Z. Wahrscheinlichkeitstheorie verw. Geb. 15, 111–130 (1970)

The Vibrating String Forced by White Noise

E. M. Cabaña

Summary. The equation of the vibrating string forced by white noise is formally solved, using stochastic integrals with respect to a *plane Brownian motion*, and it is proved that a certain process associated to the energy is a martingale. Then Doob's martingale inequality is used to furnish some probability bounds for the energy.

Such bounds provide a solution for the double barrier problem for the class of Gaussian stationary processes which can be represented as linear functionals of the positions and the velocities of the string.

0. Introduction

The (formal) stochastic differential equation

$$\dot{x}+a\,x=\dot{\beta} \tag{1}$$

in which $\dot{\beta}$ represents a Gaussian white noise, is ordinarily interpreted as equivalent to the integral equation

$$x(t)-x(0)+a\int_0^t x(\tau)\,d\tau=\beta(t)-\beta(0) \tag{2}$$

where β is a Brownian motion. More generally, a linear differential equation of higher order with a white noise in the right-hand member may be interpreted as the formally equivalent system of integral equations, where, instead of the meaningless $\dot{\beta}(t)$, we find a Brownian motion $\beta(t)$.

In the present paper we define a generalization of the ordinary Brownian motion (§ 1) which we call a *plane Brownian motion*, and we apply it to propose a stochastic integral equation for the vibrating string, forced with a *plane white noise* (that is, a forcing term which presents the randomness of the ordinary white noise through time as well as along the length of the string) (§§ 2, 3).

The equation of the vibrating string is solved formally, in the space of Fourier transforms, without restricting the domain of the operator appearing in the equation of the string to a set of functions satisfying suitable boundary conditions (see Feller [3, 4]); this should be done if the results are to be interpreted in the original space, by applying the inverse Fourier transform. Then some martingale properties of the energy associated to the string are proved in § 3, and some probabilistic bounds for the energy are obtained using Doob's martingale inequality.

As an application, stationary Gaussian processes are represented as linear combinations of the positions and velocities associated to the string (§ 4). Now the probabilistic bounds for the energy lead to analogous bounds for the absolute value of such a process, and this allows us to compute bounds for the probability that such a Gaussian stationary process $\gamma(t)$ remains bounded between a double barrier during a prescribed period of time $\left(P\{|\gamma(s)|\leq k+a\,s \text{ for all } s\in(0,t)\}\right)$.

1. Stochastic Preliminaries. Brownian Motion on the Plane and Stochastic Integrals

1.1. *Definition and construction of a Brownian motion on a plane half-strip.*

Let J be the half-line $[0, \infty)$, let I be an interval of the real line (possibly infinite at one or both ends) containing 0, and let us consider the space Ω of continuous paths β: $J \times I \to R$ such that $\beta(0, z) = \beta(t, 0) = 0$ for all $z \in I$, $t \in J$. On this space we consider the σ-field $\mathscr{B}_\infty$ generated by the events $\{\beta | a \leqq \beta(t, z) < b\}$ ($a < b$, $t \in J$, $z \in I$) and impose probabilities such that, denoting the double increment $\beta(b, d) - \beta(a, d) - \beta(b, c) + \beta(a, c)$ by $\beta(S)$ $(S = \{(t, z) | a \leqq t < b, c \leqq z < d\})$, then

(i) for each rectangle S in $J \times I$ with edges parallel to the coordinate axes and area $|S|$, the variable $\beta(S)$ is Gaussian with zero mean and variance equal to $|S|$,

(ii) if S_1, S_2 are two disjoint rectangles as above, then $\beta(S_1)$, $\beta(S_2)$ are independent.

The probability space $(\Omega, \mathscr{B}_\infty, P)$ such that P induces probabilities that satisfy (i), (ii) is called a *Brownian motion* on $J \times I$.

In order to prove that this assignment of probabilities is possible, let us construct a Brownian motion on $J \times I$, adapting for that purpose Ciesielski's construction of a Brownian motion on the line ([2, 7]).

When the interval $[0, 1)$ is substituted for J and $I = [0, 1)$, this construction may be applied with minor changes. It is based on the use of the family of Haar functions $\{\psi_r | r \in \mathscr{T} = \text{set of dyadic rationals in } (0, 1]\}$, which is a complete orthonormal set in $L^2(0, 1)$. The Schauder functions

$$\varphi_r(z) = \int_0^z \psi_r(\zeta)\, d\zeta,$$

defined by

$$\varphi_1(z) = z,$$

$$\varphi_r(z) = 2^{-(i(r)+1)/2} \max\{1 - |z - r| 2^{i(r)}, 0\} \qquad (r \neq 1)$$

where $i(r)$ is the unique positive integer such that r may be written as an irreducible fraction $r = k/2^{i(r)}$, satisfy

$$\sum_{r \in T} \varphi_r(z')\, \varphi_r(z'') = \min\{z', z''\} \qquad (z', z'' \in (0, 1]) \tag{1}$$

as a consequence of Parseval's identity applied to the characteristic functions of the intervals $(0, z')$, $(0, z'')$.

Let us define

$$\beta(t, z) = \sum_{\substack{m=0\\ n=0}}^{\infty} \Big(\sum_{\substack{i(r)=m\\ i(s)=n}} g_{rs}\, \varphi_r(t)\, \varphi_s(z) \Big) \tag{2}$$

where $(g_{rs} | r, s \in T)$ are mutually independent standard Gaussian variables $(E\{g_{rs}\} = 0, E\{g_{rs}^2\} = 1, r, s \in T)$.

The proof of the uniform convergence with probability one of the one-dimensional analogue of our double series in m, n, applies also here with the obvious modifications, proving the continuity of the paths of the process defined by (2). In fact, if

$$e_{mn} = \sup_{t,z\in I} \Big| \sum_{\substack{i(r)=m \\ i(s)=n}} g_{rs}\, \varphi_r(t)\, \varphi_s(z) \Big|,$$

it may be shown (following [7] for instance) that

$$P\Big[\bigcup_{k=1}^{\infty} \bigcap_{m+n=k}^{\infty} \{e_{mn} \leqq \theta \sqrt{2^{-m-n} \log 2^{m+n}}\}\Big] = 1$$

for $\theta > \frac{1}{2}$; this implies that $\sum_{mn} e_{mn}$ converges with probability one, hence the right-hand side in (2) converges uniformly with probability one.

Since the set of variables $\{\beta(t,z) \,|\, t, z \in [0,1)\}$ is a Gaussian family (i.e., their linear combinations are all Gaussian variables with zero means), it only remains to check that any two of those variables have the required covariance, namely

$$E\{\beta(t',z')\,\beta(t'',z'')\} = \min\{t',t''\}\,\min\{z',z''\}, \tag{3}$$

in order to show that the properties (i) and (ii) are satisfied.

Using (2) we obtain

$$E\{\beta(t',z')\,\beta(t'',z'')\} = \sum_{\substack{m=0 \\ n=0}}^{\infty} \Big(\sum_{\substack{i(r)=m \\ i(s)=n}} \varphi_r(t')\,\varphi_r(t'')\,\varphi_s(z')\,\varphi_s(z'') \Big)$$

and (3) follows from (1).

Finally, piecing together independent copies $\{\beta_k \,|\, k=1,2,\ldots\}$ of this Brownian motion on $[0,1)\times[0,1)$ we obtain a Brownian motion β on the half-plane $J\times R$; its restriction to $J\times I$ is the desired process.

1.2. *Brownian motion related to a canonical measure.*

From now on, I will be an open interval. Let m be a Borel Measure on I, which is finite on closed intervals and positive on open intervals (such a measure will be called a *canonical measure*). The symbol m will also be used to represent the distribution function

$$m(z) = m(z+) = \begin{cases} m(0,z] & \text{for } z \geqq 0 \\ -m(z,0] & \text{for } z < 0. \end{cases}$$

Let us call I_m the (possibly infinite) range of the function m and let β_m be a Brownian motion on $J\times I_m$. Then the process β on $J\times I$ defined by $\beta(t,z) = \beta_m(t, m(z))$ is said to be a *m-Brownian motion* on $J\times I$.

