [7] defined the strong lower limit s-liminf C_n and the weak upper limit w-limsup C_n. In Chap. 1 we prove that if s-liminf C_n = w-limsup $C_n = C_\infty \neq \emptyset$, Proposition (P) is satisfied, and that it is also the consequence of weak convergence. In Chap. 2 we study the strong lower limit and the weak upper limit of a sequence of subspaces of H, and in Chap. 3 we see the relation between the limit of closed convex sets and the limit of σ-fields. The lower limit and the upper limit of a sequence of σ-subfields are corresponding to the strong lower limit and the weak upper limit of the sequence of subspaces defined by the σ-subfields in $L^2(\Omega, \Sigma, \mu)$ respectively.

1. The Limit of a Sequence of Closed Convex Sets

Let H be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$, and $\mathfrak{C}$ be the set of all, non-empty, closed convex subsets of H. For any $x \in H$ and $C \in \mathfrak{C}$ there exists a unique closest point $p(x|C)$ of C to x. The following lemma is well known (see, for example, Brunk [3]).

Lemma 1.1. *Let* $C \in \mathfrak{C}$ *and* $x, y \in H$. *Then*

(i) $x' = p(x|C)$ *if and only if* $x' \in C$ *and* $\operatorname{Re}\langle x - x' | x' - z \rangle \geqq 0$ *for any* $z \in C$;

(ii) $\|p(x|C) - p(y|C)\| \leqq \|x - y\|$; *That is,* $p(\cdot|C)$ *is non-expansive.*

Let $\{C_n\}$ be a sequence in $\mathfrak{C}$. We can define a *lower limit* liminf C_n and an *upper limit* limsup C_n of $\{C_n\}$ with respect to set inclusion as

$$\operatorname{liminf} C_n = \overline{\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} C_n}, \qquad \operatorname{limsup} C_n = \bigcap_{m=1}^{\infty} \overline{co} \bigcup_{n=m}^{\infty} C_n,$$

where $\overline{co}$ means the closed convex hull. On the other hand, Mosco [7] defined a strong lower limit and a weak upper limit of a sequence of subsets of Banach spaces. Following it, we define a *strong lower limit* s-liminf C_n and a *weak upper limit* w-limsup C_n of $\{C_n\}$ as

$$\text{s-liminf } C_n = \{x \in H\colon\ x_n \to x \text{ as } n \to \infty,\ x_n \in C_n \text{ for every } n\},$$

$$\text{w-limsup } C_n = \overline{co}\{x \in H\colon\ x_{n'} \to x \text{ (weakly) as } n' \to \infty,\quad x_{n'} \in C_{n'} \text{ for every } n',\quad \{C_{n'}\} \text{ is a subsequence of } \{C_n\}\}.$$

If s-liminf C_n = w-limsup C_n, then the common value is denoted by s-lim C_n. Some elementary properties and examples are seen in [7].

Proposition 1.2. *Let $\{C_n\}$ be a sequence in $\mathfrak{C}$.*

(i) s-liminf $C_n = \{x \in H: p(x|C_n) \to x$ *as* $n \to \infty\}$;

(ii) s-liminf C_n *and* w-limsup C_n *are in* $\mathfrak{C} \cup \{\emptyset\}$;

(iii) liminf $C_n \subset$ s-liminf $C_n \subset$ w-limsup $C_n \subset$ limsup C_n.

Proof. (i): If $x \in$ s-liminf C_n, then

$$\|x - p(x|C_n)\| \leqq \|x - x_n\| \to 0,$$

as $n \to \infty$. The converse is trivial.

(ii): Let $x \in \overline{\text{s-liminf } C_n}$. Then for any $\varepsilon > 0$ there exists $x_0 \in$ s-liminf C_n with $\|x - x_0\| < \varepsilon/3$. On the other hand, by (i) we have an n_0 such that $\|x_0 - p(x_0|C_n)\| < \varepsilon/3$ for any $n \geqq n_0$. Hence for $n \geqq n_0$

$$\begin{aligned}\|x - p(x|C_n)\| &\leqq \|x - x_0\| + \|x_0 - p(x_0|C_n)\| \\ &\quad + \|p(x_0|C_n) - p(x|C_n)\| \\ &\leqq 2 \cdot \|x - x_0\| + \|x_0 - p(x_0|C_n)\| \leqq \varepsilon.\end{aligned}$$

Thus $\|x - p(x|C_n)\| \to 0$ as $n \to \infty$, and we have $x \in$ s-liminf C_n. Therefore s-liminf C_n is closed. It is clear from the definition that s-liminf C_n is convex and w-limsup C_n is closed convex. If $C_n = \{x_n\}$ and $\|x_n\| \to \infty$ as $n \to \infty$, then s-liminf C_n = w-limsup $C_n = \emptyset$.

