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A Generalized Formula of Ito and Some Other Properties of Stochastic Flows

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Summary. A stochastic differential equation with smooth coefficients is considered, which defines a continuous flow $\phi_t(\omega,\cdot)$ of C^∞ mappings of R^d in R^d. If z_t is a continuous semi-martingale, $\phi_t(\omega, z_t)$ is proved to be a semi-martingale, for which an Ito type formula is established. It is shown that a.s., for any $t, \phi_t(\omega,\cdot)$ is an onto diffeomorphism. If z_t is a continuous semi-martingale. $\phi_t^{-1}(\omega, z_t)$ is proved to be a semi-martingale, whose Ito decomposition is explicitly found.

Consider the stochastic differential equation

$$dx = X_0(x)\,dt + X_i(x)\cdot dw^i \qquad (0.1)$$
$$x(0) = x$$

where $X_0, X_1 \dots X_m$ are $m+1$ vector fields on R^d, $w = (w^1 \dots w^m)$ is a Brownian motion, and dw is its Stratonovitch differential.

Under differentiability assumptions on the vector fields $X_0, \dots, X_m$, it is easily proved that (0.1) defines a flow of C^∞ mappings of R^d in R^d, i.e. if $x_t^x(\omega)$ is the solution of (0.1), it is possible to choose a version of the mapping

$$(\omega, t, x) \to x_t^x(\omega)$$

in such a way that a.s., $x_{\cdot}(\omega)$ is C^∞ on R^d (see Malliavin [11], Elworthy [5] and our work [3]).

Let $\phi_{\cdot}(\omega,\cdot)$ be this essentially unique version, i.e.

$$\phi_t(\omega, x) = x_t^x(\omega). \qquad (0.2)$$

In Sect. 1 of this paper we recall some results concerning other known results about $\phi_{\cdot}(\omega,\cdot)$.

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In Sect. 2, we prove that if z_t is a continuous semi-martingale with values in R^d, $\phi_t(\omega, z_t)$ is a semi-martingale, whose Ito-Meyer decomposition is explicitly found, in Ito form and in Stratonovitch form.

This result can be easily extended to any general semi-martingale.

In Sect. 3, we show that a.s., for every $t, \phi_t(\omega, \cdot)$ is a diffeomorphism of R^d onto R^d. The derivation of the a.s. injectivity of $\phi_t(\omega, \cdot)$ is easy using standard differential analysis techniques. Due to the non compactness of R^d, it is much harder to prove that $\phi_t(\omega, \cdot)$ is a.s. onto. The problem is solved by treating the lack of surjectivity of $\phi_t(\omega, \cdot)$ at certain times as a singularity, then using a selection Theorem from the theory of stochastic processes [4] to arrive at a contradiction with the Markov property of the flow.

In Sect. 4, we show that if z_t is taken as in (0.2), $\phi_t^{-1}(\omega, z_t)$ is a continuous semi-martingale, which is obtained by solving a stochastic differential equation.

The results contained in this paper were announced in [2]. Let us also mention a forthcoming paper by Kunita [9], which has close connections with our paper.

1. Stochastic Flows

Let Ω be the set of continuous functions defined on R^+ with valued in R^m. A point in Ω is written ω, and the trajectory is w_t. Let F_t be the σ-field $F_t = B(w_s | s \leqq t)$.

P is the brownian measure on Ω, with $P(w_0=0)=1$. $\{F_t^+\}_{t \geqq 0}$ is the right continuous regularization of $\{F_t\}_{t \geqq 0}$, which is completed by the negligible sets in F_∞ [4].

$X_0, \dots, X_m$ are $m+1$ C^∞ vectors on R^d, which are bounded with bounded derivatives of all orders.

Following the notations of Stroock and Varadhan [13], we define

$$t_n = \frac{[2^n t]}{2^n} \qquad t_n^+ = \frac{[2^n t]+1}{2^n}$$

$$\dot{w}^{i,n}(t) = 2^n (w^i(t_n^+) - w^i(t_n)).$$

Consider the stochastic differential equation

$$\begin{aligned} dx &= X_0(x)\, dt + X_i(x) \cdot dw^i \\ x(0) &= x \end{aligned} \tag{1.1}$$

where dw is the Stratonovitch differential of w. (1.1) may be put in the equivalent Ito form

$$\begin{aligned} dx &= X_0^*(x)\, dt + X_i(x) \cdot \delta w^i \\ x(0) &= x \end{aligned} \tag{1.2}$$

where X_0^* is given by

$$X_0^*(x) = X_0(x) + \frac{1}{2} \frac{\partial X_i}{\partial x}(x)\, X_i(x) \tag{1.3}$$

and δw is the Ito differential of w [12].

Consider now the approximations of Stroock and Varadhan [13] of the solution of (1.1) by the solutions of the ordinary differential equations

$$(1.4)\qquad \begin{aligned} &dx^n = (X_0(x^n) + X_i(x^n)\dot{w}^{i,n})\,dt \\ &x^n(0) = x. \end{aligned}$$

By using standard results on ordinary differential equations, the solutions $x_t^{n,x}$ of (1.4) is easily proved to depend differentiably on x. Let $\phi_t^n(\omega, x)$ be defined by

$$\phi_t^n(\omega, x) = x_t^{n,x}.$$

For every ω, $\phi_t^n(\omega, x)$ is then jointly continuous in (t, x), C^∞ in the x variable, and for any m, $\frac{\partial^m \phi^n}{\partial x^m}{}_t(\omega, x)$ is jointly continuous in (t, x). Moreover, by using time reversal on (1.4), for every ω, and every $t \geqq 0$, $\phi_t^n(\omega, \cdot)$ is a diffeomorphism of R^d onto R^d, and $[\phi_\cdot^n]^{-1}(\omega, \cdot)$ has the same properties as $\phi_\cdot^n(\omega, \cdot)$.

We then have the fundamental:

Theorem 1.1. *There exists an essentially unique mapping $\phi_t(\omega, x)$ defined on $\Omega \times R^+ \times R^d$ with values in R^d, such that:*

a) *For any $(t, x) \in R^+ \times R^d$, $\omega \to \phi_t(\omega, x)$ is F_t^+ measurable.*

b) *A.s., the mapping $(t, x) \to \phi_t(\omega, x)$ is continuous on $R^+ \times R^d$.*

c) *A.s., for any (t, x), the differentials $\frac{\partial^m \phi}{\partial x^m}{}_t(\omega, x)$ exist, and are continuous on $R^+ \times R^d$.*

d) *A.s., for any (t, x), $\frac{\partial \phi}{\partial x}{}_t(\omega, x)$ is non-singular.*

e) *For any $x \in R^d$, $t \to \phi_t(\omega, x)$ is the essentially unique continuous solution of (1.1) with initial condition x. For any $x \in R^d$, $t \to \frac{\partial \phi}{\partial x}{}_t(\omega, x)$ is the essentially unique solution of the stochastic differential equation on (d, d) matrices*

$$(1.5)\qquad \begin{aligned} &dZ = \frac{\partial X_0}{\partial x}(\phi_t(\omega, x))\,Z\,dt + \frac{\partial X_i}{\partial x}(\phi_t(\omega, x))\,Z \cdot dw^i \\ &Z(0) = I. \end{aligned}$$

f) *$\phi_\cdot^n(\omega, \cdot)$ converges in probability uniformly on any compact set of $R^+ \times R^d$ to $\phi_\cdot(\omega, \cdot)$.*

g) *For any m, $\frac{\partial^m \phi^n}{\partial x^m}\cdot(\omega, \cdot)$ converges in probability uniformly on any compact set of $R^+ \times R^d$ to $\frac{\partial^m \phi_\cdot}{\partial x^m}(\omega, \cdot)$.*

h) *$\left[\frac{\partial \phi_\cdot^n}{\partial x}\cdot(\omega, \cdot)\right]^{-1}$ converges in probability uniformly on any compact set of $R^+ \times R^d$ to $\left[\frac{\partial \phi}{\partial x}\cdot(\omega, \cdot)\right]^{-1}$.*

Proof. Some of these results are contained in Malliavin [11] (see also Elworthy [5]). All of them are proved in Theorems I.1.2 and I.2.1 of our own work [3] (see

