Z. Wahrscheinlichkeitstheorie verw. Gebiete
62, 1–6 (1983)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Complete Characterization of the Upper and Lower Class of the Record and Inter-Record Times of an I.I.D. Sequence

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Summary. If $\{X_n,\ n \geqq 1\}$ is an i.i.d. sequence of continuously distributed random variables, and if n_k is for $k=1,2,\ldots$ the index of the k-th upper outstanding value of the sequence, the record times sequence is defined as $\{n_k,\ k \geqq 1\}$, whereas the inter-record times sequence is defined as $\{\Delta_k = n_k - n_{k-1},\ k \geqq 1\}$. We give here for $\theta_k = n_k$ or Δ_k a complete characterization of the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that $P(\theta_k \leqq \beta_k$ i.o.) or $P(\theta_k \geqq \alpha_k$ i.o.$)=0$ or 1.

1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent and identifically distributed random variables with a continuous distribution. We define the sequence $\{n_k,\ k \geqq 1\}$ of *record times* of $\{X_n,\ n \geqq 1\}$ by

$$n_1 = 1, \quad n_k = \operatorname{Min}\{n \geqq 1;\ X_n > X_{n_{k-1}}\}, \quad k = 2, 3, \ldots,$$

and the sequence $\{\Delta_k,\ k \geqq 1\}$ of *inter-record times* by

$$\Delta_1 = n_1, \quad \Delta_k = n_k - n_{k-1}, \quad k = 2, 3, \ldots.$$

The following Theorem is due to Rényi (1962):

$$\operatorname*{Lim\,Sup}_{k\infty} \frac{\operatorname{Log} n_k - k}{\sqrt{2k \operatorname{Log}_2 k}} = 1, \quad \operatorname*{Lim\,Inf}_{k\infty} \frac{\operatorname{Log} n_k - k}{\sqrt{2k \operatorname{Log}_2 k}} = -1 \text{ a.s..} \tag{1}$$

In 1970, Strawderman and Holmes obtained that

$$\operatorname*{Lim\,Sup}_{k\infty} \frac{\operatorname{Log} \Delta_k - k}{\sqrt{2k \operatorname{Log}_2 k}} = 1, \quad \operatorname*{Lim\,Inf}_{k\infty} \frac{\operatorname{Log} \Delta_k - k}{\sqrt{2k \operatorname{Log}_2 k}} = -1 \text{ a.s.,} \tag{2}$$

0044-3719/83/0062/0001/$01.20

precising a result of Neuts (1967), who had shown that

$$\Delta_k^{1/k} \xrightarrow{P} e, \quad \text{and} \quad \frac{\operatorname{Log} \Delta_k - k}{\sqrt{k}} \xrightarrow{W} N(0,1), \quad k \to \infty. \tag{3}$$

The exact distribution of Δ_k (see Neuts 1967, Tata 1969) is given by

$$P(\Delta_k \geqq r) = \int_0^{+\infty} \frac{x^k}{k!} e^{-x}(1-e^{-x})^r \, dx, \quad r \geqq 1, \; k \geqq 2, \tag{4}$$

from where it follows that $E(\Delta_k^\alpha) < \infty$ if $0 < \alpha < 1$, and $E(\Delta_k) = \infty$.

Further results were described by Shorrock (1972, a, 1972, b) who proved that the sequence $\{n_{n+1}/n_k, \; k \geqq n\}$ converges in law to the i.i.d. process $\{W_m, m \geqq 1\}$ with $P(W_1 > u) = 1/u$, $u \geqq 1$, and that there could not exist constants $\{a_k, k \geqq 1\}$ such that $\{n_k/a_k, \; k \geqq 1\}$ has a non degenerate limit distribution as $k \to \infty$.