(iii): The first and second inclusions are clear from the definition. The last one follows from the fact that closed convex sets are weakly closed. □

Lemma 1.3. *Let $\{C_n\}$ be a sequence in $\mathfrak{C}$ such that* s-liminf $C_n \neq \emptyset$. *Then for every $x \in H$*

(i) $\sup_n \|p(x|C_n)\| < \infty$;

(ii) *If $\{p(x|C_n)\}$ weakly converges to x, then $\|x - p(x|C_n)\| \to 0$ as $n \to \infty$.*

Proof. Let x_0 be an element of s-liminf C_n. Since $\{p(x_0|C_n)\}$ is a norm converging sequence by Proposition 1.2(i), $\sup_n \|p(x_0|C_n)\| < \infty$. On the other hand

$$\begin{aligned}\|p(x|C_n)\| &\leqq \|p(x|C_n) - x\| + \|x\| \\ &\leqq \|p(x_0|C_n) - x\| + \|x\| \\ &\leqq \|p(x_0|C_n)\| + 2 \cdot \|x\|,\end{aligned}$$

for every n. Hence we have (i). If $\{p(x|C_n)\}$ weakly converges to x,

$$\begin{aligned}\|x - p(x|C_n)\|^2 &\leqq \|x - p(x|C_n)\|^2 + \mathrm{Re}\langle x - p(x|C_n) | p(x|C_n) - p(x_0|C_n)\rangle \\ &= \mathrm{Re}\langle x - p(x|C_n) | x - x_0\rangle + \mathrm{Re}\langle x - p(x|C_n) | x_0 - p(x_0|C_n)\rangle \\ &\leqq \mathrm{Re}\langle x - p(x|C_n) | x - x_0\rangle \\ &\quad + (\|x\| + \|p(x|C_n)\|) \cdot \|x_0 - p(x_0|C_n)\| \to 0,\end{aligned}$$

as $n \to \infty$. Thus we have (ii). □

Theorem 1.4. *Let $\{C_n\}$ be a sequence in $\mathfrak{C}$. Then the following assertions are equivalent:*

(i) s-lim C_n *exists and is not empty;*

(ii) s-liminf C_n *is not empty and there exists* $C_\infty \in \mathfrak{C}$ *such that* $\{p(x|C_n)\}$ *weakly converges to* $p(x|C_\infty)$ *for every* $x \in H$;

(iii) $\{p(x|C_n)\}$ *is a norm convergence sequence for every* $x \in H$.

Moreover, if these assertions are satisfied, $\{p(x|C_n)\}$ *converges to* $p(x|\text{s-lim } C_n)$ *for every* $x \in H$.

Proof. (i) $\Rightarrow$ (ii): Let $C_\infty = \text{s-lim } C_n$. For any $x \in H$, by Lemma 1.3 (i), $\{p(x|C_n)\}$ is norm bounded. Hence for any subsequence $\{p(x|C_{n'})\}$ of $\{p(x|C_n)\}$ there exists a subsequence $\{p(x|C_{n''})\}$ which weakly converges to some $x' \in H$. Then $x' \in \text{w-limsup } C_n = C_\infty$. On the other hand, for any $y \in C_\infty$, since $y = \text{s-lim } p(y|C_n)$, we have

$$\|x - x'\| \leqq \lim \|x - p(x|C_{n''})\| \leqq \lim \|x - p(y|C_{n''})\| = \|x - y\|,$$

where the first inequality follows from weak lower semicontinuity of norm $\|\cdot\|$. Thus $x' = p(x|C_\infty)$ and we have (ii).