A m-Brownian motion satisfies the following properties derived from (i) and (ii):

(i′) for each rectangle S in $J\times I$ with edges parallel to the coordinate axes and $dt\times dm$ *measure* $|S|$, the variable $\beta(S)$ is Gaussian with zero mean and variance $|S|$.

(ii′) same as (ii).

As a function of (t, z), β is continuous for all t and all the points z of continuity of m.

1.3. *Stochastic integrals with respect to plane Brownian motions.*

Double stochastic integrals with respect to plane Brownian motions may be defined in regions of a very particular shape (namely on $[0, t)\times I$) but general enough to serve our present applications, following the same procedure as employed for the definition of the *simple integrals* with almost no additional trouble.

The scheme of McKean may be closely followed. For the proofs and details, we refer to [7].

Let $\mathscr{B}_t (t\in J)$ be the σ-field generated by the events $\{a\leqq\beta(t', z)<b \mid a<b, 0\leqq t'\leqq t, z\in I\}$, and $\mathscr{B}_t^+$ the σ-field generated by the process $\beta_t^+(t', z)=\beta(t+t', z)-\beta(t, z)$. An increasing family $\{\mathscr{A}_t \mid t\in J\}$ of σ-fields is given in such a way that for each $t\in J$, $\mathscr{A}_t$ is independent of $\mathscr{B}_t^+$ and includes $\mathscr{B}_t$.

A function f on $J\times I\times \Omega$ to R is said to be a *non-anticipating functional* when

(i) f is measurable in the product of the Borel field on $J\times I$ and the field $\mathscr{A}_\infty$ generated by the union of the family $\{\mathscr{A}_t \mid t\in J\}$,

(ii) for each $t\in J, z\in I, f(t, z)$ is $\mathscr{A}_t$-measurable.

In the following the symbol $\iint\limits_0^T$ will be used as an abbreviation for $\iint\limits_{(0,T)\times I}$, T denoting any positive real number or $+\infty$.

When

$$P\{\iint\limits_0^T f^2(t, z)\, dt\, dm<\infty\}=1 \tag{1}$$

the integral

$$\iint\limits_0^t f\, d\beta=\iint\limits_0^t f(\tau, z)\, d\beta(\tau, z) \qquad (0\leqq t<T) \tag{2}$$

is defined as a function of t and of the Brownian path, satisfying the following conditions[1]:

0. If f is the characteristic function of the rectangle S with edges parallel to the coordinate axes, and if $S\subset(0, T)\times I$, then $\iint\limits_0^T f\, d\beta=\beta(S)$;

1. $\iint\limits_0^t f_1\, d\beta+\iint\limits_0^t f_2\, d\beta=\iint\limits_0^t (f_1+f_2)\, d\beta$ for all $t\in(0, T)$;

2. $\iint\limits_0^t kf\, d\beta=k\iint\limits_0^t f\, d\beta$ for all $t\in(0, T)$ and any real constant k;

3. $\iint\limits_0 f\, d\beta$ is a pathwise continuous function of t with probability 1;

[1] The integrands appearing in the stochastic integrals are assumed to satisfy the analogue of (1).

4. If $\mathfrak{t}$ is a stopping time (i.e., $\mathfrak{t}$: $\Omega \to J$ is such that $\{\beta \mid \mathfrak{t}(\beta) < t\} \in \mathscr{B}_t$), for each $0 \leqq t < T$) and if $\chi_{\mathfrak{t}}$ is the characteristic function of $\{t \mid t < \mathfrak{t}\}$, then

$$\iint_0^{\mathfrak{t}} f \, d\beta = \iint_0^T f \chi_{\mathfrak{t}} \, d\beta;$$

5. If $E\left\{\iint_0^T f^2 \, d\tau \, dm\right\} < \infty$, then

(i) $E\left\{\iint_0^T f \, d\beta\right\} = 0.$

(ii) $E\left\{\iint_0^T f \, d\beta \mid \mathscr{A}_s\right\} = \iint_0^s f \, d\beta$ for $0 \leqq s \leqq t < T.$

(iii) $E\left\{\left(\iint_0^T f \, d\beta\right)^2\right\} = E\left\{\iint_0^T f^2 \, d\tau \, dm\right\}.$

There is little or no change when the integrand f takes its values on a separable Hilbert space H, instead of R. The measurability to be applied is, equivalently, the strong measurability, or the weak one. The stochastic integral satisfies properties 0 to 4. It also satisfies the following property 5′, which is the natural generalization of 5.

5′. If $E\left\{\iint_0^T \|f\|^2 \, d\tau \, dm\right\} < \infty$, then $E\left\{\iint_0^T f \, d\beta\right\} = 0$, $E\left\{\iint_0^T f \, d\beta \mid \mathscr{A}_s\right\} = \iint_0^s f \, d\beta$ for $0 \leqq s \leqq t < T$, and $E\left\{\left\|\iint_0^T f \, d\beta\right\|^2\right\} = E\left\{\iint_0^T \|f\|^2 \, d\tau \, dm\right\}.$

The integrals of Hilbert-space-valued functionals with respect to Brownian motions on the line are a very particular case of those considered in [1]. The corresponding Itô's formula for the stochastic differentiation of composite functions is proved there. One version of Itô's formula that holds in the present context reads as follows.

Lemma 1.1. *Let $\varphi = \varphi(t, x)$ be a function from $J \times H$ to R with continuous partial derivative $D_0 \varphi = \partial\varphi/\partial t$. Let φ also have first and second derivatives $D\varphi$ (on $J \times H$ to H) and $D^2 \varphi$ (on $J \times H$ to the space of bounded linear operators $H \to H$) with respect to $x \in H$, such that $D^2 \varphi$ is symmetric and continuous in the operator topology, and*

$$\varphi(t_0 + t, x_0 + x) = \varphi(t_0, x_0) + D_0 \varphi(t_0, x_0) \, t + (D\varphi(t_0, x_0), x) + \tfrac{1}{2}(D^2 \varphi(t_0, x_0) \, x, x) + o_1(t, x) + o_2(t, x)$$

where

$$\lim_{|t|, \|x\| \to 0} |t|^{-1} o_1(t, x) = 0, \quad \lim_{|t|, \|x\| \to 0} \|x\|^{-2} o_2(t, x) = 0.$$

Let β be a m-Brownian motion on $J \times I$, and let $g(t), f(t, z)$ be non-anticipating H-valued processes such that $E\left\{\int_0^T \|g\|^2 \, d\tau\right\} < \infty$, $E\left\{\iint_0^T \|f\|^2 \, d\tau \, dm\right\} < \infty.$

Now, if γ is the stochastic integral

$$\gamma(t)=a+\int_0^t g(\tau)\,d\tau+\iint_0^t f(\tau,z)\,d\beta(\tau,z), \tag{3}$$

the formula

$$\begin{aligned}\varphi(t,\gamma(t))=\varphi(0,a)&+\int_0^t D_0\,\varphi(\tau,\gamma(\tau))\,d\tau+\int_0^t (D\varphi(\tau,\gamma(\tau)),g(\tau))\,d\tau\\ &+\iint_0^t (D\varphi(\tau,\gamma(\tau)),f(\tau,z))\,d\beta(\tau,z)\\ &+\tfrac{1}{2}\iint_0^t (D^2\varphi(\tau,\gamma(\tau))f(\tau,z),f(\tau,z))\,d\tau\,dm\end{aligned} \tag{4}$$

holds with probability one for all $0\leqq t<T$, simultaneously.

Proof. As in [1], the following reductions can be made without loss. Let us introduce the stopping time

$$T_n=\sup\{t\in J\,|\,\text{for } 0\leqq\tau<t,\ \|D\varphi(\gamma(\tau))\|+\|D^2\varphi(\gamma(\tau))\|<n\}$$

which satisfies $P\{\lim_{m\to\infty} T_n\geqq T\}=1$, and prove (4) only for each stopped process

$$\gamma^{(n)}(t)=a+\int_0^t \chi_n g\,d\tau+\iint_0^t \chi_n f\,d\beta,$$

where χ_n is the characteristic function of the interval $(0, T_n)$. Since the integrals in (3) are continuous functions of t, it is enough to prove that the equality holds for each fixed t with probability one. Also since the integrands may be approximated by linear combinations of characteristic functions of rectangles with edges parallel to the coordinate axes (the so-called simple functionals), it suffices to prove (4) for constant g and f (using the additivity of the integrals).

