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Some Problems in Finite Dams with an Application to Insurance Risk

By

B. WEESAKUL and G. F. YEO

Summary

This paper considers a finite dam in continuous time fed by inputs, with a negative exponential distribution, whose arrival times form a Poisson process; there is a continuous release at unit rate, and overflow is allowed. Various results have been obtained by appropriate limiting methods from an analogous discrete time process, for which it is possible to find some solutions directly by determinantal methods.

First the stationary dam content distribution is found. The distribution of the probability of first emptiness is obtained both when overflow is, and is not allowed. This is followed by the probability the overflow before emptiness, which is then applied to determine the exact solution for an insurance risk problem with claims having a negative exponential distribution. The time-dependent content distribution is found, and the analogy with queueing theory is discussed.

1. Introduction

Since the formulation of a stochastic theory of dams by MORAN [*9*] a considerable literature has accumulated for both discrete and continuous time dam processes; solutions have been obtained for various stationary and time-dependent problems, including those of emptiness which are related to first passage times in random walks. Descriptive details and bibliographies have been given by GANI [*6*] and MORAN [*10*]. However, most results have been found for an infinite dam and little is known about the finite dam, when the problem is complicated by the possibility of overflow. For a finite dam in discrete time PRABHU [*11*] has given a method of solution for the stationary distribution, GHOSAL [*7*] and WEESAKUL [*14*] have discussed problems of first emptiness and WEESAKUL [*15*] has found the time-dependent solution when the inputs have a geometric distribution.

In this paper we are concerned with a dam of finite capacity K (a positive real number) fed by inputs, whose size is a random variable having a negative exponential distribution with distribution function (d.f.)

$$G(x) = 1 - e^{-\mu x} (0 \leqq x, \mu < \infty),$$

which occur in a time-homogeneous Poisson process with finite mean λ. The release of water is continuous at unit rate per unit time unless the dam is empty, and any input which raises the total content beyond K overflows and is lost. We denote this model A.

Let the content of the dam at time t ($\geqq 0$) for model A be denoted by $Z(t)$; $Z(t)$ is defined on the non-negative real numbers not exceeding K, and is a Markov process satisfying:

$$Z(t+dt) = \min(Z(t) + dX(t), K) - \min(Z(t) + dX(t), (1-\eta)\,dt),$$

where $\eta\, dt$ is the time during $(t, t+dt)$ that the dam is empty, and $dX(t)$ is the input in $(t, t+dt)$, this being additive and independent in non-overlapping time intervals. Define as $F(z, t \mid U, K) = \Pr\{Z(t) \leqq z \mid Z(0) = U; K\}$ the d.f. of the dam content at time t given that the non-negative number U is the content at time zero. By enumerating the possible occurences during $(t, t+dt)$ and letting $dt \to 0$ we obtain the following integro-differential equation (c.f. TAKÁCS [13] for $K = \infty$), which is the forward Kolmogorov equation of the process:

$$\frac{\partial}{\partial t} F(z,t \mid U,K) - \frac{\partial}{\partial z} F(z,t \mid U,K) = -\lambda F(z,t \mid U,K) + \lambda\mu \int\limits_{x=0}^{z} F(z-x, t \mid U,K)\, e^{-\mu x}\, dx,\quad 0 \leqq z \leqq K, \tag{1.1}$$

with $F(z,t \mid U,K) = 1$ for $z \geqq K$. We have $F(z,t \mid U,K) = 0$ for all $z < 0$. $F(z,t \mid U,K)$ is continuous everywhere in $\max(0, U-t) < z \leqq K$, but has a discontinuity at $z = \max(0, U-t)$; when $t \geqq U$ the concentration $F(0,t \mid U,K)$ is the probability of emptiness of the dam at time t; $F(z,t \mid U,K)$ has a derivative $\partial F(z,t \mid U,K)/\partial z$ continuous everywhere, except at $z = \max(0, U-t)$ where only a right derivative exists and at $z = K$ where only a left derivative exists. $F(z,t \mid U,K)$ is continuous with respect to time for all t and z ($0 \leqq z \leqq K$) except at $t = U - z$ when $0 \leqq t \leqq U$, $0 \leqq z \leqq U$ (see DOOB [4]).

We now describe a discrete dam model in which the content Z_t ($t = 0, 1, 2, \ldots$) can take only the integral values $0, 1, 2, \ldots$. In an interval $(t, t+1)$ of time there is a probability a ($0 < a < 1$) of an input occurring and a probability $b = 1 - a$ of no input occurring; the interval of time between successive inputs has the geometric distribution ab^{i-1} ($i = 1, 2, \ldots$). Whenever there is an input its size W is a random variable with the geometric distribution

$$\Pr\{W = i\} = p q^{i-1} \quad (i = 1, 2, \ldots; 0 < p < 1, q = 1 - p).$$

The inputs occur independently, and during $(t, t+1)$ the input X_t is seen to have the modified geometric distribution:

$$\Pr\{X_t = 0\} = p_0 = b; \quad \Pr\{X_t = i\} = p_i = a p q^{i-1}, \quad i = 1, 2, \ldots. \tag{1.2}$$

Let there be a unit release just before the end of each unit time interval, i.e. at $t - 0$ ($t = 1, 2, \ldots$), unless the dam is empty. Further let the dam have integral capacity k (> 0), so that there is an overflow of size $X_t + Z_t - k$ if $X_t + Z_t > k$. We label this discrete formulation model B.