[2]). In [3], the key point of the proof is the inequality

(1.6) For $s, t \leqq T, x, y \in R^d, p \geqq 2$,

$$E|\phi_t^n(\omega, x) - \phi_s^n(\omega, y)|^{2p} \leqq C_{T,p}(|x-y|^{2p} + |t-s|^p).$$

The time part of the inequality is essentially in Stroock and Varadhan [13]. The space past of the inequality is slightly more involved: we use Gronwall's lemma on each dyadic interval $\left[k/2^n, \frac{k+1}{2^n}\right]$, as well as the martingale techniques of [13]. For p large enough, (1.6) proves that the measures $\tilde{P}^n$ on $\Omega \times C(R^+ \times R^d; R^d)$ images of measure P by the mappings $\omega \to (\omega, \phi^n_{\cdot}(\omega, \cdot))$ are tight. Moreover the inequality

(1.7) For $s, t \leqq T, x, y \in R^d \quad p \geqq 2$

$$E|x_t^x - x_s^y|^{2p} \leqq C_{T,p}(|x-y|^{2p} + |t-s|^p)$$

is trivially proved for the solutions of (1.1), which implies the existence of the a.s. continuous $\phi_{\cdot}(\omega, \cdot)$ on $R^+ \times R^d$.

Using Stroock and Varadhan's result [13] on the convergence in law of the solutions of (1.4) to the solution of (1.1), the limit of the sequence $\tilde{P}^n$ is easily proved to be the measure $\tilde{P}$ image of P by the mapping $\omega \to (\omega, \phi_{\cdot}(\omega, \cdot))$. A measure theoretic argument proves then that $\phi^n_{\cdot}(\omega, \cdot)$ converges in probability uniformly on any compact set of $R^+ \times R^d$ to $\phi_{\cdot}(\omega, \cdot)$.

$\frac{\partial \phi^n_{\cdot}}{\partial x}(\omega, x)$ is given by the solution $Z^{n,x}$ of the differential equation

$$(1.8) \qquad \begin{aligned} &dZ^{n,x} = \left(\frac{\partial X_0}{\partial x}(x^{n,x}) + \frac{\partial X_i}{\partial x}(x^{n,x})\, \dot{w}^{i,n}\right) Z^{n,x}\, dt \\ &Z^{n,x}(0) = I. \end{aligned}$$

Consider the stochastic differential equation

$$(1.9) \qquad \begin{aligned} &dZ^x = \frac{\partial X_0}{\partial x}(x^x)\, Z^x\, dt + \frac{\partial X_i}{\partial x}(x^x)\, Z^x \cdot dw^i \\ &Z_0^x = I. \end{aligned}$$

Inequalities corresponding to (1.6) and (1.7) are proved for $Z^{n,x}$ and Z^x. It follows that a a.s. continuous version of the mapping $(t, x) \to Z_t^x$ exists. $Z_t^{n,x}(\omega)$ is then easily proved to converge in probability uniformly on any compact set of $R^+ \times R^d$ to $Z_t^x(\omega)$. Since $Z_t^{n,x}(\omega) = \frac{\partial \phi_t^n}{\partial x}(\omega, \cdot)$, trivially a.s., for any t, $\frac{\partial \phi_t}{\partial x}(\omega, x)$ exists and is equal to $Z_t^x(\omega, \cdot)$.

A similar method applies to prove the a.s. existence and continuity of $\frac{\partial^m \phi}{\partial x^m}\cdot(\omega, \cdot)$. To prove *d*), consider the equation

$$dZ'^x = -Z'^x \frac{\partial X_0}{\partial x}(x^x)\,dt - Z'^x \frac{\partial X_i}{\partial x}(x^x)\cdot dw^i \tag{1.10}$$
$$Z'^x_0 = I.$$

An inequality of type (1.7) is easily proved for $Z'^x_t(\omega)$, which implies the existence of an a.s. continuous version of $(\omega, t, x)\to Z'^x_t(\omega)$. Now the rules of Stratonovitch calculus allow us to see that for any $x, t\to Z^x_t Z'^x_t$ is a solution of a stochastic differential equation whose unique trivial solution is I. Then for any $x\in R^d$, a.s., for any $t\geqq 0$

$$Z^x_t Z'^x_t = I. \tag{1.11}$$

From the a.s. continuity of Z and Z' in (t, x), it follows that (1.11) holds a.s. for any (t, x). Since $Z^x_t = \frac{\partial \phi}{\partial x}{}_t(\omega, x)$, (1.11) proves d). h) follows from d) and g). □

We now express the Markov property of the flow $\phi_{\boldsymbol{\cdot}}(\omega, \cdot)$.

Theorem 1.2. *Let θ_s be the translation operator*

$$\omega = (w_t) \to \theta_s(\omega) = (w_{s+t} - w_s).$$

Then for any stopping time T, a.s. on $(T< +\infty)$, for any $(t, x)\in R^+\times R^d$, one has

$$\phi_{T+t}(\omega, x) = \phi_t(\theta_T(\omega), \phi_T(\omega, x)). \tag{1.12}$$

Proof. The proof is essentially trivial. Note that by a classical result on Brownian motion, the measure P^T defined by

$$P^T(A) = \frac{P(\theta_T^{-1}(A), (T< +\infty))}{P(T< +\infty)} \tag{1.13}$$

is equal to P. $\phi_{\boldsymbol{\cdot}}(\theta_T(\omega), \cdot)$ is then unambiguously defined a.s. on $(T< +\infty)$. Moreover for fixed $x\in R^d$, the a.s. equality expresses the Markov property of the solution of (1.1) – see e.g. [14] – Theorem 5.1.5. Using the a.s. continuity of both sides of (1.12) in (t, x) on $(T< +\infty)$, (1.12) is proved. □

2. The Generalized Ito-Stratonovitch Formula

The assumptions and notations in this section are the same as in Section 1.

We now have the following result, which generalizes the classical Ito formula of change of variables:

Theorem 2.1. *Let z_t be a continuous semi-martingale on (Ω, F_t^+, P) with values in R^d, which may be written as*

$$z_t = z_0 + A_t + \int_0^t H_i\cdot \delta w^i$$

where $z_0 \in R^d$, A_t *is a continuous adapted bounded variation process such that* $A_0 = 0$, *and* $H_1 \dots H_m$ *are adapted processes such that for any* $t \geqq 0$, $\int_0^t |H_i|^2 ds < +\infty$ a.s.

Then $\phi_t(\omega, z_t)$ *is a continuous semi martingale whose Ito decomposition is given by*

$$(2.1)\qquad \phi_t(\omega, z_t) = z_0 + \int_0^t \Big[X_0^*(\phi_u(\omega, z_u)) + \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u)) \frac{\partial \phi_u}{\partial x}(\omega, z_u) H_i + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}{}_u(\omega, z_u)(H_i, H_i) \Big] du + \int_0^t \frac{\partial \phi}{\partial x}{}_u(\omega, z_u) dA_u + \int_0^t \Big[X_i(\phi_u(\omega, z_u)) + \frac{\partial \phi}{\partial x}{}_u(\omega, z_u) H_i \Big] \cdot \delta w^i.$$

Proof. The proof is an adaptation of the now classical proof of Ito's formula given by Kunita-Watanabe in [10], and extended by Meyer in [12], but it is somewhat more involved. The difficulty comes from the necessity of controlling the flow $\phi.(\omega, \cdot)$ as well as z_t.

Note that the two sides of (2.1) define a.s. continuous processes so that we need to establish (2.1) only for a fixed $t \in R^+$. By a stopping argument, we may assume that z_s, $\int_0^s |dA|$, $\int_0^s H_i \cdot \delta w^i$ are bounded processes. Let k be an upper bound for their norm.

Define the stopping time S_l by

$$(2.2)\qquad S_l = \inf \Big\{ t \geqq 0;\ \sup_{\substack{|x| \leqq k \\ 0 \leqq m \leqq 3}} \Big| \frac{\partial^m \phi}{\partial x^m}{}_t(\omega, x) \Big| \geqq l \Big\}$$

Note that by the a.s. joint continuity of $\frac{\partial^m \phi}{\partial x^m}.(\omega, \cdot)$ in (t, x), when $l \to +\infty$, $S_l \to +\infty$ a.s. We will prove (2.1) at time $t \wedge S_l$. By making $l \to +\infty$, (2.1) will follow for t.

