Z Wahrscheinlichkeitstheorie verw Gebiete
56, 21-45 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On Regularity of Multiparameter Amarts and Martingales*

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In the case of product measures indexed by $\mathbb{N} \times \mathbb{N}$, martingales converge a.s. if they are bounded in $L\,\mathrm{Log}\,L$. This probabilistic version of the Jessen-Marcinkiewicz-Zygmund theorem [11] was established by R. Cairoli [5]. We prove here a more general result:[1] every 1-martingale of class $L\,\mathrm{Log}\,L$ converges a.s. The proof consists in showing that such a process is an amart with respect to totally ordered σ-algebras, and then applying the amart convergence theorems. Also L_1-bounded sums of independent random variables in the plane are identified as amarts with respect to totally ordered σ-algebras. In both instances these are the same σ-algebras, obtained by lumping together all the σ-algebras on the vertical line corresponding to the same first coordinate. Precise definitions are given below.

The main results of the paper are in the continuous parameter case, where we extend the amart theory [9] to the plane, and apply it to show that $L\,\mathrm{Log}\,L$ bounded 1- and 2-martingales have versions which are well-behaved in quadrants I, II, and IV. In quadrant III the existence of limits is proved only under an additional assumption satisfied by martingales with respect to Brownian motion. The same conclusions follow for martingales with respect to product σ-algebras, and, more generally, for martingales satisfying the conditional independence assumption (F4 of [6]). By a completely different method, namely by stochastic integration in the plane, D. Bakry [2] earlier proved that martingales satisfying F4 have right-continuous and left-limited versions.

The first section of this paper deals with the discrete parameter case. The second section develops the theory of descending 1- and 2-amarts. The third section considers the ascending case. Applications to continuous parameter martingales are given in the fourth, the last, section.

The authors wish to thank D. Bakry and P.A. Meyer for most pertinent comments.

* Research in part supported by the National Science Foundation USA

[1] More general, since we do not assume the conditional indepencence assumption F4 of [6]. Under this assumption, our definition of 1-martingale reduces to that of ordinary martingale, not to that of 1-martingale in the sense of [6]

0044-3719/81/0056/0021/$05.00

1. Discrete Parameter

Let I denote $\mathbb{Z}^2$ with the usual order $(s_1, s_2) \leqq (t_1, t_2)$ if $s_1 \leqq t_1$ and $s_2 \leqq t_2$; then I is filtering to the right. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, ans let $(\mathscr{F}_t)$ be a *stochastic basis* indexed by I, i.e., an increasing family of sub-sigma-algebras of $\mathscr{F}$. For every $t=(t_1, t_2)$, set $\mathscr{F}_t^1 = \bigvee_u \mathscr{F}_{t_1,u} = \mathscr{F}_{t_1,\infty}$, and $\mathscr{F}_t^2 = \bigvee_u \mathscr{F}_{u,t_2} = \mathscr{F}_{\infty,t_2}$, and set $\mathscr{F}_\infty = \bigvee_{t\in I} \mathscr{F}_t$. A process (X_t) is *adapted* if X_t is $\mathscr{F}_t$ measurable for every $t \in I$. An integrable process (X_t) is a *martingale* if $E(X_t|\mathscr{F}_s) = X_s$, whenever $s \leqq t$. An integrable process (X_t) is a 1-*martingale* if it is a martingale, and $E(X_{t_1,t_2}|\mathscr{F}_{s_1,\infty}) = X_{s_1,t_2}$, whenever $s=(s_1,s_2) \leqq (t_1,t_2)=t$. Similarly, an integrable process (X_t) is a 2-*martingale* if it is a martingale, and $E(X_{t_1,t_2}|\mathscr{F}_{\infty,s_2}) = X_{t_1,s_2}$, whenever $s \leqq t$. A map $\tau: \Omega \to I$ is a *simple* 1-*stopping time* if τ takes on finitely many values, and $\{\tau = t\} \in \mathscr{F}_t^1$ for every t. The set of simple 1-stopping times is denoted by T^1. An adapted process (X_t) is a *1-amart* if the net $(EX_\tau, \tau \in T^1)$ converges. One defines similarly *2-stopping times* and *2-amarts*. A process (X_t) is *bounded in* $L \operatorname{Log} L$ if $\sup_{t\in I} E(|X_t| \log^+ |X_t|) < \infty$. We show that $L \operatorname{Log} L$-bounded 1-martingales are 1-amarts. (It is easy to see that L_1-bounded 1-martingales, even L_1-bounded strong martingales [20] are not.)

Theorem 1.1. *Let (X_t) be an $L \operatorname{Log}$-bounded 1-martingale. Then the net $(X_\tau, \tau \in T^1)$ converges in L_1. Hence (X_t) is a 1-amart, and hence converges almost surely.*

Proof. The uniform integrability of the martingale (X_t) insures the existence of an integrable random variable $X \in \mathscr{F}_\infty$ such that $X_t = E(X|\mathscr{F}_t)$, and therefore by a theorem of L.L. Helms (see e.g. [19], p. 65), (X_t) converges to X in L_1. Set $\Phi(x) = x \operatorname{Log}^+ x$; Φ is an increasing convex function. Then by Fatou's lemma applied to a subsequence of $X_{n,n}$, $E[\Phi(|X|)] < \infty$. Now by Jensen's inequality,

$$\begin{aligned}\Phi(|X - X_t|) &\leqq \tfrac{1}{2}[\Phi(2|X|) + \Phi(2|X_t|)] \leqq \tfrac{1}{2}\Phi(2|X|) + \tfrac{1}{2}\Phi(E(2|X||\mathscr{F}_t)) \\ &\leqq \tfrac{1}{2}\Phi(2|X|) + \tfrac{1}{2}E[\Phi(2|X|)|\mathscr{F}_t].\end{aligned}$$

Therefore the net $\Phi(|X - X_t|)$ is uniformly integrable, and hence converges to zero in L^1.

Let $\varepsilon > 0$ be arbitrary. Choose an index u such that $\sup_{t \geqq u} \|X_u - X_t\|_1 \leqq \varepsilon$, and also $\sup_{t \geqq u} E[\Phi(|X_t - X_u|)] \leqq \varepsilon$. For every $t \geqq u$, set $Y_t = X_t - X_u$ and let τ be a simple 1-stopping time such that for some $v \in I$, one has $u \leqq \tau \leqq v$. For every $a \in \mathbb{N}$ with $u_1 \leqq a \leqq v_1$, set

$$S_a = \sup_{u_2 \leqq b \leqq v_2} |Y_{a,b}|, \qquad \mathscr{G}_a = \bigvee_{b \in \mathbb{N}} \mathscr{F}_{a,b}.$$

It follows from the definition of a 1-martingale that $(Y_{a,b}, \mathscr{G}_a)_{u_1 \leqq a \leqq v_1}$ is a one-parameter martingale for every fixed b with $u_2 \leqq b \leqq v_2$. Hence the one-parameter process $(S_a, \mathscr{G}_a)_{u_1 \leqq a \leqq v_1}$ is a one-parameter submartingale, and

$$|EY_\tau| \leqq E|Y_\tau| \leqq \sum E[1_{\{\tau = t\}} E(|Y_{v_1,t_2}||\mathscr{F}_t^1)] \leqq ES_{v_1}.$$

Applying Doob's inequality (see e.g. [19], p. 69) to the submartingale

$$(|Y_{v_1,b}|, \mathscr{F}_{v_1,b})_{u_2 \leqq b \leqq v_2} \text{ yields } \alpha P(S_{v_1} > \alpha) \leqq E[1_{\{S_{v_1} > \alpha\}} |Y_v|],$$

$\forall \alpha > 0$. Choose η with $0 < \eta < 1$; then, using the elementary inequality $a \operatorname{Log}^+ b \leqq a \operatorname{Log}^+ a + b/e$, one has

$$\begin{aligned} ES_{v_1} = \int_0^\infty P(S_{v_1} > \alpha)\, d\alpha &\leqq \eta + \int_\eta^\infty \alpha^{-1} [\int 1_{\{S_{v_1} > \alpha\}} |Y_v|\, dP]\, d\alpha \\ &\leqq \eta + \int |Y_v| [\operatorname{Log} S_{v_1} - \operatorname{Log} \eta]\, 1_{\{S_{v_1} > \eta\}} dP \\ &\leqq \eta + E(|Y_v| \operatorname{Log}^+ |Y_v|) + e^{-1} ES_{v_1} + |\operatorname{Log} \eta| E|Y_v| \\ &\leqq \eta + \varepsilon + \varepsilon |\operatorname{Log} \eta| + e^{-1} ES_{v_1}. \end{aligned}$$

Fix $\delta > 0$, and choose ε and η with $0 < \varepsilon < \delta$ and $0 < \eta < 1$ such that $\eta + \varepsilon + \varepsilon |\operatorname{Log} \eta| \leqq \delta(e-1)/e$. Then for every $\tau \in T^1$, $\tau > u$, $|X_\tau - X| \leqq |X_\tau - X_u| + |X - X_u|$ implies

$$E|X_\tau - X| \leqq E|Y_\tau| + E|X - X_u| \leqq ES_{v_1} + \varepsilon \leqq \delta + \varepsilon \leqq 2\delta.$$

Hence the net $(X_\tau, \tau \in T^1)$ converges to X in L^1. The stochastic basis $(\mathscr{F}_t^1)$ is totally ordered, and hence satisfies the Vitali condition V. Therefore the stochastic convergence of the net $(X_\tau, \tau \in T^1)$ implies the almost sure convergence of X_t ([13], Theorem 4.2). Another proof consists in observing that (X_t) is a 1-amart, so that the almost sure convergence of X_t follows from Astbury's amart convergence theorem (see [1] or [13], Theorem 5.1).

As a corollary, we obtain the following:

Theorem 1.2 *(Cairoli): Let $(\mathscr{F}_t)$ be a stochastic basis such that $\mathscr{F}_t^1$ and $\mathscr{F}_t^2$ are conditionally independent given $\mathscr{F}_t$. Then every $L \operatorname{Log} L$-bounded martingale $(X_t, \mathscr{F}_t)$ converges almost surely.*

Proof. The conditional independence condition on the stochastic basis insures that every martingale is both a 1- and 2-martingale, so that Theorem 1.1 applies. □

The following theorem asserts the equivalence between almost sure convergence of martingales and the validity of an asymptotic maximal inequality.

We say that a family $\mathscr{E}$ of processes $(X_t, \mathscr{F}_t)$ is *stable* if for every process $(X_t) \in \mathscr{E}$, and for every index t_0, the process $(Y_t, \mathscr{G}_t)$ defined by $Y_t = X_t - X_{t_0}$, $\mathscr{G}_t = \mathscr{F}_t$ for $t \geqq t_0$, and $Y_t = 0$, $\mathscr{G}_t = \mathscr{F}_{t_0}$ otherwise also belongs to $\mathscr{E}$.

Theorem 1.3. *Let $(\mathscr{F}_t)$ be a stochastic basis, and let $\mathscr{E}$ be a stable subfamily of the family of uniformly integrable martingales (in particular the family of $L \operatorname{Log} L$-bounded martingales). Then the following conditions are equivalent:*

(i) *For every martingale (X_t) belonging to $\mathscr{E}$, and every $\lambda > 0$,* $P(\limsup |X_t| \geqq \lambda) \leqq \frac{1}{\lambda} \lim E|X_t|$.

(ii) *Every martingale (X_t) belonging to $\mathscr{E}$ converges almost surely.*

Proof. The equivalence of (i) and (ii) in the case where $\mathscr{E}$ is the class of *all* martingales was proved in [15], and in [16]. The argument also establishes Theorem 1.3. □

The conjunction of Theorems 1.1 and 1.3 immediately gives:

Theorem 1.4. *Let (X_t) be a 1-martingale bounded in $L \operatorname{Log} L$. Then for every $\lambda>0$,*

$$P(\limsup |X_t| \geqq \lambda) \leqq \frac{1}{\lambda} \lim E|X_t|.$$

Theorem 1.4 can be also deduced from [14], Theorem 3.1, applied to the 1-amart $(X_t, \mathscr{F}_t^1)$, which implies that

$$P(\limsup |X_t| \geqq \lambda) \leqq \frac{1}{\lambda} \limsup_{\tau \in T^1} E|X_\tau| = \frac{1}{\lambda} \lim E|X_t|.$$

The following theorem shows that L^1-bounded sums of independent random variables are 1-amarts, thus giving another proof of the a.s. convergence of such sums, first shown by J.P. Gabriel [10].

Theorem 1.5. *Let $(Y_{i,j})$ be a family of independent random variables, set $X_{m,n} = \sum_{(i,j) \leqq (m,n)} Y_{i,j}$, and suppose that $(X_{m,n})$ is L^1-bounded. Then the net $(X_\tau, \tau \in T^1)$ converges in L^1, and $(X_{m,n})$ is an ascending 1-amart which converges a.s.*

Proof. The following maximal inequality, generalizing an inequality of Marcinkiewicz-Zygmund, has been proved by J.P. Gabriel [10]. There exists a constant K such that

$$\| \sup_{(m,n)} |X_{m,n}| \|_1 \leqq K \sup \| X_{m,n} \|_1$$

and K does not depend on the sequence $(Y_{i,j})$. The uniformly integrable martingale $(X_{m,n})$ converges in L^1 to a random variable X. Fix ε, and (m_0, n_0) such that $\sup_{(m,n) \geqq (m_0,n_0)} \| X_{m,n} - X_{m_0,n_0} \|_1 \leqq \varepsilon$. Let $\tau \in T^1$ satisfy $\tau \geqq (m_0, n_0)$. Then

$$\begin{aligned}
\| X_\tau - X_{m_0,n_0} \|_1 &\leqq \sum_{(m,n) \geqq (m_0,n_0)} E[1_{\{\tau=(m,n)\}} | \sum_{i \leqq m} \sum_{n_0 < j \leqq n} Y_{i,j} |] \\
&\quad + \sum_{(m,n) \geqq (m_0,n_0)} E[I_{\{\tau=(m,n)\}} | \sum_{m_0 < i \leqq m} \sum_{j \leqq n_0} Y_{i,j} |] \\
&\leqq \| \sup_{(m,n) \geqq (m_0,n_0)} | \sum_{i \leqq m} \sum_{n_0 < j \leqq n} Y_{i,j} | \|_1 \\
&\quad + \| \sup_{(m,n) \geqq (m_0,n_0)} | \sum_{m_0 < i \leqq m} \sum_{j \leqq n_0} Y_{i,j} | \|_1 \\
&\leqq K \sup_{(m,n) \geqq (m_0,n_0)} \| X_{m,n} - X_{m,n_0} \|_1 + K \sup_{(m,n) \geqq (m_0,n_0)} \| X_{m,n_0} - X_{m_0,n_0} \| \\
&\leqq 2K\varepsilon + K\varepsilon = 3K\varepsilon.
\end{aligned}$$

Hence $(X_\tau, \tau \in T^1)$ converges to X in L_1. □

We now state analogous results in the descending case. Proofs, similar to those in the ascending case, are omitted. An adapted process (X_t) is a *descending 1-amart* if the net $(EX_\tau:\tau\in T^1, \tau\to(-\infty,-\infty))$ converges (here $\mathbb{Z}^2$ is considered as a directed set filtering to *the left*).