We shall, in the following, precise these results by proving:

Theorem 1. *For any* $p \geqq 4$,

$$\begin{aligned} &P(\operatorname{Log} \Delta_k - k \geqq \sqrt{2k\{\operatorname{Log}_2 k + (3/2) \operatorname{Log}_3 k + \operatorname{Log}_4 k + \ldots + (1+\varepsilon) \operatorname{Log}_p k\}} \text{ i.o.}) \\ &= P(\operatorname{Log} \Delta_k - k \leqq -\sqrt{2k\{\operatorname{Log}_2 k + (3/2) \operatorname{Log}_3 k + \operatorname{Log}_4 k + \ldots + (1+\varepsilon) \operatorname{Log}_p k\}} \text{ i.o.}) \\ &= 0 \text{ or } 1 \text{ according as } \varepsilon > 0 \text{ or } \varepsilon \leqq 0. \end{aligned} \tag{5}$$

Theorem 2. *For any* $p \geqq 4$,

$$\begin{aligned} &P(\operatorname{Log} n_k - k \geqq \sqrt{2k\{\operatorname{Log}_2 k + (3/2) \operatorname{Log}_3 k + \operatorname{Log}_4 k + \ldots + (1+\varepsilon) \operatorname{Log}_p k\}} \text{ i.o.}) \\ &= P(\operatorname{Log} n_k - k \leqq -\sqrt{2k\{\operatorname{Log}_2 k + (3/2) \operatorname{Log}_3 k + \operatorname{Log}_4 k + \ldots + (1+\varepsilon) \operatorname{Log}_p k\}} \text{ i.o.}) \\ &= 0 \text{ or } 1 \text{ according as } \varepsilon > 0 \text{ or } \varepsilon \leqq 0. \end{aligned} \tag{6}$$

In the proof, we shall make use of strong approximation techniques, enabling us to precise the bounds (5), (6) up to a complete characterization (Theorems 4 and 6). It is to be noted that the remarkable symmetry between upper and lower bounds of (5) and (6) could not be intuitively expected from (4). because of the dissymetry of the distributions of Δ_k and n_k for finite k.

2. Complete Characterization of the Limiting Bounds

It can be remarked that the record times $\{n_k, \; k \geqq 1\}$ can be defined in an equivalent way for maxima and minima, and this, independently of the (continuous) distribution of the sequence's terms.

Accordingly, for sake of simplicity, put $U_1, U_2, \ldots$ to be an i.i.d. sequence of uniformly distributed on (0, 1) random variables, and define n_k by

$$n_1 = 1, \quad n_k = \operatorname{Min}\{n \geqq 1; \; U_n < U_{n_{k-1}}\}, \quad k = 2, 3, \ldots,$$

and

$$\Delta_1 = n_1, \quad \Delta_k = n_k - n_{k-1}, \quad k = 2, 3, \ldots.$$

The following result was proved in Deheuvels (1981):

Lemma 1. *Without loss of generality, there exists an i.i.d. sequence* $\{\omega_k, +\infty < k < +\infty\}$ *of exponentially* $E(1)$ *distributed random variables, and a normalized Poisson process* $\{N(t), -\infty < t < +\infty\}$*, independent of* $\{\omega_k\}$*, with times of arrivals* $\ldots < z_{-1} < z_0 < 0 < z_1 < z_2 < \ldots$*, such that* $N(u) = k$ *when* $z_k < u \leqq z_{k+1}$*, and such that*

$$n_k = 1 + \sum_{i=1}^{k-1} \left(\left[\frac{\omega_i}{-\operatorname{Log}(1 - e^{-z_i})} \right] + 1 \right), \quad k = 1, 2, \ldots, \tag{7}$$

(*where* $[u]$ *denotes the integer part of* u).

If we remark that, by (7),

$$\Delta_{k+1} = \left[\frac{\omega_k}{-\operatorname{Log}(1 - e^{-z_i})} \right] + 1, \quad k = 1, 2, \ldots, \tag{8}$$

and also that (see Barndorff-Nielsen 1961, Deheuvels 1974):

Lemma 2. *For any* $p \geqq 1$ *and* $\varepsilon > 0$*, there exists almost surely a* k_0 *such that for* $k \geqq k_0$,

$$\frac{1}{k(\operatorname{Log} k) \ldots (\operatorname{Log}_p k)^{1+\varepsilon}} \leqq \omega_k \leqq \operatorname{Log} k + \operatorname{Log}_2 k + \ldots + (1 + \varepsilon) \operatorname{Log}_p k.$$