Take $\varepsilon > 0$. Let $\{T_n\}$ be the stopping times

$$(2.3)\qquad \begin{aligned} T_0 &= 0 \\ T_{n+1} &= t \wedge S_l \wedge (T_n + \varepsilon) \\ &\wedge \inf \Big\{ s \geqq T_n;\ \sup \Big[|A_s - A_{T_n}|, \Big| \int_{T_n}^s H_i \cdot \delta w^i \Big|, \\ &\sup_{\substack{|x| \leqq k \\ 0 \leqq m \leqq 2}} \Big| \frac{\partial^m \phi}{\partial x^m}{}_s(\omega, x) - \frac{\partial^m \phi}{\partial x^m} T_n(\omega, x) \Big| \Big] \geqq \varepsilon \Big\}. \end{aligned}$$

Clearly, by a joint continuity argument, the sequence T_n is seen to increase a.s. to $t \wedge S_l$, and to be stationary for n large enough (of course, the smaller n when T_n becomes stationary depends on ω). We then have

$$(2.4)\qquad \phi_{t \wedge S_l}(\omega, z_{t \wedge S_l}) = z_0 + \Sigma(\phi_{T_{n+1}}(\omega, z_{T_{n+1}}) - \phi_{T_n}(\omega, z_{T_n})).$$

Moreover

$$\phi_{T_{n+1}}(\omega, z_{T_{n+1}}) - \phi_{T_n}(\omega, z_{T_n}) = (\phi_{T_{n+1}}(\omega, z_{T_{n+1}}) - \phi_{T_{n+1}}(\omega, z_{T_n})) + (\phi_{T_{n+1}}(\omega, z_{T_n}) - \phi_{T_n}(\omega, z_{T_n})). \tag{2.5}$$

In the sequel, we make ε tend to 0, i.e. take any sequence ε_m of >0 reals decreasing to 0 when $m \to +\infty$, and take the limit in the R.H.S. of (2.4).

· Limit of $\Sigma(\phi_{T_{n+1}}(\omega, z_{T_n}) - \phi_{T_n}(\omega, z_{T_n}))$

By Theorem 1.2, we know that a.s.

$$\phi_{T_n+\cdot}(\omega, z_{T_n}) = \phi_\cdot(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n})) \tag{2.6}$$

(note that T_n is always $<+\infty$).

Now since z_{T_n} is F_{T_n}-measurable, $\phi_{T_n}(\omega, z_{T_n})$ is F_{T_n}-measurable. Moreover since $\theta_{T_n}^{-1}(F_\infty)$ and F_{T_n} are independent, $\phi_s(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n}))$ is a semi-martingale whose starting point $\phi_{T_n}(\omega, z_{T_n})$ is independent of $\theta_{T_n}^{-1}(F_\infty)$. It follows that for $s \geqq 0$

$$\begin{aligned} &\phi_s(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n})) \\ &\quad = \phi_{T_n}(\omega, z_{T_n}) + \int_0^s X_0^*(\phi_u(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n})))\, du \\ &\quad + \int_0^s X_i(\phi_u(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n}))) \cdot \delta w^i(\theta_{T_n}(\omega)). \end{aligned} \tag{2.7}$$

Using Theorem 1.2, it follows that for $s \geqq T_n$, one has

$$\phi_s(\omega, z_{T_n}) = \phi_{T_n}(\omega, z_{T_n}) + \int_{T_n}^s X_0^*(\phi_u(\omega, z_{T_n}))\, du + \int_{T_n}^s X_i(\phi_u(\omega, z_{T_n})) \cdot \delta w^i. \tag{2.8}$$

For $u < t \wedge S_l$, let $n(u)$ be defined by

$$T_{n(u)} \leqq u < T_{n(u)+1} \tag{2.9}$$

From (2.8), we get that

$$\begin{aligned} &\Sigma(\phi_{T_{n+1}}(\omega, z_{T_n}) - \phi_{T_n}(\omega, z_{T_n})) \\ &\quad = \int_0^{t \wedge S_l} X_0^*(\phi_u(\omega, z_{T_{n(u)}}))\, du + \int_0^{t \wedge S_l} X_i(\phi_u(\omega, z_{T_{n(u)}})) \cdot \delta w_i. \end{aligned} \tag{2.10}$$

When $\varepsilon \to 0$, the optional process $z_{T_{n(u)}}$ converges uniformly to z_u on $[0, t \wedge S_l]$. Then $\phi_u(\omega, z_{T_{n(u)}})$ converges uniformly to $\phi_u(\omega, z_u)$ on $[0, t \wedge S_l]$. Since $X_0^*, X_1 \dots X_m$ are bounded and continuous, trivially

$$\begin{aligned} &\Sigma(\Phi_{T_{n+1}}(\omega, z_{T_n}) - \phi_{T_n}(\omega, z_{T_n})) \\ &\quad \to \int_0^{t \wedge S_l} X_0^*(\phi_u(\omega, z_u))\, du + \int_0^{t \wedge S_l} X_i(\phi_u(\omega, z_u)) \cdot \delta w^i \quad \text{in probability.} \end{aligned} \tag{2.11}$$

· Limit of $\Sigma(\phi_{T_{n+1}}(\omega, z_{T_{n+1}})-\phi_{T_{n+1}}(\omega, z_{T_n}))$

By Taylor's formula, since on $[0, S_l]$, $\left|\frac{\partial^3\phi}{\partial x^3}\cdot(\omega,\cdot)\right|$ is uniformly bounded on $\{x\in R^d;\ |x|\leqq k\}$ by l, it follows that

$$\begin{aligned}(2.12)\qquad &\phi_{T_{n+1}}(\omega, z_{T_{n+1}})-\phi_{T_{n+1}}(\omega, z_{T_n})\\ &=\frac{\partial\phi}{\partial x}_{T_{n+1}}(\omega, z_{T_n})(z_{T_{n+1}}-z_{T_n})\\ &+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})(z_{T_{n+1}}-z_{T_n}, z_{T_{n+1}}-z_{T_n})+R_n(\omega)\end{aligned}$$

with

$$(2.13)\qquad |R_n(\omega)|\leqq l|z_{T_{n+1}}-z_{T_n}|^3.$$

By (2.3), we know that

$$(2.14)\qquad |z_{T_{n+1}}-z_{T_n}|\leqq C\varepsilon$$

which implies

$$(2.15)\qquad |R_n(\omega)|\leqq C\varepsilon|z_{T_{n+1}}-z_{T_n}|^2.$$

1) We have

$$\begin{aligned}(2.16)\qquad \frac{\partial\phi}{\partial x}_{T_{n+1}}(\omega, z_{T_n})(z_{T_{n+1}}-z_{T_n})&=\frac{\partial\phi}{\partial x}_{T_{n+1}}(\omega, z_{T_n})(A_{T_{n+1}}-A_{T_n})\\ &+\frac{\partial\phi}{\partial x}_{T_{n+1}}(\omega, z_{T_n})\int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j.\end{aligned}$$

a) Obviously

$$(2.17)\qquad \Sigma\left(\frac{\partial\phi}{\partial x}_{T_{n+1}}(\omega, z_{T_n})(A_{T_{n+1}}-A_{T_n})\right)=\int_0^{t\wedge S_l}\frac{\partial\phi}{\partial x}_{T_{n(u)+1}}(\omega, z_{T_{n(u)}})\,dA.$$

Since $T_{n(u)}$ converges uniformly to u, and since $\frac{\partial\phi}{\partial x}_t(\omega, x)$ is uniformly bounded on $[0, t\wedge S_l]$ when $|x|\leqq k$, it follows immediately that

$$(2.18)\qquad \int_0^{t\wedge S_l}\frac{\partial\phi}{\partial x}_{T_{n(u)+1}}(\omega, z_{T_{n(u)}})\,dA\to\int_0^{t\wedge S_l}\frac{\partial\phi}{\partial x}_u(\omega, z_u)\,dA.$$

b) Using (1.12), we have

$$(2.19)\qquad \frac{\partial\phi}{\partial x}_{T_n+\cdot}(\omega, z_{T_n})=\frac{\partial\phi}{\partial x}\cdot(\theta_{T_n}(\omega), \phi_{T_n}(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_{T_n}(\omega, z_{T_n}).$$