It is known that $\{Z_t\}$ forms a Markov chain with the states $0, 1, \ldots, k-1$ and the transition matrix

$$\underset{\sim}{S} = \begin{bmatrix} b + ap & apq & apq^2 & \cdot\ \cdot\ \cdot & apq^{k-2} & aq^{k-1} \\ b & ap & apq & & apq^{k-3} & aq^{k-2} \\ 0 & b & ap & & apq^{k-4} & aq^{k-3} \\ & & & & & \\ 0 & 0 & 0 & & bap & aq \\ 0 & 0 & 0 & 0 & b & a \end{bmatrix}. \tag{1.3}$$

The cumulative probabilities $Q_i(t\,|\,u, k) = \Pr\{Z_t \leqq i\,|\,Z_0 = u; k\}$ of the dam content distribution given integral initial content u and capacity k satisfy the difference equations

$$Q_i(t+1\,|u, k) = \sum_{j=0}^{i+1} Q_j(t\,|\,u, k)\, p_{i+1-j}\,, \qquad i = 0, 1, \ldots, k-1\,. \tag{1.4}$$

The relation (1.1) may be obtained from the analogous discrete time difference equations (1.4) by the limiting process described below. We have been unable to solve (1.1) directly and have resorted to limiting methods from the solutions for the discrete model B. We shall pass from the discrete to the continuous problem as follows: let the discrete model B be defined in units of size $\Delta\,(>0)$ instead of unity, so that content, time, input and release are all measured in units of size Δ. We put

(a) $\quad a = \lambda\Delta + o(\Delta)\,,\quad b = 1 - \lambda\Delta + o(\Delta)\,,\quad p = \mu\Delta + o(\Delta)\,,$
$\quad\; q = 1 - \mu\Delta + o(\Delta)\,,$

(b) $\quad i = z\Delta^{-1}\,,\quad t' = t\Delta^{-1}\,,\quad u = U\Delta^{-1}\,,\quad k = K\Delta^{-1}\,, \qquad (1.5)$

where t' represents time in the discrete case. As we let $\Delta \to 0$ it is easily seen that the time interval between inputs and the size of inputs have negative exponential distributions with means λ^{-1} and μ^{-1} respectively, and the release becomes continuous at unit rate per unit time. Under (1.5) as $\Delta \to 0$ we have (1.4) tending to (1.1) and model B is thus a discrete analogue of model A.

2. The stationary distribution

We first consider some properties of model A when it has settled down to a stationary state which happens with probability one whenever K is finite. We prove

Theorem 1. *For model A the stationary d.f. $F(z\,|\,K) = \lim_{t\to\infty} F(z, t\,|\,U, K)$ of the dam content, which exists independently of the initial content, is given for $K < \infty$ by*

$$F(z\,|\,K) = \frac{\mu - \lambda\, e^{-(\mu-\lambda)z}}{\mu - \lambda\, e^{-(\mu-\lambda)K}}\,,\; 0 \leqq z \leqq K\,. \tag{2.1}$$

The Laplace-Stieltjes transform (LST) $\psi(s\,|\,K) = \int_{z=0}^{\infty} e^{-sz}\, dF(z\,|\,K)$ $(Rl\, s \geqq 0)$ *of* $F(z\,|\,K)$ *is*

$$\psi(s\,|\,K) = \frac{(\mu-\lambda)\{\mu + s - \lambda\, e^{-(\mu+s-\lambda)K}\}}{(\mu - \lambda + s)\{\mu - \lambda\, e^{-(\mu-\lambda)K}\}}\,. \tag{2.2}$$

Proof. For a finite dam in discrete time fed by independent discrete inputs a formula has been given by Prabhu [*11*] for determining the stationary dam content distribution. A generating function $V(\theta) = \sum_{i=0}^{\infty} V_i\, \theta^i\,(|\,\theta\,| \leqq 1)$ is obtained from the relation

$$V(\theta) = \frac{p_0(1-\theta)}{A(\theta) - \theta}\,,$$

where $A(\theta) = \sum_{i=0}^{\infty} p_i\, \theta^i\,(|\,\theta\,| \leqq 1)$ is the probability generating function (p.g.f.) of the input distribution. Expanding $V(\theta)$ and normalising yields the stationary probabilities $P_i = \lim_{t\to\infty} \Pr\{Z_t = i \mid u, k\}$. For model B, $A(\theta) = b + ap\theta(1 - q\theta)^{-1}$ so that we obtain

$$Q_i = \sum_{j=0}^{i} P_j = \frac{p - a(q/p)^{i+1}}{p - a(q/p)^k}\, i = 0, 1, \ldots, k-1. \tag{2.3}$$

To obtain the stationary d.f. $F(z \mid K)$ for model A from the discrete analogue we use units of size Δ and the substitution (1.5) and let $\Delta \to 0$. We have

$$\begin{aligned} F(z \mid K) &= \lim_{\Delta\to 0} \frac{\mu\Delta - \lambda\Delta\left(\frac{1-\mu\Delta}{1-\lambda\Delta}\right)^{z\Delta^{-1}+1}}{\mu\Delta - \lambda\Delta\left(\frac{1-\mu\Delta}{1-\lambda\Delta}\right)^{K\Delta^{-1}}} \\ &= \frac{\mu - \lambda e^{-(\mu-\lambda)z}}{\mu - \lambda e^{-(\mu-\lambda)K}} \qquad 0 \leqq z \leqq K, \end{aligned}$$

which we wished to prove. The LST $\psi(s \mid K)$ is found by integration of (2.1) and is given by (2.2).

