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Local Times for a Class of Purely Discontinuous Martingales

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Summary. Suppose X_t is a purely discontinuous martingale. A sufficient condition for X_t to have a local time is given in terms of the local characteristics of X. An example is constructed to show that this condition is nearly optimal.

1. Introduction

In 1975, Meyer [10] showed how to use Tanaka's formula to construct local times for martingales with nondegenerate continuous parts. He then raised the question: when does a purely discontinuous martingale have a local time? Yoeurp [16] and Yor [17] have both shown that Tanaka's formula fails badly in this case.

On the other hand, the theory of additive functionals may be used to construct local times for certain Markov processes, those for which points are regular for themselves (Blumenthal and Getoor [3]). Sufficient conditions for local time to exist in terms of the resolvent operator were given by Boylan [4] and Griego [6]. Maisonneuve [9] has extended the Markov theory approach to construct local times for martingales when points are regular for themselves.

The difficulty with the Markov theory approach is that one can very rarely check whether points are regular for themselves or whether the resolvent operators have the proper form. Kesten [8] obtained very good results in the case of processes with stationary, independent increments, but virtually no other examples are known.

The purpose of this paper is to give a condition that is sufficient for the existence of local times for purely discontinuous semimartingales. Our theorem is stated in terms of local characteristics. Hence, when a process is defined by means of a stochastic differential equation with respect to a Poisson point process, or when, in the case of real-valued Markov processes, it is defined by its infinitesimal generator, one can easily read off the local characteristics and apply our condition.

To state our results, we first need some definitions. We will consider only local times that are occupation time densities with respect to Lebesgue measure. Thus, we are interested in the existence of a jointly measurable process $L_t(x)$ such that, a.s.,

(1.1) $\int_0^t 1_B(X_s)\,ds = \int 1_B(x)\,L_t(x)\,dx$ for all t, for all Borel $B \subseteq \mathbb{R}$.

Our processes X_t will be semimartingales that have local characteristics (a_s, v_s). This means (our definition may vary slightly from other definitions) (i) a_s is adapted, (ii) v_s is adapted, and for each s and ω, a σ-finite measure on $\mathbb{R}$-$\{0\}$, (iii) for each Borel B such that $\bar{B} \subseteq \mathbb{R}$-$\{0\}$ is compact, $\sum_{s \leq t} 1_B(\Delta X_s) - \int_0^t v_s(B)\,ds$ is a local martingale, and (iv) $X_t - \sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| > 1)} - \int_0^t a_s\,ds$ is a local martingale. X_t purely discontinuous means that X_t is the uniform limit of

$$\sum_{s \leq t} \Delta X_s 1_{(|\Delta X_s| \geq \varepsilon)} - \int_0^t \int_{1 \geq |h| \geq \varepsilon} h v_s(dh)\,ds + \int_0^t a_s\,ds \quad \text{as} \quad \varepsilon \to 0.$$

If $1 < \alpha < 2$, let $\theta_\alpha(dh) = \zeta_\alpha |h|^{-(1+\alpha)}dh$ be the Lévy measure for a stable symmetric process Z_t of index α. Here ζ_α is the positive constant chosen so that $E\exp(\mathrm{is}\,Z_t) = \exp(-t|s|^\alpha)$.

Our main result is: suppose

(1.2) *X_t is a purely discontinuous semimartingale with local characteristics $(b_s a_s, b_s v_s)$ such that*

a) *b_s is measurable, and for some $K_{1.2A}$,* $\sup_s |b_s^{-1}| \leq K_{1.2A}$, a.s.,

b) *for some $K_{1.2B}$,* $\sup_s |a_s| \leq K_{1.2B}$, a.s.,

c) *for some $K_{1.2C}$,* $\sup_s |\int h^2 \wedge 1 v_s(dh)| \leq K_{1.2C}$, a.s., *and*

d) *for some* $1 < \alpha < 2$, $0 < \sigma < 1$, $\varepsilon > 0$, $K_{1.2D} > 0$,

$$\sup_s \int_{-\sigma}^{\sigma} |h|^{\alpha - \varepsilon} |v_s - \theta_\alpha|(dh) \leq K_{1.2D}, \text{ a.s.,}$$

where $|v_s - \theta_\alpha|$ is the total variation measure of $v_s - \theta_\alpha$.

If (1.2) holds, then a local time $L_t(x)$ for X_t exists (see Theorem (4.13) and Sect. 6).

At first glance, (1.2)d may seem very restrictive. However, in fact, (1.2)d is nearly optimal. By this, we mean, if $\varepsilon > 0$, then there exists a process X_t such that

(1.3) *X_t is a purely discontinuous local martingale with local characteristics* $(0, v_s)$ *such that*

a) $v_s(dh) \geq \theta_\alpha(dh)$ *for all s and ω, and all* $|h| \leq 1$, *and*

b) *for some $K_{1.3}$,*

$$\sup_s \int_{-1}^{1} |h|^{\alpha+\varepsilon}(v_s - \theta_\alpha)(dh) \leq K_{1.3},$$

but for which no local time for X_t can exist (see Theorem (8.8)).

If $\psi_s(dh) = (v_s - \theta_\alpha)(dh)$, (1.2)d says that ψ_s can be very large near 0, but no larger than θ_α itself. In fact, if we allow ψ_s to be larger than θ_α near 0, then, even if ψ_s is positive, there need not exist a local time for X_t.

Although (1.2)d is nearly optimal in the sense given above, it is by no means necessary. For example, there are processes with stationary, independent increments that do not satisfy (1.2)d but have local times. It would be of interest to get a result with θ_α in (1.2)d replaced by the Lévy measure of any process with stationary, independent increments that itself has a local time, particularly for those whose Lévy measure is close to that of a (asymmetric) Cauchy. It would be even more interesting to see if any kind of continuity condition on $v_s(dh)$, together with $v_s(dh) \geqq \theta_\alpha(dh)$, $|h| \leqq 1$, some $\alpha > 1$, would suffice for the existence of a local time.

In Sect. 2 we obtain some estimates on the density of the resolvent of a symmetric stable process of index α. In Sect. 3 we consider X_t satisfying

(1.4) *X_t is a purely discontinuous local martingale with local characteristics* $(0, v_s)$ *such that* (1.2)c *holds, for some* $1 < \alpha < 2$, $v_s(dh) = \theta_\alpha(dh)$ *if* $|h| > 1$, *and* (1.2)d *holds with this same value of* α *and with* $\sigma = 1$.

We use a perturbation argument to show that the expected time spent in sets by X_t has a bounded density with respect to Lebesgue measure. We also indicate how, if $a_s = \bar{a}(X_s)$, $b_s = \bar{b}(X_s)$, and $v_s(dh) = \bar{v}(X_s, dh)$ for functions $\bar{a}$ and $\bar{b}$ and a kernel $\bar{v}(x, dh)$, then the techniques of Section 3 may be used to prove uniqueness of a martingale problem (and hence of a Markov process) specified by $(b_s a_s, b_s v_s)$. This extends the results of Tanaka, Tsuchiya, and Watanabe [15] who discussed the case (a_s, θ_α).

Section 4 uses the results of Sect. 3 and the stochastic calculus to construct the required $L_t(x)$ when X_t satisfies (1.4). In Sect. 5 we show that $L_t(x)$ is continuous in t, and we obtain a modulus of continuity. (There is no reason to expect $L_t(x)$ to be continuous in x.) In Sect. 6 we use localization and time change to show the existence of $L_t(x)$ when (1.2) holds. After some preliminary results concerning weak convergence in Sect. 7, in Sect. 8 we construct a process satisfying (1.3) that has no local time. Since any process satisfying (1.3) spends 0 time at points (Proposition (7.6)), we accomplish our construction by finding a Cantor-like set of Lebesgue measure 0 and a process X_t that spends positive time there.

Other notation we will use is $\Delta_C^h f(x) = f(x+h) + f(x-h) - 2f(x)$, $\Delta_R^h f(x) = f(x+h) - f(x) - hf'(x)$. Let $\mathscr{G}_\alpha$ be the infinitesimal generator of a symmetric stable process of index α. Hence $\mathscr{G}_\alpha f(x) = \int \Delta_R^h f(x)\theta_\alpha(dh)$.

Let $\| \ \|$ be sup norm, C_K continuous functions with compact support, C^2 twice continuously differentiable functions. Let $\| \ \|_{L_1}$ be the L_1 norm with respect to μ, Lebesgue measure, and let $*$ denote convolution.

Let $\mathscr{F}_t^0 = \sigma(X_s; s \leqq t)$ and let $\mathscr{F}_t$ be the P-completion of $\mathscr{F}_{t+}^0$. Let $[X, X_t] = \Sigma_{s \leqq t} \Delta X_s^2$ for purely discontinuous semimartingales. Notation and terminology relative to stochastic integrals and semimartingales may be found in [5], [7], and [10]. Constants whose subscripts are proposition numbers, e.g., $c_{2.1}$, do not change, but constants with an integer as a subscript, e.g., c_1, may chance from place to place.

I would like to thank M. Cranston for many helpful conversations concerning Sect. 7 and 8.

2. Stable Densities

In this section we derive some estimates for the densities of stable processes. Although the methods used are routine, the results do not appear to be in the literature.

Let $1 < \alpha < 2$ be fixed, and let X_t be a symmetric stable process of index α with $X_0 = 0$. Let $q_t(x)$ be the density of X_t, and let us write $q(x)$ for $q_1(x)$. The characteristic function of X_1 is $E\exp(isX_1) = \exp(-|s|^\alpha)$. Since $|s|^r\exp(-|s|^\alpha)$ is integrable for all positive r, q has bounded derivatives of all orders. We need the following estimate on $q''(x)$, the proof of which closely follows [2].

(2.1) **Proposition.** *There exist positive constants $c_{2.1}$ and $M_{2.1}$ such that if $x \geqq M_{2.1}$,*

$$q''(x) = c_{2.1}\, x^{-(3+\alpha)}(1+o(1)).$$

Proof. Since q is symmetric, it suffices to consider $x > 0$. Since $s^2\exp(-|s|^\alpha)$ is integrable and real,

$$q''(x) = -(2\pi)^{-1}(u(x) + \bar{u}(x)), \tag{2.2}$$

where

$$u(x) = -\int_0^\infty e^{-isx} s^2 \exp(-|s|^\alpha)\, ds.$$

Choose $\varphi < 0$ and sufficiently small so that $|\varphi| < \pi/2$ and so that $a = \pi + \alpha\varphi$, $b = 3\pi/2 + \varphi$ lie in $(\pi/2, 3\pi/2)$. Integrate $z^2\exp(-izx - z^\alpha)$ along the contour C in the complex plane (r,θ) made up of the pieces $\theta = 0$, $r_0 \leqq r \leqq r_1$; $\theta = \varphi$, $r_0 \leqq r \leqq r_1$; $r = r_0$, $0 \geqq \theta \geqq \varphi$; and $r = r_1$, $0 \geqq \theta \geqq \varphi$. The integral around C is 0, and letting $r_0 \to 0$, $r_1 \to \infty$, we get

$$u(x) = -e^{3i\varphi}\int_0^\infty s^2 \exp(sxe^{ib} + s^\alpha e^{ia})\, ds.$$

Letting $z = s^\alpha e^{ia}$, observing that $Re(z) \leqq 0$, and using the expansion $e^z = 1 + z + 0(|z|^2)$ for such z, we get

$$u(x) = -e^{3i\varphi}\int_0^\infty s^2\exp(sxe^{ib})\, ds - e^{3i\varphi+ia}\int_0^\infty s^{2+\alpha}\exp(sxe^{ib})\, ds + R, \tag{2.3}$$

where $|R| \leqq c_1 \int_0^\infty s^{2+2\alpha} e^{-c_2 sx}\, ds = O(x^{-(3+2\alpha)})$

Now integrate the functions $z^2\exp(zxe^{ib})$ and $z^{2+\alpha}\exp(zxe^{ib})$ around the contour $\theta = 0$, $r_0 \leqq r \leqq r_1$; $\theta = -\varphi - \pi/2$, $r_0 \leqq r \leqq r_1$; $r = r_0$, $0 \geqq \theta \geqq -\varphi - \pi/2$; and $r = r_1$, $0 \geqq \theta \geqq -\varphi - \pi/2$, and let $r_0 \to 0$, $r_1 \to \infty$ to transform the first two integrals on the right side of (2.3). We then see that the first term on the right of (2.3) is purely imaginary.