Descending 2-amarts are defined analogously.

Theorem 1.6. *Let (X_t) be a 1-martingale bounded in $L\mathrm{Log}L$; then $(X_\tau$: $\tau\in T^1$, $\tau\to(-\infty,-\infty))$ converges in L_1, (X_t) is a descending amart, and (X_t), $t\to(-\infty,-\infty)$ converges almost surely.*

Finally results analogous to the theorems of this section are true by symmetry if one replaces 1-processes by the corresponding 2-processes.

2. Continuous Parameter Amarts; Quadrants I, II and IV

In this section we show that continuous parameter 1-amarts indexed by $\mathbb{R}_+^2$ have modifications which are well-behaved in quadrants I and IV. The same is true for 2-amarts in quadrants I and II.

Given $s,t\in\mathbb{R}_+^2$, set $s\ll t$ if $s_1<t_1$ and $s_2<t_2$. Given an index t, denote the four quadrants determined by t by $Q_{\mathrm{I}}(t)=\{s:s\geqq t\}$, $Q_{\mathrm{II}}(t)=\{s:s_1\leqq t_1,\ s_2\geqq t_2\}$, $Q_{\mathrm{III}}(t)=\{s:s\leqq t\}$, and $Q_{\mathrm{IV}}(t)=\{s:s_1\geqq t_1,\ s_2\leqq t_2\}$. Let $(\mathscr{F}_t)$ be an increasing family of σ-algebras indexed by $t=(t_1,t_2)\in\mathbb{R}_+^2$, and such that $\mathscr{F}_{(0,0)}$ contains all the null sets. Define $\mathscr{F}_t^1=\mathscr{F}_{t_1,\infty}=V_u\mathscr{F}_{t_1,u}$, and $\mathscr{F}_t^2=\mathscr{F}_{t_2,\infty}=V_u\mathscr{F}_{u,t_2}$. A stochastic basis $(\mathscr{G}_t)_{t\in\mathbb{R}_+^2}$ is *right-continuous* if for every index t, $\mathscr{G}_t=\cap\{\mathscr{G}_s:s\gg t\}$. A stochastic process $(X_t)_{t\in\mathbb{R}_+^2}$ is *continuous in* Q_i, $i=\mathrm{I},\dots,\mathrm{IV}$, if $X_t=\lim(X_s:s\to t,\ s\in Q_i(t))$ for every t. For every $i=\mathrm{I},\dots,\mathrm{IV}$, denote $Q_i^0(t)$ the interior of $Q_i(t)$ for the euclidean topology of the plane. The process (X_t) has *limits in* Q_i, $i=\mathrm{I},\dots,\mathrm{IV}$, if $\lim(X_s:s\to t,\ s\in Q_i^0(t))$ exists for every index t. Continuity in Q_{I} [Q_{III}] is called *right-* [*left*] *continuity*, and a process having limits in Q_{I} [Q_{III}] is said to have *right* [*left*] *limits*. A process (X_t) is *adapted* if X_t is $\mathscr{F}_t$ measurable for every t. A process (Y_t) is a *modification* of (X_t) if for every t, $X_t=Y_t$ a.s., and (Y_t) is *indistinguishable* from (X_t) if for almost every ω, $X_t(\omega)=Y_t(\omega)$ for every index t. A process (X_t) is *separable* if there exists a countable subset S of $\mathbb{R}_+^2$ (called *separant*) and a null set N such that for every $\omega\notin N$ and every $t\in\mathbb{R}_+^2$,

$$X_t(\omega)\in\bigcap_{0\in\mathcal{O}(t)}\overline{(\cup[X_s(\omega):s\in 0\cap S])}$$

where the intersection is over the family $\mathcal{O}(t)$ of all open sets 0 containing t. Doob's theorem ([7], p. 57) about the existence of separable modifications of one-parameter processes extends to the present setting. Given any subset S of $\mathbb{R}_+^2$, denote by S_1 [S_2] the set of first [second] coordinates of the elements of S.

Let J be a directed set filtering to the right, and let $(\mathscr{G}_t, t\in J)$ be an increasing family of sub-σ-algebras of $\mathscr{F}$. A map $\tau:\Omega\to J$ is a *stopping time for* $(\mathscr{G}_t)$ if $\{\tau\leqq t\}\in\mathscr{G}_t$ for every t. A *1-stopping time* is a stopping time for $(\mathscr{F}_t^1, t\in\mathbb{R}_+^2)$, and a *2-stopping time* is a stopping time for $(\mathscr{F}_t^2, t\in\mathbb{R}_+^2)$. Let $i=1,2$; an i-stopping time is called *simple* if it takes on finitely many values. T^i denotes the set of simple i-

stopping times, and for every subset S of $\mathbb{R}_+^2$, $T^i(S)$ denotes the set of elements of T^i with all the values in S. If τ is a stopping time for $(\mathscr{G}_t)$, let $\mathscr{G}_\tau = \{A \in \mathscr{F} : A \cap \{\tau \leqq t\} \in \mathscr{G}_t \text{ for all } t\}$.

A sequence $\tau(n)$ of 1-stopping times *1-decreases to τ in $Q_{\rm I}[Q_{\rm IV}]$* if $\lim \tau(n) = \tau$, the sequence $\tau(n)_1$ decreases, and $\tau(n) \geqq \tau$ for every n $[\tau(n)_1 \geqq \tau_1$ and $\tau(n)_2 \leqq \tau_2$ for every $n]$. A sequence $\tau(n)$ of 1-stopping times *1-recalls τ in $Q_{\rm I}[Q_{\rm IV}]$* if $\tau(n)$ 1-decreases to τ in $Q_{\rm I}[Q_{\rm IV}]$, and if $\tau(n) \gg \tau$ for every n $[\tau(n)_1 > \tau_1$ and $\tau(n)_2 < \tau_2$ on the set $\{\tau_2 > 0\}$ for every $n]$. The definitions for 2-stopping times are similar, exchanging everywhere the roles of the first and second coordinates. Thus one defines sequences $\tau(n)$ of 2-stopping times 2-decreasing to τ [2-recalling τ] in the quadrants $Q_{\rm I}$ and $Q_{\rm II}$.

The following proposition shows that given a 1-stopping time τ, the cluster points of the nets $X_t(\omega)$ when t approaches τ in the quadrants $Q_{\rm I}$ and $Q_{\rm IV}$ can be obtained as cluster points for some sequences $X_{\tau(n)}$, with $\tau(n) \in T^1(S)$. It generalizes Proposition 7.1 (a) [9] to processes indexed by $\mathbb{R}_+^2$.

Proposition 2.1. *Let (X_t) be an adapted process, let S be a separant set, and let τ be a bounded 1-stopping time.*

(1) There exists a bounded sequence $\tau(n)$ in $T^1(S)$ which 1-recalls τ in $Q_{\rm I}$, and such that the two nets $\{X_{\tau(n)}(\omega) : n \in \mathbb{N}\}$ and $\{X_t(\omega) : t \to \tau(\omega),\ t \gg \tau(\omega)\}$ have the same cluster points for almost every ω.

(2) Suppose that $\tau_2 > 0$. For every $\varepsilon > 0$, there exists a 1-stopping time σ with $P(\sigma = \tau) \geqq 1 - \varepsilon$, $\sigma_1 \geqq \tau_1$, $0 < \sigma_2 < \tau_2$, and there exists a bounded sequence $\sigma(n)$ in $T^1(S)$ which 1-recalls σ in $Q_{\rm IV}$, such that the two nets $\{X_{\sigma(n)}(\omega) : n \in \mathbb{N}\}$ and $\{X_t(\omega) : t \to \sigma(\omega),\ t_1 > \sigma_1(\omega),\ t_2 < \sigma_2(\omega)\}$ have the same cluster points for almost every ω.

Proof. For every $a \geqq 0$, set $\mathscr{G}_a = \mathscr{F}_{a,\infty} = \mathscr{F}^1_{a,b}$, $\forall b \geqq 0$.

(1) Let (M, M) be an upper bound of τ. For every $n \geqq 1$, and every $i, j \geqq 0$, choose an element $t(n, i, j)$ in S, and also in

$$J_n(i,j) = [i2^{-n}, (i+1)2^{-n}[\times [j2^{-n}, (j+1)2^{-n}[.$$

Set

$$\sigma(n) = \sum_{0 \leqq i \leqq M2^n} \sum_{0 \leqq j \leqq M2^n} t(n, i+1, j+1)\, 1_{\{\tau \in J_n(i,j)\}}.$$

It is easy to see that $\sigma(n) \in T^1(S)$, decreases and 1-recalls τ in $Q_{\rm I}$. Fix n; since S is a separant set, there exists a finite subset of S, $\{t(n,1), t(n,2), \ldots, t(n, r_n)\}$ such that, setting

$$A = \{\omega : \text{for every } t \text{ with } \tau(\omega) \ll t \ll \sigma(n)(\omega), \text{ there exists } i \text{ with } 1 \leqq i \leqq r_n$$
$$\text{such that } \tau(\omega) \ll t(n,i) \ll \sigma(n)(\omega) \text{ and } |X_t(\omega) - X_{t(n,i)}(\omega)| < 2^{-n}\},$$

one has $P(A) > 1 - 2^{-n}$.

Set $r_0 = 0$, fix $j \geqq 1$, and let $n = n(j)$, $i = i(j)$ be the unique pair of intergers such that

$$j = r_0 + \ldots + r_{n-1} + i, \quad n \geqq 1, \quad 1 \leqq i \leqq r_n.$$

Set $B_j=\{\tau_1<t(n,i)_1\leqq\sigma(n)_1\}\cap\{\tau_2<t(n,i)_2\leqq\tau_2+2^{-n}\}$. Since $\sigma(n)_1$ is a one-dimensional stopping time for $(\mathscr{G}_a)_{a\geqq 0}$,

$$B_j=\{t(n,i)_1\leqq\sigma(n)_1\}\cap\{\tau\ll t(n,i)\}\cap\{\tau\ll(t(n,i)_1,t(n,i)_2-2^{-n})\}^c,$$

and $B_j\in\mathscr{F}^1_{t(n,i)}$, $B_j\in\mathscr{G}_{\sigma(n)_1}$.

Define an element $v(j)$ of $T^1(S)$ by

$$v(j)=t(n,i)\quad\text{on } B_j,$$
$$v(j)=\sigma(n)\quad\text{elsewhere.}$$

Clearly $v(j)\gg\tau$, $\lim_{j\to\infty}v(j)=\tau$, and the cluster points of the sequence $\{X_{v(j)}(\omega):j\geqq 1\}$ and of the net $\{X_t(\omega):t\gg\tau(\omega),\ t\to\tau(\omega)\}$ are the same for almost all ω. However, we still must arrange the first coordinates of $v(j)$ by decreasing values. We define a sequence $\rho(n)$ of discrete (not necessarily simple) 1-stopping times as follows. Let $\rho(1)$ be the first value of $\{v(1),v(2),\ldots\}$ when arranged in decreasing order of $v(n)_1$, i.e., for every $t\in S$, we set

$$\{\rho(1)=t\}=\bigcup_{i\geqq 1}[\{v(i)=t)\cap(\bigcap_{j<i}\{v(j)_1<t_1\})\cap(\bigcap_{j>i}\{v(j)_1\leqq t_1\})].$$

Since the maps $v(j)_1$ are one-parameter stopping times for $(\mathscr{G}_a)_{a\geqq 0}$, $\rho(1)$ is a 1-stopping time. Let $\rho(2)$ be the second value of $\{v(1),v(2),\ldots\}$ when arranged in decreasing order of $v(n)_1$, i.e., for every $t\in S$,

$$\begin{aligned}\{\rho(2)=t\}=\bigcup_{i\geqq 1}\{\{v(i)=t\}&\\ \cap[\bigcup_{j\neq i}(\{v(j)_1>t_1\}&\cap(\bigcap_{\substack{k<i\\k\neq j}}\{v(k)_1<t_1\})\cap(\bigcap_{\substack{k>i\\k\neq j}}\{v(k)_1\leqq t_1\}))]\\ \cap[\bigcup_{j<i}(\{v(j)_1=t_1\}&\cap(\bigcap_{\substack{k<i\\k\neq j}}\{v(k)_1<t_1\})\cap(\bigcap_{k>i}\{v(k)_1\leqq t_1\}))]\}.\end{aligned}$$