(ii) $\Rightarrow$ (iii): Any $x \in H$ is fixed. Since $\{p(p(x|C_\infty)|C_n)\}$ weakly converges to $p(x|C_\infty)$, by Lemma 1.3 (ii), we have $p(x|C_\infty) = \text{s-lim } p(p(x|C_\infty)|C_n)$. Hence by Lemma 1.1 (i)

$$\begin{aligned}\text{limsup } \|p(x|C_n)\|^2 \leqq\ & \text{limsup } \|p(x|C_n)\|^2 \\ & + \text{liminf} \langle x - p(x|C_n) | p(x|C_n) - p(p(x|C_\infty)|C_n) \rangle \\ =\ & \|p(x|C_\infty)\|^2.\end{aligned}$$

On the other hand, because of weak lower semicontinuity of norm $\|\cdot\|$, we have $\text{liminf} \|p(x|C_n)\| \geqq \|p(x|C_\infty)\|$. Thus $\lim \|p(x|C_n)\| = \|p(x|C_\infty)\|$, and by one of the convex properties of Hilbert space norm (see, for example, Day [5]) we have $p(x|C_\infty) = \text{s-lim } p(x|C_n)$.

(iii) $\Rightarrow$ (i): We put $C_\infty = \{\text{s-lim } p(x|C_n) : x \in H\}$. It is clear that $C_\infty \subset \text{s-liminf } C_n$. Now assume that x is an element such that there exist a subsequence $\{C_{n'}\}$ of $\{C_n\}$ and $x_{n'} \in C_{n'}$ for every n' with $x_{n'} \to x$ (weakly) as $n' \to \infty$. Let $y = \text{s-lim } p(x|C_n)$. From Lemma 1.1 we have

$$\text{Re} \langle x - p(x|C_{n'}) | p(x|C_{n'}) - x_{n'} \rangle \geqq 0$$

for every n'. Tending n' to ∞, $\text{Re} \langle x - y | y - x \rangle \geqq 0$. Therefore $x = y \in C_\infty$. Thus $C_\infty = \text{s-liminf } C_n = \text{w-limsup } C_n$. □

If $\{C_n\}$ is increasing (resp. decreasing and if $\bigcap_n C_n \neq \emptyset$), then by Proposition 1.2 (iii) and the above theorem it follows that $\{p(x|C_n)\}$ converges to $p\left(x \middle| \overline{\bigcup_{n=1}^{\infty} C_n}\right)$ $\left(\text{resp. } p\left(x \middle| \bigcap_{n=1}^{\infty} C_n\right)\right)$ for every $x \in H$. See Brunk [3].

2. The Limit of a Sequence of Subspaces

Let $\{H_n\}$ be a sequence of subspaces (i.e., closed linear manifolds) of H. Then we can easily see that both s-liminf H_n and w-limsup H_n are subspaces of H.

Proposition 2.1. *Let $\{H_n\}$ be a sequence of subspaces of H. Then*

(i) s-liminf $H_n^{\perp} = (\text{w-limsup}\, H_n)^{\perp}$;
(ii) w-limsup $H_n^{\perp} = (\text{s-liminf}\, H_n)^{\perp}$.

Proof. For the simplicity we denote $p(\cdot|H_n)$ by P_n, which is the orthogonal projection onto H_n, for every n. Suppose that $x \in \text{s-liminf}\, H_n^{\perp}$. Then $\{P_n x\}$ converges to 0. Let y be any element such that there exists a subsequence $\{H_{n'}\}$ and $y_{n'} \in H_{n'}$ for every n' with $y_{n'} \to y$ (weakly) as $n' \to \infty$. Then

$$\begin{aligned}|\langle x|y\rangle| &= \lim |\langle x|y_{n'}\rangle| = \lim |\langle P_{n'} x|y_{n'}\rangle| \\ &\leqq \lim \|P_{n'} x\| \cdot \|y_{n'}\| = 0.\end{aligned}$$

Therefore $x \in (\text{s-liminf}\, H_n)^{\perp}$. Conversely suppose that $x \in (\text{w-limsup}\, H_n)^{\perp}$. It suffices to prove that $\{P_n x\}$ converges to 0. Since $\{P_n x\}$ is norm bounded, for any subsequence $\{P_{n'} x\}$ of $\{P_n x\}$ there exists a subsubsequence $\{P_{n''} x\}$ which weakly converges to some $y \in H$. Then $y \in \text{w-limsup}\, H_n$. Hence we have

$$\|P_{n''} x\|^2 = \langle P_{n''} x|x\rangle \to \langle y|x\rangle = 0,$$

as $n'' \to \infty$. Therefore $\{P_{n''} x\}$ converges to 0, and $\{P_n x\}$ does. Thus (i) is proved. (ii) follows from (i). □

Theorem 2.2. *Let $\{H_n\}$ be a sequence of subspaces of H. Then*

(i) s-liminf H_n *is the maximum subspace among subspaces H' of H with*

$$\liminf \|p(x|H_n)\| \geqq \|p(x|H')\|, \tag{1}$$

for all $x \in H$;

(ii) w-limsup H_n *is the minimum subspace among subspaces H' of H with*

$$\limsup \|p(x|H_n)\| \leqq \|p(x|H')\|, \tag{2}$$

for all $x \in H$.

