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A Note on a Theorem of Berkes and Philipp

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Summary. We give an example that shows that for strongly mixing sequences of random variables with values in infinite-dimensional spaces no analogue to an approximation theorem of Berkes and Philipp holds.

1. Introduction

Let $\{X_k, k \geqq 1\}$ be a sequence of random variables with values in the Polish space B. Let $\mathfrak{M}_a^b$ denote the σ-field generated by the random variables $X_a, X_{a+1}, \ldots, X_b$.

The sequence $\{X_k, k \geqq 1\}$ is called

i) strong mixing, if for some sequence $\alpha(n) \downarrow 0$ we have

$$|P(A \cap B) - P(A)\,P(B)| \leqq \alpha(n) \quad \text{for all } A \in \mathfrak{M}_1^k,\ B \in \mathfrak{M}_{k+n}^\infty \text{ and all } k, n \geqq 1 \tag{1}$$

ii) absolutely regular, if for some sequence $\beta(n) \downarrow 0$ we have

$$E\Big(\sup_{A \in \mathfrak{M}_{k+n}^\infty} |P(A \mid \mathfrak{M}_1^k) - P(A)|\Big) \leqq \beta(n) \quad \text{for all } k, n \geqq 1 \tag{2}$$

iii) φ-mixing or uniformly mixing, if for some sequence $\varphi(n) \downarrow 0$ we have

$$|P(A \cap B) - P(A)\,P(B)| \leqq \varphi(n)\,P(A) \quad \text{for all } A \in \mathfrak{M}_1^k,\ B \in \mathfrak{M}_{k+n}^\infty \text{ and all } k, n \geqq 1. \tag{3}$$

It is well known that every φ-mixing sequence is absolutely regular and that every absolutely regular sequence is strong mixing. On the other hand there are examples of strongly mixing sequences that are not absolutely regular and of absolutely regular sequences that are not φ-mixing. One can construct these examples by using theorems of Ibragimov and Rozanov (1978) that characterize Gaussian sequences satisfying (1)–(3).

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In a recent paper Berkes and Philipp (1979) proved some approximation theorems for sequences satisfying one of these conditions (1)–(3). Roughly speaking they could show that the sequence $\{X_k, k \geqq 1\}$ can be approximated by a sequence of independent random variables $\{Y_k, k \geqq 1\}$, such that $d(X_k, Y_k)$ converges to zero in probability if $\{X_k\}$ is φ-mixing or if $\{X_k\}$ is strong mixing and the space B is a finite-dimensional Euclidean space. As was already mentioned in Dehling and Philipp (1982) one can easily show that this approximation theorem also holds for absolutely regular sequences with values in general Polish spaces. Now of course the question arises, whether the approximation theorem also holds for strongly mixing sequences with values in an infinite-dimensional space. It is the goal of this note to show that this is impossible. In fact we shall construct examples of strongly mixing sequences of ℓ^2-valued random variables X_k, which cannot be approximated by independent random variables Y_k in such a way that $X_k - Y_k$ converges to zero in probability.

2. The Example

Let us first introduce some more notation.

If $(\Omega, \mathfrak{A}, P)$ is a probability space and $\mathfrak{F}, \mathfrak{G}$ are sub-σ-fields of $\mathfrak{A}$, then we define the following distances of $\mathfrak{F}$ and $\mathfrak{G}$

$$\alpha(\mathfrak{F}, \mathfrak{G}) = \sup_{F \in \mathfrak{F}, G \in \mathfrak{G}} |P(F \cap G) - P(F)\, P(G)|$$

$$\beta(\mathfrak{F}, \mathfrak{G}) = E(\sup_{G \in \mathfrak{G}} |P(G \mid \mathfrak{F}) - P(G)|).$$

With this notation, of course, a sequence of random variables $\{X_j, j \geqq 1\}$ is strongly mixing (absolutely regular) iff

$$\sup_{k \geqq 1} \alpha(\mathfrak{M}_1^k, \mathfrak{M}_{k+n}^\infty) \xrightarrow{n \to \infty} 0 \quad (\sup_{k \geqq 1} \beta(\mathfrak{M}_1^k, \mathfrak{M}_{k+n}^\infty) \xrightarrow{n \to \infty} 0).$$

For our purposes here we need another representation of $\beta(\mathfrak{F}, \mathfrak{G})$. We define two probability measures P_1 and P_2 on $(\Omega, \mathfrak{F}) \otimes (\Omega, \mathfrak{G})$ by $P_1(F \times G) = P(F \cap G)$ and $P_2(F \times G) = P(F)\, P(G)$ for $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$. Put

$$\beta'(\mathfrak{F}, \mathfrak{G}) = \sup_{A \in \mathfrak{F} \otimes \mathfrak{G}} |P_1(A) - P_2(A)|.$$

Lemma 1 (Volkonskii and Rozanov, 1959).

$$\beta' \equiv \beta.$$

As we mentioned in the introduction, there exists a stochastic process $\{Z_j, j \geqq 1\}$, which is strongly mixing, but not absolutely regular. Hence if we define $\mathfrak{M}_a^b$ as before, we get:

$$\begin{aligned} &\sup_k \alpha(\mathfrak{M}_1^k, \mathfrak{M}_{k+n}^\infty) \to 0 \quad \text{as } n \to \infty \\ &\sup_k \beta'(\mathfrak{M}_1^k, \mathfrak{M}_{k+n}^\infty) \not\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4}$$

Let (α_n) be any sequence of positive real numbers. Then by (4) there exists a constant $C>0$ and a sequence of σ-fields $\mathfrak{F}_k$, $\mathfrak{G}_k$, such that

$$\alpha(\mathfrak{F}_k, \mathfrak{G}_k) \leqq \alpha_{2k},$$
$$\beta'(\mathfrak{F}_k, \mathfrak{G}_k) \geqq C.$$