Using (2.2), we get

$$\begin{aligned} q''(x) &= -\pi^{-1}\operatorname{Re} u(x) \\ &= c_3\int_0^\infty s^{2+\alpha}\exp(-sx)\, ds + \text{Remainder} \\ &= c_4 x^{-(3+\alpha)} + O(x^{-(3+2\alpha)}). \quad \square \end{aligned}$$

Since X_t has the same law as $t^{1/\alpha}X_1$, $q_t(x)=t^{-1/\alpha}q(xt^{-1/\alpha})$, and $q_t''(x)=t^{-3/\alpha}q''(xt^{-1/\alpha})$.

Define

$$r_{\lambda,\varepsilon}(x)=\int_\varepsilon^\infty e^{-\lambda t}q_t(x)\,dt=\int_\varepsilon^\infty e^{-\lambda t}t^{-1/\alpha}q(xt^{-1/\alpha})\,dt, \tag{2.4}$$

and $r_\lambda(x)=r_{\lambda,0}(x)$.

(2.5) **Proposition.** *Suppose $\lambda\geqq 1$.*

a) $0\leqq r_{\lambda,\varepsilon}(x)\leqq c_{2.5A}(\lambda)$, *where $c_{2.5A}(\lambda)$ is independent of ε and x and $\to 0$ as $\lambda\to\infty$;*

b) $|r''_{\lambda,\varepsilon}(x)|\leqq c_{2.5B}|x|^{\alpha-3}$, *$c_{2.5B}$ independent of λ and ε;*

c) $|r'_{\lambda,\varepsilon}(x)|\leqq c_{2.5C}|x|^{\alpha-2}$, *$c_{2.5C}$ independent of λ and ε;*

d) *if $\delta>0$, $|x|\geqq\delta$, $|r''_{\lambda,\varepsilon}(x)|\leqq c_{2.5D}(\lambda,\delta)|x|^{\alpha-3}$, where $c_{2.5D}(\lambda,\delta)$ is independent of ε and $\to 0$ as $\lambda\to\infty$;*

e) *if $\delta>0$, $|x|\geqq\delta$, $|r'_{\lambda,\varepsilon}(x)|\leqq c_{2.5E}(\lambda,\delta)|x|^{\alpha-2}$, where $c_{2.5E}(\lambda,\delta)$ is independent of ε and $\to 0$ as $\lambda\to\infty$.*

Proof. a) $|r_{\lambda,\varepsilon}(x)|\leqq\|q\|\int_0^\infty e^{-\lambda t}t^{-1/\alpha}dt\to 0$ as $\lambda\to\infty$ by dominated convergence, since $\alpha>1$.

b) By the symmetry of q, we may suppose $x\geqq 0$. The Fourier transform of $r_{\lambda,\varepsilon}$ is $\int_\varepsilon^\infty e^{-\lambda t}\exp(-t|s|^\alpha)\,dt=\exp(-(\lambda+|s|^\alpha)\varepsilon)/(\lambda+|s|^\alpha)$, and hence $r_{\lambda,\varepsilon}$ has derivatives of all orders.

$$\begin{aligned}|r''_{\lambda,\varepsilon}(x)|&\leqq\int_\varepsilon^\infty e^{-\lambda t}t^{-3/\alpha}|q''(xt^{-1/\alpha})|\,dt\\&=x^{\alpha-3}\int_{\varepsilon x^{-\alpha}}^\infty e^{-\lambda ux^\alpha}u^{-3/\alpha}|q''(u^{-1/\alpha})|\,du\\&\leqq x^{\alpha-3}\left(c_1\int_0^{M_{2.1}^{-\alpha}}u^{-3/\alpha}(u^{-1/\alpha})^{-(3+\alpha)}du+\|q''\|\int_{M_{2.1}^{-\alpha}}^\infty u^{-3/\alpha}du\right),\end{aligned}\tag{2.6}$$

which, since $\alpha<2$, gives b).

c) If $x_1<x_2<0$,

$$|r'_{\lambda,\varepsilon}(x_2)-r'_{\lambda,\varepsilon}(x_1)|\leqq\int_{x_1}^{x_2}|r''_{\lambda,\varepsilon}(x)|\,dx\to 0$$

as $x_1,x_2\to-\infty$, using b) and the fact that $\alpha<2$. Therefore $\lim_{x\to-\infty}r'_{\lambda,\varepsilon}(x)$ exists. Of course, the limit must be 0 by a). But then, if $x<0$,

$$r'_{\lambda,\varepsilon}(x)=\int_{-\infty}^x r''_{\lambda,\varepsilon}(y)\,dy,$$

and c) follows from integrating b). The case $x>0$ follows by symmetry.

d) If $x\geqq\delta>0$, $e^{-\lambda ux^\alpha}\leqq e^{-\lambda u\delta^\alpha}$, and d) follows from (2.6) by dominated convergence.

e) This follows from d) in the same way that c) follows from b). □

(2.7) **Proposition.** *Suppose $\lambda \geqq 1$, $\varepsilon \leqq 1$, $A \geqq 1$, and $0 < \gamma < (\alpha - 1)/2$. Then*

a) $|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq c_{2.7A} |h|^{\alpha-\gamma} |x|^{\gamma-1}$, *where $c_{2.7A}$ is independent of ε and λ;*

b) *if $\delta > 0$, $|x| \geqq \delta$, $|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq c_{2.7B}(\lambda, \delta) |h|^{\alpha-\gamma} |x|^{\gamma-1}$, where $c_{2.7B}(\lambda, \delta)$ is independent of ε and $\to 0$ as $\lambda \to \infty$.*

c) *if $|x| \geqq M_{2.1}$, $|h| \leqq \max(|x|/2, A)$, $|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq c_{2.7C} |x|^{\alpha-3} |h|^{\alpha-\gamma} A^2$, where $c_{2.7C}$ is independent of ε and λ.*

Proof. a) First note that

$$\Delta_R^h r_{\lambda,\varepsilon}(x) = \int_0^h (h-s) r''_{\lambda,\varepsilon}(x+s)\, ds = h^2 \int_0^1 (1-t) r''_{\lambda,\varepsilon}(x+ht)\, dt. \tag{2.8}$$

To prove a), we need to consider a number of special cases.

(i) $|x/2| \leqq |h| \leqq |x|$. Using Proposition (2.5) b and (2.8),

$$\begin{aligned}|\Delta_R^h r_{\lambda,\varepsilon}(x)| &\leqq c_{2.5B} h^2 \int_0^1 (1-t) |x+ht|^{\alpha-3} dt \\ &= c_{2.5B} |h|^{\alpha-1} \int_0^1 (1-t) |x/h+t|^{\alpha-3} dt \\ &\leqq c_{2.5B} |h|^{\alpha-1} \int_0^1 (1-t)^{\alpha-2} dt \\ &\leqq c_1 |h|^{\alpha-\gamma} |x|^{\gamma-1}.\end{aligned}$$

(ii) $|h| \leqq |x|/2$. From (2.8) and Proposition (2.5) b,

$$\begin{aligned}|\Delta_R^h r_{\lambda,\varepsilon}(x)| &\leqq h^2 \sup_{|y| \geqq |x|/2} |r''_{\lambda,\varepsilon}(y)| \\ &\leqq c_{2.5B} h^2 (|x|/2)^{\alpha-3} \\ &\leqq c_2 |h|^{\alpha-\gamma} |x|^{\gamma-1}.\end{aligned}$$

(iii) $|h| > |x|$, $x/h > 0$. As in (i),

$$\begin{aligned}|\Delta_R^h r_{\lambda,\varepsilon}(x)| &\leqq c_{2.5B} |h|^{\alpha-1} \int_0^1 (1-t) |x/h+t|^{\alpha-3} dt \\ &\leqq c_{2.5B} |h|^{\alpha-1} \int_0^1 (x/h+t)^{\alpha-3} dt \\ &\leqq c_3 |h|^{\alpha-1} (x/h)^{\alpha-2} \\ &\leqq c_3 |h|^{\alpha-\gamma} |x|^{\gamma-1}.\end{aligned}$$

(iv) $|h| > |x|$, $x/h < 0$. By symmetry, we may suppose without loss of generality that $x > 0$, $h < 0$. Also, by symmetry, $r_{\lambda,\varepsilon}(x+h) = r_{\lambda,\varepsilon}(-x-h)$, and so

$$\begin{aligned}|\Delta_R^h r_{\lambda,\varepsilon}(x)| &\leqq |r_{\lambda,\varepsilon}(-x-h) - r_{\lambda,\varepsilon}(x) - (-2x-h) r'_{\lambda,\varepsilon}(x)| \\ &\quad + |2x+2h|\ |r'_{\lambda,\varepsilon}(x)| \\ &= |\Delta_R^{-2x-h} r_{\lambda,\varepsilon}(x)| + |2x+2h|\ |r'_{\lambda,\varepsilon}(x)|.\end{aligned} \tag{2.9}$$

Consider the first term on the right of (2.9). Since $|2x+h| \leqq |h|$, then applying either (i) or (ii), we get

$$|\Delta_R^{-2x-h} r_{\lambda,\varepsilon}(x)| \leqq c_4 |2x+h|^{\alpha-\gamma} |x|^{\gamma-1} \leqq c_4 |h|^{\alpha-\gamma} |x|^{\gamma-1}.$$

Now consider the second term on the right of (2.9).

$|x+h| = -h-x \leqq -h = |h|$, and then, using Proposition (2.5) c,

$$|2x+2h|\ |r'_{\lambda,\varepsilon}(x)| \leqq 2c_{2.5C}|x+h|\ |x|^{\alpha-2}$$
$$\leqq c_5|h|\ |x|^{\alpha-2} \leqq c_5|h|^{\alpha-\gamma}|x|^{\gamma-1}.$$

Summing, $|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq (c_3+c_4+c_5)|h|^{\alpha-\gamma}|x|^{\gamma-1}$, and so case (iv), hence a), is proved.

b) The proof is similar to a), using Proposition (2.5) d and e in place of (2.5) b and c.

c) By case (ii) of a),

$$|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq c_6 h^2 |x|^{\alpha-3}$$
$$\leqq c_6 A^{2-\alpha+\gamma}|h|^{\alpha-\gamma}|x|^{\alpha-3}. \quad \square$$

If f is bounded and Borel, define

$$R_{\lambda,\varepsilon}f(x) = \int f(y)\, r_{\lambda,\varepsilon}(x-y)\, dy. \tag{2.10}$$

An usual, write $R_\lambda f(x)$ for $R_{\lambda,0}f(x)$. It is easy to see that $f\in C^2$ implies $R_\lambda f\in C^2$. The main theorem of this section is

(2.11) **Theorem.** *Suppose $0<\gamma<(\alpha-1)/2$ is fixed, K, A positive real numbers $\geqq 1$. Then there exists a nonnegative function G and a real number $\lambda_{2.11}$ such that*

a) $\|G\|_{L_1} \leqq 1/2$;

b) *if $\lambda \geqq \lambda_{2.11}$, $f\in C_K^2$, and ν is a measure such that $\int |h|^{\alpha-\gamma}\nu(dh) \leqq K$ and support $(\nu) \subseteq [-A, A]$, then*

$$\left|\int [R_\lambda f(x+h) - R_\lambda f(x) - h(R_\lambda f)'(x)]\,\nu(dh)\right| \leqq G*|f|(x)$$

for all x.