But in that case, $\gamma(t)$ reduces to $a+g\,t+f\beta(t)$ with $\beta(t)=\beta((0,t)\times I)$ and Itô's formula for integrals with respect to Brownian motions in the line applies to this case, since β is such a process up to a constant factor[2]. This leads easily to the desired result.

2. The Vibrating String

This paragraph is just a sketch of some facts related to the formulation and solution of the equation of the damped vibrating string. Its main purpose is to establish the notations to be used below.

Instead of the classical equation

$$\partial^2 u(t,z)/\partial t^2+2b\,\partial u(t,z)/\partial t=\partial^2 u(t,z)/\partial z^2+F(t,z) \tag{1}$$

the generalized point of view of Feller ([4]) is adopted, considering the string as a family of pairs of functions $u(t,\bullet)$, $v(t,\bullet)$ defined on the interval I and depending

[2] This version of Itô's formula is not a particular case of the one proved in [1], because the function φ there considered did not depend on the first variable (i.e., it was defined on H instead of $J\times H$); its actual proof can be made following the same lines as for the scalar case (cf. [7], for instance) and dealing with the additional troubles introduced by the Hilbert-space context as in [1], but the present case is much simpler.

on the real parameter t. The function $u(t,\bullet)$ represents the position of the string at time t, and the function $v(t,\bullet)$ represents the velocity. As before, it is assumed (without loss of generality) that the interval I contains 0.

A canonical measure m is given on I and the corresponding Hilbert space $L^2(I,m)$ is called H_m, its norm being denoted by

$$\|\varphi\|_m=(\int_I \varphi^2\, dm)^{\frac{1}{2}}.$$

A Borel measure γ finite on closed intervals is given on I. An operator L is defined by

$$L\varphi\, dm=d\varphi'-\varphi\, d\gamma \tag{2}$$

acting on the set $\mathscr{D}$ of continuous functions φ with derivative $\varphi'=d\varphi/dx$ at each point where m and γ are continuous, and one-sided derivatives φ^+, φ^- at all points, such that φ' is of bounded variation; the meaning of (2) is that both terms are equal as measures.

L anihilates two independent positive convex functions ψ_1, ψ_2 on I; for any such ψ, L can be expressed as

$$L\varphi=\frac{1}{\psi}\frac{d}{dm}\left(\psi^2\frac{d}{dx}\left(\frac{\varphi}{\psi}\right)\right)$$

with $d\gamma=d\psi'/\psi$ ([4]).

The operator L induces a Fourier transform, as shown by McKean [6] (cf. also [5]), which maps functions in H_m into a new Hilbert space H_f of functions on $(-\infty,0]$ whose values are vectors of two components.

From now on, unless the contrary is specified, the indexes will run over the set $\{1,2\}$. Let $e_i(z,\mu)$ be the solution of

$$L e_i(\bullet,\mu)=\mu\, e_i(\bullet,\mu)$$
$$e_1(0,\mu)=e_2^+(0,\mu)=1,\ e_2(0,\mu)=e_1^+(0,\mu)=0.$$

Then there exists a Borel Measure $f=(f^{ij})$ from $(-\infty,0]$ to 2×2 symmetric non-negative definite matrices, such that if H_f is $L^2((-\infty,0],(df^{ij}))$ with the norm of a 2-vector function $\varphi(\bullet)=(\varphi_1(\bullet),\varphi_2(\bullet))$ denoted by

$$\|\varphi(\bullet)\|=\left(\int_{-\infty}^{0+}\varphi_i(\mu)\,\varphi_j(\mu)\,df^{ij}(\mu)\right)^{\frac{1}{2}} \text{[3]} \tag{3}$$

then the Fourier transform from H_m to H_f may be defined by

$$u(\in H_m)\to(\hat{u}_i(\bullet))=(\int_I u(z)\, e_i(z,\bullet)\, dm(z))\in H_f \tag{4}$$

which has the inverse

$$\hat{u}=(\hat{u}_i)(\in H_f)\to u(\bullet)=\int_{-\infty}^{0+}\hat{u}_i(\mu)\, e_j(\bullet,\mu)\, df^{ij}(\mu)\in H_m. \text{[3]} \tag{5}$$

Furthermore, the Plancherel theorem $\|u\|_m=\|\hat{u}\|$ holds.

[3] The repeated index denotes summation.

Now the equation of the damped vibrating string, with an external force F and given initial conditions $u(0,\bullet)$, $v(0,\bullet)$, may be written as a system of integral equations

$$\begin{aligned} u(t,\bullet)&=u(0,\bullet)+\int_0^t v(\tau,\bullet)\,d\tau \\ v(t,\bullet)&=v(0,\bullet)+\int_0^t \big(Lu(\tau,\bullet)-2b\,v(\tau,\bullet)\big)\,d\tau+\int_0^t F(\tau,\bullet)\,d\tau \end{aligned} \tag{6}$$

with formal solution

$$\begin{pmatrix} u(t,\bullet)\\ v(t,\bullet)\end{pmatrix}=e^{Bt}\begin{pmatrix} u(0,\bullet)\\ v(0,\bullet)\end{pmatrix}+\int_0^t e^{B(t-\tau)}\begin{pmatrix}0\\1\end{pmatrix}F(\tau,\bullet)\,d\tau, \tag{7}$$

where $B=\begin{pmatrix}0 & 1\\ L & -2b\end{pmatrix}$.

The transformed version of (7) is

$$\begin{aligned} \hat{u}_i(t,\mu)&=\varepsilon_1^1(t,\mu)\,\hat{u}_i(0,\mu)+\varepsilon_2^1(t,\mu)\,\hat{v}_i(0,\mu)\\ &\quad+\iint_0^t \varepsilon_j^1(t,\mu)\,\varepsilon_2^j(-\tau,\mu)\,e_i(z,\mu)\,F(\tau,z)\,d\tau\,dm(z)\\ \hat{v}_i(t,\mu)&=\varepsilon_1^2(t,\mu)\,\hat{u}_i(0,\mu)+\varepsilon_2^2(t,\mu)\,\hat{v}_i(0,\mu)\\ &\quad+\iint_0^t \varepsilon_j^2(t,\mu)\,\varepsilon_2^j(-\tau,\mu)\,e_i(z,\mu)\,F(\tau,z)\,d\tau\,dm(z) \end{aligned} \tag{8}$$

where

$$\begin{pmatrix}\varepsilon_1^1(t,\mu) & \varepsilon_2^1(t,\mu)\\ \varepsilon_1^2(t,\mu) & \varepsilon_2^2(t,\mu)\end{pmatrix}=\exp(B_\mu t),\qquad B_\mu=\begin{pmatrix}0 & 1\\ \mu & -2b\end{pmatrix}. \tag{9}$$

Because of the nature of the external force that we wish to apply, the form (8) of writing the formal solutions will be more suitable. Let us formulate the remainder of this paragraph in the space of transforms. The analogous formulae in terms of functions of $z\in I$ may be easily worked out.

Two new Hilbert spaces are introduced; H'_f is the space of functions on $(-\infty,0]$ to two-vectors, with norm given by

$$|||\varphi(\bullet)|||=\left(\int_{-\infty}^{0+}(-\mu)\,\varphi_i(\mu)\,\varphi_j(\mu)\,df^{ij}(\mu)\right)^{\frac{1}{2}},$$

and inner product denoted by $((\bullet,\bullet))$ and $H_{\mathscr{E}}$ is the space of pairs $(\varphi;\psi)$ with $\varphi=(\varphi_i)\in H'_f$, $\psi=(\psi_i)\in H_f$ and norm

$$\|(\varphi;\psi)\|_{\mathscr{E}}=(|||\varphi|||^2+\|\psi\|^2)^{\frac{1}{2}}.$$

If $(\varphi;\psi)\in H_{\mathscr{E}}$, $\frac{1}{2}\|(\varphi;\psi)\|_{\mathscr{E}}^2$ is said to be the energy of $(\varphi;\psi)$; the two terms $\frac{1}{2}|||\varphi|||^2$ and $\frac{1}{2}\|\psi\|^2$ into which the energy may be decomposed are respectively the *potential energy* and the *kinetic energy*.