Hence there exists for each k a set $C_k \in \mathfrak{F}_k \otimes \mathfrak{G}_k$ such that

$$P_1^k(C_k) - P_2^k(C_k) \geqq C.$$

A measure-theoretic argument shows that we can approximate C_k by a disjoint union $\dot{\bigcup}_{(i,j)\in I} F_i^k \times G_j^k$ where $\{F_i^k, i \geqq 1\}$, $\{G_j^k, j \geqq 1\}$ are disjoint partitions of Ω into sets from $\mathfrak{F}_k$ resp. $\mathfrak{G}_k$ and $I \subset \mathbb{N} \times \mathbb{N}$ such that

$$\sum_{(i,j)\in I} P(F_i^k \times G_j^k) - P(F_i^k)\, P(G_j^k) \geqq C/2. \tag{5}$$

Now we define $\mathfrak{F}_k$ resp. $\mathfrak{G}_k$-measurable random variables U_k and V_k with values in ℓ^2 as follows:

$$U_k = \sum_i e_i I_{F_i^k} \qquad V_k = \sum_j e_j I_{G_j^k}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots)$ is the i-th standard basis vector in ℓ^2.

Recall that the Prohorov distance π between two probability measures μ and λ on a metric space (S, d) is defined as $\pi(\mu; \lambda) = \inf\{\varepsilon > 0\colon \mu(A) < (A^\varepsilon) + \varepsilon$, for all Borel sets $A\}$. Here $A^\varepsilon = \{y \in S\colon d(x, y) < \varepsilon,\ x \in A\}$. For a random variable X let P_X denote the probability measure induced by X.

We investigate now the Prohorov-distance π of the two measures $P_{(U_k, V_k)}$ and $P_{U_k} \times P_{V_k}$ on $\ell^2 \times \ell^2$, where $\ell^2 \times \ell^2$ is equipped with the maximum-norm.

Lemma 2. $\pi(P_{(U_k, V_k)};\ P_{U_k} \times P_{V_k}) \geqq C/2$.

Proof. Let $A = \{(e_i, e_j) \mid (i,j) \in I\}$. Then

$$P_{(U_k, V_k)}(A) = P\Big(\dot{\bigcup_{(i,j)\in I}} F_i^k \cap G_j^k\Big) = \sum_{(i,j)\in I} P(F_i^k \cap G_j^k)$$

and

$$P_{U_k} \times P_{V_k}(A) = \sum_{(i,j)\in I} P(F_i^k)\, P(G_j^k).$$

Since the (e_i, e_j)'s have mutually the distance 1 we know that $A^{C/2}$ does not contain more mass than A under $P_{U_k} \times P_{V_k}$ or $P_{(U_k, V_k)}$. Hence by (5) $P_{(U_k, V_k)}(A) \geqq P_{U_k} \times P_{V_k}(A^{C/2}) + C/2$, which proves Lemma 2.

Now define $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P}) = \bigotimes_{k=1}^{\infty} (\Omega, \mathfrak{A}, P)$. On $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P})$ we can define the following sequence $\{X_j, j \geqq 1\}$ of ℓ^2-valued random variables:

$$X_j = \begin{cases} U_k & \text{if } j = 2k-1 \\ V_k & \text{if } j = 2k. \end{cases}$$

Thus the pairs (X_{2k-1}, X_{2k}) form a sequence of independent $\ell^2 \times \ell^2$ valued random variables. In order to compute the strong mixing coefficients of $\{X_k, k \geqq 1\}$ we need the following lemma, which is easily proved.

Lemma 3. *Let $\mathfrak{F}$, $\mathfrak{G}$ be two σ-algebras and let $\mathfrak{D}$ be independent of $\mathfrak{F} \vee \mathfrak{G}$. Then* $\alpha(\mathfrak{F}, \mathfrak{G} \vee \mathfrak{D}) \leqq \alpha(\mathfrak{F}, \mathfrak{G})$.

By applying Lemma 3 twice we can show that if $\mathfrak{C}$ is a 4^{th} σ-field independent of $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{D}$, then

$$\alpha(\mathfrak{F} \vee \mathfrak{D}, \mathfrak{G} \vee \mathfrak{C}) \leqq \alpha(\mathfrak{F}, \mathfrak{G}).$$

Let $\mathfrak{M}_a^b$ as always be the σ-field generated by $X_a, \ldots, X_b$. Then we get

$$\alpha(\mathfrak{M}_1^{k-1}, \mathfrak{M}_k^\infty) \leqq \begin{cases} 0 & \text{if } k=2j-1 \\ \alpha(\mathfrak{F}_j, \mathfrak{G}_j) & \text{if } k=2j \end{cases}$$
$$\leqq \alpha_k.$$

Now assume the sequence $\{X_j, j \geqq 1\}$ could be approximated by a sequence $\{Y_j, j \geqq 1\}$ of independent random variables with $L(X_j) = L(Y_j)$, such that $X_j - Y_j$ converges to zero in probability. Then there exists a sequence of nonnegative real numbers (γ_j) converging to zero such that

$$P\{\|X_j - Y_j\| > \gamma_j\} < \gamma_j.$$

Hence $P\{d((X_{2j-1}, X_{2j}); (Y_{2j-1}, Y_{2j})) \geqq \gamma_{2j-1} + \gamma_{2j}\} \leqq \gamma_{2j-1} + \gamma_{2j}$. But this contradicts Lemma 2 since by construction we have:

$$(X_{2j-1}, X_{2j}) \quad \text{has distribution } P_{(U_j, V_j)}$$

and

$$(Y_{2j-1}, Y_{2j}) \quad \text{has distribution } P_{U_j} \times P_{V_j}.$$

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