Recall that $*$ denotes convolution.

Proof. First we define G. Pick $M_1 \geqq M_{2.1} \vee 2A$, such that

$$A^2 c_{2.7C} K \int_{M_1}^{\infty} x^{\alpha-3}\,dx \leqq 1/12.$$

Pick δ small so that

$$c_{2.7A} K \int_0^{\delta} x^{\gamma-1}\,dx \leqq 1/12.$$

Now pick $\lambda_{2.11}$ sufficiently large so that if $\lambda \geqq \lambda_{2.11}$,

$$c_{2.7B}(\lambda,\delta) K \int_{\delta}^{M_1} x^{\gamma-1}\,dx \leqq 1/12.$$

Let $c_1(\delta) = \sup_{\lambda \geqq \lambda_{2.11}} c_{2.7B}(\lambda,\delta)$.

Define $G(x)$ by

$$G(x) = \begin{cases} c_{2.7A} K|x|^{\gamma-1} & \text{if } |x| \leqq \delta \\ c_1(\delta) K|x|^{\gamma-1} & \text{if } \delta < |x| \leqq M_1 \\ A^2 c_{2.7C} K|x|^{\alpha-3} & \text{if } |x| > M_1. \end{cases} \tag{2.12}$$

Clearly, a) is satisfied. Moreover, if $\lambda \geqq \lambda_{2.11}$, and $|h| \leqq A$,

$$|\Delta_R^h r_{\lambda,\varepsilon}(x)| \leqq G(x)\,|h|^{\alpha-\gamma}/K,$$

by Proposition (2.7).

It is easy to see that if $f \in C_K^2$, for each ε, $R_{\lambda,\varepsilon} f \in C^2$, and that $(R_{\lambda,\varepsilon} f)'' = R_{\lambda,\varepsilon}(f'')$ converges uniformly to $R_\lambda(f'') = (R_\lambda f)''$ as $\varepsilon \to 0$, and similarly for first derivatives. If we show

$$|\textstyle\int \Delta_R^h R_{\lambda,\varepsilon} f(x)\, v(dh)| \leqq G * |f|(x) \tag{2.13}$$

for each $\varepsilon \leqq 1$, for each x, b) will follow by dominated convergence, since

$$|\Delta_R^h R_{\lambda,\varepsilon} f(x)| \leqq (\|(R_{\lambda,\varepsilon} f)''\| + 2\,\|(R_{\lambda,\varepsilon} f)'\|)\,(h^2 \wedge h).$$

However, (2.13) holds, since

$$\begin{aligned}
&|\textstyle\int \Delta_R^h R_{\lambda,\varepsilon} f(x)\, v(dh)| \\
&\quad = |\textstyle\int\int [r_{\lambda,\varepsilon}(x-y+h) - r_{\lambda,\varepsilon}(x-y) - h r'_{\lambda,\varepsilon}(x-y)]\, f(y)\, dy\, v(dh)| \\
&\quad \leqq \textstyle\int\int |\Delta_R^h r_{\lambda,\varepsilon}(x-y)|\, v(dh)\, |f(y)|\, dy \\
&\quad \leqq \textstyle\int G(x-y)\, |f(y)| \int |h|^{\alpha-\gamma}\, v(dh)\, dy/K \\
&\quad \leqq G * |f|(x). \quad \square
\end{aligned}$$

3. Densities of Potentials

Define

$$S_\lambda f = E \int_0^\infty e^{-\lambda t} f(X_t)\, dt \tag{3.1}$$

for $f \geqq 0$. Our main goal in this section is to show that the measure S_λ has a bounded density with respect to Lebesgue measure. First we need some technical propositions.

(3.2) **Proposition.** *Suppose X_t is a local martingale with local characteristics* $(0, v)$, $X_0 = 0$, *and* $\sup_s \int_0^\infty |h|^2 \wedge |h|^\tau v_s(dh) \leqq K < \infty$, *for some* $1 < \tau < 2$. *Then, for* $0 < \varepsilon < \tau$,

$$E|X_{t_0}|^{\tau-\varepsilon} \leqq c_{3.2}(K, \varepsilon)\, t_0 + 1.$$

Proof. Let $M \geqq 1$, and let

$$Y_t^M = X_t - \sum_{s \leqq t} \Delta X_s 1_{(|\Delta X_s| > M)} + \int_0^t \int_{|h| > M} h\, v_s(dh)\, ds. \tag{3.3}$$

$Y_t = Y_t^M$ is a local martingale with bounded jumps. If $T_N = \inf\{t \colon Y_t \geqq N\}$, $Y_{t \wedge T_N}$ is square integrable, and

$$EY_{t\wedge T_N}^2 = E[Y,Y]_{t\wedge T_N} = E\int_0^{t\wedge T_N}\int_{-M}^{M} h^2 \nu_s(dh)\,ds$$
$$\leqq t\Big(K + M^{2-\tau}\sup_{s\leqq t} E \int_{1<|h|\leqq M} |h|^\tau \nu_s(dh)\Big)$$
$$\leqq Kt(1+M^{2-\tau}).$$

By Fatou,

(3.4) $$P(|Y_t|\geqq M) \leqq M^{-2}EY_t^2 \leqq M^{-2}\liminf_{N\to\infty} EY_{t\wedge T_N}^2 \leqq c_1 t M^{-\tau}.$$

(3.5) $$P\Big(\int_0^t \int_{|h|>M} |h|\,\nu_s(dh)\,ds \geqq M\Big) \leqq M^{-1}E\int_0^t\int_{|h|>M} |h|\,\nu_s(dh)\,ds$$
$$\leqq M^{-\tau} t \sup_{s\leqq t} E\int_{|h|>M} |h|^\tau \nu_s(dh)$$
$$\leqq KtM^{-\tau}.$$

And,

(3.6) $$P\Big(\Big|\sum_{s\leqq t} \Delta X_s 1_{(|\Delta X_s|>M)}\Big| \geqq M\Big) \leqq P\Big(\sum_{s\leqq t} 1_{(|\Delta X_s|>M)} \geqq 1\Big)$$
$$\leqq E\sum_{s\leqq t} 1_{(|\Delta X_s|\geqq M)}$$
$$= E\int_0^t \int_{|h|\geqq M} \nu_s(dh)\,ds$$
$$\leqq tKM^{-\tau}.$$

Adding (3.4), (3.5), and (3.6), we get

$$P(|X_t|\geqq 3M) \leqq P(|Y_t|\geqq M) + P\Big(\Big|\int_0^t\int_{|h|>M} h\,\nu_s(dh)\,ds\Big| \geqq M\Big)$$
$$+ P\Big(\Big|\sum_{s\leqq t}\Delta X_s 1_{(|\Delta X_s|>M)}\Big|\geqq M\Big)$$
$$\leqq c_2 t M^{-\tau}.$$

Then,

$$E|X_t|^{\tau-\varepsilon} = \int_0^\infty (\tau-\varepsilon)|M|^{\tau-\varepsilon-1} P(|X_t|\geqq M)\,dM$$
$$\leqq 1 + c_3 t(\tau-\varepsilon)\int_1^\infty M^{-\varepsilon-1}\,dM. \quad \square$$

(3.7) **Proposition.** *Suppose X_t satisfies the hypotheses of Proposition* (3.2) *and $f\in C_K^2$. Then*

a) $E\int_0^t f'(X_{s-})\,dX_s = 0$.

b) $E\sum_{s\leqq t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_S] = E\int_0^t \int \Delta_R^h f(X_s)\,\nu_s(dh)\,ds$.

Proof. a) Let $M \geqq 1$, and define Y_t^M by (3.3). $E(Y_t^M)^2 < \infty$ by (3.4), hence Y_t^M is a locally square integrable martingale, and $E\int_0^t f'(X_{s-})\,dY_s^M = 0$.

$$\left|\int_0^t f'(X_{s-})\,dX_s - \int_0^t f'(X_{s-})\,dY_s^M\right| \leqq \|f'\|\left(\sum_{s\leqq t} |\Delta X_s|\,1_{(|\Delta X_s|\geqq M)} + \int_0^t \int_{|h|\geqq M} |h|\,\nu_s(dh)\,ds\right). \tag{3.8}$$

The expectation of the right hand side is $\leqq 2\|f'\|\,E\int_0^t \int_{|h|\geqq M} |h|\,\nu_s(dh)\,ds \leqq 2\|f'\|\,Kt$. Hence

$$E\left|\int_0^t f'(X_{s-})\,dX_s - \int_0^t f'(X_{s-})\,dY_s^M\right| \to 0$$

as $M \to \infty$ by dominated convergence, and a) follows.

b) By a monotone class argument, if $g: \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is bounded with support on $\mathbb{R}\times(-\delta,\delta)^c \cap [-M, M]$,

$$E\sum_{s\leqq t} g(X_{s-}, \Delta X_s) = E\int_0^t \int g(X_{s-}, h)\,\nu_s(dh)\,ds. \tag{3.9}$$

Apply (3.9) with $g(x,h) = [f(x+h) - f(x) - f'(x)\,h]\,1_{(\delta\leqq|h|\leqq M)}$.

$$\begin{aligned}
&E\sum_{s\leqq t} |f(X_s) - f(X_{s-}) - f'(X_{s-})\,\Delta X_s|\,1_{(|\Delta X_s|\in[\delta,M]^c)}\\
&\leqq \|f''\|\,E\sum_{s\leqq t} |\Delta X_s|^2\,1_{(|\Delta X_s|\leqq\delta)} + (2\|f\| + \|f'\|)\,E\sum_{s\leqq t} |\Delta X_s|\,1_{(|\Delta X_s|\geqq M)}\\
&\leqq (\|f''\| + 2\|f'\|)\,E\int_0^t \int_{|h|\notin[\delta,M]} |h|^2 \wedge |h|\,\nu_s(dh)\,ds \to 0
\end{aligned}$$

by dominated convergence as $\delta\to 0$, $M\to\infty$. Similarly,

$$\begin{aligned}
&E\int_0^t \int_{|h|\notin[\delta,M]} |f(X_{s-}+h) - f(X_{s-}) - f'(X_{s-})\,h|\,\nu_s(dh)\,ds\\
&\leqq (\|f''\| + 2\|f'\|)\,E\int_0^t \int_{|h|\notin[\delta,M]} |h|^2 \wedge |h|\,\nu_s(dh)\,ds \to 0
\end{aligned}$$

by dominated convergence. Letting $\delta\to 0$, $M\to\infty$ in (3.9),

$$\begin{aligned}
&E\sum_{s\leqq t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\,\Delta X_s]\\
&= E\int_0^t \int [f(X_{s-}+h) - f(X_{s-}) - f'(X_{s-})\,h]\,\nu_s(dh)\,ds.
\end{aligned}$$

Since X_s has only countably many jumps, the right hand side is unchanged if we replace X_{s-} by X_s. □

Recall the definitions of R_λ (2.10) and S_λ (3.1). The next theorem links S_λ with R_λ.