From the independence of $\theta_{T_n}^{-1}(F_\infty)$ and F_{T_n}, and from (1.5) written in Ito's form, one sees that for $s\geqq T_n$

$$\text{(2.20)}\qquad \frac{\partial\phi}{\partial x}_s(\omega, z_{T_n}) = \frac{\partial\phi}{\partial x}_{T_n}(\omega, z_{T_n}) + \int_{T_n}^{s} \frac{\partial X_0^*}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\,du + \int_{T_n}^{s} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\cdot\delta w^i.$$

α) We have

$$\text{(2.21)}\qquad \Sigma\left(\frac{\partial\phi}{\partial x}_{T_n}(\omega, z_{T_n})\int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right) = \int_0^{t\wedge S_l} \frac{\partial\phi}{\partial x}_{T_{n(u)}}(\omega, z_{T_{n(u)}})\,H_j\cdot\delta w^j.$$

Now when $\varepsilon\to 0$, $\frac{\partial\phi}{\partial x}_{T_{n(u)}}(\omega, z_{T_{n(u)}})$ converges uniformly to $\frac{\partial\phi}{\partial x}_u(\omega, z_u)$ for $u\leq t\wedge S_l$, while staying uniformly bounded. Since $E\int_0^{+\infty}|H_j|^2\,du$ is bounded, it follows that

$$\text{(2.22)}\qquad \Sigma\left(\frac{\partial\phi}{\partial x}_{T_n}(\omega, z_{T_n})\int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right) \to \int_0^{t\wedge S_l} \frac{\partial\phi}{\partial x}_u(\omega, z_u)\,H_j\cdot\delta w^j \quad \text{in probability.}$$

β) We study the limit of

$$\text{(2.23)}\qquad \Sigma\left(\int_{T_n}^{T_{n+1}} \frac{\partial X_0^*}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\,du\left[\int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right]\right).$$

From (2.2), (2.3) and from the boundedness of $\frac{\partial X_0^*}{\partial x}$, (2.23) may be bounded in norm by $C\varepsilon t$.

When $\varepsilon\to 0$, (2.23) tends to 0.

γ) Consider now the sum

$$\text{(2.24)}\qquad \Sigma\left(\left(\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\cdot\delta w^i\right)\left(\int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right)\right).$$

Let us calculate

$$\text{(2.25)}\qquad E\left|\Sigma\left[\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\cdot\delta w^i \int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j - \int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\,H_i\,du\right]\right|^2.$$

By a martingale property, the various terms in the sum in (2.25) are mutually orthogonal. (2.25) may then be bounded by

$$\text{(2.26)}\qquad C\left\{\Sigma E\left|\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\cdot\delta w^i \int_{T_n}^{T_{n+1}} H_j\cdot\delta w^i\right|^2 + \Sigma E\left|\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial\phi}{\partial x}_u(\omega, z_{T_n})\,H_i\,du\right|^2\right\}.$$

Since $T_{n+1}-T_n$ and $\left|\int_{T_n}^{T_{n+1}} H_j \cdot \delta w^j\right|$ are bounded by ε and $\frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))$ $\frac{\partial \phi}{\partial x}_u(\omega, z_{T_n})$ is uniformly bounded, (2.25) is bounded by

$$(2.27)\qquad C\left[\varepsilon^2 E(t\wedge S_l)+\Sigma\,\varepsilon E\int_{T_n}^{T_{n+1}} |H_i|^2\,du\right] \leqq C\left(\varepsilon^2 t+\varepsilon E\int_0^{+\infty} |H_i|^2\,du\right)$$

(2.27) tends to 0 when $\varepsilon\to 0$. From (2.25)-(2.27), we get that

$$(2.28)\qquad \Sigma\left(\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial \phi}{\partial x}_u(\omega, z_{T_n})\cdot \delta w^i \int_{T_n}^{T_{n+1}} H_j\cdot \delta w^j\right)$$
$$-\int_0^{t\wedge S_l} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_{n(u)}}))\frac{\partial \phi}{\partial x}_u(\omega, z_{T_{n(u)}})\,H_i\,du\to 0 \qquad \text{in probability.}$$

Now clearly

$$(2.29)\qquad \int_0^{t\wedge S_l} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_{n(u)}}))\frac{\partial \phi}{\partial x}_u(\omega, z_{T_{n(u)}})\,H_i\,du$$
$$\to \int_0^{t\wedge S_l} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\frac{\partial \phi}{\partial x}_u(\omega, z_u)\,H_i\,du.$$

(2.28) and (2.29) imply that

$$(2.30)\qquad \Sigma\left(\int_{T_n}^{T_{n+1}} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_{T_n}))\frac{\partial \phi}{\partial x}_u(\omega, z_{T_n})\,\delta w^i \int_{T_n}^{T_{n+1}} H_j\cdot \delta w^j\right)$$
$$\to \int_0^{t\wedge S_l} \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\frac{\partial \phi}{\partial x}_u(\omega, z_u)\,H_i\,du \qquad \text{in probability.}$$

2) We have

$$(2.31)\qquad \frac{1}{2}\Sigma\left(\frac{\partial^2 \phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})(z_{T_{n+1}}-z_{T_n}, z_{T_{n+1}}-z_{T_n})\right)$$
$$=\frac{1}{2}\Sigma\left(\frac{\partial^2 \phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})(A_{t_{n+1}}-A_{T_n}, A_{T_{n+1}}-A_{T_n})\right)$$
$$+\frac{1}{2}\Sigma\left(\frac{\partial^2 \phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})\left(\int_{T_n}^{T_{n+1}} H_i\cdot \delta w^i, \int_{T_n}^{T_{n+1}} H_j\cdot \delta w^i\right)\right)$$
$$+\Sigma\left(\frac{\partial^2 \phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})\left(A_{T_{n+1}}-A_{T_n}, \int_{T_n}^{T_{n+1}} H_i\cdot \delta w^i\right)\right).$$

a) By (2.2) and (2.3), we have

$$(2.32)\qquad \left|\Sigma\left(\int_{T_n}^{T_{n+1}} \frac{\partial^2 \phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n})(A_{T_{n+1}}-A_{T_n}, A_{T_{n+1}}-A_{T_n})\right)\right| \leqq \varepsilon l\int_0^{t\wedge S_l} |dA|.$$

The L.H.S. of (2.32) tends then to 0 when $\varepsilon \to 0$.

b) First note that by (2.3), we have

$$(2.33)\quad \left|\Sigma\left(\left(\frac{\partial^2\phi}{\partial x^2}_{T_{n+1}}(\omega, z_{T_n}) - \frac{\partial^2\phi}{\partial x^2}_{T_n}(\omega, z_{T_n})\right)\left(\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i, \int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right)\right)\right| \leqq \varepsilon\Sigma\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^2.$$

Since

$$(2.34)\quad E\Sigma\left(\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^2\right) = E\int_0^{t\wedge S_l} |H_i|^2\,du$$

it follows that (2.33) converges to 0 in probability if $\varepsilon \to 0$.