Proof. We denote $p(\cdot|H')$ (resp. $p(\cdot|H_n)$) by P' (resp. P_n).

(i) We first show that $H' = \text{s-liminf}\, H_n$ satisfies (1). Since $P_n x \to x$ as $n \to \infty$ for every $x \in H'$,

$$\begin{aligned}\|P_n x\|^2 &= \|P_n P' x + P_n(1-P')x\|^2 \\ &= \langle P_n P' x|P' x\rangle + \langle P_n P' x|(1-P')x\rangle \\ &\quad + \langle (1-P')x|P_n P' x\rangle + \|P_n(1-P')x\|^2 \\ &\geqq \langle P_n P' x|P' x\rangle + \langle P_n P' x|(1-P')x\rangle \\ &\quad + \langle (1-P')x|P_n P' x\rangle \to \|P' x\|^2,\end{aligned}$$

as $n \to \infty$ for any $x \in H$. Conversely we shall show that any subspace H' of H satisfying (1) is contained in s-liminf H_n. It suffices to prove that $\{P_n P' x\}$

converges to $P'x$ for any $x \in H$. From (1) it follows that

$$\|P'x\|^2 \leqq \liminf \|P_n P'x\|^2 \leqq \limsup \|P_n P'x\|^2 \leqq \|P'x\|^2,$$

and that $\lim \|P_n P'x\| = \|P'x\|$. Hence

$$\|P'x - P_n P'x\|^2 = \|P'x\|^2 - \langle P'x | P_n P'x\rangle - \langle P_n P'x | P'x\rangle + \langle P_n P'x | P'x\rangle$$
$$= \|P'x\|^2 - \|P_n P'x\|^2 \to 0,$$

as $n \to \infty$.

(ii): Now suppose that $H' = \text{w-limsup}\, H_n$. Then $H'^{\perp} = \text{s-liminf}\, H_n^{\perp}$ and from (1) we have

$$\liminf \|(1 - P_n)x\|^2 \geqq \|(1 - P')x\|^2,$$

for any $x \in H$. Hence

$$\|x\|^2 - \limsup \|P_n x\|^2 \geqq \|x\|^2 - \|P'x\|^2,$$

and we have $\limsup \|P_n x\| \leqq \|P'x\|$ for any $x \in H$. Conversely let H' be a subspace of H with (2). Then by the converse calculation above we have $H' \supset \text{w-limsup}\, H_n$. □

3. The Limit of a Sequence of σ-fields

Let (Ω, Σ, μ) be a probability measure space. For the simplicity we assume that Σ and every σ-subfield of Σ considered in this section are μ-complete (i.e., all μ-null sets are contained). Given a σ-subfield Σ', we denote by $L^p(\Sigma')$ the Banach space of p-th integrable Σ'-measurable functions, and by $E(\cdot|\Sigma')$ the conditional expectation with respect to Σ'. The norm on $L^p(\Sigma)$ is denoted by $\|\cdot\|_p$.

Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Kudō [6] defined a *lower limit* μ-$\liminf \Sigma_n$ and an *upper limit* μ-$\limsup \Sigma_n$ as follows:

(i) μ-$\liminf \Sigma_n$ is the σ-subfield Σ_0 such that if $\Sigma' = \Sigma_0$, then for every bounded measurable function f

$$\liminf \|E(f|\Sigma_n)\|_1 \geqq \|E(f|\Sigma')\|_1, \tag{1}$$

and that any σ-subfield Σ' satisfying (1) is contained in Σ_0;

(ii) μ-$\limsup \Sigma_n$ is the σ-subfield Σ_0 such that if $\Sigma' = \Sigma_0$, then for every bounded measurable function f

$$\limsup \|E(f|\Sigma_n)\|_1 \leqq \|E(f|\Sigma')\|_1, \tag{2}$$

and that any σ-subfield Σ' satisfying (2) contains Σ_0.