(3.10) **Theorem.** *Suppose* $P(X_0 = x_0) = 1$. *Suppose* X_t *satisfies* (1.4). *If* $g \in C_K^2$,

a) $S_\lambda g = R_\lambda g(x_0) + E \int_0^\infty e^{-\lambda s} \int \Delta_R^h R_\lambda g(X_s)\,(v_s - \theta_\alpha)(dh)\, ds;$

if, in addition, $\lambda \geqq \lambda_{2.11}$,

b) $|S_\lambda g| \leqq R_\lambda |g|(x_0) + S_\lambda(G * |g|)$,

where G is defined by Theorem (2.11).

Proof. Suppose $f \in C_K^\infty$. Recall that $\mathscr{G}_\alpha$ is defined by

$$\mathscr{G}_\alpha f(x) = \int \Delta_R^h f(x)\, \theta_\alpha(dh). \tag{3.11}$$

By Ito's lemma, Proposition (3.7) (applied to $X_t - x_0$), and the fact that X_t is purely discontinuous,

$$\begin{aligned} Ef(X_t) - Ef(X_0) &= E\int_0^t f'(X_{s-})\, dX_s + \tfrac{1}{2} E \int_0^t f''(X_{s-})\, d\langle X^c, X^c\rangle_s \\ &\quad + E \sum_{s \leqq t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s] \\ &= E \int_0^t \Delta_R^h f(X_s)\, v_s(dh)\, ds \\ &= E\int_0^t \mathscr{G}_\alpha f(X_s)\, ds + E \int_0^t \int \Delta_R^h f(X_s)\,(v_s - \theta_\alpha)\,(dh)\, ds. \end{aligned} \tag{3.12}$$

Since $f \in C_K^\infty$, $\mathscr{G}_\alpha f$ is bounded, and $|\int \Delta_R^h f(X_s)(v_s - \theta_\alpha)(dh)| \leqq (\|f''\| + 2\|f'\|) \int h^2 \wedge |h|\, |v_s - \theta_\alpha|(dh)$ is bounded. Multiply (3.12) by $\lambda e^{-\lambda t}$ and integrate to get

$$\lambda S_\lambda f - f(x_0) = E \int_0^\infty e^{-\lambda s} \mathscr{G}_\alpha f(X_s)\, ds + E \int_0^\infty e^{-\lambda s} \int \Delta_R^h f(X_s)\,(v_s - \theta_\alpha)\,(dh)\, ds,$$

or

$$S_\lambda(\lambda f - \mathscr{G}_\alpha f) = f(x_0) + E \int_0^\infty e^{-\lambda s} \int \Delta_R^h f(X_s)\,(v_s - \theta_\alpha)(dh)\, ds.$$

Now let $g = \lambda f - \mathscr{G}_\alpha f$. g will be bounded and in C^∞, and so we have a) for $g \in (\lambda - \mathscr{G}_\alpha)(C_K^\infty)$.

Now let $g \in C_K^2$, and let $f = R_\lambda g$. f, f', f'' will be bounded and continuous. Moreover, it is easy to see that $f, f', f'' \to 0$ as $|x| \to \infty$. Choose $f_n \in C_K^\infty$ such that $f_n \to f$, $f_n' \to f'$, $f_n'' \to f''$ uniformly. Let $g_n = (\lambda - \mathscr{G}_\alpha) f_n$. $g_n \to g$ uniformly. $R_\lambda g_n = f_n$, and $\int \Delta_R^h f_n(x)(v_s - \theta_\alpha)(dh)$ converges to the corresponding expression with f_n replaced by f by dominated convergence, since $|\Delta_R^h f_n| \leqq (\|f_n''\| + 2\|f_n'\|)(h^2 \wedge |h|)$. Since a) holds for g_n, we see that it holds for g as well by letting $n \to \infty$.

b) follows by applying Theorem (2.11) with $A = 1$. □

If r_λ is defined by (2.4), note that

(3.13)
$$r_\lambda(0) = \int_0^\infty e^{-\lambda t} q_t(0)\,dt = \int_0^\infty e^{-\lambda t} t^{-1/\alpha} q(0)\,dt = c_{3.13}\,\lambda^{1/\alpha - 1}$$

The next theorem is the main goal of this section.

(3.14) **Theorem.** *Suppose X_t satisfies* (1.4), $\lambda \geqq \lambda_{2.11}$. *Then*

a)
$$|S_\lambda g| \leqq 2r_\lambda(0)\,\|g\|_{L_1}$$

and

b) *there exists a nonnegative function $s_\lambda(x)$ bounded by $2r_\lambda(0)$ such that if $\|g\|_{L_1} < \infty$,*
$$S_\lambda g = \int g(x)\,s_\lambda(x)\,dx.$$

Proof. First of all, $\|G*|g|\|_{L_1} \leqq \|G\|_{L_1}\|g\|_{L_1} \leqq \frac{1}{2}\|g\|_{L_1}$ and $G*|g|(x) \leqq \frac{1}{2}\|g\|$. Let $\rho_M = \sup_{\|g\|_{L_1} \leqq 1,\, \|g\| \leqq M} |S_\lambda g|$. By (3.1), $|S_\lambda g| \leqq \|g\|/\lambda$, and so $\rho_M < \infty$. Since S_λ is a measure, $\rho_M = \sup_{\|g\|_{L_1} \leqq 1,\, \|g\| \leqq M,\, g \in C_K^2} |S_\lambda g|$.

By Theorem (3.10)b, if $g \in C_K^2$, $\|g\|_{\ 1} \leqq 1$, $\|g\| \leqq M$,
$$\begin{aligned}|S_\lambda g| &\leqq R_\lambda|g|(x_0) + S_\lambda(G*|g|)\\ &\leqq \|g\|_{L_1}\|r_\lambda\| + \tfrac{1}{2}\rho_M\\ &\leqq r_\lambda(0) + \tfrac{1}{2}\rho_M.\end{aligned}$$

Taking the sup over all such g,
$$\rho_M \leqq r_\lambda(0) + \tfrac{1}{2}\rho_M,$$

or $\rho_M \leqq 2r_\lambda(0)$. Letting $M \to \infty$, $\sup_{\|g\|_{L_1} \leqq 1} |S_\lambda g| \leqq 2r_\lambda(0)$, which gives a) by the linearity of S_λ.

If A has 0 Lebesgue measure, a) shows that $S_\lambda 1_A = 0$, and hence S_λ has a density $s_\lambda(x)$ with respect to μ, Lebesgue measure. If $\varepsilon > 0$ and $A = \{x : s_\lambda(x) > 2r_\lambda(0) + \varepsilon\}$,
$$(2r_\lambda(0) + \varepsilon)\,\mu(A) \leqq S_\lambda 1_A \leqq 2r_\lambda(0)\,\mu(A),$$

or $\mu(A) = 0$. Since this is true for each ε, we may take $s_\lambda(x) \leqq 2r_\lambda(0)$ for all x. □

In the case X_t is Markov, one would want there to exist a kernel $\bar{v}(x, dh)$ such that $v_s(dh) = \bar{v}(X_s(\omega), dh)$. Theorem (3.10) a then becomes

(3.15)
$$S_\lambda g = R_\lambda g(x_0) + S_\lambda B R_\lambda g,$$

where
$$Bf(x) = \int \Delta_R^h f(x)\,(\bar{v}(x, dh) - \theta_\alpha(dh))$$

Although it is a digression from our main topic, we take a moment to show how Theorem (3.10) can be used to show uniqueness of the Markov process corresponding to the integral operator $\mathscr{G}_\alpha + B$. The key is that by Theorem (2.11), B is a relatively bounded perturbation of $\mathscr{G}_\alpha$. Let $\Omega = \{$functions on $[0, \infty)$ that are right continuous and have left limits$\}$, let $X_t(\omega) = \omega(t)$, and suppose P_1, P_2 are two probabilities on Ω for which $P_i(X_0 = x_0) = 1$ and (1.4) holds for each P_i. Let E_i denote expectation with respect to P_i, and let $S_\lambda^{(i)} f = E_i \int_0^\infty e^{-\lambda t} f(X_t)\, dt$. As before, $|S_\lambda^{(i)} f| \leqq \|f\|/\lambda$. Writing (3.15) for $S_\lambda^{(i)}$, $i = 1,2$, and then taking the difference, we get

$$(S_\lambda^{(1)} - S_\lambda^{(2)}) g = (S_\lambda^{(1)} - S_\lambda^{(2)})(B R_\lambda g). \tag{3.16}$$

Let $\rho = \sup_{\|g\| \leqq 1,\, g \in C_K^2} |(S_\lambda^{(1)} - S_\lambda^{(2)}) g|$. If $g \in C_K^2$, $\|g\| \leqq 1$, and λ is sufficiently large,

$$|B R_\lambda g(x)| \leqq G * |g|(x) \leqq \tfrac{1}{2} \|g\| \leqq \tfrac{1}{2}.$$

So,

$$|(S_\lambda^{(1)} - S_\lambda^{(1)}) g| \leqq \tfrac{1}{2} \rho,$$

and taking sups, $\rho \leqq \frac{1}{2}\rho$. Since $\rho \leqq 2/\lambda < \infty$, we must have $\rho = 0$, or $S_\lambda^{(1)} g = S_\lambda^{(2)} g$ for bounded g. From this one may use techniques from [14, Chap. 6] to conclude there is a unique solution to the "martingale problem" given by (1.4), and hence a unique Markov process corresponding to the generator $\mathscr{G}_\alpha + B$. Using techniques of [14, 13], and Sect. 6, one can then show uniqueness when (ba, bv) satisfies the weaker condition (1.2), where $b_s = \bar{b}(X_s)$, $a_s = \bar{a}(X_s)$ for functions $\bar{b}$, $\bar{a}$.

4. Construction of Local Time

In this section we construct our local times by first looking at their potentials $U_t(\lambda, x)$. In the Markov case, we would just define $U_t(\lambda, x)$ in terms of s_λ, but in the general martingale case a more complicated construction is needed. Throughout this section we suppose (1.4) holds.

Let $Q_t(\omega, \cdot)$ be a regular conditional probability distribution for $\mathscr{F}_t$. That is, for each $A \in \mathscr{F}$, $Q_t(\cdot, A)$ is $\mathscr{F}_t$-measurable; for each ω, $Q_t(\omega, \cdot)$ is a probability measure on $\mathscr{F}$; and

$$Q_t(\cdot, A) = P(A \mid \mathscr{F}_t), \text{ a.s.},$$

for each $A \in \mathscr{F}$. Q_t exists since $\mathscr{F}$ is the completion of a countably generated σ-field and X_t is real-valued. Let us write $Q_t Y(\omega)$ for $\int Y(\omega') Q_t(\omega, d\omega')$.

If $A = \{\omega : t \mapsto X_t(\omega)$ is right continuous with left limits$\}$, $E(Q_t 1_A) = P(A) = 1$, or, for each t, $Q_t(\cdot, A) = 1$, a.s.

