Z. Wahrscheinlichkeitstheorie verw. Gebiete
65, 627-636 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Influence of Non-Commutativity on Limit Theorems

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Summary. We show that the distribution of the composition $g_1 \dots g_n$ of random elements $g_1, \dots, g_n$ of the group $\mathbf{SO}(3)$ tends to the uniform distribution in far more general situations, than in the commutative case.

1. Introduction

The convolution of probability measures $\mu_1, \dots, \mu_n, \dots$ on a compact connected group $\mathbf{G}$ tends under quite general conditions to the Haar measure of this group (see [1] and references therein). For example, if we restrict ourselves to the class of measures, absolutely continuous with respect to Haar measure χ, whose densities p_i are bounded from above, $p_i \leqq c_i$, then the general result of [1] tells us, that for

$$q_n = p_1 * \dots * p_n$$

one has

$$\max_{g \in \mathbf{G}} |q_n(g) - 1| \to 0 \tag{1}$$

provided the series

$$\sum_1^n (c_i)^{-2} \to \infty \quad \text{as} \quad n \to \infty. \tag{2}$$

It is also proven in [1], that in terms of constants c_i the condition (2) can not be improved:

$$\text{if } \sum_1^n (c_i)^{-2} < C \qquad \text{for all } n$$

and *the group G has non-trivial one-dimensional representation*, then there exist a sequence of probability measures $\mu_1, \dots, \mu_n, \dots$ with densities $p_1, \dots, p_n, \dots$ which satisfy $p_i \leqq c_i$ and $q_n \nrightarrow \chi$ even in a weak sense.

The purpose of this paper is to show, that for groups $\mathbf{G}$, which have *no low-dimensional representations*, the situation is quite different. In this paper we are dealing with the simplist group of this type, $\mathbf{G}=\mathbf{SO}(3)$ - the group of proper rotations of three-dimensional euclidean space - in which case condition (2) can be replaced by far more weaker ones: to have the convergence (1) it is enough to have

$$\sum_1^n (c_i)^{-1} \to \infty \quad \text{as} \quad n \to \infty. \tag{3}$$

This result shows, that "*algebraic*" *mixing properties* of non-commutative groups are better, then those of commutative groups.

The paper is organized as follows: In § 2 we formulate our general result and recall some facts from [1]. In § 3 we make necessary changes in the strategy of [1] in order to adapt it to our case. In § 4 we study unitary representations of $\mathbf{SO}(3)$. The final § 5 contains further hypothesis.

2. The Main Theorem

In [1] there were introduced some useful objects. Here we recall some of them. Let μ be a probability measure on $\mathbf{G}$, and $\mu=\lambda+\nu$ be its decomposition into singular and absolutely continuous according to χ parts respectively. Let

$$p=d\nu/d\chi,$$

and

$$N_\mu(x)=\sup_{\mathbf{E},\ \chi(\mathbf{E})=x} \inf\{p(g),\ g\in\mathbf{E}\subset\mathbf{G}\}.$$

Put

$$S_k(\mu)=\int_0^1 x^{2/k}\, N_\mu(x)\, dx$$

(the quantity $S(\mu)$ of [1] coincides with our $S_1(\mu)$).

The main result of [1] was the following

Theorem 1. *Let $\mu_1, \ldots, \mu_n$, $n \geqq 2$ be a sequence of probability measures on a compact connected group $\mathbf{G}$. Suppose for some i, j, $1 \leqq i < j \leqq n$ the measures μ_i, μ_j are absolutely continuous with respect to Haar measure χ of $\mathbf{G}$ with density functions p_i, p_j belonging to $\mathbf{L}^2(\mathbf{G}, \chi)$. Then the measure $\mu_1 * \ldots * \mu_n$ has a density function q_n satisfying the inequality*

$$\sup_{g\in\mathbf{G}} |q_n(g)-1| \leqq \|p_i-1\|_{L^2}\, \|p_j-1\|_{L^2} \prod_{\substack{s=1\\ s\neq i,j}}^n (1-C_1 S_1(\mu_s)). \tag{4}$$

Here $C_1>0$ is some absolute constant.

Now we can formulate the main result of the present paper:

Theorem 2. *In the case $\mathbf{G}=\mathbf{SO}(3)$ the Theorem 1 remains valid after replacing the estimation (4) by the improved one:*

$$\sup_{g\in\mathbf{SO}(3)} |q_n(g)-1| \leqq \|p_i-1\|_{L^2}\, \|p_j-1\|_{L^2} \prod_{\substack{s=1\\ s\neq i,j}}^n (1-C_2 S_2(\mu_s)) \tag{5}$$

where $C_2>0$ is some absolute constant.

Remark 1. It is easy to see that for μ having density function p the bound $p \leqq c$ implies

$$\mathbf{S}_k(\mu) \geqq (k/k+2)\, c^{-(2/k)}$$

hence the results mentioned in introduction follow from Theorems 1, 2.

Remark 2. About the use of the quantities $\mathbf{S}_k(\mu)$, $k>2$ the reader is referred to §5.