From (2.2) the probability $F(0 \mid K)$ that the dam is empty is

$$F(0 \mid K) = (\mu - \lambda)\{\mu - \lambda e^{-(\mu-\lambda)K}\}^{-1},$$

while when $\lambda = \mu$

$$F(z \mid K) = (1 + \lambda z)(1 + \lambda K)^{-1} \qquad 0 \leqq z \leqq K.$$

The moments of the content distribution may be found by differentiation of (2.2); the mean is

$$E(z) = \begin{cases} \dfrac{\lambda\{1 - (1 + (\mu - \lambda)K)e^{-(\mu-\lambda)K}\}}{(\mu-\lambda)\{\mu - \lambda e^{-(\mu-\lambda)K}\}} & \lambda \neq \mu \\ \dfrac{\lambda K^2}{2(1+\lambda K)} & \lambda = \mu. \end{cases}$$

3. First emptiness and overflow

Our main concern in this section is to obtain for model A the improper d.f's.

$$\begin{aligned} G^*(0, t \mid U, K) &= \Pr\{Z(\tau) = 0 \quad \text{for some } \tau \text{ in} \quad 0 < \tau \leqq t; \\ &\qquad 0 < Z(v) < K, 0 < v < \tau \mid U, K\}, \\ G^*(K, t \mid U, K) &= \Pr\{Z(\tau) = K \quad \text{for some } \tau \text{ in} \quad 0 < \tau \leqq t; \\ &\qquad 0 < Z(v) < K, 0 < v < \tau \mid U, K\} \end{aligned}$$

that the dam with initial content U and capacity K empties (overflows) for the first time before overflow (emptiness). We also consider the zero and overflow avoiding (improper) distribution

$$G^*(z, t \mid U, K) = \Pr\{Z(t) \leqq z; 0 < Z(\tau) < K, 0 < \tau < t \mid U, K\} \quad 0 \leqq z < K,$$

where continuity properties are similar to that for $F(z, t \mid U, K)$ for $0 \leqq z < K$ although there is not now a concentration of probability at $z = 0$ except for $t = U$. We shall determine the LST's

$$\begin{aligned} \varphi^*(0, s \mid U, K) &= \int_{t=0}^{\infty} e^{-st}\, dG(0, t \mid U, K) \\ \varphi^*(K, s \mid U, K) &= \int_{t=0}^{\infty} e^{-st}\, dG(K, t \mid U, K) \qquad Rl\, s \geqq 0 \\ \varphi^*(z, s \mid U, K) &= \int_{t=0}^{\infty} e^{-st}\, G(z, t \mid U, K)\, dt \qquad 0 < z < K, \end{aligned}$$

and show how they may be inverted.

Theorem 2. *For model A*

$$\varphi^*(0, s \mid U, K) = \frac{e^{-U(s+\lambda)}\{\eta_1 e^{-(K-U)\eta_2} - \eta_2 e^{-(K-U)\eta_1}\}}{\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1}} \tag{3.1}$$

$$\varphi^*(K, s \mid U, K) = \frac{\lambda e^{-(K-U)\mu}\{e^{-U\eta_2} - e^{-U\eta_1}\}}{\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1}} \tag{3.2}$$

$$\varphi^*(z, s \mid U, K) = \begin{cases} e^{-U(s+\lambda)}\{\eta_1 e^{-(K-U)\eta_2} - \eta_2 e^{-(K-U)\eta_1}\} \times \\ \left\{\dfrac{\mu\nu - \eta_1(\eta_1 - s - \lambda) e^{-z(\eta_2 - s - \lambda)} + \eta_2(\eta_2 - s - \lambda) e^{-z(\eta_1 - s - \lambda)}}{\nu(\eta_1 - s - \lambda)(\eta_2 - s - \lambda)(\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1})}\right\} & 0 < z \leqq U \\ \varphi^*(U, s \mid U, K) + \lambda\mu e^{\mu U}\{e^{-U\eta_2} - e^{-U\eta_1}\} \times \\ \left\{\dfrac{(\eta_1 - \mu)e^{-K\eta_2}(e^{z(\eta_2-\mu)} - e^{U(\eta_2-\mu)}) - (\eta_2 - \mu) e^{-K\eta_1}(e^{z(\eta_1-\mu)} - e^{U(\eta_1-\mu)})}{\nu(\eta_1 - \mu)(\eta_2 - \mu)(\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1})}\right\} & U \leqq z < K \end{cases}$$

where (3.3)

$$\nu = \{(\lambda + \mu + s)^2 - 4\lambda\mu\}^{1/2}, \quad 2\eta_{1,2} = \lambda + \mu + s \pm \nu.$$