(4.1) **Proposition.** *Suppose $(Y_t, \mathscr{F}_t, P)$ is a uniformly integrable martingale whose paths are right continuous with left limits. Fix t_0. Then $(Y_{t+t_0}, \hat{\mathscr{F}}_{t+t_0}, Q_{t_0})$ is a right continuous martingale,* a.s. (P).

Here $\hat{\mathscr{F}}_{t+t_0}$ denotes the Q_{t_0} completion of $\mathscr{F}^0_{t+t_0+}$.

Proof. As above, Y_{t+t_0} is right continuous with left limits, a.s. (Q_{t_0}). Since Y_t is uniformly integrable, there exists an even positive function h with $h(x)/x \to \infty$ as $x \to \infty$ such that $Eh(Y_\infty) < \infty$. However, then $EQ_{t_0}h(Y_\infty) = Eh(Y_\infty) < \infty$, or $Q_{t_0}h(Y_\infty) < \infty$, a.s. (P). Let $N_1 = \{\omega : Q_{t_0}h(Y_\infty) = \infty\}$.

Fix $s < t$. Pick $A \in \mathscr{F}^0_{s+t_0}$ and $B \in \mathscr{F}_{t_0}$.

$$\begin{aligned} E(Q_{t_0}(Y_{t+t_0}1_A); B) &= E(Y_{t+t_0}1_A 1_B) \\ &= E(Y_{s+t_0}1_A 1_B) \\ &= E(Q_{t_0}(Y_{s+t_0}1_A); B). \end{aligned}$$

Since this holds for arbitrary $B \in \mathscr{F}_{t_0}$,

$$Q_{t_0}(Y_{t+t_0}; A) = Q_{t_0}(Y_{s+t_0}; A), \text{ a.s. } (P). \tag{4.2}$$

Let $N(s,t,A)$ be the set of ω's for which (4.2) fails to hold. Let A^s_n be a sequence that generates $\mathscr{F}^0_{s+t_0}$, and let $N_2 = \bigcup_{s,t \text{ rational}} \bigcup_{n+1}^{\infty} N(s,t,A^s_n)$.

By a similar argument, we can find a null set N_3 outside of which $Q_{t_0}h(Y_{t+t_0}) \leqq Q_{t_0}h(Y_\infty)$, t rational.

Fix $\omega \notin N_1 \cup N_2 \cup N_3$. Y_{t+t_0} is then uniformly integrable with respect to $Q_{t_0}(\omega,\cdot)$, t rational. Pick $s < t$ real, $s_m > s$, $t_m > t$ rational $\downarrow s, t$, respectively. Pick $A \in \mathscr{F}^0_{s+t_0+}$. Then $A \in \mathscr{F}^0_{s_m+t_0}$, and so from (4.2), by a monotone class argument,

$$Q_{t_0}(Y_{t_m+t_0};\ A) = Q_{t_0(Y_{s_m+t_0}};A).$$

Use the right continuity and uniform integrability of Y and let $m \to \infty$ to get

$$Q_{t_0}(Y_{t+t_0}; A) = Q_{t_0}(Y_{s+t_0}; A) \tag{4.3}$$

for $A \in \mathscr{F}^0_{s+t_0+}$. The proof of the proposition is now immediate. □

By applying Proposition (4.1) to $X_{t\wedge u_0}$, u_0 fixed, we see that (X_{t+t_0}, Q_{t_0}) is a locally uniformly integrable martingale. If A_n is a sequence of compact subsets of $\mathbb{R}$-$\{0\}$ that generate the Borel σ-field of $\mathbb{R}$, and

$$Y^{A_n}_t = \sum_{s \leqq t} 1_{A_n}(\Delta X_s) - \int_0^t \int 1_{A_n}(h)\, \nu_s(dh)\, ds,$$

applying Proposition (4.1) allows us to show easily that X_{t+t_0} is purely discontinuous with local characteristics $(0, \nu_{s+t_0})$. (To get the uniform integrability of $Y^{A_n}_{t\wedge u_0}$, use Proposition (3.2).)

For fixed ω, if $A \subseteq \mathbb{R}$,

$$Q_{t_0}(\omega, X_{t_0} \in A) = P(X_{t_0} \in A \mid \mathscr{F}_{t_0}) = 1_A(X_{t_0}),$$

which is 0 or 1. So for almost all $\omega(P)$, X_{t_0} is constant, a.s. (Q_{t_0}).

Fix t_0 and fix $\lambda \geqq \lambda_{2.11}$. If ω is not in any of the null sets, we may apply Theorem (3.14) to see that there exists a Borel mesurable function of x, bounded by $2r_\lambda(0)$, which we will denote by $V_{t_0}(\lambda, x)(\omega)$, such that

$$Q_{t_0}\left(\int_0^\infty e^{-\lambda t} 1_A(X_{t+t_0})\, dt\right)(\omega) = \int_A V_{t_0}(\lambda, x)(\omega)\, dx, \tag{4.4}$$

for all Borel $A \subseteq R$. Our potential $U_t(\lambda, x)$ will be a regularized version of $V_t(\lambda, x)$.

(4.5) **Proposition.** *There exists a set $N_{4.5}$ with $\mu(N_{4.5})=0$ such that if $x \notin N_{4.5}$, $(e^{-\lambda t} V_t(\lambda, x), \mathscr{F}_t^0, P)$, t rational, is a supermartingale.*

Proof. Fix $s<t$. We will first prove

$$E(e^{-\lambda t} V_t(\lambda, x); A) \leqq E(e^{-\lambda s} V_s(\lambda, x); A), \text{ a.e. } (\mu) \tag{4.6}$$

if $A \in \mathscr{F}_s^0$. If $B \subseteq R$ is Borel

$$\begin{aligned} \int_B E(e^{-\lambda t} V_t(\lambda, x); A)\, dx &= E\left(e^{-\lambda t} \int_B V_t(\lambda, x)\, dx; A\right) \\ &= E\left(e^{-\lambda t} Q_t \int_0^\infty e^{-\lambda r} 1_B(X_{t+r})\, dr; A\right) \\ &= E\left(\int_t^\infty e^{-\lambda r} 1_B(X_r)\, dr; A\right), \end{aligned} \tag{4.7}$$

using Fubini and the boundedness of $V_t(\lambda, x)$. Applying (4.7) with s in place of t,

$$\begin{aligned} \int_B E(e^{-\lambda s} V_s(\lambda, x); A)\, dx &= E\left(\int_s^\infty e^{-\lambda r} 1_B(X_r)\, dr; A\right) \geqq E\left(\int_t^\infty e^{-\lambda r} 1_B(X_r)\, dr; A\right) \\ &= \int_B E(e^{-\lambda t} V_t(\lambda, x); A)\, dx. \end{aligned}$$

Since B was arbitrary, (4.6) follows.

Let $N(s, t, A)$ be the null set of x's for which (4.6) fails. Let A_n^s be a sequence of sets generating $\mathscr{F}_s^0$, and let

$$N_{4.5} = \bigcup_{s, t \text{ rational}} \bigcup_{n+1}^{\infty} N(s, t, A_n^s). \tag{4.8}$$

The proposition now follows. □

If $x \notin N_{4.5}$, $e^{-\lambda t} V_t(\lambda, x)$, t rational, has left and right limits, a.s. Let

$$U_t(\lambda, x) = \limsup_{t_m \text{ rational},\, t_m > t,\, t_m \downarrow t} V_{t_m}(\lambda, x). \tag{4.9}$$

$V_t(\lambda, x)$ is measurable in x, for each t. Hence $U_t(\lambda, x)$ is jointly measurable in t and x.

(4.10) **Proposition.** *If $x \notin N_{4.5}$, $(e^{-\lambda t} U_t(\lambda, x), \mathscr{F}_t, P)$ is a supermartingale. Furthermore, if $B \subseteq \mathbb{R}$ is Borel,*

$$\int_B e^{-\lambda t} U_t(\lambda, x)\, dx = E\left[\int_t^\infty e^{-\lambda r} 1_B(X_r)\, dr \,|\, \mathscr{F}_t\right], \text{ a.s. } (P). \tag{4.11}$$

Proof. The proof of the first assertion is routine and is omitted. If $f \in C_K$,

$$\int f(x) V_t(\lambda, x)\, dx = E\left[\int_0^\infty e^{-\lambda r} f(X_{t+r})\, dr \,|\, \mathscr{F}_t\right], \text{ a.s.} \tag{4.12}$$

by (4.4). (4.11) follows from (4.12) by a limiting argument and then using the monotone class theorem. □

The main theorem of this paper is the following.

(4.13) **Theorem.** *Suppose X_t satisfies* (1.4). *Then there exist jointly measurable right continuous increasing processes $L_t(x)$ such that*

a) *for each t and x, $L_t(x)$ has moments of all orders,*

b) *there exists a set $N_{4.13}$ such that $P(N_{4.13}) = 0$, and if $\omega \notin N_{4.13}$ and $B \subseteq \mathbb{R}$ is Borel, then*

$$\int_B L_t(x)\,dx = \int_0^t 1_B(X_s)\,ds$$

for all t.

Proof. If $x \notin N_{4.5}$, $e^{-\lambda t} U_t(\lambda, x)$ is a supermartingale bounded by $2r_\lambda(0)$. In this case, if $\lambda \geqq \lambda_{2.11}$, let $L_t^\lambda(x)$ be the predictable increasing part of $e^{-\lambda t} U_t(\lambda, x)$. If $x \in N_{4.5}$, let $L_t^\lambda(x) = 0$. By [12], we may take $L_t^\lambda(x)$ to be jointly measurable in x and t. Fix λ sufficiently large, and let

(4.14)
$$L_t(x) = \int_0^t e^{\lambda s}\,dL_s^\lambda(x).$$

By [5, p. 188]

$$E[L_t^\lambda(x)]^p \leqq p!\,[2r_\lambda(0)]^p, \; p = 1,2,\ldots,$$

since the potential of $L_t^\lambda(x)$ is bounded. Then

$$E[L_t(x)]^p \leqq e^{p\lambda t} E[L_t^\lambda(x)]^p < \infty,$$

which proves a).

By a) and the definition of $L_t^\lambda(x)$, $e^{-\lambda t} U_t(\lambda, x) + L_t^\lambda(x)$ is a square integrable martingale (if $t \leqq u_0 < \infty$). Integrating by parts,

$$\begin{aligned} L_t(x) &= e^{\lambda t} L_t^\lambda(x) - \int_0^t L_{s-}^\lambda(x)\,\lambda e^{\lambda s}\,ds \\ &= e^{\lambda t} L_t^\lambda(x) - \int_0^t L_s^\lambda(x)\,\lambda e^{\lambda s}\,ds, \end{aligned}$$

since $L_s^\lambda(x)$ has at most countably many discontinuities. If $f \in C_K$,

$$\int e^{-\lambda t} U_t(\lambda, x) f(x)\,dx + \int f(x) L_t^\lambda(x)\,dx$$

is a square integrable martingale. On the other hand, by (4.11)

$$\int e^{-\lambda t} U_t(\lambda, x) f(x)\,dx + \int_0^t e^{-\lambda r} f(X_r)\,dr = E\left[\int_0^\infty e^{-\lambda r} f(X_r)\,dr \,|\, \mathscr{F}_t\right]$$

is a martingale. Hence, since $L_t^\lambda(x)$ is predictable for each x, $\int f(x) L_t^\lambda(x)\,dx - \int_0^t e^{-\lambda r} f(X_r)\,dr$ is a predictable martingale that has paths of bounded variation and is 0 at time 0; therefore it is identically 0.