Clearly $\rho(2)$ is a 1-stopping time, and the procedure can be continued by induction. Since $\lim\rho(n)=\tau$ and $\rho(n)\gg\tau$, the cluster points of the sequence $\{X_{\rho(n)}:n\geqq 1\}$ and of the net $\{X_t(\omega):t\to\tau(\omega),\ t\gg\tau(\omega)\}$ are the same for almost every ω. By the definition of $v(j)$, for every $n\geqq 1$ and every $j\geqq 1$ with $r_0+r_1+\ldots+r_{n-1}<j\leqq r_0+\ldots+r_n$, we have $v(j)_1\leqq\sigma(n)_1$; hence we also have $\rho(j)_1\leqq\sigma(n)_1$. To obtain the promised sequence $\tau(n)$, we now keep only finitely many values from each 1-stopping time $\rho(n)$. For every $j\geqq 1$, let $R(j)$ be a finite subset in the range of $\rho(j)$ such that if $C_j=\{\rho(j)\in R(j)\}$, then $P(C_j)>1-2^{-j}$. For every j, let n satisfy $r_0+\ldots+r_{n-1}<j\leqq r_0+\ldots+r_n$, and set

$$\tau(j)=t\quad\text{on}\quad\{\rho(j)=t\}\cap(\bigcap_{k>j}C_k)\qquad\text{for every } t\in R(j),$$
$$\tau(j)=\sigma(n)\quad\text{on}\quad(\bigcap_{k\geqq j}C_k)^c.$$

Let $\omega\in\{\rho(j)=t\}\cap(\bigcap_{k>j}C_k)$ for some $t\in R(j)$; then

$$\tau(j)(\omega)=\rho(j)(\omega),\quad\text{and}\quad\tau(j+1)(\omega)=\rho(j+1)(\omega).$$

Hence $\tau(j+1)_1(\omega) \leqq \tau(j)_1(\omega)$. Let $\omega \in (\bigcap_{k \geqq j} C_k)^c$; then $\tau(j+1)(\omega)$ can be either $\rho(j+1)(\omega)$, or $\sigma(n)(\omega)$, or $\sigma(n+1)(\omega)$ (the last value being possible only if $j=r_0+\ldots+r_n$). Since $\rho(j+1)_1(\omega) \leqq \sigma(n)_1(\omega)$, we also have $\tau(j+1)_1(\omega) \leqq \tau(j)_1(\omega)$. Clearly $\tau(j) \gg \tau$, $\lim \tau(j)=\tau$, and $\tau(j)_1$ decreases. For almost every ω, there exists an integer N such that $\omega \in \bigcap_{j \geqq N} C_j$. Since the sequences $(X_{\rho(j)}: j \geqq N)$ and $(X_{\tau(j)}: j \geqq N)$ agree on $\bigcap_{j \geqq N} C_j$, it suffices to check that each $\tau(j)$ is a 1-stopping time (necessarily in $T^1(S)$). This follows easily from the fact that $\rho(j)_1$ decreases, and that $\bigcap_{k \geqq j} C_k$ is $\mathscr{G}_{\sigma(n)_1}$-measurable.

(2) Fix $\varepsilon>0$, and choose $\alpha>0$ such that $P(\tau_2>2\alpha) \geqq 1-\varepsilon$. Let K be an integer such that $\tau \leqq (K\alpha, K\alpha)$. Let $s(n)$ be a bounded sequence in S such that $K\alpha<s(1)_2<s(2)_2<\ldots$, and $s(1)_1>s(2)_1>\ldots>K\alpha$.

For every $n \geqq 1$, $i \geqq 0$ and $j \geqq 0$, set

$$L(n,i,j)=\left[\frac{i\alpha}{2^n}, \frac{(i+1)\alpha}{2^n}\right[\times \left[\frac{j\alpha}{2^n}, \frac{(j+1)\alpha}{2^n}\right[,$$

and choose $t(n,i,j)$ in $S \cap L(n,i,j)$. For every $n \geqq 0$, every i,j with $0 \leqq i \leqq K2^n$, $2^{n+1} \leqq j \leqq K2^n$, set $v(n)=t(n,i+2,j-2)$ on $\{\tau \in L(n,i,j)\}$, $v(n)=s(n)$ on $\{\tau_2 \leqq 2\alpha\}$. Then $v(n) \in T^1(S)$, $v(n)_1$ strictly decreases, $v(n)_2$ strictly increases, and $v(n)$ 1-recalls $\sigma=\lim v(n)$ in Q_{IV}; furthermore $P(\tau=\sigma) \geqq 1-\varepsilon$.

An argument analogous to the one given in the proof of (1) allows us to transform $v(n)$ into a sequence $\sigma(n)$ in $T^1(S)$ having the required cluster points. □

Let $(\mathscr{F}_n, n \in \mathbb{N})[\mathscr{F}_n, n \in -\mathbb{N}]$ be an increasing sequence of σ-algebras, and let $T[-T]$ be the set of simple stopping times for $(\mathscr{F}_n)$, taking values in $\mathbb{N}[-\mathbb{N}]$. The set $(T, \leqq)[(-T, \leqq)]$ is filtering to the right [left]. An integrable process $(X_n, \mathscr{F}_n, n \in \mathbb{N})$ $[(X_n, \mathscr{F}_n, n \in -\mathbb{N})]$ is an *ascending* [*descending*] *amart* if the net $(EX_\tau, \tau \in T)$ $[(EX_\tau, \tau \in -T)]$ converges. If $\mathscr{F}_n \downarrow$ for $n \geqq 0$, $Y_{-n}=X_n$, $\mathscr{G}_{-n}=\mathscr{F}_n$, then $(X_n, \mathscr{F}_n, n \in \mathbb{N})$ is called a *descending amart* if $(Y_n, \mathscr{G}_n, n \in -\mathbb{N})$ is one. Every L_1-bounded ascending amart, and every descending amart converges a.s. ([9], Theorem 2.3). An integrable one-parameter process $(X_t, \mathscr{F}_t, t \geqq 0)$ is an *ascending* [*descending*] *amart*, if for every stopping time τ for $(\mathscr{F}_t)$, and for every sequence $(\tau(n), n \in \mathbb{N})$ $[(\tau(n), n \in -\mathbb{N})]$ of simple stopping times that increases to τ, the process $(X_{\tau(n)}, \mathscr{F}_{\tau(n)}, n \in \mathbb{N})$ $[(X_{\tau(n)}, \mathscr{F}_{\tau(n)}, n \in -\mathbb{N})]$ is an ascending [descending] amart. An integrable process $(X_t, \mathscr{F}_t, t \geqq 0)$ is an *amart* if it is both an ascending and a descending amart. A process $(X_t, \mathscr{F}_t, t \geqq 0)$ is of *class* (*AL*) if for every uniformly bounded increasing sequence of simple stopping times $\tau(n)$, $\sup E|X_{\tau(n)}|<\infty$. If $(X_t, \mathscr{F}_t, t \geqq 0)$ is a separable amart of class (*AL*), then almost all trajectories have right and left limits ([9] Theorem 7.3). The following definitions give generalizations of the notion of one-parameter amart to processes indexed by $\mathbb{R}_+ \times \mathbb{R}_+$. They are formulated with 1 and 2-stopping times in the four quadrants. An integrable process $(X_t)_{t \in \mathbb{R}_+^2}$ in a *1-amart* in $Q_{\mathrm{I}}[Q_{\mathrm{IV}}]$

if for every bounded 1-stopping time τ, and every uniformly bounded sequence $(\tau(n), n\in\mathbb{N})$ in T^1 which 1-recalls τ in $Q_{\mathrm{I}}[Q_{\mathrm{IV}}]$, the process $(X_{\tau(n)}, \mathscr{F}^1_{\tau(n)}, n\in\mathbb{N})$ is a descending amart. The process $(X_t)_{t\in\mathbb{R}^2_+}$ is a *descending 1-amart* if it is a 1-amart in Q_{I} and Q_{IV}, and if for every $b\geqq 0$, the one-parameter process $(X_{t,b}, \mathscr{F}^1_{t,b}, t\geqq 0)$ is a descending amart.

Similar definitions are given for descending 2-amarts, the first and second coordinates being exchanged everywhere.

The following proposition is a local result at one-dimensional stopping times for $(\mathscr{F}_{t,\infty})_{t\geqq 0}$. The idea of establishing uniformity in the second coordinate is due to J.B. Walsh [20].

Proposition 2.2. *Let $(X_t)_{t\in\mathbb{R}^2_+}$ be a separable descending 1-amart with separant set S. Suppose that for every fixed $b\in S_2$ the one-parameter process $(X_{a,b}, \mathscr{F}^1_{a,b}, a\geqq 0)$ is separable. Suppose that the family of σ-algebras $(\mathscr{F}_t^1)_{t\in\mathbb{R}^2_+}$ is right-continuous, and let σ be a bounded one-parameter stopping time for $(\mathscr{F}_{a,\infty})_{a\geqq 0}$. Fix $M>0$; then for almost every ω, $\lim(X_{a,b}(\omega)\colon a>\sigma(\omega), a\to\sigma(\omega))$ exists uniformly in $b\in S_2$, $b\leqq M$.*

Proof. Adding if necessary countably many countable sets to S, one may assume that for every fixed $b\in S_2$ the set $\{(a,b)\in S\,|\,a\geqq 0\}$ is a dense separant set for the one-parameter process $(X_{a,b}, \mathscr{F}^1_{a,b}, a\geqq 0)$. For every $a\geqq 0$, set $\mathscr{G}_a=\mathscr{F}_{a,\infty}$, and let K be an upper bound for σ. Suppose Proposition 2.2 is false. Then there exists a measurable set A with $P(A)>0$ such that for every $\omega\in A$, there exists $\varepsilon(\omega)>0$ such that for every $\alpha>0$, there exists $s, t\in S$ with $s_2=t_2\leqq M$, $\sigma(\omega)<s_1<t_1<\sigma(\omega)+\alpha$, and $|X_s(\omega)-X_t(\omega)|>\varepsilon(\omega)$. Replacing A by a non-null subset, one may assume that for every $\omega\in A$, $\varepsilon(\omega)$ remains larger than a constant $\varepsilon>0$. For every $t\in S$, set

$$\begin{aligned}A(1,t)=\{\omega\colon &\sigma(\omega)<t_1<\sigma(\omega)+1, \exists s\in S, s_2=t_2\leqq M,\\ &\sigma(\omega)<s_1<t_1, |X_s(\omega)-X_t(\omega)|>\varepsilon\}\cap A;\end{aligned}$$

then $A(1,t)\in\mathscr{F}_t^1\cap\mathscr{G}_{\sigma+1}$. Choose finitely many indices $t(i)$ in S, $1\leqq i\leqq n_1$, with $t(1)_1\leqq t(2)_1\leqq\ldots\leqq t(n_1)_1$, such that if $A_1=\bigcup_{i\leqq n_1} A(1,t(i))$, then $P(A_1)>P(A)/2$. Set

$$\begin{aligned}&\sigma(1)=t(i) \quad \text{on} \quad A(1,t(i))\setminus\bigcup_{j<i} A(1,t(j)),\\ &\sigma(1)=(K+1,0) \quad \text{on} \quad A_1^c.\end{aligned}$$

Then $\sigma(1)$ is a simple 1-stopping time. Suppose that $\sigma(k-1)$ has been defined with $\sigma\leqq\sigma(k-1)_1\leqq K+1$, and that $P(A_1\cap\ldots\cap A_{k-1})>P(A)/2$. For every $t\in S$, set

$$\begin{aligned}A(k,t)=\Big\{\omega\colon &\sigma(\omega)<t_1<\inf\Big(\sigma(k-1)_1(\omega), \sigma(\omega)+\frac{1}{k}\Big), \exists s\in S,\\ &s_2=t_2\leqq M, \sigma(\omega)<s_1<t_1, |X_s(\omega)-X_t(\omega)|>\varepsilon\Big\}\cap A.\end{aligned}$$

Choose finitely many indices $t(i)$ in S, $1\leqq i\leqq n_k$, with $t(1)_1\leqq t(2)_1\leqq\ldots\leqq t(n_k)_1$, such that if $A_k=\bigcup_{i\leqq n_k} A(k,t(i))$, then $P(A_1\cap\ldots\cap A_{k-1}\cap A_k)>P(A)/2$. Then each set $A(k,t)\in\mathscr{F}_t^1\cap\mathscr{G}_{\sigma+(1/k)}\cap\mathscr{G}_{\sigma(k-1)_1}$. Set

$$\begin{aligned}&\sigma(k)=t(i) \quad \text{on} \quad A(k,t(i))\setminus\bigcup_{j<i} A(k,t(j)),\\ &\sigma(k)=\sigma(k-1) \quad \text{on} \quad A_k^c.\end{aligned}$$