Theorem 3.1 (Kudō [6]). *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then*

$$\mu\text{-}\liminf \Sigma_n = \{A \in \Sigma: \text{ there exist } A_n \in \Sigma_n \text{ for every } n \text{ with } \mu(A_n \triangle A) \to 0 \text{ as } n \to \infty\}.$$

On the other hand, since $\{L^2(\Sigma_n)\}$ is a sequence of subspace of Hilbert space $L^2(\Sigma)$, we can define s-liminf $L^2(\Sigma_n)$ and w-limsup $L^2(\Sigma_n)$ as is defined in the previous section.

Theorem 3.2. *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then* s-liminf $L^2(\Sigma_n)=L^2(\mu$-liminf $\Sigma_n)$.

Proof. We first show that there exists a σ-subfield Σ' such that $L^2(\Sigma')=$s-liminf $L^2(\Sigma_n)$. It suffices to prove that s-liminf $L^2(\Sigma_n)$ is a lattice and has constant functions (see Schaefer [10, Proposition 11.2]). Let f and g belong to s-liminf $L^2(\Sigma_n)$. Then there exist f_n, $g_n \in L^2(\Sigma_n)$ for every n such that $f_n \to f$ and $g_n \to g$ as $n\to\infty$ in the square mean. Since $(a\vee b - c\vee d)^2 \leqq (a-c)^2+(b-d)^2$ for any $a,b,c,d\in\mathbb{R}$, where $a\vee b$ (resp. $c\vee d$) is the maximum of $\{a,b\}$ (resp. $\{c,d\}$), we have that $\|f_n\vee g_n - f\vee g\|_2^2 \leqq \|f_n - f\|_2^2 + \|g_n - g\|_2^2 \to 0$ as $n\to\infty$. Hence we have $f\vee g\in$s-liminf $L^2(\Sigma_n)$, because $f_n\vee g_n\in L^2(\Sigma_n)$ for every n. It is clear that s-liminf $L^2(\Sigma_n)$ has constant functions. Hence we define Σ' as the σ-subfield such as $L^2(\Sigma')=$s-liminf $L^2(\Sigma_n)$. If $\Sigma'=\mu$-liminf Σ_n is shown, the theorem is proved.

Let A belong to Σ'. Then there exist $f_n\in L^2(\Sigma_n)$ for every n such that $f_n\to 1_A$ as $n\to\infty$ in the square mean and hence in measure. For each n we define $A_n=\{\omega\in\Omega\colon f_n(\omega)\geqq 1/2\}$. Then $\mu(A_n\triangle A)\to 0$ as $n\to\infty$, because $A_n\triangle A=\{\omega\in\Omega\colon (1_A(\omega)=0$ and $f_n(\omega)\geqq 1/2)$ or $(1_A(\omega)=1$ and $f_n(\omega)<1/2)\}\subset\{\omega\in\Omega\colon |1_A(\omega)-f_n(\omega)|\geqq 1/2\}$ for every n. Since $A_n\in\Sigma_n$ for every n, by Theorem 3.1 we have $A\in\mu$-liminf Σ_n.

Conversely let $A\in\mu$-liminf Σ_n. Then by Theorem 3.1 there exist $A_n\in\Sigma_n$ for every n such that $\mu(A_n\triangle A)\to 0$ as $n\to\infty$. Hence

$$\|1_{A_n}-1_A\|_2^2=\int |1_{A_n}-1_A|^2\,d\mu=\int 1_{A_n\triangle A}\,d\mu=\mu(A_n\triangle A)\to 0,$$

as $n\to\infty$. Thus $1_A\in$s-liminf $L^2(\Sigma_n)$ and $A\in\Sigma'$. □

Theorem 3.3. *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then μ-liminf Σ_n is the maximum σ-subfield among σ-subfields Σ' of Σ with*

$$\liminf \|E(f|\Sigma_n)\|_2 \geqq \|E(f|\Sigma')\|_2, \tag{3}$$

for every $f\in L^2(\Sigma)$.

Proof. Since $E(\cdot|\Sigma_n)$ is the orthogonal projection onto $L^2(\Sigma_n)$ for every n and from Theorem 3.2 $E(\cdot|\mu$-liminf $\Sigma_n)$ is the one onto s-liminf $L^2(\Sigma_n)$, it follows from Theorem 2.2 (i) that $\Sigma'=\mu$-liminf Σ_n satisfies (3).