Therefore

(4.15)
$$\int f(x) L_t^\lambda(x)\,dx = \int_0^t e^{-\lambda r} f(X_r)\,dr.$$

It follows easily that for each t,

$$\int f(x)\, L_t(x)\, dx = \int_0^t f(X_r)\, dr, \text{ a.s.} \tag{4.16}$$

Since both sides are right continuous, we can find a single null set $N(f)$ independent of t for which (4.16) holds. Taking a countable sequence f_n of functions in C_K that generate the Borel σ-field of $\mathbb{R}$, we obtain b) by letting $N_{4.13} = \bigcup_{n=1}^{\infty} N(f_n)$. □

If we were to choose a different value of λ, say $\hat{\lambda}$, and let

$$L_t(x) = \int_0^t e^{\hat{\lambda}s}\, dL_s^{\hat{\lambda}}(x),$$

the argument leading to (4.15) shows that

$$\int f(X)\, \hat{L}_t(x)\, dx = \int f(x)\, L_t(x)\, dx, \text{ a.s.}$$

Therefore, for almost all ω, $\hat{L}_t(x) = L_t(x)$ for almost all x. By Fubini, for almost all x, $\hat{L}_t(x) = L_t(x)$, a.s.

5. Continuity of Local Times

The local times $L_t(x)$ that we constructed in Theorem (4.13) turn out to have the nice property that they are continuous in t. Offhand, there is no reason to expect them to be continuous in x.

(5.1) **Theorem.** *There exists a set $N_{5.1}$ of Lebesgue measure 0 such that if $x \notin N_{5.1}$, $L_t(x)$ is continuous in t,* a.s.

Proof. Fix t, and let

$$D_s(t, x) = L_{t+s}(x) - L_t(x).$$

Then, if $f \in C_K$, $f \geqq 0$,

$$\int D_s(t, x)\, f(x)\, dx = \int_0^s f(X_{r+t})\, dr,$$

and so

$$\int f(x) \int_0^\infty e^{-\lambda s}\, dD_s(t, x)\, dx = \int_0^\infty e^{-\lambda s} f(X_{t+s})\, ds. \tag{5.2}$$

Integrating by parts,

$$\begin{aligned} \int_0^u \lambda e^{-\lambda s} D_s(t, x)\, ds &= \int_0^u D_{s-}(t, x)\, \lambda e^{-\lambda s}\, ds \\ &= -e^{-\lambda u} D_u(t, x) + \int_0^u e^{-\lambda s}\, dD_s(t, x) \\ &\leqq \int_0^\infty e^{-\lambda s}\, dD_s(t, x). \end{aligned} \tag{5.3}$$

Then, letting $u \to \infty$ in (5.3) and using (5.2) and (4.4),

$$(5.4)\qquad \int f(x)\, Q_t \int_0^\infty \lambda e^{-\lambda s} D_s(t,x)\, ds\, dx \leqq Q_t \int_0^\infty e^{-\lambda s} f(X_{t+s})\, ds$$
$$= \int f(x)\, V_t(\lambda, x)\, dx$$
$$\leqq 2 r_\lambda(0)\, \|f\|_{L_1}, \text{ a.s.}$$

It follows that, except for x in a set $N(t)$ of Lebesgue measure 0,

$$Q_t \int_0^\infty \lambda e^{-\lambda s} D_s(t,x)\, ds \leqq 2 r_\lambda(0) \leqq 2 c_{3.13}\, \lambda^{1/\alpha - 1},$$

by (3.13). $D_s(t,x)$ is increasing in s, and so if $h = 1/\lambda$,

$$Q_t D_h(t,x) \leqq e^{\lambda h} \int_h^\infty \lambda e^{-\lambda s} Q_t D_s(t,x)\, ds$$
$$\leqq c_1 h^{1-1/\alpha},$$

or, for each t, if $x \notin N(t)$,

$$(5.5)\qquad E(L_{t+h}(x) - L_t(x) \mid \mathscr{F}_t) \leqq c_1 h^{1-1/\alpha}, \text{ a.s.}$$

By Fubini applied to $\{(t, \omega, x, h)$: (5.5) does not hold$\}$, there is a null set N_1 such that if $x \notin N_1$, (5.5) holds for almost all t, h, and ω. Fix $x \notin N_1$, fix u_0 small, and let

$$A_u = L_{t+u \wedge u_0}(x) - L_t(x); \qquad \mathscr{G}_u = \mathscr{F}_{t+u}.$$

The potential of A with respect to $\mathscr{G}$ is

$$E(A_{u_0} - A_u \mid \mathscr{G}_u) = E(L_{t+u_0}(x) - L_{t+u}(x) \mid \mathscr{F}_{t+u}) \leqq c_1 u_0^{1-1/\alpha}$$

for almost all u, and hence for all u by right continuity of L and $\mathscr{F}$.

Apply [5, p. 188] to A; we get

$$(5.6)\qquad E(L_{t+u_0}(x) - L_t(x))^p = E A_{u_0}^p \leqq p!\,(c_1 u_0^{1-1/\alpha})^p.$$

By right continuity, the fact that L_t is increasing, and Theorem (4.13) a, (5.6) holds for all t. Take p large enough so that $p(1-1/\alpha) > 1$. By Kolmogorov's criterion, there is a version of $L_t(x)$ that is uniformly continuous in t, a.s. Since $L_t(x)$ is right continuous, there is no need to take versions and so $L_t(x)$ is continuous, a.s. □

Henceforth, let us assume that $L_t(x) = 0$ if $x \in N_{5.1}$.

The estimates in the proof of Theorem (5.1) can be used to get a modulus of continuity for $L_t(x)$. By the Taylor expansion for e^x and (5.6),

$$E(\exp(|L_{t+u_0}(x) - L_t(x)|/2c_1 u_0^{1-1/\alpha}) \leqq 2.$$

Letting $G_0(y) = e^{|y|}$ and $m(y) = 2c_1 |y|^{1-1/\alpha}$,

$$(5.7)\qquad \int_0^1 \int_0^1 G_0\left(\frac{L_t(x) - L_s(x)}{m(t-s)}\right) ds\, dt < \infty, \text{ a.s.}$$

since the expectation of the left side of (5.7) is finite. Then by the lemma of Garsia, Rodemich, and Rumsey (for example, see [11]),

$$|L_t(x) - L_s(x)| \leqq c_2 \int_0^{|t-s|} G_0^{-1}(c_3 u^{-2})\, dm(u), \text{ a.s., } 0 \leqq s,\ t \leqq 1,$$

or

$$\limsup_{0 \leqq s, t \leqq 1, |t-s| \leqq h, h \to 0} \frac{|L_t(x) - L_s(x)|}{|ln h| h^{1-1/\alpha}} \leqq c_4. \tag{5.8}$$

If X_t is Markov, one can do slightly better and replace the denominator of (5.8) by $|ln h|^{1/\alpha} h^{1-1/\alpha}$. See Millar [11] for the proof.

6. Extensions

In this section we show that if X_t is a semimartingale with (1.2) holding, there exists a local time for X_t. First suppose Y_t is a semimartingale with local characteristics (a_s, ν_s^Y) satisfying (1.2) with $b_s \equiv 1$. Then

$$Z_t = Y_t - \sum_{s \leqq t} \Delta Y_s 1_{(|\Delta Y_s| > \sigma)} + \int_0^t \int_{1 \geqq |h| \geqq \sigma} h \nu_s^Y(dh)\, ds - \int_0^t a_s\, ds. \tag{6.1}$$

is a local martingale.

Let a_s^+, a_s^- be the positive and negative parts, respectively, of a_s, and let

$$A_t^+ = \int_0^t \int_\sigma^1 h \nu_s^Y(dh)\, ds + \int_0^t a_s^-\, ds, \quad A_t^- = -\int_0^t \int_{-1}^{-\sigma} h \nu_s^Y(dh)\, ds + \int_0^t a_s^+\, ds.$$

Let P_t^+, P_t^- be two standard Poisson processes independent of each other and of Y. It is easy to check that $P^+_{A_t^+} - A_t^+$ and $P^+_{A_t^-} - A_t^-$ are martingales with respect to the appropriate σ-fields.

Let R_t be a process with stationary, independent increments, independent of Y_t, P^+, and P^-, with Lévy measure

$$\nu^R(dh) = \begin{cases} \theta_\alpha(dh) & |h| > 1 \\ 0 & |h| \leqq 1 \end{cases}$$

Let

$$X_t = Z_t + P^+_{A_t^+} - P^-_{A_t^-} + R_t - A_t^+ + A_t^-. \tag{6.2}$$

Note that ν_s^X satisfies (1.4).

If $T_0 = 0$, and $T_{i+1} = \inf\{t > T_i : |\Delta X_t| > \sigma \text{ or } |\Delta Y_t| > \sigma\}$, then because X_t and Y_t are right continuous with left limits,

$$T_i \nearrow +\infty, \text{ a.s.}$$

Fix i. Let $X_t^{(i)} = X_{t+T_i} - X_{T_i}$. $(X_t^{(i)}, \mathscr{F}_{t+T_i}, P)$ is a local martingale, $X_0^{(i)} = 0$, a.s. and $\nu_s^{X^{(i)}}$ satisfies (1.4). Observe that

$$Y_s - Y_{T_i} = X_s - X_{T_i} = X^{(i)}_{s-T_i} \tag{6.3}$$

if $T_i \leqq s < T_{i+1}$.

By Theorems (4.13) and (5.1), there exists a continuous process $L_t^{(i)}(x)$ such that if $f \in C_K$,

$$\int_0^t f(X_s^{(i)})\, ds = \int f(x) L_t^{(i)}(x)\, dx, \text{ a.s.} \tag{6.4}$$

Fix ω not in the null set of (6.4), and let

$$g(x) = f(x + Y_{T_i}(\omega)).$$

Then

$$\text{(6.5)} \quad \int f(x) L^{(i)}_{t \wedge (T_{i+1} - T_i)}(x - Y_{T_i})\, dx = \int g(x) L^{(i)}_{t \wedge (T_{i+1} - T_i)}(x)\, dx$$

$$= \int_0^{(T_{i+1} - T_i) \wedge t} g(X_s^{(i)})\, ds$$

$$= \int_{T_i}^{T_{i+1} \wedge (t + T_i)} f(X_s - X_{T_i} + Y_{T_i})\, ds$$

$$= \int_{T_i}^{T_{i+1} \wedge (t + T_i)} f(Y_s)\, ds,$$

using (6.4).

Now define $L_t(x)$ by induction as follows:

$$L_0(x) = 0;$$

$$L_t(x) = L_{T_i}(x) + L^{(i)}_{t - T_i}(x - Y_{T_i}) \quad \text{if } T_i \leqq t < T_{i+1}.$$

$L_t(x)$ is continuous, since each $L^{(i)}$ is. Summing (6.5) over $i = 0, 1, \ldots$, we see that $L_t(x)$ is an occupation time density for Y_s.