Then $\sigma(k)$ is a simple 1-stopping time, and $\sigma \leqq \sigma(k)_1 \leqq K+1$. The sequence $\sigma(k)_1$ is clearly decreasing. Since $(\mathcal{G}_a)_{a\geqq 0}$ is right-continuous, the set $A_\infty = \liminf A_n \in \cap \mathcal{G}_{\sigma+(1/k)} = \mathcal{G}_\sigma$; furthermore $P(A_\infty) \geqq P(A)/2$. Set $\tau(\mathrm{k}) = \sigma(k)$ on A_∞, and $\tau(\mathrm{k}) = \left(\sigma + \frac{1}{\mathrm{k}}, 1\right)$ on A_∞^c. Clearly the $\tau(k)$ are bounded 1-stopping times such that $\lim \searrow \tau(\mathrm{k})_1 = \sigma$, and, for every k, $\sigma < \tau(k)_1$. It is easy to see that for every k the simple random variable $\tau(k)_2$ is $\mathcal{G}_{\tau(k)_1}$ measurable; furthermore, the right-continuity of the family $(\mathcal{G}_a)_{a\geqq 0}$ implies that $\cap \mathcal{G}_{\tau(k)_1} = \mathcal{G}_\sigma$, hence $\tau_2 = \liminf \tau(k)_2$ is $\mathcal{G}_\sigma$ measurable. Therefore $\tau = (\sigma, \tau_2)$ is a bounded 1-stopping time. Since $P(A_\infty) > 0$, at least one of the three $\mathcal{G}_\sigma$-measurable sets

$$C_1 = A_\infty \cap \limsup \{\tau(k)_2 = \tau_2\},$$
$$C_2 = A_\infty \cap \bigcap_{\varepsilon>0} [\limsup \{\tau_2 - \varepsilon < \tau(k)_2 < \tau_2\}],$$
$$C_3 = A_\infty \cap \bigcap_{\varepsilon>0} [\limsup \{\tau_2 < \tau(k)_2 < \tau_2 + \varepsilon\}],$$

the union of which is A_∞, is non-null. If $P(C_1) > 0$, then on C_1 the second coordinate of τ clearly takes on values in S_2. Since S_2 is countable, this contradicts the fact that for every fixed $b \in S_2$, almost every trajectory of the one-parameter descending amart $(X_{a,b}, \mathcal{F}^1_{a,b}, a \geqq 0)$ has right limits. Now suppose that $P(C_2) > 0$, and define a 1-stopping time ν by $\nu = \tau$ on C_2, $\nu = (K+1, 1)$ on C_2^c. Then for every $\omega \in C_2$, ω belongs to all but finitely many of the sets A_n, while for a subsequence $[\tau(k_n)](\omega)$ converging to $\tau(\omega)$, one has $[\tau(k_n)_2](\omega) < \nu_2(\omega)$. Therefore the net $\{X_t(\omega): t \in S, t \to \nu(\omega), t_1 > \nu_1(\omega), t_2 < \nu_2(\omega)\}$ fails to converge for each ω belonging to C_2. Apply Proposition 2.1 (2) to the 1-stopping time ν, let π be a 1-stopping time such that $P(\nu = \pi) > 1 - P(C_2)/2$, and let $\pi(n)$ be a sequence in $T^1(S)$ 1-recalling π in Q_{IV}, such that the two nets $\{X_{\pi(n)}(\omega): n \in \mathbb{N}\}$ and $\{X_t(\omega): t \to \pi(\omega), t \in S, t_1 > \pi_1(\omega), t_2 < \pi_2(\omega)\}$ have the same cluster point for almost all ω. Since the net $\{X_t(\omega): t \to \pi(\omega), t \in S, t_1 > \pi_1(\omega), t_2 < \pi_2(\omega)\}$ fails to converge on a set of measure at least $P(C_2)/2$, so does the net $(X_{\pi(n)}, n \in \mathbb{N})$. This brings a contradiction because the process $(Y_{-n} = X_{\pi(n)})_{n\in\mathbb{N}}$ is a descending discrete amart for the sequence of σ-algebras $\mathcal{H}_{-n} = \mathcal{G}_{\pi(n)_1}$, and hence converges a.s. In the case where $P(C_3) > 0$, one obtains a contradiction in a similar but easier way, applying Proposition 2.1 (1) directly to τ. □

The following Proposition is a global analog of Proposition 2.2, since it asserts the existence of horizontal limits a.s., uniformly in the second coordinate. The proof of this Proposition is similar to the proof of Theorem 7.2 (b), (b′) [9].

Proposition 2.3. *Let $(X_t)_{t\in\mathbb{R}^2_+}$ be a separable process with separant set S, and for every $a \geqq 0$, let $\mathcal{G}_a = \mathcal{F}_{a,\infty}$. Suppose that for every fixed $b \in S_2$, the one-parameter process $(X_{a,b}, a \geqq 0)$ is separable with separant set S_1. Fix $M \geqq 0$;*

(i) *Suppose that for every bounded one-dimensional predictable stopping time σ for $(\mathcal{G}_a)_{a\geqq 0}$, and for almost every ω, the nets $(X_{a,b}(\omega): a \to \sigma(\omega), a < \sigma(\omega))$ converge uniformly in $b \in S_2, b \leqq M$. Then for almost every ω the functions $a \mapsto X_{a,b}(\omega)$ have left limits uniformly in $b \in S_2$, $b \leqq M$.*

(ii) *Suppose that the family $(\mathcal{F}^1_t)_{t\in\mathbb{R}^2_+}$ is right-continuous. Suppose that for every bounded one-dimensional stopping time σ for $(\mathcal{G}_a)_{a\geqq 0}$, and for almost every ω,* *the nets* $(X_{a,b}(\omega): a \mapsto \sigma(\omega), a > \sigma(\omega))$ *converge uniformly in* $b \in S_2$, $b \leqq M$. *Then for almost every* ω *the functions* $a \mapsto X_{a,b}(\omega)$ *have right limits uniformly in* $b \in S_2$, $b \leqq M$.

Proof. For $a_1 < a_2$, set

$$Y(a_1, a_2, M) = \sup\{|X_{c,b} - X_{d,b}| : b \in S_2, b \leqq M, c \in S_1, d \in S_1, a_1 < c < d < a_2\}.$$

(i) Suppose that (i) fails, and for every $a > 0$, set

$$Z_a = \inf\{Y(a-\alpha, a, M): \alpha > 0\}.$$

The set

$$\bar{A} = \{\omega: \exists \bar{\varepsilon} > 0, \exists a > 0, Z_a(\omega) > \bar{\varepsilon}\}$$

is measurable, and by assumption $P(\bar{A}) > 0$. Replacing $\bar{A}$ by a non-null subset A with $P(A) = 2p > 0$, one may assume that $\bar{\varepsilon}$ is larger than a constant $\varepsilon > 0$ independent of ω, and a is less than a constant K independent of ω. Thus $A = \{\omega: \exists a, 0 < a < K, Z_a(\omega) > \varepsilon\}$. We now construct a bounded predictable stopping time σ for $(\mathscr{G}_a)_{a \geqq 0}$ such that the assumption (i) is contradicted. Set $\sigma_0 = 0$, and suppose that the stopping time σ_{k-1} for $(\mathscr{G}_a)_{a \geqq 0}$ has been defined, and that if

$$B_{k-1} = \{\omega: \exists a, \sigma_{k-1}(\omega) < a < K, Z_a(\omega) > \varepsilon\},$$

then $P(B_{k-1}) > p$. It is possible to define in terms of finitely many elements of $S_1 \times S_1$ a subset A_k of B_{k-1} of measure arbitrarily close to the measure of B_{k-1}. Choose finitely many pairs of elements (c_i, d_i) in $S_1 \times S_1$, $i = 1, \ldots, n_k$, such that $c_i < d_i < K$, and, setting

$$\begin{aligned} A_k = \{\omega: \exists i \leqq n_k, c_i > \sigma_{k-1}(\omega), \\ \sup(|X_{c_i,b}(\omega) - X_{d_i,b}(\omega)| : b \in S_2, b \leqq M) > \varepsilon, \exists a, d_i < a < K, Z_a(\omega) > \varepsilon\}, \end{aligned}$$

one has $P(A_k) > p$. Define σ_k by

$$\begin{aligned} \sigma_k = \inf\{d_i: 1 \leqq i \leqq n_k, \sigma_{k-1} < c_i, \\ \sup(|X_{c_i,b} - X_{d_i,b}| : b \in S_2, b \leqq M) > \varepsilon\} \wedge (K + 1 - 2^{-k}). \end{aligned}$$

Clearly σ_k is a simple one-dimensional stopping time for $(\mathscr{G}_a)_{a \geqq 0}$, $\sigma_{k-1} < \sigma_k < K + 1$, $B_k \supset A_k$, and $P(B_k) > p$. The bounded sequence (σ_n) of simple stopping times for $(\mathscr{G}_a)_{a \geqq 0}$ has been constructed by induction, so that the sequence of sets

$$C_n = \{\sup(|X_{\sigma_n,b} - X_{a,b}| : b \in S_2, b \leqq M, a \in S_1, \sigma_{n-1} < a < \sigma_n < K) > \varepsilon\}$$

is a decreasing sequence of sets each of probability at least p. Hence the existence of the predictable bounded stopping tine for $(\mathscr{G}_a)_{a \geqq 0}$ $\sigma = \lim \sigma_n$ contradicts the assumption (i).

(ii) The proof is similar. Suppose that the conclusion of (ii) fails, and construct by induction a bounded sequence (σ_n) of simple stopping times for $(\mathscr{G}_a)_{a \geqq 0}$ such that the existence of $\sigma = \lim \sigma_n$ contradicts the assumption in (ii). For every $a > 0$, set

$$U_a = \inf\{Y(a, a+\alpha, M): \alpha > 0\}.$$

The process $(U_a, \mathcal{G}_a, a \geqq 0)$ is measurable. Hence as in the proof of (i) choose $\varepsilon > 0$ and $K > 0$ such that, setting $A = \{\omega : \exists a,\ 0 < a < K,\ U_a(\omega) > \varepsilon\}$, one has $P(A) = 2p > 0$. Set $\sigma_0 = K$, and suppose that the simple 1-stopping time σ_{k-1} for $(\mathcal{G}_a)_{a \geqq 0}$ has been defined, and is such that if

$$B_{k-1} = \{\omega : \exists a, 0 < a < \sigma_{k-1}(\omega), U_a(\omega) > \varepsilon\},$$

then $P(B_{k-1}) > p$. Choose finitely many pairs of elements (c_i, d_i) in $S_1 \times S_1$, $i = 1, \ldots, n_k$, such that $c_i < d_i < K$, and if

$$A_k = \{\omega : \exists i \leqq n_k, d_i \leqq \sigma_{k-1}(\omega), \sup(|X_{c_i, b}(\omega) - X_{d_i, b}(\omega)| : b \in S_2, b \leqq M) > \varepsilon,$$
$$\exists a < c_i, U_a(\omega) > \varepsilon\},$$

then $P(A_k) > p$. Set

$$\sigma_k = \inf\{d_i : d_i \leqq \sigma_{k-1}, \sup(|X_{c_i, b} - X_{d_i, b}| : b \in S_2, b \leqq M) > \varepsilon,$$
$$\exists a < c_i, U_a > \varepsilon\} \wedge (K+1).$$

Then σ_k is a simple stopping time for $(\mathcal{G}_a)_{a \geqq 0}$, and $P(B_k) > p$. The sequence (σ_k) has been defined by induction so that the sequence of sets

$$C_k = \{\omega : \sup(|X_{\sigma_n, b}(\omega) - X_{a, b}(\omega)| : b \in S_2, b \leqq M, a \in S_1,$$
$$\sigma_n(\omega) < a < \sigma_{n-1}(\omega) \leqq K) > \varepsilon\}$$

is a decreasing sequence of sets each of probability at least p. Furthermore (σ_k) converges to the one-dimensional bounded stopping time σ for $(\mathcal{G}_a)_{a \geqq 0}$ by the right continuity of the family of σ-algebras $(\mathcal{G}_a)_{a \geqq 0}$. However the assumption in (ii) is clearly violated by σ, and this brings a contradiction. □

Combining the statements of the Propositions 2.2, 2.3 (ii), and the existence of limits for one-parameter amarts [9], one obtains that descending 1-amarts possess modifications which have limits in Q_{I} and Q_{IV}. The conditions (α) and (β) below assure right-continuity of one-parameter processes.

Theorem 2.4. *Suppose that $(X_t)_{t \in \mathbb{R}_+^2}$ is a descending 1-amart, and that $(\mathcal{F}_t^1)$ is right-continuous.*

(i) *Suppose that for every $a \geqq 0$, the one-parameter family $(\mathcal{F}_{a,b})_{b \geqq 0}$ is right-continuous, and the one-parameter process $(X_{a,b}, \mathcal{F}_{a,b}, b \geqq 0)$ is a descending amart. Suppose that either (α) or (β) holds.*

(α) *For every index $t = (t_1, t_2)$ and for every sequence $\tau(n)$ of simple one-dimensional stopping times for $(\mathcal{F}_{t_1, b})_{b \geqq 0}$, $t_2 = \lim \searrow \tau(n)$ implies $EX_t = \lim EX_{t_1, \tau(n)}$.*

(β) *For every index t and for every sequence $\tau(n)$ of simple one-dimensional stopping times for $(\mathcal{F}_{a, \infty})_{a \geqq 0}$, $t_1 = \lim \searrow \tau(n)$ implies $EX_t = \lim EX_{\tau(n), t_2}$.*

Then (X_t) has a modification almost every trajectory of which has limits in Q_{I}, i.e., *has right limits.*

(ii) *Suppose that for every $a \geqq 0$, the one-parameter process $(X_{a,b}, \mathcal{F}_{a,b}, b \geqq 0)$ is an ascending amart of class (AL). Suppose also that (β) holds. Then (X_t) has a modification almost every trajectory of which has limits in Q_{IV}.*

(iii) *If the assumptions* (i) *and* (ii) *are satisfied simultaneously, then the modification can be chosen to have limits a.s. both in Q_{I} and Q_{IV}.*