3. The Strategy of the Proof

Now we'll outline the necessary changes in the proof of the Theorem 1 of [1] in order to arrive to the proof of the Theorem 2.

The proof of the Theorem 1 was based on Lemmas 1–3. Lemma 1 remains unchanged, the substitute of Lemma 2 will be proven in the next paragraph, the minor changes in proof of Lemma 3 will be presented in this section.

Let $\Sigma = \{T\}$ be the set of irreducible unitary representations of $\mathbf{G}$. If μ is a measure on $\mathbf{G}$, then its Fourier transform $\hat{\mu}$ is the operator-valued function, given by

$$\hat{\mu}(T) = \int_G T(g)\, d\mu(g).$$

Let

$$\|\mu\| = \sup_{T \in \Sigma \setminus \mathrm{id}} \|\hat{\mu}(T)\|.$$

Then we have the following

Lemma 1 (*from* [1]). *Let the sequence* $\mu_1, \ldots, \mu_n$ *satisfies the conditions of the Theorem 1. Then*

$$\sup_{g \in \mathbf{G}} |q_n(g) - 1| \leqq \|p_i - 1\|_{L^2} \|p_j - 1\|_{L^2} \prod_{\substack{s=1 \\ s \neq i,j}}^{n} \|\mu_s\|.$$

So to prove the theorem one has to estimate $\|\mu_s\|$. Let $\mathbb{C}^n$ be the complex n-dimensional space, endowed with usual (complex-valued) scalar product $\langle\, ,\rangle$, $S^{2n-1} \subset \mathbb{C}^n$ be its unit sphere, for $x, y \in S^{2n-1}$ put $\rho(x,y) = \arccos \operatorname{Re}\langle x, y\rangle$,

$$\mathscr{D}_r(x) = \{y \in S^{2n-1},\ \rho(x,y) < r\}, \tag{6}$$

for $T\colon \mathbf{SO}(3) \to U(n)$ being irreducible representation, $x \in \mathbb{C}^n$ put

$$gx = T(g) \cdot x, \tag{7}$$

finally, let τ be any $\mathbf{SO}(3)$ – invariant (under the action (7)) probability measure on S^{2n-1}. Then one has

Lemma 2. *There exist* $C>0$, $r_0>0$, *such that* $\tau(\mathscr{D}_r(x)) \leqq Cr^2$ *for* $r \leqq r_0$ *uniformly in* x, T, n.

(Here lies the main difference between $\mathbf{SO}(3)$ and general $\mathbf{G}$, because in the general case the bound $\tau(\mathscr{D}_r(x)) \leqq r$ (Lemma 2 from [1]) cannot be improved.)

Since the proof of Lemma 2 is a little lengthy, we'll postpone it until the next paragraph.

Now we are able to obtain the last estimation one needs to complete the proof of the Theorem 2.

Lemma 3. *Let μ be a probability measure on* $\mathbf{SO}(3)$, *and* $\mathbf{T}$ *be any non-trivial irreducible representation of* $\mathbf{SO}(3)$. *Then* $\|\hat{\mu}(T)\|<1-C_2 S_2(\mu)$ *for some* $C_2>0$.

Proof. The arguments are very close to those of the proof of Lemma 3 in [1], so we'll be brief.

Let $x, y\in\mathbf{S}^{2n-1}$, and ν is the absolute continuous part of μ. Then

$$1-\operatorname{Re}\langle x, \hat{\mu}(T)\, y\rangle \geqq \int_0^\pi [1-\cos u]\, d\psi(u),$$

where

$$\psi(A)=\nu\{g\in\mathbf{SO}(3);\ \rho(x, T(g)\, y)\in A\}, \qquad A\subset[0,\pi].$$

Let's define the measure $\bar{\psi}$ on $[0,\pi]$ by

$$\bar{\psi}([0,r])=\begin{cases}\int_0^{Cr^2} N_\mu(x)\, dx, & r\leqq r_0,\\ \nu(\mathbf{SO}(3)), & r>r_0.\end{cases}$$

It is almost evident, that for $\mathbf{E}\subset\mathbf{SO}(3)$, $\chi(\mathbf{E})<r$ implies

$$\nu(\mathbf{E})\leqq\int_0^r N_\mu(x)\, dx$$

(if not, see [1, Lemma 3]). Now define a measure on S^{2n-1}:

$$\tau_y(B)=\chi\{g\in\mathbf{SO}(3),\ T(g)\, y\in B\}, \qquad B\subset S^{2n-1}.$$

It's easy to check that τ_y is $\mathbf{SO}(3)$-invariant. Hence, according to Lemma 2,

$$\chi\{g\in\mathbf{SO}(3),\ \rho(x, T(g)\, y)<r\}=\tau_y(\mathscr{D}_r(x))\leqq Cr^2 \quad \text{for } r\leqq r_0,$$

which, in turn, implies,

$$\psi([0,r])\leqq\bar{\psi}([0,r]) \quad \text{for all } r,\ 0\leqq r\leqq\pi.$$

Because $(1-\cos u)$ is monotone increasing,

$$\begin{aligned}\int_0^\pi (1-\cos u)\, d\psi(u) &\geqq \int_0^\pi (1-\cos u)\, d\bar{\psi}(u)\\ &\geqq \int_0^{r_0} (1-\cos u)[2CuN_\mu(Cu^2)]\, du\\ &\geqq \check{C}\int_0^{r_0} (u^3) N_\mu(Cu^2)\, du \geqq C_2\int_0^1 u N_\mu(u)\, du\end{aligned}$$

for some $C_2>0$, and the proof follows.