Let us calculate

$$(2.35)\quad E\left|\Sigma\left[\frac{\partial^2\phi}{\partial x^2}_{T_n}(\omega, z_{T_n})\left(\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i, \int_{T_n}^{T_{n+1}} H_j\cdot\delta w^j\right) - \int_{T_n}^{T_{n+1}} \frac{\partial^2\phi}{\partial x^2}_{T_n}(\omega, z_{T_n})(H_i, H_i)\,du\right]\right|^2.$$

Classically, the various terms in the sum appearing in (2.35) are mutually orthogonal. (2.35) may then be bounded by

$$(2.36)\quad C\left(\Sigma\left(E\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^4 + E\left|\int_{T_n}^{T_{n+1}} |H_i|^2\,du\right|^2\right)\right).$$

By Burkholder-Davis-Gundy inequalities, we have

$$(2.37)\quad \begin{aligned} E\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^4 &\leqq \varepsilon^2 E\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^2 = \varepsilon^2 E\int_{T_n}^{T_{n+1}} |H_i|^2\,du \\ E\left|\int_{T_n}^{T_{n+1}} |H_i|^2\,du\right|^2 &\leqq CE\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^4 \leqq C\varepsilon^2 E\int_{T_n}^{T_{n+1}} |H_i|^2\,du. \end{aligned}$$

(2.36) may then be bounded by $C\varepsilon^2$. (2.35) tends then to 0 when $\varepsilon \to 0$. Now

$$(2.38)\quad \Sigma\left(\int_{T_n}^{T_{n+1}} \frac{\partial^2\phi}{\partial x^2}_{T_n}(\omega, z_{T_n})(H_i, H_i)\,du\right) = \int_0^{t\wedge S_l} \frac{\partial^2\phi}{\partial x^2}_{T_{n(u)}}(\omega, z_{T_{n(u)}})(H_i, H_i)\,du$$

which trivially converges to

$$(2.39)\quad \int_0^{t\wedge S_l} \frac{\partial^2\phi}{\partial x^2}_u(\omega, z_u)(H_i, H_i)\,du.$$

The second sum in the R.H.S. of (2.31) converges then to

$$(2.40)\quad \frac{1}{2}\int_0^{t\wedge S_l} \frac{\partial^2\phi}{\partial x^2}_u(\omega, z_u)(H_i, H_i)\,du.$$

c) One has

$$(2.41)\qquad \left|\Sigma\left(\frac{\partial^2\phi}{\partial x^2}{}_{T_{n+1}}(\omega, z_{T_{n+1}})\left(A_{T_{n+1}}-A_{T_n}, \int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right)\right)\right| \leqq \varepsilon l \int_0^{t\wedge S_l} |dA|.$$

The third sum in the R.H.S. of (2.31) tends to 0 when $\varepsilon\to 0$.

3) Obviously

$$(2.42)\qquad \begin{aligned}\Sigma|z_{T_{n+1}}-z_{T_n}|^2 &\leqq C\left[\Sigma\left(|A_{T_{n+1}}-A_{T_n}|^2+\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^2\right)\right]\\ &\leqq C\left(\varepsilon\int_0^{+\infty}|dA|+\Sigma\left|\int_{T_n}^{T_{n+1}} H_i\cdot\delta w^i\right|^2\right)\end{aligned}$$

which implies

$$(2.43)\qquad E(\Sigma|z_{T_{n+1}}-z_{T_n}|^2)\leqq C\left(\varepsilon k+E\int_0^{t\wedge S_l}|H_i|^2\,du\right).$$

From (2.15) and (2.43), we get that

$$(2.44)\qquad R_n\to 0 \quad \text{in probability.}$$

Using (2.18), (2.22), (2.30), (2.40), (2.44), we get that

$$(2.45)\qquad \begin{aligned}&\Sigma(\phi_{T_{n+1}}(\omega, z_{T_{n+1}})-\phi_{T_{n+1}}(\omega, z_{T_n}))\\ &\to\int_0^t \frac{\partial\phi}{\partial x}{}_u(\omega, z_u)\,dA+\int_0^{t\wedge S_l}\frac{\partial\phi}{\partial x}{}_u(\omega, z_u)\,H_i\cdot\delta w^i\\ &+\int_0^{t\wedge S_l}\frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\frac{\partial\phi}{\partial x}{}_u(\omega, z_u)\,H_i\,du\\ &+\frac{1}{2}\int_0^{t\wedge S_l}\frac{\partial^2\phi}{\partial x^2}{}_u(\omega, z_u)(H_i, H_i)\,du \qquad \text{in probability.}\end{aligned}$$

(2.1) follows from (2.11) and (2.45). □

With no extra cost, the following result may also be proved.

Theorem 2.2. *Let $(\tilde\Omega, \tilde F, \tilde P)$ be a complete probability space with a right continuous increasing family of σ-fields $\{\tilde F_t\}_{t\geqq 0}$ which contain the negligible sets of $\tilde F$. Let $w=(w^1\dots w^m)$ be a m dimensional brownian martingale on $(\tilde\Omega, \tilde F, \tilde P)$, and $\phi_{\cdot}(\tilde\omega, \cdot)$ be the flow defined by*

$$\phi_{\cdot}(\tilde\omega, \cdot)=\phi_{\cdot}(\omega(\tilde\omega), \cdot)$$

where $\omega(\tilde\omega)$ is given by: $\omega(\tilde\omega)=(w_{\cdot}(\tilde\omega))$.

Let z_t be a semi-martingale on $(\tilde\Omega, \tilde F, \tilde P)$ with values in R^d which may be written

$$z_t=z_0+A_t+M_t$$

where z_0 is F_0-measurable, A is an adapted a.s. right-continuous bounded variation process with $A=0$ and M is a local martingale with $M_0=0$. Then $\phi_t(\tilde{\omega}, z_t)$ is a semi-martingale which may be written

$$(2.46)\qquad \phi_t(\tilde{\omega}, z_t)=z_0+\int_0^t X_0^*(\phi_u(\tilde{\omega}, z_u))\,du+\int_0^t X_i(\phi_u(\tilde{\omega}, z_u))\cdot\delta w^i$$
$$+\int_0^t \frac{\partial X_i}{\partial x}(\phi_u(\tilde{\omega}, z_u^-))\frac{\partial \phi}{\partial x}{}_u(\tilde{\omega}, z_u^-)\,d\langle z, w^i\rangle_c$$
$$+\frac{1}{2}\int_0^t \frac{\partial^2 \phi}{\partial x^2}{}_u(\tilde{\omega}, z_u^-)\,d\langle z, z\rangle_c+\int_0^t \frac{\partial \phi}{\partial x}{}_u(\tilde{\omega}, z_u^-)\cdot\delta z$$
$$+\sum_{0\leq s\leq t}\left(\phi_s(\tilde{\omega}, z_s)-\phi_s(\tilde{\omega}, z_s^-)-\frac{\partial \phi}{\partial x}{}_s(\tilde{\omega}, z_s^-)\,\Delta z_s\right)$$

(δz is the Ito differential of z, $\langle z_i, z_j\rangle_c$ and $\langle z_j, w^i\rangle_c$ are defined by

$$(2.47)\qquad \begin{aligned}\langle z_i, z_j\rangle_c&=\langle M_i^c, M_j^c\rangle\\ \langle z_j, w^i\rangle_c&=\langle M_j^c, w^i\rangle\end{aligned}$$

where M^c is the continuous part of M and $\langle\ \rangle$ denotes quadratic variation [12]).

Proof. The proof follows the same lines as the proof of Theorem 2.1 and uses the techniques of [12]. □

Remark 1: The C^∞ assumption on $X_0 \dots X_m$ is too strong and may easily be weakened.

We now express formula (2.1) in Stratonovitch form, i.e. with the help of Stratonovitch integrals. Namely

Theorem 2.3. *Under the assumptions of Theorem* 2.1, *if dz is the Stratonovitch differential of z, then formula* (2.1) *may be written*

$$(2.48)\qquad \phi_t(\omega, z_t)=z_0+\int_0^t X_0(\phi_u(\omega, z_u))\,du$$
$$+\int_0^t X_i(\phi_u(\omega, z_u))\cdot dw^i+\int_0^t \frac{\partial \phi}{\partial x}{}_u(\omega, z_u)\cdot dz.$$

Proof. By formula (2.1) and the definition of Stratonovitch integral [12], we have

$$(2.49)\qquad \int_0^t X_i(\phi_u(\omega, z_u))\cdot dw^i=\int_0^t X_i(\phi_u(\omega, z_u))\cdot\delta w^i$$
$$+\tfrac{1}{2}\int_0^t \frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\left[X_i(\phi_u(\omega, z_u))+\frac{\partial \phi}{\partial x}{}_u(\omega, z_u)\,H_i\right]du.$$