Proof. We at first prove the existence of a separable modification of (X_t) having the properties stated in Proposition 2.2. Suppose that the assumption (β) holds. Let (Y_t) be a separable modification of (X_t) with separant set S; we may and do assume that $S=S_1\times S_2$, where S_1 and S_2 are dense in $\mathbb{R}_+$. For every $a\in S_1$, choose a separable modification of the one-parameter process $(Y_{a,b})_{b\geqq 0}$, say $(V_{a,b})_{b\geqq 0}$, with separant set $S(a)$ which may and will be assumed to contain S_2. Denote by $(Y_t')_{t\in\mathbb{R}_+^2}$ the process defined by $Y_t'=V_t$ if $t_1\in S_1$, $Y_t'=Y_t$ otherwise. Since the processes $(Y_s)_{s\in S}$ and $(Y_s')_{s\in S}$ are indistinguishable, S separates the process $(Y_s', s\in(\mathbb{R}_+\setminus S_1)\times\mathbb{R}_+)$. Hence $S'=S_1\times(\bigcup_{a\in S_1} S(a))$ separates the process $(Y_t')_{t\in\mathbb{R}_+^2}$. Fix $b\in S_2'=\{s_2: \exists s=(s_1,s_2)\in S'\}$; Theorem 7.4 [9] applied to the one-parameter descending amart $(Y_{a,b}', \mathscr{F}_{a,b}^1, a\geqq 0)$ gives the existence of a right-continuous modification of this process, say $(W_{a,b})_{a\geqq 0}$. Define the process $(Z_t)_{t\in\mathbb{R}_+^2}$ by $Z_t=W_t$ if $t_2\in S_2'$, $Z_t=Y_t'$ otherwise. Then S' separates the process $(Z_t, t\in\mathbb{R}_+\times(\mathbb{R}_+\setminus S_2'))$, and since the processes $(W_{a,b})_{a\geqq 0}$ are right-continuous for every fixed $b\in S_2'$, the set $S_1'\times\{b\}$, which is dense in $\mathbb{R}_+\times\{b\}$, separates this process. Hence S' separates $(Z_t)_{t\in\mathbb{R}_+^2}$. Furthermore, for every $a\in S_1'$ $[b\in S_2']$, the processes $(Z_{a,b})_{b\geqq 0}$ $[(Z_{a,b})_{a\geqq 0}]$ are separable. Since (Z_t) is a modification of (X_t), the process (Z_t) has the same amart properties as the original process (X_t). So far we have assumed (β); under the alternate assumption (α) in (i), one obtains the existence of such a modification (Z_t) by exchanging the roles of the coordinates. Therefore, under either assumption (i) or (ii), one can apply Proposition 2.2, Proposition 2.3(ii) and Theorem 7.3 [9] about the existence of limits for one-parameter separable amarts. It clearly suffices to show the existence of limits in every square $[0,M[\times[0,M[$. Fix $M\geqq 0$; let N be a null set such that for every $\omega\notin N$, the functions $(a\mapsto Z_{a,b}(\omega))$ have right limits uniformly in $b\in S_2'$, $b\leqq M$. Fix $\omega\notin N$, $t\in\mathbb{R}_+^2$ with $t_2<M$. Given $\varepsilon>0$, choose $\alpha>0$ such that

$$\sup(|Z_{a,b}(\omega)-Z_{c,b}(\omega)|: b\in S_2', b\leqq M, (a,b)\in S', (c,b)\in S', \\ t_1<a<c<t_1+\alpha)<\varepsilon.$$

(i) Let $N'\supset N$ be a null set such that for every $\omega\notin N'$, and for every $a\in S_1'$, the functions $b\to Z_{a,b}(\omega)$ have right limits. Suppose $\omega\notin N'$; fix $a\in S_1'$ with $t_1<a<t_1+\alpha$, and choose $\beta>0$ such that $\beta<M-t_2$, and $\sup(|Z_{a,b}(\omega)-Z_{a,c}(\omega)|: t_2<b<c<t_2+\beta)<\varepsilon$. For any pair of indices $u,v\in S'\cap(]t_1,t_1+\alpha[\times]t_2,t_2+\beta[)$, one has

$$|Z_u(\omega)-Z_v(\omega)|\leqq|Z_u(\omega)-Z_{a,u_2}(\omega)|+|Z_v(\omega)-Z_{a,v_2}(\omega)| \\ +|Z_{a,u_2}(\omega)-Z_{a,v_2}(\omega)|\leqq 3\varepsilon.$$

This shows the existence of limits in Q_{I} along the elements of S' in each square $[0,M[^2$, and completes the proof of (i) since S' is a separant set for (Z_t). The proof if (ii) is similar. □

The following theorem, asserting the existence of right-continuous modifications of descending 1-amarts, is a two-parameter version of Theorem 7.4 [9].

Theorem 2.5. *Suppose that* $(\mathscr{F}_t^1)$ *is a right-continuous family of σ-algebras, and let* (X_t) *be a descending* 1*-amart.*

(i) *Suppose that for every* $a \geqq 0$, *the one-parameter family* $(\mathscr{F}_{a,b})_{b \geqq 0}$ *is right-continuous, and the one-parameter process* $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ *is a descending amart. Suppose that for every* $t \in \mathbb{R}_+^2$, *and every sequence* $\tau(n)$ *of simple 1-stopping times 1-decreasing to t in* Q_{I}, *one has* $\lim EX_{\tau(n)} = EX_t$. *Then* (X_t) *has a modification every trajectory of which is continuous in* Q_{I}, *i.e., right-continuous.*

(ii) *Suppose that for every* $a \geqq 0$, *the one-parameter process* $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ *is an ascending amart of class (AL). Suppose that for every* $t \in \mathbb{R}_+^2$, *and every sequence* $\tau(n)$ *of simple 1-stopping times 1-decreasing to t in* Q_{IV}, *one has* $\lim EX_{\tau(n)} = EX_t$. *Then* (X_t) *has a modification every trajectory of which is continuous in* Q_{IV}.

Proof. (i) By Theorem 2.4(i), the process (X_t) has a modification (Y_t) having limits in Q_{I} a.s. Define $Z_t = \lim(Y_s, s \gg t)$; it is easy to see that $Z_t = \lim(Z_s, s \geqq t)$. To prove that (Z_t) is a modification of (X_t), it suffices to prove that for every t, $Z_t = Y_t = X_t$ a.s. Fix t, let $A \in \mathscr{F}_t^1$, and for every $n \geqq 0$, define $\tau(-n) = t$ on A^c, $\tau(-n) = \left(t_1 + \frac{1}{n}, t_2 + \frac{1}{n}\right)$ on A. Then $\lim EY_{\tau(-n)} = \lim EX_{\tau(-n)} = EX_t = EY_t$, and it is easy to see that $(Y_{\tau(n)}, \mathscr{F}_{\tau(n)}^1, n \in -\mathbb{N})$ is a descending amart, and hence is uniformly integrable [9, Theorem 2.9]. Since $\lim Y_{\tau(-n)} = Y_t 1_{A^c} + Z_t 1_A$ a.s., one has $E[1_A Z_t] = E[1_A Y_t]$. Since (Y_t) and (Z_t) are $\mathscr{F}_t^1$-measurable, it follows that (Z_t) is a modification of (X_t). This concludes the proof of (i).

The proof of (ii) is similar. □

Corollary 2.6. *Suppose that* $(\mathscr{F}_t^1)$ *is right-continuous, and let* (X_t) *be a descending 1-amart. Assume that for every* $a \geqq 0$, *the family* $(\mathscr{F}_{a,b})_{b \geqq 0}$ *is right-continuous, and the process* $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ *is both a descending amart and an ascending amart of class (AL).*

(i) *Suppose that for every t and every sequence* $\tau(n)$ *in* T^1 *1-decreasing to t in* Q_{I}, $\lim EX_{\tau(n)} = EX_t$. *Then* (X_t) *has a modification* (Y_t) *continuous in* Q_{I} *and such that for almost every ω,* $\lim(Y_s\colon s_1 \geqq t_1, s_2 < t_2)$ *exists for every t with* $t_2 > 0$.

(ii) *Suppose that for every t and every sequence* $\tau(n)$ *in* T^1 *1-decreasing to in* Q_{IV}, $\lim EX_{\tau(n)} = EX_t$. *Then* (X_t) *has a modification* (Y_t) *continuous in* Q_{IV}, *such that for almost every ω,* $\lim(Y_s\colon s_1 \geqq t_1, s_2 > t_2)$ *exists for every t.*

Corollary 2.7. *Suppose that* $(\mathscr{F}_t^1)$ *and* $(\mathscr{F}_t^2)$ *are right-continuous. Let* (X_t) *be an integrable process having simultaneously the descending 1- and 2-amart properties. Suppose that for every* $a \geqq 0$, *both processes* $(X_{a,t}, \mathscr{F}_{a,t}, t \geqq 0)$ *and* $(X_{t,a}, \mathscr{F}_{t,a}, t \geqq 0)$ *are ascending amarts of class (AL). Then* (X_t) *has a modification* (Y_t) *such that for almost every ω,* $(Y_t(\omega))$ *has limits in* $Q_{\mathrm{I}}, Q_{\mathrm{II}}$, *and* Q_{IV}. *If furthermore for either* $i = 1$ *or* $i = 2$, $\lim EX_{\tau(n)} = EX_t$ *for every t and every sequence* $\tau(n)$ *in* T^i *i-decreasing to t in* Q_{I}, *then* (X_t) *has a modification* (Y_t) *continuous in* Q_{I} *and having the following limits for almost every ω and every t:* $\lim(Y_s(\omega)\colon s_1 \geqq t_1,\ s_2 < t_2)$, *and* $\lim(Y_s(\omega)\colon s_1 < t_1,\ s_2 \geqq t_2)$.

3. Continuous Parameter Amarts: The Ascending Case

Finally, we study the existence of limits in Q_{III}. We assume in this section that the families $(\mathscr{F}_t)$, $(\mathscr{F}_t^1)$ and $(\mathscr{F}_t^2)$ are right-continuous. Since the third quadrant corresponds to the "ascending" case for both the first and the second coordinate, the methods of this paper will require more stringent assumptions on the process (X_t).

A sequence $\tau(n)$ of 1-stopping times *1-increases* to τ if $\lim \tau(n)=\tau$, and the sequence $\tau(n)_1$ increases. A sequence $\tau(n)$ *1-increases to* τ *in* $Q_{\mathrm{III}}[Q_{\mathrm{II}}]$ if $\tau(n)$ 1-increases to τ, and $\tau(n)\leqq\tau[\tau(n)_2\geqq\tau_2]$ for every n. A sequence $\tau(n)$ of 1-stopping times *1-announces* τ if $\tau(n)$ 1-increases to τ, and if $\tau(n)_1<\tau_1$ on $\{\tau_1>0\}$ for every n. A sequence $\tau(n)$ *1-announces* τ *in* $Q_{\mathrm{III}}[Q_{\mathrm{II}}]$ if $\tau(n)$ 1-announces τ, and if for every n, $\tau(n)\ll\tau$ on the set $\{(0,0)\ll\tau\}$ $[\tau(n)_2>\tau_2$ on the set $\{\tau_1>0\}]$. A stopping time τ is *1-predictable* [*in* Q_{III}/*in* Q_{II}] if it can be 1-announced [in Q_{III}/in Q_{II}] by some sequence $\tau(n)$; it is easy to see that a 1-stopping time 1-predictable in $Q_{\mathrm{III}}[Q_{\mathrm{II}}]$ can be announced by a sequence $\tau(n)$ such that $\tau(n)_2$ is monotone increasing [decreasing]. The definitions for 2-stopping times are obtained by exchanging everywhere the roles of the first and second coordinate. Thus one defines sequences $\tau(n)$ of 2-stopping times 2-increasing to τ or 2-announcing τ [in Q_{III}/in Q_{IV}].

The following proposition is a left analog of Proposition 2.1, and a two-parameter analog of Proposition 7.1(c) [9].

Proposition 3.1. *Let* (X_t) *be an adapted process, let* S *be a separant set, and let* τ *be a bounded 1-stopping time.*

(1) *Suppose that* $\tau_1>0$, *and that* τ *is 1-predictable in* Q_{II}. *Then given any number* $\varepsilon>0$, *there exists a bounded 1-predictable 1*-stopping time σ with $P(\sigma=\tau)\geqq 1-\varepsilon$, $\sigma\geqq\tau$, and a bounded sequence $\sigma(n)$ in $T^1(S)$ which 1-announces σ (not necessarily in Q_{II}), such that the set of cluster points of the net $\{X_t(\omega): t\to\sigma(\omega),\ t_1<\sigma_1(\omega),\ t_2>\sigma_2(\omega)\}$ is contained in the set of cluster points of the net $\{X_{\sigma(n)}(\omega): n\in\mathbb{N}\}$ for almost all ω. If τ is 1-announced in Q_{II} by a sequence $\tau(n)$ in $T^1(S)$, then σ can be chosen equal τ.

(2) Suppose that $\tau\gg(0,0)$ is 1-predictable in Q_{III}. Then given any $\varepsilon>0$, there exists a bounded 1-predictable 1-stopping time σ with $\sigma\geqq\tau$, $P(\sigma=\tau)\geqq 1-\varepsilon$, and there exists a bounded sequence $\sigma(n)$ in $T^1(S)$ which 1-announces σ, such that the set of cluster points of the net $\{X_t(\omega): t\to\sigma(\omega),\ t\ll\sigma(\omega)\}$ is contained in the set of cluster points of the net $\{X_{\sigma(n)}(\omega): n\in\mathbb{N}\}$ for almost all ω. If τ is 1-announced in Q_{III} by a sequence $\tau(n)$ in $T^1(S)$, then σ can be chosen equal τ.

Proof. For every $a\geqq 0$, set $\mathscr{G}_a=\mathscr{F}_{a,\infty}=\mathscr{F}^1_{a,b}$, $\forall b\geqq 0$. For every $n\geqq 0$, set $D(n)=\{i2^{-n}: i\geqq 0\}$, and let $D=\bigcup_{n\geqq 0} D(n)$ be the set of positive dyadic numbers. In order to simplify the proof, we will assume in the argument below that S contains $D\times D$.

(1) Suppose that $\tau_1>0$, and τ is 1-announced in Q_{II} by a bounded sequence $\tau(n)$ such that $\lim\searrow\tau(n)_2=\tau_2$. Let (M,M) be an upper bound of τ and $\tau(n)$. One first replaces in several steps $\tau(n)$ by elements of $T^1(S)$. For every fixed n, the map $\tau(n)_2$ is clearly $\mathscr{G}_{\tau(n)_1}$-measurable. Hence replacing $\tau(n)_2$ by its n-th dyadic approximation from above, we may and do assume that $\tau(n)_2$ is a simple $\mathscr{G}_{\tau(n)_1}$-measurable random variable taking on values in $D(n)$. Also $\tau(n)_1$ announces the one-dimensional stopping time τ_1 for $(\mathscr{G}_a)_{a\geqq 0}$. Therefore, τ_1 is announced by an increasing sequence $\alpha(n)$ of discrete dyadic (not necessarily simple) stopping

times for $(\mathscr{G}_a)_{a\geqq 0}$ (see e.g. [7], p. 132). Fix $\varepsilon>0$; for every $n\geqq 1$, choose an integer N_n such that $P[\alpha(n)\in D(N_n)]>1-\varepsilon\cdot 2^{-n}$, and set

$$\beta(n)=\alpha(n) \quad \text{on} \quad \bigcap_{i\leqq n}\{\alpha(i)\in D(N_i)\},$$
$$\beta(n)=M+1-2^{-n} \qquad \text{otherwise.}$$

The maps $\beta(n)$ are simple 1-stopping times for $(\mathscr{G}_a)_{a\geqq 0}$, and announce a stopping time β such that $P[\beta=\tau_1]>1-\varepsilon$. For every n, for every ω such that $\tau(1)_1(\omega)<\beta(n)(\omega)\leqq M$, set $\gamma(n)(\omega)=\tau(i_n)_2(\omega)$, where $i_n=\sup\{j\colon \tau(j)_1(\omega)<\beta(n)(\omega)\}$. On $\{\tau(1)_1<\beta(n)\leqq M\}^c$, set $\gamma(n)=M+1-2^{-n}$. Then $\gamma(n)$ is a discrete (not necessarily simple) dyadic map. We now check that $\gamma(n)$ is $\mathscr{G}_{\beta(n)}$-measurable. For every $b\leqq M$, for every $c<M+1$,

$$\{\beta(n)=b\}\cap\{\gamma(n)=c\}=\sum_{j\geqq 1}\{\beta(n)=b\}\cap\{\tau(j)_1<b\}\cap\{\tau(j+1)_1\geqq b\}\cap\{\tau(j)_2=c\},$$

hence $\{\beta(n)=b\}\cap\{\gamma(n)=c\}\in\mathscr{G}_b$.