Conversely let Σ' be a σ-subfield satisfying (3). Then by Theorem 2.2 (i) and Theorem 3.2 we have $L^2(\mu$-liminf $\Sigma_n)\supset L^2(\Sigma')$. Hence μ-liminf $\Sigma_n\supset\Sigma'$. □

Theorem 3.4. *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then for any $f\in L^1(\Sigma)$ the following assertions are equivalent:*

(i) *f is μ-liminf Σ_n-measurable;*

(ii) *$\|E(f|\Sigma_n)-f\|_1\to 0$ as $n\to\infty$.*

Proof. (i) $\Rightarrow$ (ii): Since $f\in L^1(\mu$-liminf $\Sigma_n)$, for any $\varepsilon>0$ there exists $f_0\in L^2(\mu$-liminf $\Sigma_n)$ with $\|f-f_0\|_1\leqq\varepsilon/3$. By Theorem 3.2 and Proposition 1.2 (i) we have that $\|E(f_0|\Sigma_n)-f_0\|_2\to 0$ as $n\to\infty$. Therefore there exists n_0 such that for any $n\geqq n_0$

$$\|E(f|\Sigma_n)-f\|_1 \leqq \|E(f|\Sigma_n)-E(f_0|\Sigma_n)\|_1 + \|E(f_0|\Sigma_n)-f_0\|_1 + \|f_0-f\|_1 \leqq 2\cdot\|f-f_0\|_1 + \|E(f_0|\Sigma_n)-f_0\|_2 < \varepsilon.$$

Thus $\|E(f|\Sigma_n)-f\|_1 \to 0$ as $n\to\infty$.

(ii) $\Rightarrow$ (i): We define $M=\{f\in L^1(\Sigma)\colon \|E(f|\Sigma_n)-f\|_1\to 0$ as $n\to\infty\}$. Let f, $g\in M$. Then in the same way of the proof of Theorem 3.2 we have $\|E(f|\Sigma_n)\vee E(g|\Sigma_n)-f\vee g\|_1\to 0$ as $n\to\infty$. Therefore

$$\begin{aligned}\|E(f\vee g|\Sigma_n)-f\vee g\|_1 &\leqq \|E(f\vee g|\Sigma_n)-E(f|\Sigma_n)\vee E(g|\Sigma_n)\|_1 \\ &\quad + \|E(f|\Sigma_n)\vee E(g|\Sigma_n)-f\vee g\|_1 \\ &= \|E(f\vee g-E(f|\Sigma_n)\vee E(g|\Sigma_n)|\Sigma_n)\|_1 \\ &\quad + \|E(f|\Sigma_n)\vee E(g|\Sigma_n)-f\vee g\|_1 \\ &\leqq 2\cdot\|f\vee g-E(f|\Sigma_n)\vee E(g|\Sigma_n)\|_1\to 0,\end{aligned}$$

as $n\to\infty$. Thus $f\vee g\in M$. It can be easily seen that M is closed linear and has constant functions. Hence there exists a σ-subfield Σ' such that $M=L^1(\Sigma')$. For any $f\in L^\infty(\Sigma')$

$$\|E(f|\Sigma_n)-f\|_2^2 \leqq \|E(f|\Sigma_n)-f\|_\infty\cdot\|E(f|\Sigma_n)-f\|_1 \leqq 2\cdot\|f\|_\infty\cdot\|E(f|\Sigma_n)-f\|_1\to 0,$$

as $n\to\infty$. Therefore $f\in L^2(\mu\text{-liminf}\,\Sigma_n)$. Thus $L^\infty(\Sigma')\subset L^2(\mu\text{-liminf}\,\Sigma_n)$. Taking here the closure in L^1-norm for both sides, we have $M\subset L^1(\mu\text{-liminf}\,\Sigma_n)$. □

Let $\{\Sigma_n\}$ be a sequence of σ-subfields of Σ. We define $(\Sigma)_p$ as the family of σ-subfields Σ' of Σ with

$$\limsup\|E(f|\Sigma_n)\|_p \leqq \|E(f|\Sigma')\|_p,$$

for every bounded measurable function f $(p=1,2)$.