Proof. We begin with the discrete model B; we define

$$G(i, t \mid u, k) = \Pr\{Z_t \leqq i;\ 0 < Z_j < k,\ 0 < Z_j + X_j \leqq k \quad \text{for all} \quad 0 < j < t \mid Z_0 = u; k\},$$

$$G(k, t \mid u, k) = \Pr\{Z_{t-1} + X_{t-1} > k;\ Z_j > 0,\ 0 < j < t;\ Z_i + X_i \leqq k,\ 0 \leqq i \leqq t-2 \mid Z_0 = u; k\}$$

where $G(0, t \mid u, k)$ and $G(k, t \mid u, k)$ are respectively the probabilities of first emptiness (overflow) at t before overflow (emptiness). Following the argument of WEESAKUL [*14*] it may be shown that

$$G(i, t \mid u, k) = \underline{P}(u)\, \underline{Q}^{t-2}\, \underline{G}(i) \qquad i = 0, 1, \ldots, k, \tag{3.4}$$

where $\underline{P}(u)$ is the row vector $(0 \ldots 0\ b\ ap \ldots apq^{k-u-1})$ with b in the $(u-1)$-th position, $\underline{G}(i)$ is the column vector of elements $G(i, 1 \mid j, k)$ $(j = 1, 2, \ldots, k)$ and $\underline{Q}$ is a matrix similar to $\underline{S}$ except that the first element of the first row is ap and the last column of $\underline{S}$ is replaced by $(apq^{k-2}\ apq^{k-3} \ldots apq\ ap)$. We have

$$G(i, 1 \mid j, k) = \begin{cases} aq^{-j}(1 - q^{i+1}) & 1 \leqq j \leqq i \\ b & j = i + 1 \quad i < k \\ 0 & j > i + 1 \\ aq^{k-j} & 1 \leqq j \leqq k; \quad i = k. \end{cases} \tag{3.5}$$

Introducing the time transform $\varphi(i, \theta \mid u, k) = \sum_{t=0}^{\infty} G(i, t \mid u, k)\ \theta^t\ (|\theta| \leqq 1)$ it follows from (3.4) that

$$\varphi(i, \theta \mid u, k) = \theta^2\, \underline{P}(u)\, (\underline{I} - \theta\, \underline{Q})^{-1}\, \underline{G}(i).$$

By writing $(\underline{I} - \theta\, \underline{Q})^{-1}$ in terms of the cofactors of its elements we can show that

$$\varphi(i, \theta \mid u, k) = \theta^2 \{|\underline{I} - \theta\, \underline{Q}|\}^{-1} \sum_{j=1}^{k-1} D_j\, G(i, 1 \mid j, k), \tag{3.6}$$

where D_j is the determinant of the matrix $(\underline{I} - \theta \underline{Q})$ with the j-th row replaced by $\underline{P}(u)$. Each of the determinants in (3.6) is evaluated by using recursive relations after reduction into a continuant form; e.g. for $|\underline{I} - \theta \underline{Q}|$ we carry out the reduction by subtracting from each row the row immediately below it multiplied by q. We quote the following results:

$$|\underline{I} - \theta \underline{Q}| = (\lambda_1 - \lambda_2)^{-1}\{(1-\lambda_2)\lambda_1^k - (1-\lambda_1)\lambda_2^k\},$$

$$D_i = \begin{cases} \theta(\theta b)^{u-i} B_{k-u-1} B_{i-1} & i = 1, 2, \ldots, u \\ q^{i-u} A_{u-1}\{(ap - bq) B_{k-i-1} + bq B_{k-i-2}\} & i = u+1, \ldots, k \end{cases}$$

$$B_n = (\lambda_1 - \lambda_2)^{-1}\{(1-\lambda_2)\lambda_1^{n+1} - (1-\lambda_1)\lambda_2^{n+1}\} \quad n \geqq 1$$

$$A_n = (\lambda_1 - \lambda_2)^{-1}\{\lambda_1^{n+1} - \lambda_2^{n+1}\} \quad n \geqq 1$$

$$2\lambda_{1,2} = \gamma \pm (\gamma^2 - 4\theta b q)^{1/2}, \quad \gamma = 1 - \theta(ap - bq),$$

where $B_0 = B_{-1} = A_0 = 1$. Using these results we now obtain

$$\varphi(0, \theta \,|\, u, k) = (\theta b)^u B_{k-u-1} B_{k-1}^{-1} \tag{3.7}$$

$$\varphi(k, \theta \,|\, u, k) = \frac{a\theta}{B_{k-1}}\{B_{k-u-1} \sum_{j=1}^{u} (\theta b)^{u-j} q^{k-j} B_{j-1} +$$
$$+ \theta a p q^{k-u} A_{u-1} \sum_{j=u+1}^{k-2} B_{k-j-1} + \theta b q^{k-u+1} A_{u-1}(1 - B_{k-u-1})\} \tag{3.8}$$

$$\varphi(i, \theta \,|\, u, k) = \begin{cases} \dfrac{\theta B_{k-u-1}}{B_{k-1}} \sum\limits_{j=1}^{i} \{a(\theta b)^{u-j}(1-q^{i-j+1}) B_{j-1} + b(\theta b)^{u-1-j} B_j\} & 1 \leqq i \leqq u \\ \varphi(u, \theta \,|\, u, k) + \dfrac{\theta}{B_{k-1}}[a B_{k-u-1} \sum\limits_{j=1}^{u} (\theta b q)^{u-j}(1 - q^{i-u+1}) + \\ \quad \theta A_{u-1} \sum\limits_{j=u+1}^{i} \{a^2 p q^j (1 - q^{i-j+1}) B_{k-j-1} - \\ a b q^j (1 - q^{i-j+1}) B_{k-u-1} + ap(1 + bq^3) \sum\limits_{r=j}^{i} B_{k-r-2} + bq\}] \\ & u+1 \leqq i \leqq k-1. \end{cases} \tag{3.9}$$