Then, since by (8), we have

$$\Delta_{k+1} = \omega_k e^{z_k} - \tfrac{1}{2} \omega_k + O(1) \text{ a.s.} \quad \text{as } k \to \infty, \tag{9}$$

the law of the iterated logarithm applied to $\{z_k\}$ enables to obtain (2) in a direct way. Such an argument was worked out by M. Berkane, who derived a new proof of the results of Strawderman and Holmes (1970). We intend here to precise this argument.

It will be necessary to use a Theorem of Komlós, Major, and Tusnády (1975, 1976), which we state in the following:

Lemma 3. *If* $\{\eta_k, k \geqq 1\}$ *is an i.i.d. sequence of random variables such that* $E(e^{t\eta_1})$ *exists in a neighborhood of* $t = 0$*, then, without loss of generality, there exists a Wiener process* $\{W(t), t \geqq 0\}$ *such that, if* $S_n = \eta_1 + \ldots + \eta_n$, $E(\eta_1) = 0$, $E(\eta_1^2) = 1$, *then*

$$|S_n - W(n)| = O(\operatorname{Log} n) \text{ a.s.} \quad \text{as } n \to \infty. \tag{10}$$

Taking here $\eta_k = z_k - z_{k-1} - 1$, $S_k = z_k - k$, we get easily from (8)–(10):

Theorem 3. *Without loss of generality, there exists a Wiener process* W *such that*

$$\Delta_{k+1} = \exp(k + W(k) + O(\operatorname{Log} k)) \text{ a.s.} \quad \text{as } k \to \infty. \tag{11}$$

It may be remarked that, since $W(k) - W(k-1) = O(\sqrt{\operatorname{Log} k})$, (11) can be written without modification with Δ_k in place of Δ_{k+1}.

We shall now use a result of Erdős (1942), who proved that, if $p \geqq 4$, and if $h_{p,\varepsilon}(t) = \sqrt{2t\{\mathrm{Log}_2 t + (3/2)\,\mathrm{Log}_3 t + \mathrm{Log}_4 t + \ldots + (1+\varepsilon)\,\mathrm{Log}_p t\}}$, then, if $\varepsilon > 0$, there exists a.s. a t_ε such that, for any $t \geqq t_\varepsilon$, $-h_{p,\varepsilon}(t) \leqq W(t) \leqq +h_{p,\varepsilon}(t)$, whereas a.s. for any t_0, there exists a $t' \geqq t_0$ such that $W(t') \leqq -h_{p,0}(t')$, and a $t'' \geqq t_0$ such that $W(t'') \geqq +h_{p,0}(t'')$.

Lemma 4. *If* $p \geqq 4$,

$$h_{p,\varepsilon}(t) - h_{p,0}(t) \sim \frac{\varepsilon\sqrt{t}(\mathrm{Log}_p t)}{\sqrt{2\,\mathrm{Log}_2 t}} \quad \text{as } t \to \infty.$$

Proof. Straightforward.

We are now ready to prove Theorem 1. By (11), if $\varepsilon > 0$,

$$\Delta_k \leqq \exp(k + h_{p,\varepsilon}(k) + O(\mathrm{Log}\,k)) \leqq \exp(k + h_{p,2\varepsilon}(k)) \text{ a.s.} \quad \text{as } k \to \infty,$$

making use of the fact (Lemma 4) that $\mathrm{Log}\,k = o(h_{p,2\varepsilon}(k) - h_{p,\varepsilon}(k))$. Hence, $\varepsilon > 0$ being arbitrary, and using the same argument for $-W(k)$, we get (5) for $\varepsilon > 0$.

For $\varepsilon = 0$, we may note that $\sup_{1 \leqq k \leqq n} \sup_{0 \leqq t \leqq 1} |W(k) - W(k+t)| = O(\mathrm{Log}\,n)$ a.s., and that $\mathrm{Log}\,k = o(h_{p+1,0}(k) - h_{p,0}(h))$. This finishes the proof.