For every $b\leqq M$,

$$\{\beta(n)=b\}\cap\{\gamma(n)=M+1+2^{-n}\}=\{\tau(1)_1\geqq b\}\in\mathscr{G}_b.$$

Finally,

$$\{\beta(n)=M+1-2^{-n}\}\cap\{\gamma(n)=M+1+2^{-n}\}=\{\beta(n)=M+1-2^{-n}\}\in\mathscr{G}_{M+1-2^{-n}},$$

which concludes the proof that $\gamma(n)$ is $\mathscr{G}_{\beta(n)}$-measurable. For every n, keep a finite subset $R(n)$ of the range of $\gamma(n)$ such that $P[\gamma(n)\in R(n)]\geqq 1-\varepsilon\cdot 2^{-n}$. Set

$$\pi(n)=(\beta(n),\gamma(n)) \quad \text{on} \quad \bigcap_{i\leqq n}\{\gamma(i)\in R(i)\},$$
$$\pi(n)=(M+1-2^{-n}, M+1+2^{-n}) \qquad \text{otherwise.}$$

Then $\pi(n)\in T^1(D\times D)\subset T^1(S)$, $\pi(n)$ 1-announces in Q_{II} a bounded 1-stopping time π such that $\pi\geqq\tau$, and $P(\pi=\tau)\geqq 1-2\varepsilon$.

We now have to change the sequence $\pi(n)$ to a sequence of elements of $T^1(S)$ having the required cluster points. Replacing $\pi(n)$ by a subsequence, we may and do assume that if $F_n=\bigcup_{k\geqq n}\{\pi(k)_2-\pi_2>2^{-n}\}$, then $P(F_n)<\varepsilon\cdot 2^{-n}$. Choose a finite subset of S, say $\{t(1,1),\ldots,t(1,r_1)\}$, such that, setting

$$\begin{aligned}A_1=\{\omega\colon \forall t \text{ with } &\pi(1)_1(\omega)<t_1<\pi_1(\omega) \text{ and } \pi(1)_2(\omega)>t_2>\pi_2(\omega),\\ &\text{there exists } i \text{ with } 1\leqq i\leqq r_1 \text{ such that } \pi(1)_1(\omega)<t(1,i)_1<\pi_1(\omega),\\ &\pi(1)_2(\omega)>t(1,i)_2>\pi_2(\omega), \text{ and } |X_t(\omega)-X_{t(1,i)}(\omega)|<2^{-1}\},\end{aligned}$$

one has $P(A_1)>1-\varepsilon\cdot 2^{-1}$. Choose an integer $n_2\geqq 2$ such that, if

$$C_1=\bigcup_{1\leqq i\leqq r_1}(\{t(1,i)_1<\pi_1\}\setminus\{t(1,i)_1<\pi(n_2)_1\}),$$

then $P(C_1)<\varepsilon\cdot 2^{-1}$. For every i with $1\leqq i\leqq r_1$, set

$$B(1,i)=\{\pi(1)_1<t(1,i)_1<\pi(n_2)_1\}\cap\{\pi(1)_2-2^{-1}<t(1,i)_2<\pi(1)_2\};$$

since $\pi(1)_2$ is $\mathscr{G}_{\pi(1)_1}$-measurable, and since $\pi(1)_1$ and $\pi(n_2)_1$ are one-parameter stopping times for $(\mathscr{G}_a)_{a \geqq 0}$, the sets $B(1, i)$ are $\mathscr{G}_{t(1,i)_1}$-measurable. For every i with $1 \leqq i \leqq r_1$, define an element $v(1, i)$ of $T^1(S)$ as follows:

$$v(1,i)=t(1,i) \quad \text{on} \quad B(1,i).$$
$$v(1,i)=\pi(n_2) \quad \text{on} \quad B(1,i)^c.$$

We now order the 1-stopping times $v(1,i)$, $1 \leqq i \leqq r_1$, by increasing values of the first coordinates, thus obtaining r_1 elements of $T^1(S)$, say $\sigma(i)$, $1 \leqq i \leqq r_1$, such that $\sigma(i)_1$ increases, and $\sigma(r_1)_1 \leqq \pi(n_2)_1$. The construction of the sequence $\sigma(n)$ is continued by induction as follows: Suppose that the stopping times $\sigma(i)$ such that $\sigma(i)_1$ increases, have been defined for $1 \leqq i \leqq r_1 + \ldots + r_{k-1}$, and let n_k be an integer larger than k such that

$$\sigma(r_1 + \ldots + r_{k-1})_1 \leqq \pi(n_k)_1.$$

Choose a finite subset of S, say $\{t(k,1), \ldots, t(k,r_k)\}$, such that, setting

$$\begin{aligned} A_k = \{\omega\colon &\forall t \text{ with } \pi(n_k)_1(\omega) < t_1 < \pi_1(\omega) \text{ and } \pi(n_k)_2(\omega) > t_2 > \pi_2(\omega),\\ &\text{there exists } i \text{ with } 1 \leqq i \leqq r_k \text{ such that } \pi(n_k)_1(\omega) < t(k,i)_1 < \pi_1(\omega),\\ &\pi(n_k)_2(\omega) \geqq t(k,i)_2 > \pi_2(\omega), \text{ and } |X_t(\omega) - X_{t(k,i)}(\omega)| < 2^{-k}\},\end{aligned}$$

one has $P(A_k) > 1 - \varepsilon \cdot 2^{-k}$. Choose an integer $n_{k+1} \geqq k+1$, $n_{k+1} \geqq n_k$, such that if

$$C_k = \bigcup_{1 \leqq i \leqq r_k} (\{t(k,i)_1 < \pi_1\} \setminus \{t(k,i)_1 < \pi(n_{k+1})_1\}),$$

then $P(C_k) < \varepsilon \cdot 2^{-k}$. For every i with $1 \leqq i \leqq r_k$, set

$$B(k,i) = \{\pi(n_k)_1 < t(k,i)_1 < \pi(n_{k+1})_1\} \cap \{\pi(n_k)_2 - 2^{-k} < t(k,i)_2 < \pi(n_k)_2\}.$$

Then $B(k,i) \in \mathscr{F}^1_{t(k,i)} \cap \mathscr{F}^1_{\pi(n_{k+1})}$. Define $v(k,i)$ in $T^1(S)$ by:

$$v(k,i) = t(k,i) \quad \text{on} \quad B(k,i),$$
$$v(k,i) = \pi(n_{k+1}) \quad \text{on} \quad B(k,i)^c.$$

Order the 1-stopping times $v(k,i)$ by increasing values of the first coordinates, thus obtaining r_k elements of $T^1(S)$, say $\sigma(j)$ with $r_1 + \ldots + r_{k-1} + 1 \leqq j \leqq r_1 + \ldots + r_k$. The finite sequence $\sigma(j)$, $1 \leqq j \leqq r_1 + \ldots + r_k$ clearly satisfies the induction hypothesis. The sequence $(\sigma(n))_{n=1}^{\infty}$ converges to π. The cluster points of the nets $\{X_t(\omega)\colon t_1 < \pi_1(\omega),\ t_2 > \pi_2(\omega),\ t \to \pi(\omega)\}$ also are cluster points of the net $\{X_{\sigma(n)}(\omega)\colon n \in \mathbb{N}\}$ if ω belongs to the set $C = \liminf(A_n \cap C_n^c \cap F_n^c)$; since $P(C) = 1$ this completes the proof of (1).

The second part of the proposition is proved in a similar way, only changing the inequalities and monotonicity for the second coordinate. □

An integrable process $(X_t)_{t \in \mathbb{R}^2_+}$ is an *ascending 1-amart* [of *class AL*] if for every bounded sequence $\tau(n)$ of simple 1-stopping times such that $\tau(n)$ converges to a 1-stopping τ, $\lim \nearrow \tau(n)_1 = \tau_1$, and $\tau(n)_1 < \tau_1$ for every n, one has that

$\lim EX_{\tau(n)}$ exists [and $\sup_n E|X_{\tau(n)}| < \infty$]. Similar definitions are given for ascending 2-amarts. The following proposition is a left analog of Proposition 2.2.

Proposition 3.2. *Let $(X_t)_{t\in\mathbb{R}^2_+}$ be a separable ascending 1-amart of class (AL), with separant set $S=S_1\times S_2$. Suppose that for every fixed $b\in S_2$, the one-parameter process $(X_{a,b}, \mathscr{F}^1_{a,b}, a\geqq 0)$ is separable. Fix $M\geqq 0$ and a bounded, predictable one-parameter stopping time σ for $(\mathscr{F}_{a,\infty})_{a\geqq 0}$. Then for almost all ω, $\lim(X_{a,b}(\omega)$: $a<\sigma(\omega)$, $a\to\sigma(\omega))$ exists uniformly in $b\in S_2$, $b\leqq M$.*

Proof. Enlarging S if necessary, one may assume that for every $b\in S_2$, $\{(a,b)\in S$: $a\geqq 0\}$ is a separant set for the process $(X_{a,b}, a\geqq 0)$. For every $a\geqq 0$, set $\mathscr{G}_a=\mathscr{F}_{a,\infty}$, let K be an upper bound for σ, and let $\sigma(n)$ be a sequence of stopping times for $(\mathscr{G}_a)_{a\geqq 0}$ 1-announcing σ. Suppose that the stated uniform convergence fails to hold for almost all ω. Let $\varepsilon>0$, and let A be a non-null measurable set such that for every $\omega\in A$ and for every $\alpha>0$, there exist $s,t\in S$ with $s_2=t_2\leqq M$, $\sigma(\omega)-\alpha<s_1<t_1<\sigma(\omega)$, and $|X_s(\omega)-X_t(\omega)|>\varepsilon$. Denote by S_1 the set of the first coordinates of the elements of S. For every $t\in S$, set

$$A(1,t)=\{\omega\colon t_1<\sigma(\omega), \exists s\in S, \text{ such that } s_2=t_2\leqq M, [\sigma(1)](\omega)<s_1<t_1, \\ |X_s(\omega)-X_t(\omega)|>\varepsilon\};$$

then $A(1,t)\in\mathscr{F}^1_t, \forall t$. Choose finitely many indices $t(i)$ in S, $1\leqq i\leqq n_1$, with $t(1)_1\leqq t(2)_1\leqq\ldots\leqq t(n_1)_1$, such that if $A_1=\bigcup_{i\leqq n_1} A(1,t(i))$, then $P(A_1)>P(A)/2$. Set

$$\tau(1)=t(i) \quad \text{on} \quad A(1,t(i))\setminus\bigcup_{j<i} A(1,t(j)),$$
$$\tau(1)=(K+\tfrac{1}{2},0) \quad \text{on} \quad A_1^c.$$

Then $\tau(1)$ is a 1-stopping time, and $P[\tau(1)_1<\sigma]>P(A)/2$. Suppose that the simple 1-stopping time $\tau(k-1)$ has been defined, and that $P[\tau(k-1)_1<\sigma]>P(A)/2$. For every $t\in S$, set

$$A(k,t)=\{\omega\colon t_1<\sigma(\omega), \exists s\in S, s_2=t_2\leqq M, \\ [\sigma(k)](\omega)\vee[\tau(k-1)_1](\omega)<s_1<t_1, |X_s(\omega)-X_t(\omega)|>\varepsilon\}.$$

Choose finitely many indices $t(i)$ in S, $1\leqq i\leqq n_k$, with $t(1)_1\leqq t(2)_1\leqq\ldots\leqq t(n_k)_1$, such that if $A_k=\bigcup_{i\leqq n_k} A(k,t(i))$, then $P(A_k)>P(A)/2$. Set

$$\tau(k)=t(i) \quad \text{on} \quad A(k,t(i))\setminus\bigcup_{j<i} A(k,t(j)),$$
$$\tau(k)=(K+1-2^{-k},0) \quad \text{on} \quad A_k^c.$$

Then $\tau(k)$, $k\geqq 1$ is a bounded sequence of simple 1-stopping times such that $\tau(k)_1\nearrow\sigma$ on $A_\infty=\bigcap A_k$.