Moreover, using (1.5), it is easily seen that $\left(\phi_\cdot(\omega,\cdot), \frac{\partial \phi}{\partial x}\cdot(\omega,\cdot)\right)$ defines a new flow on $R^d\times(R^d\otimes R^d)$, to which formula (2.1) is applicable: note that the coefficients in (1.5) and their derivatives are unbounded, but a stopping argument carries over the case to such a flow. By formula (2.1), the martingale part of the process $\frac{\partial \phi}{\partial x}{}_t(\omega, z_t)$ may be written

$$\int_0^t \left[\frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\frac{\partial \phi}{\partial x}{}_u(\omega, z_u)) + \frac{\partial^2 \phi}{\partial x^2}{}_u(\omega, z_u))H_i\right] \cdot \delta w^i. \tag{2.50}$$

It follows that

$$\begin{aligned}\int_0^t \frac{\partial \phi}{\partial x}{}_u(\omega, z_u) \cdot dz_u = \int_0^t \frac{\partial \phi}{\partial x}{}_u(\omega, z_u) \cdot \delta z_u \\ +\tfrac{1}{2}\int_0^t \left[\frac{\partial X_i}{\partial x}(\phi_u(\omega, z_u))\frac{\partial \phi}{\partial x}{}_u(\omega, z_u) H_i \right. \\ \left. + \frac{\partial^2 \phi}{\partial x^2}{}_u(\omega, z_u)(H_i, H_i)\right] du.\end{aligned} \tag{2.51}$$

From (2.1), (2.49), (2.51), we obtain (2.48). □

Remark 2: It is no surprise that formula (2.48) is similar to the corresponding formula for deterministic flows and differentiable z, since we use here Stratonovitch calculus. Note that writing (2.48) formally, (2.1) may be easily derived. However, (2.48) may only be proved using the techniques of Theorem 2.1.

3. The Diffeomorphism Property of the Flow

Let T be $a > 0$ real number. Set

$$\begin{aligned}&\tilde{X}_0 = -X_0,\ \tilde{X}_1 = -X_1 \dots \tilde{X}_m = -X_m \\ &\tilde{w}_s^T = w_T - w_{T-s} \qquad 0 \leq s \leq T.\end{aligned} \tag{3.1}$$

Let $\tilde{\phi}_.(\omega, \cdot)$ be the flow associated to $(\tilde{X}_0, \tilde{X}_1, \dots, \tilde{X}_m)$.

Note the trivial fact that the mapping $w \to \tilde{w}^T$ defines a mapping of (Ω, F_T) on (Ω, F_T): $\omega \to \tilde{\omega}^T$, which preserves P.

We now give the key time reversal argument of Ito [8], Malliavin [11] which proves that a.s. $\phi_T(\omega, \cdot)$ is a diffeomorphism of R^d onto R^d.

Theorem 3.1. *For any $T > 0$, there is a negligible set $\mathcal{N}_T$ in Ω such that if $\omega \notin \mathcal{N}_T$, then*

$$\phi_T(\omega, \tilde{\phi}_T(\tilde{\omega}^T, \cdot)) = \tilde{\phi}_T(\tilde{\omega}^T, \phi_T(\omega, \cdot)) = \text{identity on } R^d. \tag{3.2}$$

Proof. Assume first that $T=1$. Note the trivial

$$\phi_T^n(\tilde{\phi}_T^n(\tilde{\omega}^T, \cdot)) = \tilde{\phi}_T^n(\tilde{\omega}^T, \phi_T^n(\omega, \cdot) = \text{identity on } R^d \tag{3.3}$$

which follows from the possibility of time reversal on the differential equation (1.4). Now since $\omega \to \tilde{\omega}^T$ preserves measure P, we may apply f) in Theorem 1.1 to the flows $\phi^n_{\cdot}(\tilde{\omega},\cdot)$ and $\tilde{\phi}^n_{\cdot}(\tilde{\omega}^T,\cdot)$, and see that (3.2) holds. For $T \neq 1$, we do the time change $s \to s/T$ in Equation (1.1). □

From Theorem 3.1, we get

Theorem 3.2. *A.s., for any* $t \in Q^+$, $\phi_t(\omega,\cdot)$ *is a diffeomorphism of* R^d *onto* R^d, *and for any* $t \in R^+$, $\phi_t(\omega,\cdot)$ *is a diffeomorphism of* R^d *on the open set* $\phi_t(\omega, R^d)$.

Proof. Eliminating the countable $\{\mathcal{N}_t\}_{t\in Q^+}$, the first part of the Theorem 3.2 is a trivial consequence of Theorem 3.1. Assume that $\omega \notin \bigcup \{\mathcal{N}_t\}_{t\in Q^+}$ is such that all the properties mentioned in Theorem 1.1 hold.

Let $s \in R^+$. Suppose there exists $x, x' \in R^d$, with $x \neq x'$ such that $\phi_s(\omega,x) = \phi_s(\omega,x') = y$. Now $\frac{\partial \phi_s}{\partial x}(\omega,x)$ and $\frac{\partial \phi_s}{\partial x}(\omega,x')$ are non-singular. Since $(t,y) \to \frac{\partial \phi}{\partial x}{}_t(\omega,y)$ is continuous, by the implicit function Theorem, there exists disjoint neighborhoods V and V' of x and x' such that for $t \in R^+$ close enough to s, the equation in z $\phi_t(\omega,z) = y$ has one solution in V and one solution in V'. In particular if such a t is taken in Q^+, $\phi_t(\omega,\cdot)$ is not injective and this contradicts the assumption made on ω. Finally $\phi_s(\omega, R^d)$ is open since $\frac{\partial \phi}{\partial x}{}_s(\omega,\cdot)$ is continuous and non-singular. □

Note that if instead of considering R^d, we assume that it is replaced by a compact connected manifold N, it is trivial that a.s., for any $t \geqq 0$, $\phi_t(\omega,N) = N$ since $\phi_t(\omega,N)$ is open and closed in N.

The difficulty comes here from the non-compactness of R^d and more specifically from the lack of control at infinity of $\|\phi_t(\omega,x) - x\|$. Using (1.7), it is proved in [3] (also see [2]) that for $\beta > 1$, $T > 0$, $L_{\beta,T}(\omega)$ exists such that

(3.4) for $t \leqq T$, $x \in R^d$, $|\phi_t(\omega,x) - x| \leqq L_{\beta,T}(\omega)(1 + |x|^\beta)$

but this is clearly an insufficient bound.

Finally note the trivial fact that the uniform consequence in probability of $\phi^n_{\cdot}(\omega,\cdot)$ on compact sets does not suffice to prove the a.s. onto property of $\phi_{\cdot}(\omega,\cdot)$, even though the $\phi^n_{\cdot}(\omega,\cdot)$ are themselves onto for any ω, t.

From now on, we assume that all the properties of Theorem 1.1b), c), d) *and* the properties listed in Theorem 3.2 hold for *every* $\omega \in \Omega$. To do this, it suffices to set

$$\phi_t(\omega,\cdot) = \text{identity for any } t \in R^+$$

when ω belongs to the negligible set where one of these properties does not hold.

We then have the key result

Theorem 3.3. *A.s., for any* $t \in R^+$, $\phi_t(\omega,\cdot)$ *is a diffeomorphism of* R^d *onto* R^d.

Proof. Let A be the random set in $\Omega \times R^+$

$$A = \{(\omega,t);\ \phi_t(\omega,R^d) = R^d\}. \tag{3.5}$$

We now prove that A is optional [4]. Let B and C be the random sets

$$B=\{(\omega,t);\ \overline{\phi_t(\omega,R^d)}=R^d\} \tag{3.6}$$
$$C=\{(\omega,t);\ \lim_{\|x\|\to+\infty}\|\phi_t(\omega,x)\|=+\infty\}.$$

We first prove that

$$A=B\cap C. \tag{3.7}$$

Clearly if $(\omega,t)\in A, (\omega,t)\in B$. Moreover when $(\omega,t)\in A$, since $\frac{\partial\phi}{\partial x}{}_t(\omega,\cdot)$ is non-singular and is continuous in x, and since $\phi_t(\omega,\cdot)$ is injective, the mapping $\phi_t^{-1}(\omega,\cdot)$ is well defined, is continuous and differentiable – it is in fact C^∞ – on R^d. If $(\omega,t)\in A$, $(\omega,t)\notin C$, there is a sequence x_n such that

$$\|x_n\|\to+\infty \tag{3.8}$$
$$y_n=\phi_t(\omega,x_n)\to y\in R^d.$$

Clearly

$$x_n=\phi_t^{-1}(\omega,y_n)$$

and since $y_n\to y$, we have

$$x_n\to x=\phi_t^{-1}(\omega,y) \tag{3.9}$$

which is a contradiction to (3.8). We have then proved that $A\subset B\cap C$. Conversely if $(\omega,t)\in B\cap C$, take $y\in R^d$. Since y is in the closure of $\phi_t(\omega,R^d)$, there exists x_n such that $\phi_t(\omega,x_n)\to y$. Now because $(\omega,t)\in C$, the sequence x_n is bounded and there is a subsequence $x_{n_k}\to x$. This implies that $\phi_t(\omega,x)=y$. $\phi_t(\omega,\cdot)$ is then onto and $(\omega,t)\in A$. (3.7) is then proved.