Lemma 2.1. *The function* $\hat{U}^0=(\hat{u}^0;\hat{v}^0)$, *defined by*

$$\begin{aligned} \hat{u}^0(t)&=\hat{u}^0(t,\bullet)=(\hat{u}_i^0(t,\bullet))=(\varepsilon_1^1(t,\bullet)\,\hat{u}_i(0,\bullet)+\varepsilon_2^1(t,\bullet)\,\hat{v}_i(0,\bullet))\\ \hat{v}^0(t)&=\hat{v}^0(t,\bullet)=(\hat{v}_i^0(t,\bullet))=(\varepsilon_1^2(t,\bullet)\,\hat{u}_i(0,\bullet)+\varepsilon_2^2(t,\bullet)\,\hat{v}_i(0,\bullet)) \end{aligned} \tag{10}$$

has the following properties:

(i) *if* $b=0$, *the energy of* $\hat{U}^0$ *is constant*

(ii) *for any* $b \geqq 0$, $\frac{1}{2}\|\hat{U}^0(t,\bullet)\|_{\mathscr{E}}^2 + 2b\int_0^t \|\hat{v}^0(\tau,\bullet)\|^2\, d\tau$ *is constant.*

Property (i) expresses the conservation of energy, when no external forces act on the system, and property (ii) describes how the damping reduces the energy. Since (i) is a particular case of (ii), we only prove the latter property.

The matrix $(\varepsilon_j^i(t,\mu))$ may be written as follows. For $-\infty<\mu<-b^2$ and $r=\sqrt{-\mu-b^2}$,

$$\varepsilon_j^i(t,\mu)=\begin{pmatrix} e^{-bt}\left(\cos rt+\dfrac{b}{r}\sin rt\right) & e^{-bt}\dfrac{\sin rt}{r} \\ e^{-bt}\dfrac{\mu}{r}\sin rt & e^{-bt}\left(\cos rt-\dfrac{b}{r}\sin rt\right) \end{pmatrix} \tag{11}$$

while, for $-b<\mu\leqq 0$ and $r'=\sqrt{\mu+b^2}$,

$$\varepsilon_j^i(t,\mu)=\begin{pmatrix} e^{-bt}\left(\cosh r't+\dfrac{b}{r'}\sinh r't\right) & e^{-bt}\dfrac{\sinh r't}{r'} \\ e^{-bt}\dfrac{\mu}{r'}\sinh r't & e^{-bt}\left(\cosh r't-\dfrac{b}{r'}\sinh r't\right) \end{pmatrix}; \tag{11'}$$

finally

$$\varepsilon_j^i(t,-b^2)=\begin{pmatrix} e^{-bt}(1+bt) & t\,e^{-bt} \\ \mu t\,e^{-bt} & e^{-bt}(1-bt) \end{pmatrix}. \tag{11''}$$

The estimates

$$\begin{aligned} (\varepsilon_i^i(t,\mu))^2 &\leqq 4e^{4t^-} \qquad (i=1,2), \\ (-\mu)(\varepsilon_1^1(t,\mu))^2 &\leqq e^{4t^-}, \\ (-\mu)^{-1}(\varepsilon_1^2(t,\mu))^2 &\leqq e^{4t^-}, \end{aligned} \tag{12}$$

with $t^-=\max\{0,-t\}$, allow us to conclude that

$$\begin{aligned} &\varepsilon_1^1(t,\bullet)\,\hat{u}(0,\bullet)\in H_f', \\ &\varepsilon_2^1(t,\bullet)\,\hat{v}(0,\bullet)\in H_f', \\ &\varepsilon_1^2(t,\bullet)\,\hat{u}(0,\bullet)\in H_f, \end{aligned}$$

and

$$\varepsilon_2^2(t,\bullet)\,\hat{v}(0,\bullet)\in H_f.$$

Therefore, $\hat{u}^0\in H_f'$, $\hat{v}^0\in H_f$, and hence $\hat{U}^0\in H_{\mathscr{E}}$.

From

$$\begin{pmatrix} \dot{\varepsilon}_1^1 & \dot{\varepsilon}_2^1 \\ \dot{\varepsilon}_1^2 & \dot{\varepsilon}_2^2 \end{pmatrix} = \frac{\partial}{\partial t}\exp(B_\mu t)=B_\mu\exp(B_\mu t)=\begin{pmatrix} \varepsilon_1^2 & \varepsilon_2^2 \\ \mu\varepsilon_1^1-2b\,\varepsilon_1^2 & \mu\varepsilon_2^1-2b\,\varepsilon_2^2 \end{pmatrix}$$

it follows that

$$\frac{\partial \hat{u}^0(t,\mu)}{\partial t} = \hat{v}^0(t,\mu),$$

$$\frac{\partial \hat{v}^0(t,\mu)}{\partial t} = \mu \hat{u}^0(t,\mu) - 2b\, \hat{v}^0(t,\mu).$$

Hence

$$\frac{\partial}{\partial t}\left(-\mu \hat{u}_i^0(t,\mu)\, \hat{u}_j^0(t,\mu) + \hat{v}_i^0(t,\mu)\, \hat{v}_j^0(t,\mu)\right) = -4b\, \hat{v}_i^0(t,\mu)\, \hat{v}_j^0(t,\mu)$$

and therefore

$$\frac{\partial}{\partial t}\left(|||\hat{u}^0(t)|||^2 + \|\hat{v}^0(t)\|^2 + 4b \int_0^t \|\hat{v}^0(\tau)\|^2\, d\tau\right) = 0.$$

This proves (ii).

3. Some Properties of the Formal Stochastic Solution of the Equation of the Vibrating String

If the external force F that appears in the system § 2 (6) is replaced by the formal mixed second derivative $\partial^2 \beta(t,z)/\partial t\, \partial m(z)$ of a given m-Brownian motion β on $J \times I$, the equations § 2 (8) are replaced by

$$\begin{aligned} \hat{u}(t) &= \hat{u}^0(t) + \varepsilon_j^1(t,\bullet)\, \gamma^j(t), \\ \hat{v}(t) &= \hat{v}^0(t) + \varepsilon_j^2(t,\bullet)\, \gamma^j(t) \end{aligned} \tag{1}$$

with $\hat{u}^0$, $\hat{v}^0$ given by § 2 (10) and

$$\gamma_i^j(t) = \iint_0^t \varepsilon_2^j(-\tau,\bullet)\, e_i(z,\bullet)\, d\beta(\tau,z), \qquad i=1,2;$$

the formal differential $\partial^2 \beta/\partial t\, \partial m$ has been replaced by the true stochastic differential $d\beta(\tau,z)$. We shall say that (1) is the solution of the equation of the string forced with *plane white noise:*

$$\partial^2 u/\partial t^2 + 2b\, \partial u/\partial t = Lu + \partial^2 \beta/\partial t\, \partial m.$$

It will be convenient to apply a slightly modified forcing term, in order to assure some required convergences, and that is the reason why a factor $G(\bullet)$ will be introduced in the definition of γ^j.

Let us assume that $\hat{U}(0) = (\hat{u}(0), \hat{v}(0))$ is a given $\mathscr{A}_0$-measurable random variable such that $P\{\|\hat{U}(0)\|_{\mathscr{E}} < \infty\} = 1$, and let us define the process $\hat{U}(t) = (\hat{u}(t), \hat{v}(t))$ by means of (1), where the stochastic integrals γ_i^j are defined in (3) as the integrals of the functions $\varepsilon_2^j(-\tau,\bullet)\, e_i(z,\bullet)\, G(\bullet)$ for which the assumption

$$k = \int_I \int_{-\infty}^{0+} G^2(\mu)\, e_i(z,\mu)\, e_j(z,\mu)\, df^{ij}(\mu)\, dm(z) < \infty \tag{2}$$

is made. The estimates in § 2 (12) imply (for all t simultaneously) that

$$\begin{aligned}\gamma_i^1(t)&=\iint_0^t \varepsilon_2^1(-\tau,\bullet)\, G e_i(z,\bullet)\, d\beta(\tau,z)\in H_f',\\ \gamma_i^2(t)&=\iint_0^t \varepsilon_2^2(-\tau,\bullet)\, G e_i(z,\bullet)\, d\beta(\tau,z)\in H_f,\end{aligned} \tag{3}$$

with probability one.