We shall now precise (5) by the use of Kolmogorov's test (see Ito-McKean, 1965, p. 33):

Lemma 5. *If* $H(t)$ *is a positive function defined for* $0 < t \leqq \varepsilon$, *such that* $H\uparrow$ *and* $t^{-1/2} H\downarrow$, *then,* a.s. $t H(1/t)$ *belongs to the upper or lower class of* $W(t)$, $t \to \infty$, *according as*

$$I = \int_{0+} t^{-3/2} H(t) \exp(-H^2(t)/2t)\, dt$$

converges or diverges.

Theorem 4. *If* $H(t)$ *is a positive function defined for* $0 < t \leqq \varepsilon$, *such that* $H\uparrow$ *and* $t^{-1/2} H\downarrow$, *then*

$$\begin{aligned} &P(\mathrm{Log}\,\Delta_k - k \geqq k\,H(1/k) \text{ i.o.}) = \\ &P(\mathrm{Log}\,\Delta_k - k \leqq -k\,H(1/k) \text{ i.o.}) = 0 \text{ or } 1, \end{aligned} \tag{12}$$

according as $U = \int_{0+} t^{-3/2} H(t) \exp(-H^2(t)/2t)\, dt$ *converges or diverges.*

Proof. Put $H(t) = \left(2t\left\{\mathrm{Log}_2 \frac{1}{t} + \frac{1}{2}\mathrm{Log}_3 \frac{1}{t} + \mathrm{Log}\,\phi(t)\right\}\right)^{1/2}$. Then, I converges or diverges according as $J = \int_{0+} \frac{dt}{t(\mathrm{Log}(1/t))\,\phi(t)}$ converges or diverges. Furthermore, if we put $\alpha(t) = \phi(t)\left(\mathrm{Log}\frac{1}{t}\right)\left(\mathrm{Log}_2 \frac{1}{t}\right)^{1/2}$, then $H\uparrow$ and $t^{-1/2} H\downarrow$ is equivalent to $\alpha(t)\uparrow$ and $t^{1/2}\alpha(t)\downarrow$ as $t\downarrow 0$. Let us assume that these conditions are satisfied, and let us consider first the case where $\alpha(t)\uparrow\infty$ and $t^{1/2}\alpha(t)\downarrow 0$ as $t\downarrow 0$. Put $H_q(t) = \left(2t\left\{\mathrm{Log}_2 \frac{1}{t} + \frac{1}{2}\mathrm{Log}_3 \frac{1}{t} + \mathrm{Log}\,\phi(t)\,(1+q(t))\right\}\right)^{1/2}$, and chose $q(t) \to 0$ as

$t\to\infty$. Then $K=\int_{0_+} \frac{dt}{t(\mathrm{Log}(1/t))\,\phi(t)\,(1+q(t))}$ converges or diverges according as the same happens for J. Put now $q(t)=C(t)\sqrt{2t}\left(\mathrm{Log}\frac{1}{t}\right)\left(\mathrm{Log}_2\frac{1}{t}\right)$; since $\alpha(t)\uparrow\infty$ and $t^{1/2}\alpha(t)\downarrow 0$, as $t\downarrow 0$, it is possible to chose $C(t)$ such that simultaneously: $\alpha(t)(1+q(t))\uparrow\infty$, $t^{1/2}\alpha(t)(1+q(t))\downarrow 0$, $q(t)\to 0$ as $t\downarrow 0$, and either $\mathrm{Lim\,Sup}_{t\infty}\, C(t)\leqq A$, either $\mathrm{Lim\,Inf}_{t\infty}\, C(t)\geqq A$, where A is any given constant. This implies that H_q satisfies the conditions of Lemma 5. If we make also the assumption that $\mathrm{Log}\,\phi(t)=o\left(\mathrm{Log}_2\frac{1}{t}\right)$, $t\downarrow 0$, then $H_q(t)=H(t)+t\,C(t)\left(\mathrm{Log}\frac{1}{t}\right)(1+o(1))$. Now, by (10), Lemma 5, and the preceding argument, if $\mathrm{Lim\,Sup}_{t\infty}\, C(t)\leqq A$, and if I converges, then $\mathrm{Log}\,\Delta_k-k+O(\mathrm{Log}\,k)\leqq k\,H(1/k)+A\,\mathrm{Log}\,k$ a.s. as $k\to\infty$. By a convenient choice of A, it gives $\mathrm{Log}\,\Delta_k-k\leqq k\,H(1/k)$. A similar argument gives (12) by discussing the different possibilities. It remains to treat the case where $\alpha(t)\uparrow L<\infty$, or $t^{1/2}\alpha(t)\downarrow M>0$, or $\mathrm{Log}\,\phi(t)\neq o\left(\mathrm{Log}_2\frac{1}{t}\right)$. It can be seen that any of this cases can be described by (5) so that (12) is valid in all cases. The proof of Theorem 4 is achieved.