Lemma 3.5. *If $\Sigma'\in(\Sigma)_1\cup(\Sigma)_2$, then*

$$\limsup\|E(E(f|\Sigma')|\Sigma_n)\|_p=\limsup\|E(f|\Sigma_n)\|_p,$$

for every bounded measurable function f and $p=1,2$.

Proof. Let $\Sigma'\in(\Sigma)_1$. Then for every bounded measurable function f we have

$$\limsup\|E(E(f|\Sigma')-f|\Sigma_n)\|_1 \leqq \|E(E(f|\Sigma')-f|\Sigma')\|_1=0.$$

Hence

$$|\limsup\|E(E(f|\Sigma')|\Sigma_n)\|_1-\limsup\|E(f|\Sigma_n)\|_1| \leqq \limsup\|E(E(f|\Sigma')|\Sigma_n)-E(f|\Sigma_n)\|_1=0.$$

On the other hand, for every bounded measurable function f

$$\begin{aligned}&\limsup\|E(E(f|\Sigma')-f|\Sigma_n)\|_2^2 \\ &\quad\leq \limsup\|E(E(f|\Sigma')-f|\Sigma_n)\|_\infty\cdot\|E(E(f|\Sigma')-f|\Sigma_n)\|_1 \\ &\quad\leqq \|E(f|\Sigma')-f\|_\infty\cdot\limsup\|E(E(f|\Sigma')-f|\Sigma_n)\|_1=0.\end{aligned}$$

Hence

$$|\limsup \|E(E(f|\Sigma')|\Sigma_n)\|_2 - \limsup \|E(f|\Sigma_n)\|_2|$$
$$\leqq \limsup \|E(E(f|\Sigma')|\Sigma_n) - E(f|\Sigma_n)\|_2 = 0.$$

The lemma is proved for $\Sigma' \in (\Sigma)_2$ similarly. □

Lemma 3.6. *If $\Sigma' \in (\Sigma)_1$ and $\Sigma'' \in (\Sigma)_2$, then $\Sigma' \cap \Sigma'' \in (\Sigma)_1 \cap (\Sigma)_2$.*

Proof. We denote $E(\cdot|\Sigma')$ and $E(\cdot|\Sigma'')$ by E' and E'' respectively. Let f be a bounded measurable function. We show that $\limsup \|E(f|\Sigma_n)\|_1 \leqq \|(E' E'')^m f\|_1$ for every $m \in \mathbb{N}$. This is true for $m=1$. We assume that it is true for $m=k$. Then using Lemma 3.5 twice, we have that

$$\|(E' E'')^{k+1} f\|_1 = \|(E' E'')^k E' E'' f\|_1 \geqq \limsup \|E(E' E'' f|\Sigma_n)\|_1$$
$$= \limsup \|E(f|\Sigma_n)\|_1.$$

Hence it is true for $m=k+1$. Tending here $m \to \infty$, we have

$$\limsup \|E(\ |\Sigma_n)\|_1 \leqq \|E(f|\Sigma' \cap \Sigma'')\|_1$$

(see [6, Lemma 3.2]). We can prove this inequality for norm $\|\cdot\|_2$ in the same way. Thus we have the lemma. □

Theorem 3.7. *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then μ-limsup Σ_n is the minimum σ-subfield among σ-subfields Σ' of Σ with*

$$\limsup \|E(f|\Sigma_n)\|_2 \leqq \|E(f|\Sigma')\|_2, \tag{4}$$

for every $f \in L^2(\Sigma)$.

Proof. We can easily see that (4) is satisfied for every $f \in L^2(\Sigma)$ if and only if it is so for every bounded measurable function f. On the other hand, by Lemma 3.6 we have $(\Sigma)_1 = (\Sigma)_2$. Hence the theorem is proved. □

Theorem 3.8. *Let $\{\Sigma_n\}$ be a sequence of σ-subfields. Then the closed linear lattice generated by* w-limsup $L^2(\Sigma_n)$ *is equal to* $L^2(\mu\text{-limsup}\,\Sigma_n)$.

Proof. It is the direct consequence of Theorem 2.2(ii) and Theorem 3.7. □

We discuss in [13] the relations between the results of Sect. 1 and 2 and the geometrical properties of Banach space norms. On the other hand, a referee pointed out in the light of his unpublished works that the theorems in Sect. 3 hold on general L^p spaces ($1 \leqq p < \infty$). The author would like to express his gratitude to this referee for his very useful comments and also to Professor H. Umegaki for his valuable advice and constant encouragement.

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