A priori the values of $\hat{U}$ are just functions on $(-\infty,0]$, but using (3) and the bounds in § 2 (12) as in the first part of the proof of Lemma 2.1, it follows that $\hat{u}(t)\in H_f'$, $\hat{v}(t)\in H_f$, and hence $\hat{U}(t)\in H_{\mathscr{E}}$ simultaneously for all t, with probability one.

Lemma 3.1. *The process*

$$z(t)=\|\hat{U}(t)\|_{\mathscr{E}}^2+4b\int_0^t \|\hat{v}(\tau)\|^2\, d\tau-k\,t, \tag{4}$$

where k $(<\infty)$ is defined by (2), *is given by the stochastic integral*

$$z(t)=z(0)+2\iint_0^t \big(v(\tau), G e(z,\bullet)\big)\, d\beta(\tau,z) \tag{5}$$

(the symbol $e(z,\bullet)$ denotes the vector-valued function $\big(e_i(z,\bullet)\big)$).

Proof. Let us consider the function

$$\varphi(t,\gamma^1,\gamma^2)=\|\hat{u}^0(t)+\varepsilon_j^1(t,\bullet)\,\gamma^j\|^2+\|\hat{v}^0(t)+\varepsilon_j^2(t,\bullet)\,\gamma^j\|^2$$

defined for $t\in J$ and $(\gamma^1,\gamma^2)\in H_{\mathscr{E}}$. Its derivative with respect to t,

$$D_0\,\varphi(t)=-4b\|\hat{v}(t)\|^2,$$

may be computed as in Lemma 2.1. Lemma 1.1 applied to φ gives

$$\begin{aligned}\|\hat{U}(t)\|_{\mathscr{E}}^2&=\|\hat{U}(0)\|_{\mathscr{E}}^2+\int_0^t(-4b)\|\hat{v}(\tau)\|^2\, d\tau\\ &\quad+2\iint_0^t \big((\hat{u}(\tau),\varepsilon_j^1(\tau,\bullet)\,\varepsilon_j^2(-\tau,\bullet)\,G(\bullet)\,e(z,\bullet))\big)\, d\beta(\tau,z)\\ &\quad+2\iint_0^t \big(\hat{v}(\tau),\varepsilon_j^2(\tau,\bullet)\,\varepsilon_2^j(-\tau,\bullet)\,G(\bullet)\,e(z,\bullet)\big)\, d\beta(\tau,z)\\ &\quad+\iint_0^t \|\varepsilon_j^2(\tau,\bullet)\,\varepsilon_2^j(-\tau,\bullet)\,G(\bullet)\,e(z,\bullet)\|^2\, d\tau\, dm(z).\end{aligned} \tag{6}$$

Since $\varepsilon_j^i(\tau,\bullet)\,\varepsilon_k^j(-\tau,\bullet)=\delta_k^i$, (6) reduces to

$$\|\hat{U}(t)\|_{\mathscr{E}}^2=\|\hat{U}(0)\|_{\mathscr{E}}^2-4b\int_0^t\|\hat{v}(\tau)\|^2\, d\tau+k\,t+2\iint_0^t\big(\hat{v}(\tau), G(\bullet)\,e(z,\bullet)\big)\, d\beta(\tau,z)$$

and the required result follows.

Corollary 1. *The process $z(t)$ is a martingale, with respect to the family of σ-fields $\mathscr{A}_t$ $(t\in J)$.*

Corollary 2. *The following estimates hold for $|2k\lambda t|<1$:*

(i)
$$E\{(z(t)+kt)^n|\mathscr{A}_0\}\leqq\sum_{j=0}^{n}\frac{(2n)!}{(n-j)!\,2^{n-j}(2j)!}(z(0))^j(kt)^{n-j}.$$

(ii)
$$E\{\exp(\lambda z(t))|\mathscr{A}_0\}\leqq e^{-\lambda kt}\exp\left(\frac{\lambda z(0)}{1-2\lambda kt}\right)\frac{1}{\sqrt{1-2\lambda kt}}.$$

Corollary 3. *For each positive α and $0<\lambda<1/2kt$,*

$$P\{\|\hat{U}(s)\|^2\leqq\alpha+ks \text{ for all } s\in(0,t)\}>1-\frac{e^{-\lambda(\alpha+kt)}}{\sqrt{1-2\lambda kt}}E\left\{\exp\frac{\lambda\|\hat{U}(0)\|_{\mathscr{E}}^2}{1-2\lambda kt}\right\}.$$

Proof of Corollary 1. Use property 5 (ii) (§ 1.3) of the stochastic integral.

Proof of Corollary 2. Let $c_n(t)$ be an upper bound of $E\{(z(t)+kt)^n|\mathscr{A}_0\}$. Obviously we may take $c_0(t)=1$, $c_1(t)=z(0)+kt$.

From Lemma 3.1,

$$z(t)+kt=z(0)+\int_0^t k\,d\tau+2\iint_0^t(\hat{v}(\tau),Ge(z,\bullet))\,d\beta(\tau,z),$$

so that Itô's formula implies

$$\begin{aligned}(z(t)+kt)^n=(z(0))^n&+\int_0^t n(z(\tau)+k\tau)^{n-1}k\,d\tau\\&+\iint_0^t n(z(\tau)+k\tau)^{n-1}2(\hat{v}(\tau),Ge(z,\bullet))\,d\beta(\tau,z)\\&+\frac{n(n-1)}{2}\iint_0^t(z(\tau)+k\tau)^{n-2}4(\hat{v}(\tau),Ge(z,\bullet))^2\,d\tau\,dm(z),\end{aligned}$$

and hence

$$\begin{aligned}E\{(z(t)+kt)^n|\mathscr{A}_0\}&\leqq(z(0))^n+\int_0^t nk\,c_{n-1}(\tau)\,d\tau\\&\quad+2n(n-1)\,E\left\{\iint_0^t(z(\tau)+k\tau)^{n-2}\|\hat{v}(\tau)\|^2k\,d\tau\,dm(z)|\mathscr{A}_0\right\}\\&\leqq(z(0))^n+nk\int_0^t c_{n-1}(\tau)\,d\tau+2n(n-1)k\int_0^t E\{(z(\tau)+k\tau)^{n-2}\|\hat{v}(\tau)\|^2|\mathscr{A}_0\}\,d\tau\\&\leqq(z(0))^n+kn(2n-1)\int_0^t c_{n-1}(\tau)\,d\tau,\end{aligned}$$

because $\|\hat{v}(t)\|^2\leqq z(t)+kt$. We may therefore take

$$c_n(t)=(z(0))^n+kn(2n-1)\int_0^t c_{n-1}(\tau)\,d\tau.$$

From this and the fact that $c_0(t)=1$, (i) is obtained.

As for (ii), it follows from (i) that

$$E\{e^{\lambda z(t)}|\mathscr{A}_0\} \leqq e^{-\lambda k t} \sum_{n=0}^{\infty} \frac{\lambda^n c_n(t)}{n!}$$
$$= e^{-\lambda k t} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{\lambda^n (2n)! (z(0))^j (k t)^{n-j}}{n!(n-j)!\, 2^{n-j} (2j)!} \tag{7}$$
$$= e^{-\lambda k t} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^{j+m} [2(m+j)]!}{(m+j)!\, m!\, 2^m (2j)!} (z(0))^j (k t)^m .$$

On the other hand, for $|y|<\frac{1}{2}$,

$$\frac{e^{x/(1-2y)}}{1-2y} = \sum_{j=0}^{\infty} \frac{x^j}{j!} (1-2y)^{-(j+\frac{1}{2})} = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^j (2j+2m)!\, y^m}{(j+m)!(2j)!\, m!\, 2^m} \tag{8}$$

and combining (7) with (8), (ii) is readily obtained.