These results are the first step towards finding those for the analogous continuous time process. We make the substitution (1.5) and the required results (3.1)—(3.3) follow from (3.7)—(3.9) from the following passages to the limit, which may be justified by the continuity theorem in CRAMÉR [2, pp 96—100]:

$$\varphi^*(0, s \,|\, U, K) = \lim_{\Delta \to 0} \varphi(0, e^{-s\Delta} \,|\, U\Delta^{-1}, K\Delta^{-1}),$$

$$\varphi^*(K, s \,|\, U, K) = \lim_{\Delta \to 0} \varphi(K\Delta^{-1}, e^{-s\Delta} \,|\, U\Delta^{-1}, K\Delta^{-1}),$$

$$\varphi^*(z, s \,|\, U, K) = \lim_{\Delta \to 0} \Delta\, \varphi(z\Delta^{-1}, e^{-s\Delta} \,|\, U\Delta^{-1}, K\Delta^{-1}).$$

As a consequence of Theorem 2 we have that the probabilities $\pi_K(U, K)$ $(\pi_0(U, K))$ that overflow (emptiness) occurs before emptiness (overflow) are

$$\pi_K(U, K) = \int_{t=0}^{\infty} dG^*(K, t \,|\, U, K) = \lim_{s \to 0} \varphi^*(K, s \,|\, U, K)$$
$$= \frac{\lambda e^{-\mu K} - \lambda e^{-\mu(K-U) - \lambda U}}{\lambda e^{-\mu K} - \mu e^{-\lambda K}}, \tag{3.10}$$

and $\pi_0(U, K) = 1 - \pi_K(U, K)$, indicating that eventual emptiness or overflow is certain. First emptiness and first overflow in a finite dam are analogous to the problems of absorption in random walk between two absorbing barriers.

Each of (3.1), (3.2) and (3.3) may be inverted; as an example we consider (3.1). We may write this as

$$\varphi^*(0, s \mid U, K) = e^{\mu U - U(s+\lambda+\mu)/2}\left[e^{-\nu U} + \sum_{j=0}^{\infty} \varrho^{2j+2}\left\{e^{-((2j+2)K+U)\nu} - e^{-((2j+2)K-U)\nu}\right\}\right], \quad (3.11)$$

where $\varrho = (\lambda\mu)^{1/2}/\eta_1$. The inversion of each of the terms of (3.11) may be determined from ERDELYI [5], so that we obtain

$$\begin{aligned} G^*(0, t \mid U, K) &= e^{-\lambda t} + U(\lambda\mu)^{1/2} e^{-\mu U} \int_{\tau=U}^{t} e^{-(\lambda+\mu)\;(\tau-U/2)} \frac{I_1\{2(\lambda\mu\tau(\tau-U))^{1/2}\}}{\tau(\tau-U)} d\tau + \\ &+ e^{\mu U} \sum_{j=0}^{[(t-U)/K]-1} \Bigg\langle \int_{\tau=(j+1)K+U}^{t} e^{-(\lambda+\mu)(\tau-U/2)} \left[\left(\frac{\tau-(j+1)K-U}{\tau+(j+1)K}\right)^{j+1/2} I_{2j+1} - \right. \\ &\left. - (\lambda+\mu)\left(\frac{\tau-(j+1)K-U}{\tau+(j+1)K}\right)^{j+1} I_{2j+2}\right] d\tau - \\ &- e^{-(\lambda+\mu)(t-U/2)} \left(\frac{t-(j+1)K-U}{t+(j+1)K}\right)^{j+1} I_{2j+2} \\ &\qquad \{(4\lambda\mu\,(t+(j+1)K)(t-(j+1)K-U))^{1/2}\} + \Bigg\rangle \\ &+ e^{\mu U} \sum_{j=0}^{[t/K]-1} \Bigg\langle \int_{\tau=(j+1)K}^{t} e^{-(\lambda+\mu)(\tau-U/2)} \left[\left(\frac{\tau-(j+1)K}{\tau+(j+1)K-U}\right)^{j+1/2} I_{2j+1} - \right. \\ &\left. - (\lambda+\mu)\left(\frac{\tau-(j+1)K}{\tau+(j+1)K-U}\right)^{j+1} I_{2j+2}\right] d\tau \\ &- e^{-(\lambda+\mu)(t-U/2)} \left(\frac{t-(j+1)K}{t+(j+1)K-U}\right)^{j+1} I_{2j+2}\{(4\lambda\mu(t-(j+1)K)(t+ \\ &\qquad + (j+1)K-U))^{1/2}\} \Bigg\rangle \end{aligned}$$

where in the first summation I_{2j+1}, I_{2j+2} are modified Bessel functions with argument $2[\lambda\mu\{\tau+(j+1)K\}\{\tau-(j+1)K-U\}]^{1/2}$ while in the second summation, I_{2j+1}, I_{2j+2} have argument

$$2[\lambda\mu\{\tau-(j+1)K\}\{\tau+(j+1)K-U\}]^{1/2};$$

$$G^*(0, U \mid U, K) = e^{-\lambda U},$$

this being the discrete probability that first emptiness occurs at time $t = U$.