To prove A is optional, we need only to prove that B and C are optional. Trivially

$$B=\bigcap_{x\in Q^d}\{(\omega,t);\ \inf_{y\in Q^d}\|x-\phi_t(\omega,y)\|=0\}. \tag{3.10}$$

Since for any $y\in Q^d$, $\phi_t(\omega,y)$ is an optional process, B is optional.

Similarly, we have

$$C=\{(\omega,t);\ \sup_n\ \inf_{\|x\|\geqq n\, x\in Q^d}\|\phi_t(\omega,x)\|=+\infty\} \tag{3.11}$$

which implies C is optional. A is then optional.

Assume now ${}^cA$ is a non vanishing set, i.e. that $P(\pi({}^cA))\neq0$, where π is the projection mapping $\Omega\times R^+$ on Ω. By the optional selection Theorem [4] IV.84 there is a stopping time T whose graph $[T]$ is such that $[T]\subset{}^cA$, and moreover $P(T<+\infty)>0$.

Now for $n\in N$ large enough $(T\leqq n)$ is non negligible. From Theorem 1.2, we know that

$$\text{(a.s.), on } (T\leqq n),\ \phi_n(\omega,\cdot)=\phi_{n-T}(\theta_T(\omega),\ \phi_T(\omega,\cdot)). \tag{3.12}$$

Since for any ω, all the $\phi_s(\omega,\cdot)$ are injective, $\phi_{n-T}(\theta_T(\omega),\cdot)$ is injective. Since $[T]\subset{}^cA$, on $(T<+\infty)$ $\phi_T(\omega,R^d)\neq R^d$. From this, we conclude that

(3.13) a.s. on $(T \leqq n)$ $\phi_n(\omega, R^d) \neq R^d$.

Since $n \in Q^+$, (3.13) is a contradiction to Theorem 3.2, which asserts that $\phi_n(\omega, \cdot)$ is onto.

^cA is then a vanishing set. Theorem 3.2 is proved. □

Remark 1. The key argument in the proof is that w is an independent increment propagator, such that when a singularity appears in the flow $\phi_{\cdot}(\omega, \cdot)$, it remains for the whole future. This of course prevents $\phi_t(\omega, \cdot)$ from not being onto at certain times.

Remark 2. Since a.s., for any t, $\phi_t(\omega, \cdot)$ is a nonsingular C^∞ diffeomorphism of R^d onto R^d, $\phi_t^{-1}(\omega, \cdot)$ is then a well defined flow of C^∞ diffeomorphisms of R^d onto R^d, such that $\phi_t^{-1}(\omega, \cdot), \ldots, \dfrac{\partial^m [\phi_t^{-1}]}{\partial x^m}(\omega, x) \ldots$ are all a.s. jointly continuous in (t, x).

4. Action of the Reciprocal Flow on Continuous Semi-Martingales

By Remark 2.2, we know that if z_t is a continuous process, the process $\phi_t^{-1}(\omega, z_t)$ is well defined and is a.s. continuous.

Assume that z_t is a continuous semi-martingale defined on (Ω, F, F_t^+, P) with values in R^d, which may be written

$$z_t = z_0 + A_t + \int_0^t H_i \cdot \delta w^i \tag{4.1}$$

where $z_0 \in R^d$, A is a continuous adapted bounded variation process such that $A_0 = 0$, $H_1 \ldots H_m$ are adapted processes such that $\int_0^t |H_i|^2 \, ds < +\infty$ a.s.

We then have

Theorem 4.1. *$y_t = \phi_t^{-1}(\omega, z_t)$ is a continuous semi-martingale which is the unique solution of the stochastic differential equation*

$$\begin{aligned} dy &= \left[\frac{\partial \phi}{\partial x}{}_t(\omega, y_t)\right]^{-1} [dz - X_0(z_t)\, dt - X_i(z_t) \cdot dw^i] \\ y(0) &= z_0 \end{aligned} \tag{4.2}$$

where dz is the Stratonovitch differential of z. (4.2) *may be also written*

$$\begin{aligned} dy = {} & \left[\frac{\partial \phi}{\partial x}{}_t(\omega, y_t)\right]^{-1} [\delta z - X_0(z_t)\, dt - X_i(z_t) \cdot \delta w^i] \\ & + \left[\frac{\partial \phi}{\partial x}{}_t(\omega, y_t)\right]^{-1} \left[-\frac{1}{2}\frac{\partial X_i}{\partial x}(z_t) H_i - \frac{1}{2}\frac{\partial X_i}{\partial x}(\phi_t(\omega, y_t))\,(H_i - X_i(z_t)) \right. \\ & - \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}{}_t(\omega, y_t) \left(\left[\frac{\partial \phi}{\partial x}{}_t(\omega, y_t)\right]^{-1} (H_i - X_i(z_t)), \right. \\ & \left.\left. \left[\frac{\partial \phi}{\partial x}{}_t(\omega, y_t)\right]^{-1} (H_i - X_i(z_t)) \right) \right] dt \\ y(0) = {} & z_0 \end{aligned} \tag{4.2'}$$

where δz is the Ito differential of z.

Proof. Consider the stochastic differential equation (4.2′). Note first that the growth bounds (3.4) which may also be proven for $\frac{\partial \phi}{\partial x}.(\omega,\cdot)$ and $\left[\frac{\partial \phi}{\partial x}.(\omega,\cdot)\right]^{-1}$ are insufficient to ensure existence and uniqueness of a solution of (4.2′) on $[0,+\infty[$, since β in (3.4) is always >1.

Put (4.2′) in the form

$$\begin{aligned} &dy = f_{0_t}(\omega, y_t)\cdot\delta h^0 + f_{1_t}(\omega, y_t).\delta h^1 + \dots + f_{l_t}(\omega, y_t)\cdot\delta h^l \\ &y(0) = y \end{aligned} \tag{4.3}$$

where $h^0 \dots h^l$ are continuous semi-martingales.

Let $T^{n,k}$ be the stopping time

$$T^{n,k} = \inf\left\{t \geqq 0 \colon \sup_{|y| \leqq n} \left[\left|\left[\frac{\partial \phi}{\partial x}{}_t(\omega, y)\right]^{-1}\right| \vee \left|\frac{\partial^2 \phi}{\partial x^2}{}_t(\omega, y)\right| \vee \left|\frac{\partial^3 \phi}{\partial x^3}{}_t(\omega, y)\right|\right] \geqq k\right\}. \tag{4.4}$$

Clearly by a joint continuity argument

$$\lim_{k \to +\infty} T^{n,k} = +\infty \quad \text{a.s.} \tag{4.5}$$

Let π_n be the operator of projection on the ball $\bar{B}_n$ of center 0 and radius n in R^d.

Consider now the stochastic differential equation

$$\begin{aligned} &dy = f_{i_t}(\omega, \pi_n(y))\cdot\delta h^i \\ &y(0) = y. \end{aligned} \tag{4.6}$$

Now using (4.4) and the fact that π_n is uniformly Lipschitz, for $t \leqq T^{n,k}$, the mappings

$$y \to f_{i_t}(\omega, \pi_n(y))$$

are easily seen to be bounded and uniformly Lipschitz. By the results of Doléans-Dade, Protter, Emery [6], (4.6) has a unique solution $y^{n,k}$ on $[0, T^{n,k}]$. Clearly $y^{n,k+1}_{t\wedge T^{n,k}} = y^{n,k}_{t\wedge T^{n,k}}$. A unique solution y^n of (4.6) is then defined on $[0,+\infty[$.