Proof of Corollary 3. Since $z(t)$ is a martingale, $\exp(\lambda z(t))$ is a sub-martingale for $\lambda \geqq 0$, and Doob's inequality gives

$$e^{\lambda\alpha} P\{\sup_{0\leqq s\leqq t} \exp(\lambda z(s)) > e^{\lambda\alpha}\} \leqq E\{\exp(\lambda z(t))\},$$

hence

$$P\{\sup_{0\leqq s\leqq t} z(s) > \alpha\} \leqq e^{-\lambda\alpha} E\{\exp(\lambda z(t))\}$$

and using the estimate (ii) of Corollary 2 and the definition of $z(t)$, it follows that

$$P\left\{\sup_{0\leqq s\leqq t} \|\hat{U}(s)\|_{\mathscr{E}}^2 + 4b\int_0^s \|\hat{v}(\tau)\|^2\, d\tau - k s > \alpha\right\} \leqq \frac{e^{-\lambda(\alpha+kt)}}{1-2\lambda k t} E\left\{\exp\left(\frac{\lambda z(0)}{1-2\lambda k t}\right)\right\}.$$

Therefore

$$P\left\{\|\hat{U}(s)\|_{\mathscr{E}}^2 + 4b\int_0^s \|\hat{v}(\tau)\|^2\, d\tau \leqq \alpha + k s \text{ for all } s\in(0,t)\right\}$$
$$> 1 - \frac{e^{-\lambda(\alpha+kt)}}{1-2\lambda k t} E\left\{\exp \frac{\lambda z(0)}{1-2\lambda k t}\right\}. \tag{9}$$

This last inequality implies the required conclusion.

4. Application to a Two-Sided Barrier Problem

The preceding results may be used to derive an estimate for the probability that a certain stationary Gaussian process $y(s)$ remains bounded in absolute value by a function of the form $\sqrt{A+Bs}$, during an interval of a prescribed length. The covariance function $\Gamma(s) = E\{y(t+s)\, y(t)\}$ of the process must have the representation (5) indicated below, in order that such an estimate can be obtained.

4.1. *Representation of stationary Gaussian processes.*

If the initial conditions $\hat{U}(0,\bullet)$ appearing in the definition of $\hat{U}(t)=(\hat{u}(t);\hat{v}(t))$ (§ 3 (1)) are suitably chosen, the process $\hat{U}(t)$ can be made stationary. In fact, that is the case when

$$\begin{aligned}\hat{u}_i(0,\mu)&=\iint_0^\infty \varepsilon_1^2(\tau,\mu)\,G(\mu)\,e_i(z,\mu)\,d\beta'(\tau,z)\\ \hat{v}_i(0,\mu)&=\iint_0^\infty \varepsilon_2^2(\tau,\mu)\,G(\mu)\,e_i(z,\mu)\,d\beta'(\tau,z),\end{aligned}\tag{1}$$

where β' is a new m-Brownian motion on $J\times I$, independent of β. The process $\hat{U}(t)$ is then given by

$$\hat{u}_i(t,\mu)=\varepsilon_j^1(t,\mu)\,\bar{\gamma}_i^j(t,\mu),$$
$$\hat{v}_i(t,\mu)=\varepsilon_j^2(t,\mu)\,\bar{\gamma}_i^j(t,\mu),$$

with

$$\bar{\gamma}_i^j(t,\mu)=\iint_0^\infty \varepsilon_2^j(\tau,\mu)\,G(\mu)\,e_i(z,\mu)\,d\beta'(\tau,z)+\iint_0^t \varepsilon_2^j(-\tau,\mu)\,G(\mu)\,e_i(z,\mu)\,d\beta(\tau,z),$$

or simply

$$\bar{\gamma}_i^j(t,\mu)=\iint_{-\infty}^t \varepsilon_2^j(-\tau,\mu)\,G(\mu)\,e_i(z,\mu)\,d\bar{\beta}(\tau,z),$$

where $\bar{\beta}$ pieces together β and β' in the obvious manner.

Now, given $\hat{\Phi}=(\hat{\varphi};\hat{\psi})\in H_{\mathscr{E}}$, let us compute the covariances of the stationary process

$$y(t)=(\hat{U}(t),\hat{\Phi})_{\mathscr{E}}$$

when $\hat{U}(0)$ is chosen as above [(1)]:

$$\begin{aligned}\Gamma(s)&=E\{y(t)\,y(t+s)\}=E\{y(0)\,y(s)\}\\ &=E\Big\{\int_{-\infty}^{0+}\int_{-\infty}^{0+}\big(\varepsilon_{i'}^2(0,\mu)\,\bar{\gamma}_i^{i'}(0,\mu)\,\hat{\psi}_j(\mu)-\mu\,\varepsilon_{i'}^1(0,\mu)\,\bar{\gamma}_i^{i'}(0,\mu)\,\hat{\varphi}_j(\mu)\big)\\ &\quad\cdot\big(\varepsilon_{k'}^2(s,\nu)\,\bar{\gamma}_k^{k'}(s,\nu)\,\hat{\psi}_l(\nu)-\nu\,\varepsilon_{k'}^1(s,\nu)\,\bar{\gamma}_k^{k'}(s,\nu)\,\hat{\varphi}_l(\nu)\big)\,df^{ij}(\mu)\,df^{kl}(\nu)\Big\}.\end{aligned}$$

Now insert

$$E\{\bar{\gamma}_i^{i'}(0,\mu)\,\bar{\gamma}_k^{k'}(s,\nu)\}=\iint_{-\infty}^0 \varepsilon_2^{i'}(-\tau,\mu)\,\varepsilon_2^{k'}(-\tau,\nu)\,G(\mu)\,G(\nu)\,e_i(z,\mu)\,e_k(z,\nu)\,d\tau\,dm(z)$$

in the above formula, to obtain

$$\begin{aligned}\Gamma(s)=&\int_{-\infty}^{0+}\int_{-\infty}^{0+}\iint_{-\infty}^{0}\big(\varepsilon_{i'}^2(0,\mu)\,\hat{\psi}_j(\mu)-\mu\,\varepsilon_{i'}^1(0,\mu)\,\hat{\varphi}_j(\mu)\big)\\ &\cdot\big(\varepsilon_{k'}^2(s,\nu)\,\hat{\psi}_l(\nu)-\nu\,\varepsilon_{k'}^1(s,\nu)\,\hat{\varphi}_l(\nu)\big)\,\varepsilon_2^{i'}(-\tau,\mu)\,\varepsilon_2^{k'}(-\tau,\nu)\,G(\mu)\,G(\nu)\\ &\cdot\,e_i(z,\mu)\,e_k(z,\nu)\,d\tau\,dm(z)\,df^{ij}(\mu)\,df^{kl}(\nu).\end{aligned}\tag{2}$$

Using

$$\int_{-\infty}^{0+}\int_{-\infty}^{0+}\int_I h_j'(\mu)\,h_l''(\nu)\,e_i(z,\mu)\,e_k(z,\nu)\,df^{ij}(\mu)\,df^{kl}(\nu)\,dm(z)=\int_{-\infty}^{0+} h_j'(\mu)\,h_l''(\mu)\,df^{jl}(\mu),$$

which expresses the isometry between H_m and H_f implied by the Plancherel Theorem, (2) reduces to

$$\begin{aligned}\Gamma(s)=\int_{-\infty}^{0+}\int_{-\infty}^{0} &\left(\varepsilon_{i'}^2(0,\mu)\,\hat{\psi}_j(\mu)-\mu\,\varepsilon_{i'}^1(0,\mu)\,\hat{\varphi}_j(\mu)\right)\\ &\cdot\left(\varepsilon_{k'}^2(s,\mu)\,\hat{\psi}_l(\mu)-\mu\,\varepsilon_{k'}^1(s,\mu)\,\hat{\varphi}_l(\mu)\right)\varepsilon_2^{i'}(-\tau,\mu)\,\varepsilon_2^{k'}(-\tau,\mu)\,G^2(\mu)\,df^{jl}(\mu)\,d\tau.\end{aligned}\tag{3}$$