4. An application to an insurance risk problem

In this section we consider a special case of a problem in insurance risk as described by SEGERDAHL [12], BARTLETT [1] and CRAMÉR [3]. This model C assumes that the capital $Y(t)$ $(0 \leq t,\ Y(t) < \infty)$ of an insurance company increases from an initial value c (>0) at a uniform rate, which without loss of generality we take to be unity, due to collected premiums. Claims, whose size have a negative exponential distribution with mean μ^{-1}, occur in a homogeneous Poisson process with parameter λ to reduce the capital. The company goes bankrupt whenever a claim reduces the capital to zero. By solving an integral equation of the Volterra type Segerdahl showed that the probability π_0 of ultimate ruin is not greater than $\alpha\, e^{-c\theta_0}$ where α is a constant and θ_0 is the smallest positive root of the equation

$$e^{-\lambda\{M(\theta)-1\}-\theta} = 1\,, \tag{4.1}$$

where $M(\theta)$ is the LST of the distribution of claim size, which may be of a general form. BARTLETT [1] has derived the result with $\alpha = 1$ by making use of Wald's identity. CRAMÉR [3] has given a comprehensive survey of the problems in collective risk theory; by using another integral equation technique he has obtained the exact solution for a negative exponential claim size distribution which agrees with our result (4.2).

The elapse of time T before ruin occurs, known as the prosperous period, is a random variable, which is proper only when $\mu < \lambda$, as may be seen by analogy with queueing theory; we define $H(t\,|\,c) = \Pr\{T \leqq t\,|\,c\}$ with LST

$$\zeta(s\,|\,c) = \int_0^\infty e^{-st}\, dH(t\,|\,c)\ (Rl\, s \geqq 0)\,.$$

We prove

Theorem 3. *For model C the probability π_c that ruin ever occurs is*

$$\pi_c = \begin{cases} (\lambda/\mu)\, e^{-c(\mu-\lambda)} & \lambda < \mu \\ 1 & \lambda \geqq \mu\,, \end{cases} \tag{4.2}$$

and

$$\zeta(s\,|\,c) = (\lambda/\eta_1)\, e^{-c(\mu-\eta_2)}. \tag{4.3}$$

Proof. We can connect this insurance risk problem with the dam model A; we identify the uniform release in the dam with the steady inflow of premiums, the inputs with the claims, and the initial dam content U with the initial capital $c = K - U$. If we allow the dam to become infinitely deep then the probability of ruin is analogous to the probability of overflow in the dam. Hence we obtain π_c from (3.10) as

$$\pi_c = \lim_{K\to\infty} \pi_K(K-c, K) = \begin{cases} (\lambda/\mu)\, e^{-c(\mu-\lambda)} & \lambda < \mu \\ 1 & \lambda \geqq \mu\,, \end{cases}$$

which we wished to show. Similarly (4.3) follows from (3.2) as

$$\lim_{K\to\infty} \varphi^*(K, s\,|\,K-c, K)\,.$$

The non-zero root of (4.1) for claims having a negative exponential distribution is $\theta_0 = \mu - \lambda$ so that the upper bound for π_c found by BARLETT [1] when $\lambda < \mu$ is $e^{-c(\mu-\lambda)}$. This is inexact by a factor λ/μ independent of the initial capital c; however, if c is large we might expect that a company will accept a small margin of profit, i.e. $\mu - \lambda$ is small, so that $e^{-c(\mu-\lambda)}$ would be a reasonable approximation.

When ruin occurs in a finite time its mean may be obtained by differentiation of (4.3):

$$E(T \mid T < \infty) = \begin{cases} \frac{1}{2\mu}\{1 - c\mu + (\lambda^2 - \mu^2)(1 + c\mu)\} & \lambda > \mu \\ \frac{1}{2\mu}\{1 - c\mu + (\mu^2 - \lambda^2)(1 + c\mu)\} & \lambda < \mu . \end{cases}$$

We can invert (4.3) to give

$$H(t \mid c) = \lambda e^{c\mu}\Big[\int_{\tau=0}^{t} 2e^{-(\mu+\lambda)(\tau+c/2)}\{I_0\{(4\mu\lambda\tau(\tau+c))^{1/2}\} -$$
$$- \frac{(\lambda+\mu)}{2}\left(\frac{\tau}{\lambda\mu(\tau+c)}\right)^{1/2} I_1\{(4\mu\lambda\tau(\tau+c))^{1/2}\}\,d\tau -$$
$$- \frac{1}{2}\left(\frac{t}{\lambda\mu(t+c)}\right)^{1/2} e^{-(\mu+\lambda)(t+c/2)} I_1\{(4\mu\lambda t(t+c))^{1/2}\}\Big]. \qquad (4.4)$$

We define $H(z, t \mid c) = \Pr\{Y(t) \leqq z;\ Y(\tau) > 0,\ 0 < \tau < t \mid Y(0) = c > 0\}$ as the (improper) d.f. for the capital before ruin occurs. This is analogous to the zero and overflow avoiding distribution discussed in the previous section and we can prove

Theorem 4. For model C the LST $\zeta(z, s \mid c) = \int_{t=0}^{\infty} e^{-st} H(z, t \mid c)\,dt$ $(Rl\, s \geqq 0)$ is given by

$$\zeta(z, s \mid c) = \begin{cases} \frac{e^{-z(s+\lambda-\eta_2)}\{\eta_1 e^{-c(\eta_2-s-\lambda)} - \eta_2 e^{-c(\eta_1-s-\lambda)}\}}{\nu(s+\lambda-\eta_2)} & z > c \\ \zeta(c, s \mid c) + \lambda\mu e^{c(\eta_2-\mu)}[(\eta_1-\mu)\{e^{-z(\eta_2-\mu)} - e^{-c(\eta_2-\mu)}\} - \\ \quad - (\eta_2-\mu)\{e^{-z(\eta_1-\mu)} - e^{-c(\eta_1-\mu)}\}]\{\nu(\eta_1-\mu)(\eta_2-\mu)\eta_1\}^{-1} \\ \qquad 0 < z \leqq c. \end{cases} \qquad (4.5)$$

Proof. The proof follows immediately from (3.3) with the limit

$$\zeta(z, s \mid c) = \lim_{K\to\infty} \varphi^*(K - z, s \mid K - c, K).$$

This may be inverted to give $H(z, t \mid c)$ explicitly; as in the last two sections we do not show the results, although they may be obtained from the authors.