Let T'^n be the stopping time

$$T'^n = \inf\{t;\ |y^n_t| \geqq n\}. \tag{4.7}$$

On $[0, T'^n]$, y^n is clearly a solution of (4.3), and moreover $y^{n+1}_{t\wedge T'^n} = y^n_{t\wedge T'^n}$.

Now on $[0, T'^n]$, y^n is a solution of (4.2). To see this note that for any x, $\left[\frac{\partial \phi}{\partial x}{}_t(\omega, x)\right]^{-1}$ is a solution of Equation (1.10). Now Theorem 2.1 may be used to describe the process $\left[\frac{\partial \phi}{\partial x}{}_t(\omega, y^n_t)\right]^{-1}$ on $[0, T'^n]$, which is a semi-martingale. More precisely, by (2.1), we know that the local martingale part of the process $\left[\frac{\partial \phi}{\partial x}{}_t(\omega, y^n_t)\right]^{-1}$ on $[0, T'^n]$ is

$$(4.8)\qquad -\int_0^t \left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} \frac{\partial X_i}{\partial x}(\phi_s(\omega, y_s^n)) \cdot \delta w^i$$
$$+\int_0^t \frac{\partial}{\partial y}\left(\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1}\right)\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} (H_i - X_i(z_s)) \cdot \delta w^i.$$

Using the rules of transformation of a Stratonovitch integral into an Ito integral [12] and the trivial

$$(4.9)\qquad \frac{\partial}{\partial y}\left(\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1}\right) = -\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1}\left[\frac{\partial^2 \phi}{\partial x^2}{}_s(\omega, y_s^n)\right]\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1}$$

we see that

$$(4.10)\qquad \int_0^t \left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} [dz - X_i(z_s) \cdot dw^i]$$
$$= \int_0^t \left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} [\delta z - X_i(z_s)\, \delta w^i]$$
$$+\frac{1}{2}\int_0^t \left\{\left[-\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} \frac{\partial X_i}{\partial x}(z_s) H_i\right.$$
$$-\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} \frac{\partial X_i}{\partial x}(\phi_s(\omega, y_s^n))(H_i - X_i(z_s))$$
$$-\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} \frac{\partial^2 \phi}{\partial x^2}{}_s(\omega, y_s^n)\left(\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} (H_i - X_i(z_s)),\right.$$
$$\left.\left.\left[\frac{\partial \phi}{\partial x}{}_s(\omega, y_s^n)\right]^{-1} (H_i - X_i(z_s))\right)\right\} ds.$$

Using (4.10), y_t^n is seen to be a solution of (4.2) on $[0, T'^n]$.

By Theorem 2.3, $\phi_t(\omega, y_t^n)$ is a semi-martingale on $[0, T'^n]$ such that

$$(4.11)\qquad \phi_t(\omega, y_t^n) = z_0 + \int_0^t X_0(\phi_u(\omega, y_u^n))\, du + \int_0^t X_i(\phi_u(\omega, y_u^n))\, dw^i + \int_0^t \frac{\partial \phi}{\partial x}{}_u(\omega, y_u^n) \cdot dy^n.$$

Using the explicit form of dy^n on $[0, T'^n]$, we obtain

$$(4.12)\qquad \phi_t(\omega, y_t^n) = z_t + \int_0^t [X_0(\phi_u(\omega, y_u^n)) - X_0(z_u)]\, du + \int_0^t [X_i(\phi_u(\omega, y_u^n)) - X_i(z_t)]\, dw^i.$$

If $x_t^n = \phi_t(\omega, y_t^n)$, on $[0, T'^n]$, we have

$$(4.13)\qquad x_t^n = z_t + \int_0^t (X_0(x_u^n) - X_0(z_u))\, du + \int_0^t [X_i(x_u^n) - X_i(z_u)] \cdot dw^i$$

which may be written in Ito form

$$(4.14)\qquad x_t^n = z_t + \int_0^t \left(X_0(x_u^n) - X_0(z_u) + \frac{1}{2}\frac{\partial X_i}{\partial x}(x_u^n)[H_i + X_i(x_u^n) - X_i(z_u)]\right.$$
$$\left.-\frac{1}{2}\frac{\partial X_i}{\partial x}(z_u) H_i\right) du + \int_0^t (X_i(x_u^n) - X_i(z_u)) \cdot \delta w^i.$$

Now (4.14) is a stochastic differential equation of the type

$$dx^n = g_i(x^n) \cdot \delta k^i \tag{4.15}$$

where $g_0 \dots g_l$ are Lipschitz in the variable x^n, and $k_0 \dots k_l$ are continuous semi-martingales. Using [6] again, (4.14) is seen to have a unique solution which is trivially $x_t^n = z_t$. We have then shown

(4.16) On $[0, T'^n]$ $\phi_t(\omega, y_t^n) = z_t$

i.e.

(4.17) on $[0, T'^n]$ $y_t^n = \phi_t^{-1}(\omega, z_t)$.

Since $\phi_t^{-1}(\omega, z_t)$ is a continuous process, $T'^n \to +\infty$ a.s. Existence of the solutions of (4.2), (4.2′) has then been proved on $[0, +\infty[$. Uniqueness is trivial from (4.17), or from uniqueness on $[0, T^{n,k}]$ of the solutions of (4.6). □

Remark 1. Formula (4.2) is algebraically trivial since, when we use Stratonovitch calculus, we know that the formula must look like the corresponding formula of classical differential calculus. All the calculus (4.11), (4.12) is strictly identical to what is done for deterministic flows.

References

1. Baxendale, P.: Wiener processes on manifolds of maps. To appear
2. Bismut, J.M.: Flots stochastiques et formule de Ito-Stratonovitch généralisée. C.R. Acad. Sci. Paris, série A, **290**, 483–486 (1980)
3. Bismut, J.M.: Principes de mécanique aléatoire. Lecture Notes in Mathematics. To appear
4. Dellacherie, C., Meyer, P.A.: Probabilités et potentiels. 2° édition. Paris: Hermann 1975
5. Elworthy, K.D.: Stochastic dynamical systems and their flows. Stochastic Analysis, A. Friedman and M. Pinsky Ed., 79–95. New York: Acad. Press 1978
6. Emery, M.: Equations différentielles lipschitziennes: étude de la stabilité. Séminaire de Probabilités n° 13, p. 281–293. Lecture Notes in Mathematics n° **721**. Berlin-Heidelberg-New York: Springer 1979
7. Gihman, I.I., Skorohod, A.V.: Stochastic differential equations. Ergebnisse der Mathematik und ihrer Grenzgebiete. Band **72**. Berlin-Heidelberg-New York: Springer 1972
8. Ito, K.: Extension of stochastic integrals. Proceedings of the International symposium on Stochastic differential equations of Kyoto (1976), 95–109. Tokyo: Kinokuniya and New York: Wiley 1978
9. Kunita, H.: On an extension of Ito's formula. To appear
10. Kunita, H., Watanabe, S.: On square-integrable martingales. Nagoya Math. Journal **30**, 209–245 (1967)
11. Malliavin, P.: Stochastic Calculus of variations and hypoelliptic operators. Proceedings of the International symposium on Stochastic differential equations of Kyoto (1976), 195–263. Tokyo: Kinokuniya and New York: Wiley 1978
12. Meyer, P.A.: Un cours sur les intégrales stochastiques. Séminaire de Probabilités n° **10**, 245–400. Lecture Notes in Mathematics n° **511**. Berlin-Heidelberg-New York: Springer 1976
13. Stroock, D., Varadhan, S.R.S.: On the support of diffusions with applications to the strong maximum principle. 6th Berkeley Symposium on Probability and Statistics, Vol. III, 333–359. Berkeley-Los Angeles: University of California Press 1972
14. Stroock, D., Varadhan, S.R.S.: Multidimensional diffusion processes. Grundlehren der Mathematischen Wissenschaften **233**. Berlin-Heidelberg-New York: Springer 1979

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