It is not hard to obtain

$$\int_{-\infty}^{0}\varepsilon_2^{i'}(-\tau,\mu)\,\varepsilon_2^{k'}(-\tau,\mu)\,d\tau=\frac{-1}{4b\mu}\left(\varepsilon_1^{i'}(0,\mu)\,\varepsilon_1^{k'}(0,\mu)-\mu\,\varepsilon_2^{i'}(0,\mu)\,\varepsilon_2^{k'}(0,\mu)\right)\tag{4}$$

and replacing (4) in (3) it follows that

$$\begin{aligned}\Gamma(s)=\int_{-\infty}^{0+}\Bigg[&\frac{-1}{4b\mu}\left(\varepsilon_1^2(0,\mu)\,\hat{\psi}_j(\mu)-\mu\,\varepsilon_1^1(0,\mu)\,\hat{\varphi}_j(\mu)\right)\left(\varepsilon_1^2(s,\mu)\,\hat{\psi}_l(\mu)-\mu\,\varepsilon_1^1(s,\mu)\,\hat{\varphi}_l(\mu)\right)\\ &+\frac{1}{4b}\left(\varepsilon_2^2(0,\mu)\hat{\psi}_j(\mu)-\mu\,\varepsilon_2^1(0,\mu)\,\hat{\varphi}_j(\mu)\right)\left(\varepsilon_2^2(s,\mu)\hat{\psi}_l(\mu)-\mu\,\varepsilon_2^1(s,\mu)\hat{\varphi}_l(\mu)\right)\Bigg]G^2(\mu)\,df^{ij}(\mu)\\ =\int_{-\infty}^{0+}&\frac{1}{4b}\left[\hat{\varphi}_j(\mu)\left(\varepsilon_1^2(s,\mu)\,\hat{\psi}_l(\mu)-\mu\,\varepsilon_1^1(s,\mu)\,\hat{\varphi}_l(\mu)\right)\right.\\ &\left.+\hat{\psi}_j(\mu)\left(\varepsilon_2^2(s,\mu)\,\hat{\psi}_l(\mu)-\mu\,\varepsilon_2^1(s,\mu)\,\hat{\varphi}_l(\mu)\right)\right]G^2(\mu)\,df^{jl}(\mu).\end{aligned}$$

But also $\varepsilon_1^2(s,\mu)=\mu\,\varepsilon_2^1(s,\mu)$ (because B_μ and $e^{B_\mu s}$ commute), so that

$$\begin{aligned}\Gamma(s)=&\int_{-\infty}^{0+}(-\mu)\,\varepsilon_1^1(s,\mu)\,G^2(\mu)\,\hat{\varphi}_j(\mu)\,\hat{\varphi}_l(\mu)\,df^{jl}(\mu)\\ &+\int_{-\infty}^{0+}\varepsilon_2^2(s,\mu)\,G^2(\mu)\,\hat{\psi}_j(\mu)\,\hat{\psi}_l(\mu)\,df^{jl}(\mu)\end{aligned}\tag{5}$$

and, in particular,

$$\Gamma(0)=\|G(\cdot)(\hat{\varphi}(\cdot);\hat{\psi}(\cdot))\|_{\mathscr{E}}^2.\tag{6}$$

Example. Let us set $I=(-\pi,\pi)$ and $dm(z)=e^{2b|z|}\,dz$, so that (with the abbreviation $r=\sqrt{-\mu-b^2}$)

$$e_1(z,\mu)=e^{-b|z|}\left(\cos rz+\frac{b}{r}\sin rz\right)\qquad(r>0),$$

$$e_1(z,-b^2)=(1+bz)\,e^{-b|z|},$$

$$e_2(z,\mu)=e^{-b|z|}\,\frac{\sin rz}{r}\qquad(r>0),$$

$$e_2(z,-b^2)=z\,e^{-b|z|},$$

$$df(\mu)=\frac{dn(r)}{\pi}\begin{pmatrix}1 & -b\\ -b & -\mu\end{pmatrix}\qquad\text{for } \mu\leqq -b^2,$$

where n is a jump function with jumps of magnitude 1 at the positive integers, and a jump of magnitude $\frac{1}{2}$ at $r=0$, and

$$df(\mu)=0 \quad \text{for } -b^2<\mu\leqq 0.$$

For $\mu\leqq -b^2$, $s\geqq 0$, the components ε_i^i of $\exp(B_\mu s)$ (§ 2(11)) may be written as follows

$$\varepsilon_1^1(s,\mu)=e_1(s,\mu), \qquad \varepsilon_2^2(s,\mu)=e_1(s,\mu)-2b\,e_2(s,\mu)$$

so that (5) leads to

$$e^{bs}\Gamma(s)=\frac{1}{\pi}\sum_{r=0}^{\infty}{}' G_r^2(\hat\psi_r^*+\hat\varphi_r^*(b^2+r^2))\cos rs-b\,G_r^2(\hat\psi_r^*-\hat\varphi_r^*(b^2+r^2))\frac{\sin rs}{r}, \tag{7}$$

where

$$\begin{aligned}\hat\psi_r^*&=(\hat\psi_1(\mu),\hat\psi_2(\mu))\begin{pmatrix}1 & -b\\ -b & -\mu\end{pmatrix}\begin{pmatrix}\hat\psi_1(\mu)\\ \hat\psi_2(\mu)\end{pmatrix}\\ \hat\varphi_r^*&=(\hat\varphi_1(\mu),\hat\varphi_2(\mu))\begin{pmatrix}1 & -b\\ -b & -\mu\end{pmatrix}\begin{pmatrix}\hat\varphi_1(\mu)\\ \hat\varphi_2(\mu)\end{pmatrix}\end{aligned} \tag{8}$$

and $\sum_{r=0}^{\infty}{}' a_r$ means $\frac{1}{2}a_0+\sum_{r=1}^{\infty}a_r=\int_{0-}^{\infty}a_r\,dn(r)$.

4.2. *The barrier problem.*

Theorem 4.1. *Let $y(t)$ be a stationary Gaussian process with covariance function $\Gamma(s)=E\{y(t)\,y(t+s)\}$ given by* (5), *for suitable $\hat\Phi\in H_{\mathscr{E}}$. Then, if b is the positive damping associated to the representation* (5), *and k is the constant defined by* § 3 (2), *for each $\alpha>3k/2b$,*

$$P\{|y(s)|^2\leqq(\alpha+ks)\|\Phi\|_{\mathscr{E}}^2 \text{ for all } s\in(0,t)\}>1-e^{-\frac{1}{2}\frac{2b\alpha-3k}{2bkt+3k}}\sqrt{1+\frac{2b\alpha-3k}{2bkt+3k}}. \tag{9}$$

Remark. In order to simplify the dependence on α of the right-hand term in (9), one may replace it by the lower bound

$$1-e^{-\frac{\lambda\alpha b}{2bkt+3k}}\sqrt{\frac{e^{-\lambda}}{1-\lambda}}$$

for $0\leqq\lambda<1$.

In order to prove the theorem, let us prepare the following lemma.

Lemma 4.1. *If $\hat U(0)=(\hat u(0),\hat v(0))$ is the random variable defined in* (1) *and the spectral measure df is concentrated on $(-\infty,-b^2]$, then*

$$E\{\|\hat U(0)\|_{\mathscr{E}}^{2n}\}\leqq\frac{(2n)!}{n!}(3k/4b)^n.$$

Proof. Let us introduce the (random) function

$$g(t) = \left|\!\left|\!\left| \iint_0^t \varepsilon_2^1(\tau,\mu)\, G(\mu)\, e(z,\mu)\, d\beta'(\tau,z) \right|\!\right|\!\right|^2 + \left\| \iint_0^t \varepsilon_2^2(\tau,\mu)\, G(\mu)\, e(z,\mu)\, d\beta'(\tau,z) \right\|^2 = |\!|\!|\gamma^1(t)|\!|\!|^2 + \|\gamma^2(t)\|^2,$$

with

$$f^i(\tau,z) = \varepsilon_2^i(\tau,\bullet)\, G(\bullet)\, e(z,\bullet),$$

$$\gamma^i(t) = \iint_0^t f^i(\tau,z)\, d\beta'(\tau,z),$$

and let $k_n(t)$ be an upper bound of $E\{g^n(t)\}$.