Set $\tau_2=\limsup\tau(k)_2$, and set $\tau=(\sigma,\tau_2)$. Each random variable $\tau(k)_2$ is clearly $\mathscr{G}_\sigma$-measurable, and τ is a 1-stopping time. It is easy to see that A_∞ and τ_2 are measurable with respect to $\vee \mathscr{G}_{\tau(k)_1}$, and $P(A_\infty) \geqq P(A)/2$. At least one of the three sets

$$C_1 = A_\infty \cap \limsup \{\tau(k)_2 = \tau_2\},$$
$$C_2 = A_\infty \cap \bigcap_{\varepsilon > 0} [\limsup \{\tau_2 < \tau(k)_2 < \tau_2 + \varepsilon\}],$$
$$C_3 = A_\infty \cap \bigcap_{\varepsilon > 0} [\limsup \{\tau_2 - \varepsilon < \tau(k)_2 < \tau_2\}],$$

the union of which is A_∞, is non-null. The ascending 1-amart property of (X_t) implies that for every $b \in S_2$, and for every increasing sequence of one-dimensional stopping times $\beta(n)$ for $(\mathscr{G}_a)_{a \geqq 0}$ which announces some stopping time β, the sequence $(X_{\beta(n),b}, \mathscr{F}^1_{\beta(n),b}, n \in \mathbb{N})$ is an L^1-bounded ascending amart, and hence converges a.s. Since S_2 is countable, the definition of A_∞ clearly implies that $P(C_1) = 0$. Now suppose that $P(C_2) = 2\alpha > 0$, and choose an integer n_1 such that if $D_1 = C_2 \cap (\bigcup_{j \leqq n_1} \{\tau(j)_2 > \tau_2\})$, then $P(D_1) > \alpha$. Since $D_1 \in \vee \mathscr{G}_{\tau(k)_1}$, there exists $m_1 \geqq n_1$ and a set $F_1 \in \mathscr{G}_{\tau(m_1)_1}$, such that $P(D_1 \Delta F_1) < [P(D_1) - \alpha] \wedge 2^{-1}$. Then $P(F_1 \cap C_2) > \alpha$. Set

$$v(1) = (\tau(m_1)_1, \sup_{j \leqq n_1} \tau(j)_2) \quad \text{on} \quad F_1,$$
$$v(1) = (K + 2 - \tfrac{1}{2}, 2^{-1}) \quad \text{on} \quad F_1^c.$$

Suppose that the integers m_i, n_i, the sets F_i, and the stopping times $v(i)$ have been defined for $i \leqq k-1$. Suppose that $m_i \geqq n_i > m_{i-1}$, and $P[C_2 \cap F_1 \cap \ldots \cap F_{k-1}] > \alpha$. Choose an index $n_k > m_{k-1}$ such that if

$$D_k = C_2 \cap (\bigcap_{i \leqq k-1} F_i) \cap (\bigcup_{m_{k-1} < j \leqq n_k} \{\tau(j)_2 > \tau_2\}),$$

then $P(D_k) > \alpha$. Since $D_k \in \vee \mathscr{G}_{\tau(k)_1}$, there exists $m_k \geqq n_k$, and a set $F_k \in \mathscr{G}_{\tau(m_k)_1}$, such that $P(D_k \Delta F_k) < [P(D_k) - \alpha] \wedge 2^{-k}$. Then $P[(\bigcap_{i \leqq k} F_k) \cap C_2] > \alpha$. Set

$$v(k) = (\tau(m_k)_1, \sup_{m_{k-1} < j \leqq n_k} \tau(j)_2) \quad \text{on} \quad \bigcap_{i \leqq k} F_i,$$
$$v(k) = (K + 2 - 2^{-k}, 2^{-k}) \quad \text{on} \quad (\bigcap_{i \leqq k} F_i)^c.$$

Set $F = \bigcap_k F_k$; then $P(F \cap C_2) \geqq \alpha > 0$, and since $\sum P(D_k \Delta F_k) < \infty$, the Borel-Cantelli lemma implies that almost every ω belonging to F also belongs to $\liminf D_n$. Hence for almost every ω in F, we have $\liminf v(k)_2(\omega) \geqq \tau_2(\omega)$. Since $\limsup v(k)_2 \leqq \limsup \tau(n)_2 = \tau_2$, the sequence $v(k)$ converges to τ on F. Clearly $v(k)$ converges to $(K+2, 0)$ on F^c. Set $v = \lim v(k)$; then for almost every ω, there exists an integer $k(\omega)$ such that for every $n \geqq k(\omega)$, $[v(n)_2](\omega) > v_2(\omega)$. This implies that if $B_k = F \cap C_2 \cap (\bigcap_{n \geqq k} \{v(n)_2 > v_2\})$, then $\lim \nearrow B_k = F \cap C_2$. Choose $m > 0$ such that $P(B_m) > p > 0$. Since B_m is in $\vee \mathscr{G}_{v(k)_1}$, we can choose $n \geqq m$ and $B \in \mathscr{G}_{v(n)_1}$ such that $P[B_m \Delta B] < P(B_m) - p$. Set $i_1 = n$, $\rho(1) = v(i_1)$ on B, $\rho(1) = (K + 3 - 2^{-1}, 2^{-1})$ on B^c. Suppose that the integer i_{k-1} and the 1-stopping time $\rho(k-1)$ have been defined, and that $P[\rho(k-1) \leqq (K+2, M)] > p$. For every fixed ω, if

$[\rho(k-1)_1](\omega)=[v(i_{k-1})_1](\omega)$, let $i_k=\inf\{j>i_{k-1}: [v(j)_2](\omega)<[v(i_{k-1})_2](\omega)\}$ (where $\inf\emptyset=+\infty$), and if $[\rho(k-1)](\omega)>(K+2,0)$, set $i_k=+\infty$. If $i_k<\infty$, set $[\rho(k)](\omega)=[v(i_k)](\omega)$; if $i_k=+\infty$, set $[\rho(k)](\omega)=(K+3-2^{-k},2^{-k})$. Then $\rho(k)$ is a discrete 1-stopping time. It is easy to see that $\rho(k)$ 1-announces a 1-stopping time ρ in Q_{II}, that $\rho=v$ on $B\cap B_m$, and that $P[F\cap C_2\cap\{\rho=\tau\}]\geqq p>0$. Since $F\subset\liminf D_n$, the net $\{X_t(\omega): t\to\rho(\omega),\ t_1<\rho_1(\omega),\ t_2>\rho_2(\omega)\}$ fails to converge for every ω in $F\cap C_2\cap\{\rho=\tau\}$. Applying Proposition 3.1(1), let π be a predictable 1-stopping time such that $P(\pi=\rho)>1-\frac{p}{2}$, and let $\pi(n)$ be a bounded sequence in $T^1(S)$ 1-announcing π, and such that the set of cluster points of the net $\{X_t(\omega): t\to\pi(\omega),\ t_1<\pi_1(\omega),\ t_2>\pi_2(\omega)\}$ is contained in the set of cluster points of the net $\{X_{\pi(n)}(\omega): n\geqq 1\}$ for almost every ω. Then the L^1-bounded ascending amart $(X_{\pi(n)}, \mathscr{F}^1_{\pi(n)}, n\in\mathbb{N})$ does not converge on a set of probability larger than $p/2$, which brings a contradiction. Hence $P(C_2)=0$, and for almost every $\omega\in A_\infty$, there exists an integer $N(\omega)$ such that for every $n\geqq N(\omega)$, one has $[\tau(n)_2](\omega)<\tau_2(\omega)$. This implies that if $G_k=A_\infty\cap(\bigcap_{n\geqq k}\{\tau(n)_2<\tau_2\})$, then $\lim\nearrow G_k=A_\infty$. Choose $n>0$ such that $P(G_n)>q>0$. An argument similar to the one given above for B_k and v_k now shows that one can construct a sequence of discrete 1-stopping times $\rho'(k)$ which 1-announces a 1-stopping time ρ' in Q_{III}, and such that the net $\{X_t(\omega): t\to\rho'(\omega),\ t\ll\rho'(\omega)\}$ fails to converge for all points ω that belong to a subset of A_∞ of probability at least q.

Hence Proposition 3.1(2) and the ascending 1-amart property of the process (X_t) bring a contradiction (by an argument similar to the one given above to show that $P(C_2)=0$). □

Applying Propositions 2.3(i) and 3.2, we obtain the following:

Proposition 3.3. *Let $(X_t)_{t\in\mathbb{R}^2_+}$ be a separable ascending amart of class (AL) with separant set S. Suppose that for every fixed $b\in S_2$, the one parameter process $(X_{a,b}: a\geqq 0)$ is separable with separant set S_1. Fix $M>0$; then for almost every ω the functions $a\mapsto X_{a,b}(\omega)$ have left limits uniformly in $b\in S_2$, $b\leqq M$.*

The following theorems state the existence of left-limited modifications for some classes of amarts. The proof is similar to the proof of Theorem 2.4, and will be omitted.

Theorem 3.4. *Let $(X_t)_{t\in\mathbb{R}^2_+}$ be an ascending 1-amart of class* (AL), *such that for every index t and for every sequence $\tau(n)$ of simple one-dimensional stopping times for $(\mathscr{F}_{t_1,b})_{b\geqq 0}$, $t_2=\lim\searrow\tau(n)$ implies $EX_t=\lim EX_{t_1,\tau(n)}$. Then (X_t) has a modification almost every trajectory of which has limits in Q_{II}. If furthermore for every fixed $a\geqq 0$, the one-parameter process $(X_{a,b}, \mathscr{F}_{a,b}, b\geqq 0)$ is an ascending amart of class* (AL), *then the modification of (X_t) can be chosen to have limits in Q_{II} and Q_{III}.*

We say that a process has *limits in the left half-plane* $Q_{\mathrm{II}}\cup Q_{\mathrm{III}}$ [is *continuous in the left half-plane*] if for every index t, $\lim(X_s: s\to t,\ s_1<t_1)$ exists [$\lim(X_s: s\to t, s_1\leqq t_1)=X_t$]. Similar definitions are given for the *right half-plane* $Q_{\mathrm{I}}\cup Q_{\mathrm{IV}}$, the *upper half-plane* $Q_{\mathrm{I}}\cup Q_{\mathrm{II}}$, and the *lower half-plane* $Q_{\mathrm{III}}\cup Q_{\mathrm{IV}}$.

Theorem 3.5. *Let $(X_t)_{t\in\mathbb{R}_+^2}$ be an ascending 1-amart of class* (AL), *such that for every $a \geqq 0$, the one-parameter process $(X_{a,b}: b \geqq 0)$ has a continuous modification. Then (X_t) has a modification almost every trajectory of which has limits in the left half-plane.*

Proof. As in the proof of Theorem 2.4, let (U_t) be a separable modification of (X_t) with separant set $S = S_1 \times S_2$, where S_1 and S_2 are dense in $\mathbb{R}_+$. For every $b \in S_2$, choose a separable modification of the one-parameter process $(U_{a,b})_{a\geqq 0}$, say $(V_{a,b})_{a\geqq 0}$ with separant set $S(b) \supset S_1$. Let (U'_t) be the process defined by $U'_t = V_t$ if $t_2 \in S_2$, $U'_t = U_t$ otherwise. Since S_2 is countable for every $a \in S'_1 = \bigcup_{b\in S_2} S(b)$, the one-parameter processes $(V_{a,b})_{b\geqq 0}$ and $(U_{a,b})_{b\geqq 0}$ are indistinguishable. Hence $(V_{a,b})_{b\geqq 0}$ has a continuous modification, say $(W_{a,b})_{b\geqq 0}$. The process $Z_t = W_t$ if $t_1 \in S'_1$, $Z_t = V_t$ if $t_1 \notin S'_1$ is an ascending 1-amart, and satisfies the assumptions of Proposition 3.2. Now an argument similar to the one given in Theorem 2.4 shows that the process (Z_t) has limits in the left half-plane. □

Theorem 3.6. *Let (X_t) be a descending 1-amart, and an ascending 1-amart of class* (AL).

(i) *Assume that $(\mathscr{F}_t^1)$ is right-continuous, that for every $a \geqq 0$, the family $(\mathscr{F}_{a,b})_{b\geqq 0}$ is right-continuous, and that the process $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ is both a descending and an ascending amart of class* (AL). *Suppose that for every index t and for every sequence $\tau(n)$ in T^1 1-decreasing to t in Q_{I}, one has $\lim EX_{\tau(n)} = EX_t$. Then (X_t) has a modification (Y_t) continuous in Q_{I}, and having limits in the three other quadrants.*

(ii) *Assume that the family $(\mathscr{F}_t^1)$ is right-[right- and left-] continuous, and that for every $a \geqq 0$, the process $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ has a continuous modification. Suppose that for every index t and every sequence $\tau(n)$ in T^1 converging to t with $\tau(n)_1 \geqq t_1$ $[\tau(n)_1 \leqq t_1]$, one has $\lim EX_{\tau(n)} = EX_t [\lim EX_{\tau(n)} = EX_t$, and $(X_{\tau(n)})$ is uniformly integrable]. Then (X_t) has a modification continuous in the right [left] half-plane such that almost every trajectory has limits in the left [right] half-plane.*

(iii) *Assume that the family $(\mathscr{F}_t^1)$ is right-continuous, and that for every $a \geqq 0$, the processes $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$, and $(X_{b,a}, \mathscr{F}_{b,a}, b \geqq 0)$ have continuous modifications. Suppose that for every index t, and every sequence $\tau(n)$ in T^1 converging to t in Q_{I}, one has $\lim EX_{\tau(n)} = EX_t$. Then (X_t) has a continuous modification.*

Proof. (i) The statement is an eady consequence of Corollary 2.6 and Theorem 3.4.

(ii) The existence of a modification (Y_t) of (X_t) having limits in the half-planes is an easy consequence of Theorem 3.5 and its analogue in the right half-plane. The existence of modifications continuous in the right half-plane under the unbracketed assumptions follows from Theorem 2.5. Under the bracketed assumptions, let $Z_t = \lim(Y_s: s \to t, s_1 < t_1)$; clearly (Z_t) is continuous in the left half-plane. We show that (Z_t) is a modification of (Y_t) (cf. [9] for the one-parameter case). Fix t with $t_1 > 0$, and let s be such that $s_1 < t_1$. Let $F \in \mathscr{F}_s^1$; for every $n > 1/(t_1 - s_1)$, set $\tau(n) = t$ on F^c, $\tau(n) = \left(t_1 - \frac{1}{n}, t_2\right)$ on F. Then $\tau(n)$ converges to t, $\tau(n)_1 \leqq t_1$, and $\tau(n) \in T^1$. Hence $\lim EY_{\tau(n)} = EY_t$. Clearly $Y_{\tau(n)}$ converges pointwise to $Z_t 1_F + Y_t 1_{F^c}$. Since $(Y_{\tau(n)})$ is uniformly integrable, $\lim EY_{\tau(n)} = E[1_F Z_t] + E[1_{F^c} Y_t]$. Hence $E[1_F Y_t] = E[1_F Z_t]$ for every $F \in \mathscr{F}_s^1$, $s_1 < t_1$.