Now we want to show it suffices for (1.2) to hold. Suppose B_t is a strictly increasing continuous process, $B_0 = 0$, $B_\infty = \infty$, and $dB_t/dt = b_t \geqq \delta_{6.6} > 0$. Let $T_t = \inf\{s : B_s \geqq t\}$, and suppose $X_t = Y_{B_t}$. Suppose, Y_t is a process with a local time $L_t(x)$. Hence for $\omega \notin N$, a null set,

$$\int_0^t f(Y_s)\, ds = \int f(x) L_t(x)\, dx \quad \text{for all } t.$$

Let $M_t(x) = L_{B_t}(x)$. Then

$$\text{(6.6)} \quad \int f(x) M_t(x)\, dx = \int_0^{B_t} f(Y_s)\, ds = \int_0^t f(X_u)\, dB_u$$

$$= \int_0^t f(X_u)\, b_u\, du.$$

Now define

$$\text{(6.7)} \quad N_t(x) = \int_0^t b_u^{-1}\, dM_u(x).$$

(6.8) **Proposition.** $\int f(x) N_t(x)\, dx = \int_0^t f(X_u)\, du.$

Proof. Fix $\omega \notin N$. Let d_u be a step process such that d_u^{-1} is bounded. So for fixed times t_i, $d_u = d_{t_i}$ if $t_i \leqq u < t_{i+1}$. Let $M_t^d(x) = \int_0^t d_u^{-1}\, dM_t(x)$. Then, using (6.6),

$$\text{(6.9)} \quad \int f(x) M_t^d(x)\, dx = \sum_{i=0}^{\infty} \int f(x)\, (M^d_{t \wedge t_{i+1}}(x) - M^d_{t \wedge t_i}(x))\, dx$$

$$= \sum_{i=0}^{\infty} d_{t_i}^{-1} \int_{t \wedge t_i}^{t \wedge t_{i+1}} f(X_u)\, b_u\, du$$

$$= \int_0^t f(X_u)\, (b_u/d_u)\, du.$$

A monotone class argument shows that (6.9) holds for any d_u provided d_u^{-1} is bounded, in particular with $d_u = b_u$. □

Now suppose X_t is a semimartingale satisfying (1.2). It is routine to check that there exists a Y_t satisfying (1.2) with local characteristics (a_s, v_s) such that $X_t = Y_{B_t}$. Hence X_t has a local time.

7. Weak Convergence

We now wish to construct a counterexample to show that a condition such as (1.2) is necessary. In this section we collect a number of facts related to weak convergence in $D[0,\infty)$ that will be necessary for the construction, which is carried out in Sect. 8.

Let $\Omega = D[0,\infty) = \{$functions from $[0,\infty)$ to $\mathbb{R}$ that are right continuous with left limits$\}$. Define $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. Throughout, α and β are fixed with $1 < \alpha < \beta < 2$.

We will want to consider probability measures P' such that

(7.1) a) *(X_t, P') is a purely discontinuous local martingale with local characteristics* $(0, v_s)$ and $P'(X_0 = 0) = 1$.

b) $\theta_\alpha(dh) \leqq v_s(dh) \leqq c_{7.1A}\theta_\beta(dh)$ *if* $|h| \leqq 1$,

c) $v_s(dh) = \theta_\beta(dh)$ *if* $|h| > 1$.

Note that (1.3) is satisfied if $\alpha < \beta < \alpha + \varepsilon$.

One could, in the construction that follows, define P so that $v_s(dh) = \theta_\alpha(dh)$ if $|h| > 1$, and hence so that $v_s(dh) \geqq \theta_\alpha(dh)$ for all h; however, the existence or nonexistence of local times depends only on the behavior of v_s near 0 (cf. Sect. 6), and so we do not do that here.

The proof of the following is standard.

(7.2) **Lemma.** *Suppose (Y_t, P_n) is a submartingale (supermartingale, martingale) and for each t, $\sup_n E_n|Y_t|^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$. Suppose for each $j \geqq 1$ and $t_1 < \ldots < t_j$ with $P(\Delta Y_{t_i} \neq 0) = 0$ for $i = 1, \ldots, j$, the distribution of $(Y_{t_1}, \ldots, Y_{t_j})$ under P_n converges to that of $(Y_{t_1}, \ldots, Y_{t_j})$ under P. Then (Y_t, P) is a submartingale (supermartingale, martingale).*

(7.3) **Proposition.** *Suppose P_n is a sequence of probability measures satisfying* (7.1). *Then there is a subsequence which converges weakly in $D[0,\infty)$ to P, and P satisfies* (7.1).

Proof. By (7.1) b and c, $\sup_n \int h^2 \wedge 1\, v_s^{(n)}(dh) < \infty$. By the argument of [13, pp. 237–238], the P_n's can be shown to be tight, and so there exists a subsequence $P_{n'}$ converging weakly to P. By Lemma (7.2) and Proposition (3.2), (X_t, P) is a local martingale.

We now show (X_t, P) is purely discontinuous. Let

(7.4) $$Z_t = X_t - \sum_{s \leqq t} \Delta X_s 1_{(|\Delta X_s| > 1)}$$

Since $P_{n'}$ converges weakly on $D[0,\infty)$ to P, it is not hard to show that if $t_1 < \ldots < t_j, j \geqq 1$, the distribution of $(Z_{t_1}, \ldots, Z_{t_j})$ under P_n, converges to the distribution under P, provided $P(\Delta Z_{t_i} \neq 0) = 0, i = 1, \ldots, j$. As above, (Z_t, P)

is a local martingale. To show X^c, the continuous part of X, is 0, it suffices to show $Z^c = 0$.

Z has jumps bounded by 1, and so by (3.4), $\sup_n E_n Z_t^2 \leqq c_1$. Let $f(x) = x^2 \wedge c_2$. Then

$$Ef(Z_t) = \lim_{n' \to \infty} E_{n'} f(Z_t) \leqq \limsup_{n'} E_{n'} Z_t^2 .$$

Letting $c_2 \to \infty$, by monotone convergence,

$$EZ_t^2 \leqq \limsup_{n'} E_{n'} Z_t^2 .$$

Let $h \in C_K$ with support in $[-2,2] - (-\delta/2, \delta/2)$, with $0 \leqq h(x) \leqq x^2$, and $h(x) = x^2$ if $\delta \leqq |x| \leqq 1$. Since $P_{n'} \to P$ weakly, we can show that

$$E \sum_{s \leqq t} \Delta Z_s^2 \geqq E \sum_{s \leqq t} h(\Delta Z_s) = \lim_{n' \to \infty} E_{n'} \sum_{s \leqq t} h(\Delta Z_s) .$$

Since

$$\sup_n E_n \sum_{s \leqq t} \Delta Z_s^2 1_{(|\Delta Z_s| \leqq \delta)} = \sup_n E_n \int_0^t \int_{-\delta}^{\delta} h^2 \nu_s^{(n)}(dh)\, ds \to 0$$

as $\delta \to 0$, we get

$$E \sum_{s \leqq t} \Delta Z_s^2 \geqq \liminf_{n' \to \infty} E_{n'} \sum_{s \leqq t} \Delta Z_s^2 .$$

We then have, since $(Z_t, P_{n'})$ is purely discontinuous,

$$\begin{aligned} E\langle Z^c, Z^c \rangle_t &= E\left([Z,Z]_t - \sum_{s \leqq t} \Delta Z_s^2 \right) = EZ_t^2 - E \sum_{s \leqq t} \Delta Z_s^2 \\ &\leqq \limsup_{n' \to \infty} E_{n'} \left(Z_t^2 - \sum_{s \leqq t} \Delta Z_s^2 \right) = \limsup_{n' \to \infty} E_{n'} \langle Z^c, Z^c \rangle_t = 0 . \end{aligned}$$

Finally, we show (7.1) b and c hold. If $f \in C_K$ with support contained in $[-1,1] - \{0\}$, and

$$Y_t = \sum_{s \leqq t} f(\Delta Z_s) - t c_{7.1A} \int f(h)\, \theta_\beta(dh),$$

$(Y_t, P_{n'})$ is a supermartingale. Again, the finite dimensional distributions of Y_t can be shown to converge, and so by Lemma (7.2), (Y_t, P) is a supermartingale. Repeating this argument with the support of $f \subseteq [-1,1]^c$, we see that X_t has local characteristics $(0, \nu_s)$ with

$$\nu_s(dh) \leqq c_{7.1A} 1_{(|h| \leqq 1)} \theta_\beta(dh) + 1_{(|h| > 1)} \theta_\beta(dh) . \tag{7.5}$$

The remainder of (7.1) b is similar. □

Recalling the definition of S_λ, we need

(7.6) **Proposition.** *Suppose (X_t, P) satisfies (7.1). Then*

$$\sup_{|a| \leqq 2} S_\lambda 1_{[a-\delta, a+\delta]} \leqq c_{7.6}(\delta),$$

where $c_{7.6}(\delta)$ depends only on $c_{7.1A}$, α, β, and λ, and tends to 0 as $\delta \to 0$.

Incidentally, this proposition shows that X_t spends 0 time at points, and that to construct a P for X_t has no local time, one needs a somewhat complicated construction.

Proof. Let $T_N = \inf\{t: |X_t| \geqq N\}$. By Ito's lemma (cf. (3.12)). if $f \in C^2$,

$$Ef(X_{t \wedge T_N}) - f(0) = E \int_0^{t \wedge T_N} \int \Delta_R^h f(X_s) \nu_s(dh)\, ds. \tag{7.7}$$

Since $|\Delta_R^h f(X_s)| \leqq c_1(\|f'\| + \|f''\| + \sup_{|x| \leqq N} |f(x)|)\ (h^2 \wedge |h|)$ if $s < T_N$, a limiting argument, Proposition (3.2), and the fact that $|f(x)| \leqq f(0) + |x|\, \|f'\|$ shows that (7.7) holds for f convex such that $\|f'\| + \|f''\| < \infty$. Since f is convex, $\Delta_R^h f(x) \geqq 0$. Letting $N \to \infty$, we may use monotone convergence and (7.1) b on the right of (7.7) and uniform integrability on the left of (7.7) to get

$$Ef(X_t) - f(0) \geqq E \int_0^t \int_{-1}^1 \Delta_R^h f(X_s)\, \theta_\alpha(dh)\, ds. \tag{7.8}$$

Taking a limit and applying Fatou, (7.8) holds when $f(x) = |x-a|$. Direct calculation shows that

$$\int_{-1}^1 \Delta_R^h f(x)\, \theta_\alpha(dh) \geqq c_1 |x-a|^{1-\alpha} \geqq c_1 \delta^{1-\alpha} 1_{[a-\delta, a+\delta]}(x) \tag{7.9}$$

if $|x-a| < 1/4$. Substituting (7.9) in (7.8) and again using Proposition (3.2),

$$E \int_0^t 1_{[a-\delta, a+\delta]}(X_s)\, ds \leqq c_2 \delta^{\alpha-1} (t + |a|). \tag{7.10}$$

if $\delta \leqq 1/4$. Multiplying both sides of (7.10) by $\lambda e^{-\lambda t}$ and integrating t from 0 to ∞ completes the proof. □

Let F be the finite union of disjoint closed intervals contained in $[-1, 1]$. Let R_λ be the resolvent operator for a symmetric stable process of index β. That is, R_λ is defined by (2.10) and (2.4), where now q_t is the density of a stable process of index β. Let

$$\psi(dh) = \begin{cases} \theta_\alpha(dh) - \theta_\beta(dh) & \text{if } |h| \leqq 1 \\ 0 & \text{if } |h| > 1 \end{cases} \tag{7.11}$$

and let B be the operator defined on $f \in C^2$ by

$$Bf(x) = 1_F(x) \int \Delta_C^h f(x)\, \psi(dh). \tag{7.12}$$

One of the main results we will need in the next section is

(7.13) **Proposition.** *There exists a probability P satisfying* (7.1) *such that if* $S_\lambda f = E\int_0^\infty e^{-\lambda t} f(X_t)\,dt$ *and* $f\in C_K^2$, *then*

$$S_\lambda f = R_\lambda f(0) + S_\lambda(BR_\lambda f).$$

Proof. Let $\psi_m(dh) = \psi(dh)$ if $|h| \geqq 1/m$, 0 otherwise. Let B_m be defined analogously to B in (7.12). Let P_m be a probability satisfying (7.1) such that (X_t, P_m) has local characteristics $(0, v_s^m)$ with $v_s^m(dh) = 1_F(X_s)\,\psi_m(dh) + \theta_\beta(dh)$. Such P_m may be shown to exist by [1] or the techniques of [13]. An alternate argument would be to construct probabilities corresponding to $v_s^{m,k} = f_k(X_s)\,\psi_m(dh) + \theta_\beta(dh)$, f_k Lipschitz, using techniques of stochastic differential equations, and then to take weak limits as $f_k \to 1_F$.