An application of Lemma 1.1 gives

$$g(t) = 2\iint_0^t [((\gamma^1(\tau), f^1(\tau,z))) + (\gamma^2(\tau), f^2(\tau,z))]\, d\beta'(\tau,z) + \iint_0^t (|\!|\!|f^1|\!|\!|^2 + \|f^2\|^2)\, d\tau\, dm,$$

and therefore

$$\begin{aligned} g^n(t) = \iint_0^t n\, g^{n-1}(\tau)\big(2[((\gamma^1(\tau), f^1(\tau,z))) + (\gamma^2(\tau), f^2(\tau,z))]\, d\beta' \\ + (|\!|\!|f^1(\tau,z)|\!|\!|^2 + \|f^2(\tau,z)\|^2)\, d\tau\, dm\big) \\ + \tfrac{1}{2} n(n-1) \iint_0^t g^{n-2}\, 4\big(((\gamma^1(\tau), f^1(\tau,z))) + (\gamma^2(\tau), f^2(\tau,z))\big)^2\, d\tau\, dm. \end{aligned} \tag{10}$$

From (10) and Cauchy-Schwarz inequality it follows that

$$\begin{aligned} E\{g^n(t)\} &\leqq \iint_0^t n\, k_{n-1}(\tau) \|(f^1; f^2)\|_{\mathscr{E}}^2\, d\tau\, dm \\ &\quad + E\left\{\tfrac{1}{2} n(n-1) \iint_0^t 4 g^{n-1}(\tau) \|(f^1; f^2)\|_{\mathscr{E}}^2\, d\tau\, dm\right\} \\ &\leqq n(2n-1) \iint_0^t k_{n-1}(\tau) \|(f^1; f^2)\|_{\mathscr{E}}^2\, d\tau\, dm. \end{aligned} \tag{11}$$

Now, using the definition of f^1, f^2, (11) leads to

$$\begin{aligned} E\{g^n(t)\} \leqq n(2n-1) \iint_{\mu z} G^2(\mu)\, e_i(z,\mu)\, e_j(z,\mu)\, df^{ij}(\mu) \\ \cdot \int_0^t k_{n-1}(\tau)\, [(\varepsilon_2^2(\tau,\mu))^2 - \mu(\varepsilon_2^1(\tau,\mu))^2]\, d\tau\, dm(z). \end{aligned} \tag{12}$$

On the other hand, for $\mu < -b^2$ and $r = \sqrt{-\mu - b^2}$, § 2 (11) applies and the fact that df vanishes in $(-b^2, 0]$ allows us to use the estimate

$$(\varepsilon_2^2(\tau,\mu))^2 - \mu(\varepsilon_2^1(\tau,\mu))^2 \leqq e^{-2b\tau}(1 + 2b\tau + 2b^2\tau^2). \tag{13}$$

Combining (12) and (13), and recalling the definition § 3 (2) of k, we obtain

$$E\{g^n(t)\} \leqq n(2n-1)\, k \int_0^t k_n(\tau)\, e^{-2b\tau}(1+2b\tau+2b^2\tau^2)\, d\tau$$

which leads to a recursive computation of k_n, namely

$$k_0(t)=1,\; k_n(t)=n(2n-1)\, k \int_0^t k_{n-1}(\tau)\, e^{-2b\tau}(1+2b\tau+2b^2\tau^2)\, d\tau. \tag{14}$$

Since (14) implies

$$k_n(t)=\frac{(2n)!}{n!}\left(\frac{k}{2}\right)^n \left(\int_0^t e^{-2b\tau}(1+2b\tau+2b^2\tau^2)\, d\tau\right)^n$$

the result follows by putting $t=\infty$ and using

$$\int_0^\infty e^{-2b\tau}(1+2b\tau+2b^2\tau^2)\, d\tau = \frac{3}{2b}.$$

Corollary 1. *If the assumptions of Lemma* 4.1 *hold, then*

$$E\left\{\exp \frac{\lambda \|\hat{U}(0)\|_{\mathscr{E}}^2}{1-2\lambda k t}\right\} \leqq \left(1-\frac{3k\lambda}{b(1-2\lambda k t)}\right)^{-\frac{1}{2}}$$

for $0<3k\lambda<b(1-2\lambda k t)$.

Proof.

$$E\left\{\exp \frac{\lambda \|\hat{U}(0)\|_{\mathscr{E}}^2}{1-2\lambda k t}\right\} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{1-2\lambda k t}\right)^n \frac{E\{\|\hat{U}(0)\|_{\mathscr{E}}^{2n}\}}{n!}$$
$$\leqq \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}\left(\frac{3k\lambda}{4b(1-2\lambda k t)}\right)^n = \left(1-\frac{3k\lambda}{b(1-2\lambda k t)}\right)^{-\frac{1}{2}}.$$

Corollary 2. *If the assumptions of Lemma* 4.1. *hold, and* $\hat{U}(t)$ *is the process defined in* § 3, *then for each* $\alpha>3k/2b$,

$$P\{\|\hat{U}(s)\|_{\mathscr{E}}^2 \leqq \alpha+ks \textit{ for all } s\in(0,t)\} > 1-e^{-\frac{1}{2}\frac{\alpha-3k/2b}{kt+3k/2b}} \sqrt{1+\frac{\alpha-3k/2b}{kt+3k/2b}}. \tag{15}$$

Proof. Using Corollary 1, and Corollary 3 of Lemma 3.1, the inequality

$$P\{\|\hat{U}(s)\|_{\mathscr{E}}^2 \leqq \alpha+ks \text{ for all } s\in(0,t)\} > 1-\frac{e^{-\lambda(\alpha+kt)}}{\sqrt{1-\lambda(2kt+3k/b)}} \tag{16}$$

follows readily for

$$0<\lambda<b/(2bkt+3k) \tag{17}$$

and taking the maximum of the right-hand term in (16) for λ satisfying (17), the inequality (15) is obtained for $\alpha>3k/2b$. If $\alpha<3k/2b$, the cited maximum is 0 (for $\lambda=0$) and the corresponding result is trivial.

Proof of Theorem 4.1.

The process $y(t)=(\hat{U}(t),\hat{\Phi})_{\mathscr{E}}$ has covariance Γ, so that Cauchy-Schwarz inequality and Corollary 2 lead to the desired result.

Example. Let $\gamma(s)$ be a function with Fourier expansion

$$\gamma(s)=\frac{1}{\pi}\sum_{r=0}^{\infty}{}' \left(a_r \cos r s - b\, b_r \frac{\sin r s}{r}\right),$$

where $\sum'$ has the same meaning as in § 4 (Example), and the coefficients satisfy, for $r=0,1,\ldots,$

(a) $a_r \geqq 0$,

(b) $|b_r| \leqq a_r$,

(c) there exist non-negative numbers a'_r, a''_r such that $a_r = a'_r\, a''_r$, $a'=\sum' a'_r<\infty$, $a''=\sum' a''_r<\infty$.

Then, for each number $\alpha>3a''/b$, the stationary Gaussian process y with covariance function

$$\Gamma(s)=e^{-bs}\gamma(s)$$

satisfies $\quad P\{|y(s)|^2 \leqq a'(\alpha+2a''s)$ for all s such that $0\leqq s\leqq t\leqq\pi\}$

$$>1-e^{-\frac{1}{2}\frac{\alpha-3a''/b}{2a''t+3a''/b}}\; 1+\frac{\alpha-3a''/b}{2a''t+3a''/b}. \tag{18}$$

In order to prove the preceding statement, let us represent a process with covariance Γ, choosing I, dm as in the example of § 4.1. This is accomplished by setting [see (7)]

$$a'_r=\hat{\psi}^*_r+\hat{\varphi}^*_r(b^2+r^2), \qquad a''_r=G^2_r$$
$$b_r=G^2_r(\hat{\psi}^*_r-\hat{\varphi}^*_r(b^2+r^2)),$$

thus defining $\hat{\psi}^*_r, \hat{\varphi}^*_r$ in such a way that

$$\|\Phi\|^2_{\mathscr{E}}=\|\hat{\varphi}\|^2+\|\hat{\psi}\|^2=\sum{}' \hat{\psi}^*_r+\hat{\varphi}^*_r(b^2+r^2)=a'. \tag{19}$$

On the other hand,

$$\begin{aligned} k&=\frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{r=0}^{\infty}{}' G^2_r(e_1(z,\mu),e_2(z,\mu))\begin{pmatrix}1 & -b\\ -b & b^2+r^2\end{pmatrix}\begin{pmatrix}e_1(z,\mu)\\ e_2(z,\mu)\end{pmatrix} e^{2b|z|}\,dz\\ &=2\sum_{r=0}^{\infty}{}' G^2_r=2a''. \end{aligned} \tag{20}$$

Now Theorem 4.1 implies the required conclusion.

Acknowledgement. It is my pleasure to thank Henry P. McKean to whom I owe many substantial suggestions regarding this paper.

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Prof. E. M. Cabaña
Instituto de Matemática y Estadística
Universidad de la República
Montevideo, Uruguay

(Received November 12, 1968)