In the proof of Theorem 2, we use the following results: Let, for $t\geqq 0$,

$$Y(t)=\sum_{-\infty}^{N(\mathrm{Log}\,t)} \omega_k\, e^{z_k}, \qquad \text{and put, for } k\geqq 1,\ v_k=\sum_{-\infty}^{k-1} \omega_i\, e^{z_i}.$$

Then (see [4, 5]) Y is the inverse of an extremal process, and:

Lemma 9. *The sequence* $\{\mathrm{Log}\,v_k-\mathrm{Log}\,v_{k-1}, k\geqq 1\}$ *is an i.i.d. sequence of exponentially* $E(1)$ *distributed random variables. Furthermore,*

$$|v_k-n_k|=O(k) \text{ a.s.} \quad \textit{as } k\to\infty.$$

Proof. The first assertion (see Deheuvels, 1982) follows from Dwass, 1964, Theorem 4.1; the second was proved in Deheuvels, 1981.

Theorem 5. *Without loss of generality, there exists a Wiener process* $\tilde{W}$ *such that*

$$n_k=\exp(k+\tilde{W}(k)+O(\mathrm{Log}\,k)) \text{ a.s.} \quad \textit{as } k\to\infty. \tag{13}$$

Proof. By Lemma 3 and Lemma 9, it is clear that we can put $\mathrm{Log}\,v_k=k+\tilde{W}(k)+O(\mathrm{Log}\,k)$ a.s.. Since $\mathrm{Log}\,n_k=\mathrm{Log}\,v_k+\mathrm{Log}\left(1+\left\{\frac{n_k-v_k}{v_k}\right\}\right)=\mathrm{Log}\,v_k+O(k\,e^{-k(1-\varepsilon)})$, (13) follows readily.

It may now be remarked that the whole proof of Theorem 1 and of Theorem 5 relies entirely in (11). Hence, we can derive from (13) the same conclusions for n_k as those we obtained from (11). This proves Theorem 2, and:

Theorem 6. *If* $H(t)$ *is a positive function defined for* $0<t<\varepsilon$, *such that* $H\uparrow$ *and* $t^{-1/2}H\downarrow$, *then*

$$\begin{aligned} &P(\operatorname{Log} n_k - k \geqq k H(1/k) \text{ i.o.}) \\ &= P(\operatorname{Log} n_k - k \leqq k H(1/k) \text{ i.o.}) = 0 \text{ or } 1, \end{aligned} \tag{14}$$

according as $I = \int_{0+} t^{-3/2} H(t) \exp(-H^2(t)/2t)\, dt$ *converges or diverges.*

3. Conclusion and Comments

The logical explanation for the similarities in behavior of n_k and Δ_k is that both sequences are of the form $\exp(k + \bar{W}(k) + O(\operatorname{Log} k))$, where $\bar{W}$ is a Wiener process. The optimality of the bound in the Theorem of Komlós, Major, and Tusnády (Lemma 3) shows clearly that no improvement of (11) and (13) can be hoped for.

Most results which were previously proved on these sequences can be obtained as direct corollaries of Theorem 3 and Theorem 6.

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Received May 5, 1982