5. First emptiness with overflow

We now consider the probability distribution of first emptiness regardless of how often overflow occurs. We define

$$L(t \mid U, K) = \Pr\{Z(\tau) = 0 \quad \text{for some } \tau \text{ in} \quad 0 < \tau \leqq t \mid U, K\}$$

$$\Phi^*(s \mid U, K) = \int_{t=0}^{\infty} e^{-st}\,dL(t \mid U, K) \qquad (Rl\, s \geqq 0)$$

and prove

Theorem 5. *For model A*

$$\Phi^*(s\,|\,U, K) = \frac{e^{-(s+\lambda)U}\{(\eta_1-\lambda)\,e^{-(K-U)\eta_2} - (\eta_2-\lambda)\,e^{-(K-U)\eta_1}\}}{(\eta_1-\lambda)\,e^{-K\eta_2} - (\eta_2-\lambda)\,e^{-(K-U)\eta_2}}\,. \tag{5.1}$$

Proof. We begin with the discrete model B and define

$$f(t\,|\,u, k) = \Pr\{Z_t = 0;\, 0 < Z_j,\, 0 < j < t\,|\,Z_0 = u;\, k\}\,.$$

Following WEESAKUL [14]

$$f(t\,|\,u, k) = \underline{R}(u)\,\underline{Q}_1^{t-2}\,\underline{F}(1)\,,$$

where $\underline{R}(u)$ is the row vector $(0\ldots b\; ap \ldots ap\; q^{k-u-2}\; aq^{k-u-1})$, Q_1 is the matrix differing from $\underline{S}$ only in the first element of the first row which is ap, and $\underline{F}(1)$ is the column vector of elements $f(1\,|\,i, k)$ $(i = 1, 2, \ldots, k)$. Clearly $\underline{F}(1)$ has only one non-zero element which is b in the first place. As in § 3 we can determine the generating function of the probabilities of first emptiness; we obtain

$$\Phi(\theta\,|\,u, k) = \sum_{t=0}^{\infty} \theta^t f(t\,|\,u, k)$$

$$= \frac{(\theta b)^u\{(1-a\theta-\lambda_2)\,\lambda_1{}^{k-u-1} - (1-a\theta-\lambda_1)\,\lambda_2{}^{k-u-1}\}}{(1-a\theta-\lambda_2)\,\lambda_1{}^k - (1-a\theta-\lambda_1)\,\lambda_2{}^k}\,. \tag{5.2}$$

Making the substitution (1.5) and proceeding to the limit $\Delta \to 0$ we obtain

$$\Phi^*(s\,|\,U, K) = \lim_{\Delta\to 0} \Phi(e^{-s\Delta}\,|\,U\Delta^{-1}, K\Delta^{-1})\,,$$

which yields the required result (5.1).

We note that $\lim\limits_{s\to 0} \Phi^*(s\,|\,U, K) = 1$, verifying that eventual emptiness is certain.

6. The time-dependent distribution of the dam content

We wish to find the LST $\psi^*(z, s\,|\,U, K) = \int\limits_{t=0}^{\infty} F(z, t\,|\,U, K)\,e^{-st}\,dt$ $(Rl\,s \geqq 0)$ of the time-dependent solution of the dam content; we do this in

Theorem 6. *For model A*

$$\psi^*(z, s\,|\,U, K) = \begin{cases} e^{-(s+\lambda)U}\{(\eta_1-\lambda)\,e^{-(K-U)\eta_2} - (\eta_2-\lambda)\,e^{-(K-U)\eta_1}\} \times \\ \left\{\dfrac{\eta_1 e^{-z(\eta_2-s-\lambda)} - \eta_2 e^{-z(\eta_1-\lambda-s)}}{\nu s(\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1})}\right\} \quad 0 \leqq z \leqq U \\ \psi^*(U, s\,|\,U, K) + \{(s+\lambda-\eta_2)\,e^{-U\eta_2} - (s+\lambda-\eta_1)\,e^{-U\eta_1}\} \times \\ \left\{\dfrac{\eta_2(\eta_1-\lambda)\,e^{-K\eta_2}}{s+\lambda-\eta_1}(e^{z(\eta_2-\mu)+\mu U} - e^{U\eta_2}) - \dfrac{\eta_1(\eta_2-\lambda)\,e^{-K\eta_1}}{s+\lambda-\eta_2} \times \right. \\ \left.(e^{z(\eta_1-\mu)+\mu U} - e^{-U\eta_1})\right\}\{\nu s(\eta_1 e^{-K\eta_2} - \eta_2 e^{-K\eta_1})\}^{-1} U \leqq z \leqq K. \end{cases} \tag{6.1}$$