Since the family $(\mathscr{F}_t^1)$ is left-continuous, the equality of expectations holds for every $F \in \mathscr{F}_t^1 = \sigma(\mathscr{F}_s^1: s_1 < t_1)$, and hence $Y_t = Z_t$ a.s.

(iii) Let (Z_t) be a separable modification of (X_t) with separant set $S = S_1 \times S_2$, such that for every $a \in S_1 [b \in S_2]$ the process $(Z_{a,b})_{b \geqq 0} [(Z_{a,b})_{a \geqq 0}]$ is continuous (hence separable). Fix $M \geqq 0$; by Propositions 2.2, 2.3 and 3.2, for almost every ω the functions $a \to Z_{a,b}(\omega)$ are continuous uniformly in $b \in S_2$. Since the processes $(Z_{a,b})_{b \geqq 0}$ are continuous for every $a \in S_1$, we deduce that for almost every ω, $\lim (Z_s(\omega): s \to t, s \in S)$ exists for every t. Since S is a separant set for (Z_t), this defines a continuous modification of (Z_t), and hence of (X_t). $\square$

4. Regularity of L Log L-bounded Martingales Indexed by $\mathbb{R}_+^2$

In this section we show that under natural regularity assumptions on the σ-algebras, $L \operatorname{Log} L$-bounded 1-martingales indexed by $\mathbb{R}_+^2$ are descending 1-amarts. Hence processes which are simultaneously 1- and 2-martingales have modifications which are continuous in the first quadrant, and have limits in the two quadrants Q_{II} and Q_{IV}.

Recall that if $(\mathscr{F}_t)_{t \in \mathbb{R}_+^2}$ is a stochastic basis, then

$$\mathscr{F}_t^1 = \bigvee_{u \geqq 0} \mathscr{F}_{t_1, u}, \quad \text{and} \quad \mathscr{F}_t^2 = \bigvee_{u \geqq 0} \mathscr{F}_{u, t_2}.$$

By definition a *1-martingale* (X_t) is a martingale such that $E(X_t | \mathscr{F}_s^1) = X_{s_1, t_2}$ for every $s \leqq t$, while a *2-martingale* (X_t) is a martingale such that $E(X_t | \mathscr{F}_s^2) = X_{t_1, s_2}$ for every $s \leqq t$.

Proposition 4.1. *Suppose that there exists a sequence of real numbers $a(n) \to +\infty$ such that the families $(\mathscr{F}_{a(n), b})_{b \geqq 0}$ are right-continuous, and suppose that $(\mathscr{F}_t^1)_{t \in \mathbb{R}_+^2}$ is right-continuous. Let (X_t) be a 1-martingale bounded in $L \operatorname{Log} L$.*

(1) *Let τ be a bounded 1-stopping time, and let $\tau(n)$ be a uniformly bounded sequence in T^1 such that*

(i) *$\tau(n)_2 \geqq \tau_2$ for every n,*

(ii) $\lim_n \tau(n)_2 = \tau_2$ a.s.

Then $EX_{\tau(n)}$ converges. If τ_2 is a constant, then $\lim EX_{\tau(n)} = EX_{0,0}$.

(2) *Let τ be a bounded 1-stopping time, and let $\tau(n)$ be a uniformly bounded sequence in T^1 such that*

(i) *$\tau(n)_2 < \tau_2$ for every n,*

(ii) $\lim \tau(n)_2 = \tau_2$ a.s.

Then $EX_{\tau(n)}$ converges, and the sequence $X_{\tau(n)}$ is uniformly integrable.

Proof. Fix n; the one-parameter martingale $(X_{a(n), b}, \mathscr{F}_{a(n), b})_{b \geqq 0}$ has a right-continuous modification which has left limits, say $(Z_{a(n), b})_{b \geqq 0}$. Let τ and $\tau(n)$ be

1-stopping times bounded by $(a(M), a(M))$ for some M. Then

$$\begin{aligned} EX_{\tau(n)} &= \sum_{a,b} E[1_{\{\tau(n)=(a,b)\}} X_{a,b}] \\ &= \sum_{a,b} E[1_{\{\tau(n)=(a,b)\}} E(X_{a(M),b} | \mathcal{F}_{a,\infty})] \\ &= \sum_{b} E[1_{\bigcup_a \{\tau(n)=(a,b)\}} X_{a(M),b}] \\ &= \sum_{b} E[1_{\{\tau(n)_2=b\}} Z_{a(M),b}] = E[Z_{a(M),\tau(n)_2}]. \end{aligned}$$

By the choice of $(Z_{a(M),b})_{b \geqq 0}$, the sequence $(Z_{a(M),\tau(n)_2})_{n \geqq 0}$ converges pointwise to $Z_{a(M),\tau_2}$ under the assumption (1), and to a limit denoted $Z_{a(M),\tau_2}$ - under the assumption (2). In both cases, $\sup_n |Z_{a(M),\tau(n)_2}| \leqq \sup_{b \leqq a(M)} |X_{a(M),b}|$ a.s., and the right hand-side is integrable (see (Sect. 1). This allows us to integrate and deduce the convergence of $EX_{\tau(n)}$. Under the assumption (1) for a 1-stopping time τ, one clearly has $\lim EX_{\tau(n)} = EZ_{a(M),\tau_2}$; hence if $\tau_2 \equiv b \in \mathbb{R}_+$, $\lim EX_{\tau(n)} = EX_{0,0}$. Let $\sigma(n)$ be a sequence of 1-stopping times uniformly bounded by (K, K). For every a with $0 \leqq a \leqq K$, set $Y_a = \sup\{|X_{a,b}|: b \leqq K\}$. Since $EY_a \leqq E\{\sup\{|X_{K,b}|: b \leqq K\}\}$, the process $(Y_a, \mathcal{F}_{a,\infty})_{a \leqq K}$ is an L^1-bounded submartingale. Furthermore, letting $\mathcal{G}_a = \mathcal{F}_{a,\infty}$, one has

$$\begin{aligned} |X_{\sigma(n)}| &\leqq \sum_a 1_{\{\sigma(n)_1=a\}} Y_a \\ &\leqq \sum_a 1_{\{\sigma(n)_1=a\}} E(Y_K | \mathcal{G}_a) = E(Y_K | \mathcal{G}_{\sigma(n)_1}), \end{aligned}$$

which shows the uniform integrability of the sequence $X_{\sigma(n)}$. □

The proposition above shows that under the usual assumptions on the σ-algebras, $L \operatorname{Log} L$-bounded 1-martingales are descending 1-amarts such that for every $a \geqq 0$ the one-parameter process $(X_{a,b}, \mathcal{F}_{a,b}, b \geqq 0)$ is a descending amart and an ascending amart of class (AL). The analogue of Proposition 3.1 for 2-martingales together with Corollaries 2.6 and 2.7 thus allow us to deduce the following:

Theorem 4.2. *Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+^2}$ be a stochastic basis such that the family $(\mathcal{F}_t)$ is right-continuous. If $(\mathcal{F}_t^1)$ is right-continuous, then every $L \operatorname{Log} L$-bounded 1-martingale is a descending 1-amart, and admits a modification almost every trajectory of which is continuous in Q_{I}, and has limits in Q_{IV}. Similarly, if $(\mathcal{F}_t^2)$ is right-continuous, then every $L \operatorname{Log} L$-bounded 2-martingale is a descending 2-amart, and admits a modification almost every trajectory of which is continuous in Q_{I}, and has limits in Q_{II}.*

Corollary 4.3. *Let $(\mathcal{F}_t)_{t \in \mathbb{R}_+^2}$ be a stochastic basis such that the families $(\mathcal{F}_t)$, $(\mathcal{F}_t^1)$, and $(\mathcal{F}_t^2)$ are right-continuous. Then every process bounded in $L \operatorname{Log} L$ which is both a 1-martingale and a 2-martingale admits a modification (Y_t) almost every trajectory of which is continuous in Q_{I}, and has* a.s. *the following limits for every t: $\lim(Y_s: s_1 < t_1, s_2 \geqq t_2)$, and $\lim(Y_s: s_1 \geqq t_1, s_2 < t_2)$.*

Hence one deduces the following theorem (see also D. Bakry [2], [3]).

Theorem 4.4. *Let $(\mathscr{F}_t)$ be a stochastic basis such that the families $(\mathscr{F}_t)$, $(\mathscr{F}_t^1)$ and $(\mathscr{F}_t^2)$ are right-continuous, and such that $(\mathscr{F}_t^1)$ and $(\mathscr{F}_t^2)$ are conditionally independent with respect to $(\mathscr{F}_t)$ for every index t. Then every $L\operatorname{Log}L$-bounded martingale (X_t) has a modification (Y_t) continuous in Q_{I}, and having limits in the quadrants Q_{II} and Q_{IV}.*

Trivial examples (e.g., a constant process) show that the assumptions of Corollary 4.3 are strictly weaker than those of Theorem 4.4.

Finally let us observe that a natural example of a family of σ-algebras $(\mathscr{F}_t, P)$ satisfying the conditions stated in Theorem 4.4 is $(\mathscr{F}_{t_1,t_2} = \mathscr{A}_{t_1} \otimes \mathscr{B}_{t_2},\ P = P_1 \otimes P_2)$, where $(\mathscr{A}_t)_{t \geqq 0}$ and $(\mathscr{B}_t)_{t \geqq 0}$ are right-continuous one-parameter increasing families of σ-algebras on (Ω_1, P_1) and (Ω_2, P_2).

The following theorem shows that if the vertical one-parameter processes are continuous, then 1-martingales have limits in the left and right half-planes.

Theorem 4.5. *Let $(\mathscr{F}_t)$ be a stochastic basis such that the family $(\mathscr{F}_t^1)$ is right-continuous, and let (X_t) be an $L\operatorname{Log}L$-bounded 1-martingale. Suppose that for every $a \geqq 0$, the process $(X_{a,b}, \mathscr{F}_{a,b}, b \geqq 0)$ has a continuous modification. Then (X_t) is a descending 1-amart and an ascending 1-amart of class* (AL). *Hence (X_t) has a modification (Y_t) almost every trajectory of which is continuous in the right half-plane, and has limits in the left half-plane.*

Proof. The argument given in the proof of Proposition 4.1 clearly shows the 1-amart property stated. The regularity of the trajectories follows from Theorem 3.6(ii). □

As an application of Theorem 3.6(iii) we recover a result first shown by Cairoli and Walsh [6], about martingales for the σ-algebras generated by the two-parameter Wiener process. Our method is very different from theirs, since it does not use two-parameter stochastic integrals. Let W be the random measure on $\mathbb{R}_+^2$ which assigns to each Borel set A a Gaussian random variable of mean zero and variance equal to the Lebesgue measure of A, and which assigns independent random variables to disjoint sets. Define a process $(W_t, t \in \mathbb{R}_+^2)$ by $W_t = W(R_t)$, where $R_t = \{s\colon 0 \leqq s \leqq t\}$, and set $\mathscr{F}_t = \sigma(W_s\colon s \leqq t)$. Then for every t the σ-algebras $(\mathscr{F}_t^1)$ and $(\mathscr{F}_t^2)$ are conditionally independent with respect to $(\mathscr{F}_t)$.

Theorem 4.6. *Let $\mathscr{F}_t = \sigma(W_s\colon s \leqq t)$ be defined as above, and let (X_t) be an $L\operatorname{Log}L$-bounded martingale for the family $(\mathscr{F}_t)$. Then (X_t) has a modification (Y_t) which is continuous.*

Proof. The process (X_t) is an ascending 1-amart of class (AL) and a descending 1-amart. For every $a \geqq 0$ $[b \geqq 0]$, the one-parameter martingale $(X_{a,b},\ b \geqq 0)$ $[(X_{a,b}, a \geqq 0)]$ is an L_1-bounded martingale for the σ-algebras generated by the one-parameter Brownian motion $(W_{a,b},\ b \geqq 0)$ $[(W_{a,b},\ a \geqq 0)]$; hence it has continuous modifications. Theorem 3.6(iii) concludes the proof. □

Finally, there is a local theorem at infinity (or at any index t in Q_{III}).

Theorem 4.7. *Let (X_t) be an ascending 1-amart at* ∞, i.e., *suppose that the net $(EX_\tau\colon \tau \in T^1,\ \tau \to \infty)$ converges. Suppose that* $\limsup E|X_t| < \infty$. *Then X_t converges* *essentially when $t \to \infty$. Hence $L\operatorname{Log}L$-bounded separable 1-martingales converge essentially at ∞. If the processes are separable,* a.s. *convergence holds.*

Proof. Let s be such that $\sup_{t \geqq s} E|X_t| < \infty$. The process $(X_t, \mathscr{F}_t^1, t \geqq s)$ is an L^1-bounded amart for the totally ordered family of σ-algebras $(\mathscr{F}_t^1)$. Hence by Astbury's theorem [1], X_t converges essentially at ∞. The proof that if (X_t) is an $L\operatorname{Log}L$-bounded martingale, then X_τ converges in L_1, hence EX_τ converges is similar to the proof of Theorem 1.1 above, and is omitted. Finally, a.s. convergence in the presence of a separant set follows by standard arguments. □

Added in Proof. We wish to use the present opportunity to correct an omission in the statement of Theorem 4.1 [12], p. 97, pointed out to us by Professor Alexandra Bellow. We consider a family $f(\sigma, \tau)$ of $\mathscr{F}_\sigma$-measurable functions. We assume the 'localization' in the first variable σ, but the proof also uses the localization in the second variable. Hence one should add the following assumption: For each s and each $\mathscr{F}_s$-measurable set A, $\tau = \tau'$ on A implies $f(s, \tau) = f(s, \tau')$ on A. This assumption is obviously satisfied in the application of the theorem.

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Received June 20, 1980