By Theorem (3.10) a, if $f\in C_K^2$,

$$S_\lambda^{(m)} f = R_\lambda f(0) + S_\lambda^{(m)} B_m R_\lambda f, \tag{7.14}$$

where $S_\lambda^{(m)} g = E_m\int_0^\infty e^{-\lambda t} g(X_t)\,dt$, since now R_λ is the resolvent of a stable process of index β.

Taking a subsequence if necessary, we may assume $P_m \to P$, weakly, and by Proposition (7.3), P satisfies (7.1). Since $f\in C_K^2$, $R_\lambda f\in C^2$, and note then that

$$\|B_m R_\lambda f - BR_\lambda f\| \leqq \|(R_\lambda f)''\| \int_{|h|<1/m} h^2\,\psi(dh) \to 0$$

as $m\to\infty$. Since $\int \Delta_C^h R_\lambda f(x)\,\psi(dh)$ is continuous, $BR_\lambda f(x)$ is discontinuous only at the finitely many endpoints of intervals of F. By Proposition (7.6), we can find g continuous such that $S_\lambda^{(m)}|BR_\lambda f - g|$, $S_\lambda|BR_\lambda f - g| \leqq \varepsilon$.

Since $P_m \to P$ weakly, $S_\lambda^{(m)} f \to S_\lambda f$ and $S_\lambda^{(m)} g \to S_\lambda g$.

$$\begin{aligned} |S_\lambda^{(m)} B_m R_\lambda f - S_\lambda BR_\lambda f| &\leqq |S_\lambda^{(m)} B_m R_\lambda f - S_\lambda^{(m)} BR_\lambda f| \\ &\quad + |S_\lambda^{(m)} BR_\lambda f - S_\lambda^{(m)} g| + |S_\lambda^{(m)} g - S_\lambda g| \\ &\quad + |S_\lambda g - S_\lambda BR_\lambda f| \\ &\leqq \|B_m R_\lambda f - BR_\lambda f\|/\lambda + 2\varepsilon + |S_\lambda^{(m)} g - S_\lambda g|. \end{aligned} \tag{7.15}$$

The proof of the proposition follows from letting $m\to\infty$ in (7.14), since ε was arbitrary. □

8. Counterexamples

In this section we construct a probability P satisfying (7.1) for which X_t does not have a local time. The idea is to construct a Cantorlike set D of Lebesgue measure 0, to let $v_s(dh) = \theta_\alpha(dh)$ if $X_s\in D$, $|h|\leqq 1$, $v_s(dh) = \theta_\beta(dh)$ if $X_s\notin D$, and to show that X_t spends a positive amount of time in D. In view of Proposition (7.6), letting D be a countable set would not work.

Throughout this section $1<\alpha<\beta<2$ are fixed, R_λ is the resolvent of a symmetric stable process of index β, r_λ the density of R_λ, and S_λ (or $S_\lambda^{(n)}$) as in the statement of Proposition (7.13). Let $\gamma = \min((\beta-\alpha)/2, (\alpha-1)/2)$, and let

$$H(x) = |x|^{\gamma-1}. \tag{8.1}$$

(8.2) **Proposition.** *Let F be the finite union of disjoint closed intervals contained in* $[-1,1]$. *Suppose $\mu(F)>0$, and let $w=\sup_{y\in\mathbb{R}}\int_F H(x-y)\,dx/\mu(F)$. Let ψ and B be defined by* (7.11) *and* (7.12), *and let P be the probability given by Proposition* (7.13). *Then $S_\lambda 1_F \geqq c_{8.2}(w)$, where $c_{8.2}(w)$ depends only on α, β, and w, is decreasing in w, and strictly positive if $w<\infty$.*

Proof. Since $\int_{|h|\leqq 1} |h|^{\beta-\gamma}\theta_\alpha(dh)<\infty$, by Theorem (2.11) and (2.12), we can fix a λ such that if $f\in C_K^2$,

$$\left|\int_{|h|\leqq 1} \Delta_C^h R_\lambda f(x)\,\theta_\alpha(dh)\right| \leqq c_1 H*|f|(x). \tag{8.3}$$

Since $\int \Delta_C^h g\,\theta_\beta(dh)=\mathcal{G}_\beta g$, the infinitesimal generator for a symmetric stable process of index β, and so $\mathcal{G}_\beta R_\lambda f=\lambda R_\lambda f-f$, then

$$\begin{aligned}\int \Delta_C^h R_\lambda f(x)\,\psi(dh) = &\int_{|h|\leqq 1} \Delta_C^h R_\lambda f(x)\,\theta_\alpha(dh) + \int_{|h|>1} \Delta_C^h R_\lambda f(x)\,\theta_\beta(dh)\\ &-\lambda R_\lambda f(x)+f(x).\end{aligned} \tag{8.4}$$

Then if support $(f)\subseteq F$ and $0\leqq f\leqq 1$, $\|R_\lambda f\|\leqq \|r_\lambda\|\,\mu(F)$, and

$$\begin{aligned}|f(x)-BR_\lambda f(x)| &\leqq 1_F(x)\,(\lambda R_\lambda f(x)+c_1 H*f(x)+c_2\|R_\lambda f\|)\\ &\leqq 1_F(x)\,((\lambda+c_2)\|r_\lambda\|+c_1 w)\,\mu(F).\end{aligned} \tag{8.5}$$

On F^c, $f-BR_\lambda f=0$, and so, by Proposition (7.13),

$$R_\lambda f(0)=S_\lambda(f-BR_\lambda f)\leqq (c_1 w+c_3)\,\mu(F)\,S_\lambda 1_F. \tag{8.6}$$

Use monotone convergence to show (8.6) holds for $f=1_F$, and then observe that

$$R_\lambda 1_F(0)\geqq\left(\inf_{y\in[-1,1]} r_\lambda(y)\right)\mu(F)=c_4\,\mu(F).$$

This and (8.6) shows that $S_\lambda 1_F\geqq c_4/(c_1 w+c_3)$. □

(8.7) **Lemma.** *Suppose $[a,b]$ is a closed interval, $\varepsilon>0$. Then there exists $k_0(\varepsilon)$ such that if $k\geqq k_0(\varepsilon)$ is even,*

$$\delta=(b-a)/(4k+2),\ s_i=a+(4i+1)\,\delta,$$

and

$$I_k[a,b]=\bigcup_{i=0}^{k}[s_i+\delta,s_i-\delta],$$

then

$$\int_{I_k[a,b]} H(x)\,dx\leqq\left(\frac{1}{2}+\varepsilon\right)\int_a^b H(x)\,dx.$$

The proof of Lemma (8.7), which relies only on the fact that $H(x)$ is integrable and continuous except at 0, will be omitted. Note that

$$\mu(I_k[a,b]) = \frac{k+1}{2k+1}\,\mu([a,b]) > \frac{1}{2}\mu([a,b]), \text{ while if } k \geqq 2,\ \mu(I_k[a,b]) \leqq \frac{3}{5}\mu([a,b]).$$

Our main theorem is

(8.8) **Theorem.** *There exists a probability P satisfying* (7.1) *such that S_λ is not absolutely continuous with respect to μ. In particular, (X_t, P) cannot have a local time.*

Proof. Let $D_0 = [-1,1]$.

$$\int_{D_0} H(x)\,dx = \mu(D_0)/\gamma.$$

We will first construct by induction a sequence of closed sets D_n such that $D_n \supseteq D_{n+1}$, each D_n is the finite union of disjoint closed intervals of equal length, each D_n is symmetric about 0 and contains 0, $\frac{1}{2}\mu(D_n) \leqq \mu(D_{n+1}) \leqq \frac{3}{5}\mu(D_n)$, and

$$\sup_{y\in\mathbb{R}} \int_{D_n} H(x-y)\,dx < (2/\gamma)\,\mu(D_n).$$

Suppose $D_n = \bigcup_{j=1}^{N_n} [a_j, b_j]$ has been constructed. Choose ε small enough so that

$$\int_{D_n} H(x)\,dx/\mu(D_n) < 2/\gamma(1+2\varepsilon).$$

Choose $k \geqq 2$ even and large enough so that for each j,

$$\int_{I_k[a_j,b_j]} H(x)\,dx \leqq \left(\frac{1}{2}+\varepsilon\right)\int_{a_j}^{b_j} H(x)\,dx. \tag{8.9}$$

Let $D_{n+1} = \bigcup_{j=1}^{N_n} I_k[a_j,b_j]$. Summing (8.9) over $j=1,\dots,N_n$,

$$\int_{D_{n+1}} H(x)\,dx \leqq \left(\frac{1}{2}+\varepsilon\right)\int_{D_n} H(x)\,dx < \mu(D_n)/\gamma \leqq 2\mu(D_{n+1})/\gamma.$$

Since k is even, $0 \in D_{n+1}$, and by symmetry considerations,

$$\sup_{y\in\mathbb{R}} \int_{D_{n+1}} H(x-y)\,dx = \int_{D_{n+1}} H(x)\,dx.$$

Next, for each n, apply Proposition (8.2) with $F = D_n$ to obtain P_n satisfying (7.1) and $S_\lambda^{(n)} 1_{D_n} \geqq c_{8.2}(2/\gamma)$. By Proposition (7.3), a subsequence of the P_n's converges weakly to P satisfying (7.1). If $m \geqq n$, $S_\lambda^{(m)} 1_{D_n} \geqq S_\lambda^{(m)} 1_{D_m} \geqq c_{8.2}(2/\gamma)$. Since D_n is closed, by weak convergence $S_\lambda 1_{D_n} \geqq c_{8.2}(2/\gamma)$.

Let $D = \bigcap_{n=0}^{\infty} D_n$. D is closed, $\mu(D) = 0$, and since S_λ is a measure, $S_\lambda 1_D = \lim_{n\to\infty} S_\lambda 1_{D_n} \geqq c_{8.2}(2/\gamma)$.

If (X_t, P) had a local time $L_t(x)$, for each t

$$\int_0^t 1_D(X_s)\,ds = \int_D L_t(x)\,dx = 0, \text{ a.s.};$$

but then $S_\lambda 1_D$ would be 0, a contradiction. □

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Note added in proof. The author and M. Cranston have recently shown, using the Malliavin calculus, that a purely discontinuous martingale will have a jointly continuous local time provided the local characteristics are sufficiently smooth as well as sufficiently large.