Proof. For model B we obtain by iteration of (1.4) that

$$Q_i(t\,|\,u, k) = \underline{R}(u)\,\underline{S}^{t-2}\,\underline{L}(i)\,, \tag{6.2}$$

where $\underline{L}(i)$ is the column vector with elements $Q_i(1\,|\,j, k)$ $(i = 0, 1, \ldots, k-1)$, which are given by

$$Q_i(1, |\, j, k) = \begin{cases} b + a q^{-j}(1 - q^{i+1}) & j = 0, 1, \ldots, i+1 \\ 0 & j > i+1\,. \end{cases} \tag{6.3}$$

It follows from (6.2) and (6.3) that the transform

$$\psi(i, \theta\,|\,u, k) = \sum_{t=0}^{\infty} Q_i(t\,|\,u, k)\,\theta^t (|\theta| \leqq 1)$$

is given by

$$\psi(i, \theta\,|\,u, k) = \frac{a\,\theta^2 \sum_{j=1}^{i+1} C_j (q^{-j} - q^{i+1-j}) + b\,\theta^2 \sum_{j=1}^{i+2} C_j}{|\underline{I} - \theta \underline{S}|}, \tag{6.4}$$

where C_j is similar to $|\underline{I} - \theta \underline{S}|$ except that the jth row of $(\underline{I} - \theta \underline{S})$ is replaced by $\underline{R}(u)$. Equation (6.4) is similar to (3.6) and we evaluate the determinants in the same way; we obtain

$$|\underline{I} - \theta \underline{S}| = (1-\theta)(\lambda_1 - \lambda_2)^{-1}\{(1-\lambda_2)\lambda_1^k - (1-\lambda_1)\lambda_2^k\},$$

$$C_n = \begin{cases} \theta^{u-n} b^{u-n+1} E_{k-u} G_{n-1} & n = 1, 2, \ldots, u \\ q^{n-u-1}(ap - bq) F_u E_{k-n+1} + bq E_{k-n} & n = u+1, \ldots \end{cases}$$

$$E_n = (\lambda_1 - \lambda_2)^{-1}\{(1 - a\theta - \lambda_2)\lambda_1^n - (1 - a\theta - \lambda_1)\lambda_2^n\} \qquad n \geqq 1$$

$$F_n = (\lambda_1 - \lambda_2)^{-1}\{(\lambda_1 - \theta b)\lambda_1^n - (\lambda_2 - \theta b)\lambda_2^n\} \qquad n \geqq 1$$

$$G_n = (\lambda_1 - \lambda_2)^{-1}[(1-\lambda_2)\lambda_1^n - (1-\lambda_1)\lambda_2^n - \theta b\{(1-\lambda_2)\lambda_1^{n-1} - (1-\lambda_1)\lambda_2^{n-1}\}] \qquad n \geqq 1,$$

with $E_0 = E_{-1} = F_0 = G_0 = 1$. We make the substitution (1.5) and obtain (6.1) from (6.4) from the limit process

$$\psi^*(z, s\,|\,U, K) = \lim_{\Delta \to 0} \Delta \psi(z\Delta^{-1}, e^{-s\Delta}\,|\,U\Delta^{-1}, K\Delta^{-1}).$$

The d.f. $F(z\,|\,K)$ of the stationary content distribution may be found from (6.1) by using an extension of Abel's theorem (Widder [*16*], Chapter 5) as

$$F(z\,|\,K) = \lim_{s \to 0} s\,\psi^*(z, s\,|\,U, K),$$

which agrees with (2.1)

7. Analogy with queueing theory

In a finite queueing system customers arriving at the end of a queue depart without waiting for service when the delay that will be caused is large; the queueing system $M/M/1$ with customers departing without joining the queue if the waiting time is greater than $L(>0)$ may be approximated by the finite dam in continuous time by an appropriate choice of K.

When the dam has infitine capacity we have the queueing system $M/M/1$ with the dam content being identified with the waiting time. The stationary distribution of the waiting time, which exists as a proper distribution if and only if $\lambda < \mu$ may be obtained by letting $K \to \infty$ in (2.1):

$$F(z) = \lim_{K \to \infty} F(z\,|\,K) = 1 - (\lambda/\mu)\,e^{-(\mu-\lambda)z} \qquad 0 \leqq z < \infty\,.$$

Letting $K \to \infty$ in (5.1) yields the LST of the time taken for the waiting time to first reduce to zero as

$$\Phi^*(s \mid U) = \lim_{K \to \infty} \Phi^*(s \mid U, K) = e^{-U(\eta_2 - \mu)},$$

which has been found by KENDALL [8]. Finally the LST $\psi^*(z, s \mid U)$ of the time-dependent distribution of the waiting time may be obtained from (6.1) as

$$\psi^*(z, s \mid U) = \lim_{K \to \infty} \psi^*(z, s \mid U, K)$$
$$= \begin{cases} \dfrac{(\eta_1 - \lambda)}{v s \eta_1} e^{-(s+\lambda)(U-z) + U\eta_2} \{\eta_1 e^{-z\eta_2} - \eta_2 e^{z\eta_1}\} & 0 \leqq z \leqq U \\ \psi^*(U, s \mid U, K) + \dfrac{\eta_2(\eta_1 - \lambda)}{v s \eta_1 (s + \lambda - \eta_1)} \{(s + \lambda - \eta_2) e^{-U\eta_2} \\ - (s + \lambda - \eta_1) e^{U\eta_1}\} \{e^{z(\eta_2 - \mu) + \mu U} - e^{U\eta_2}\} & z \geqq U. \end{cases}$$

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National Statistical Offices, Bangkok
and
Department of Mathematics
Stanford University
Stanford, California

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