4. The Geometry of the Orbits of the Unitary Representations of SO(3)

In this paragraph we'll use the following notations: $u, v, \ldots$ be unit vectors in $\mathbb{R}^3$,

$$S^2=\{u, v, \ldots\};$$

$r(u, v)$ be the angle between u, v; $R_u(\theta)\in \mathbf{SO}(3)$ be the rotation around u on the angle θ;

$$\mathscr{R}_u=\{R_u(\theta), \theta\in[0, 2\pi]\}\subset \mathbf{SO}(3)$$

be one-parameter subgroup; for e_1, e_2, e_3 – being ortonormal in $\mathbb{R}^3$ we'll use $R_1(\theta)$, $\mathscr{R}_1, \ldots$ instead of $R_{e_1}(\theta)$, $\mathscr{R}_{e_1}, \ldots$, finally,
T: $\mathbf{SO}(3)\to U(n)$ be some fixed unitary representation, not necessarily irreducible, but obeying the following weaker property: for any $x\in S^{2n-1}\subset \mathbb{C}^n$ there exist $R\in\mathbf{SO}(3)$ with $Rx\neq x$ (we suppress T in expressions, like $T(R)x$).

For $x\in S^{2n-1}$ consider the orbit

$$\mathcal{O}(x)=\{y;\ y=Rx,\ R\in\mathbf{SO}(3)\}\subset S^{2n-1}.$$

This orbit is a smooth submanifold in S^{2n-1}. The dimension of $\mathcal{O}(x)$ can be 2 or 3 according to x and T. To see it, consider the stationary subgroup $H(x)$ of x,

$$H(x)=\{R\in\mathbf{SO}(3), Rx=x\}.$$

Then the manifold $\mathcal{O}(x)$ is diffeomorphic to the manifold of the left cosets of $\mathbf{SO}(3)$ with respect to subgroup $H(x)$, hence

$$\dim \mathcal{O}(x)=\dim\mathbf{SO}(3)-\dim H(x),$$

which is 3 when $H(x)$ is discrete and 2 when $\dim H(x)=1$. (Those are the only possibilities.)

The representation T and the vector x provide us with the probabilistic measure τ_x on S^{2n-1},

$$\tau_x(B)=\chi\{(R;\ Rx\in B)\}, \tag{8}$$

which is $\mathbf{SO}(3)$-invariant. The bound: $\dim\mathcal{O}(x)\geqq 2$ implies the following estimation:

$$\tau_x(\mathscr{D}_r(y))\leqq Cr^2 \tag{9}$$

where $C=C(x, T)$, $r\leqq r_0=r_0(x, T)$. To see it, let $y'\in S^{2n-1}$, and

$$\rho(y', \mathcal{O}(x))\equiv \max\{\rho(y', Rx);\ R\in\mathbf{SO}(3)\}>r.$$

Then $\tau_x(D_r(y))=0$. Otherwise $D_r(y')\subset D_{2r}(y)$ for some $y\in\mathcal{O}(x)$, hence it is enough to consider the case $y\in\mathcal{O}(x)$. But the manifold $\mathcal{O}(x)$ is locally diffeomorphic to two- or three-dimensional disc B. Consider the number $P_{2r}(B)$ – the maximal possible amount of non-overlapping discs $D_{2r}(y_i)$, centered in B. It's clear, that $P_{2r}(B)\lesssim r^{-2}$. The unitarity of T implies:

$$\tau_x(D_{2r}(y_i))=\tau_x(D_{2r}(y)),$$

which, together with $\sum_i \tau_x(\mathscr{D}_{2r}(y_i)) \leqq 1$ implies the bound

$$\tau_x(D_r(y)) \leqq P_{2r}^{-1}(\mathcal{O}(x)) \leqq P_{2r}^{-1}(B) \sim r^2. \tag{10}$$

The main result of this paragraph states, that the bound (9) holds uniformly in T, x:

Theorem 3 (≡*Lemma 2*). *There exist two constants C, r_0, such that for any n, any representation T: $\mathbf{SO}(3) \to U(n)$ without invariant vectors and any $x, y \in \mathbb{C}^n$,*

$$\tau_x(D_r(y)) \leqq Cr^2 \quad \textit{for} \quad r \leqq r_0.$$

Here the measure τ_x is given by (8), and the ball $D_r(y)$ – by (6).

The statement of the theorem is not at all evident. Apriori, it is possible that for large n the orbit $\mathcal{O}(x)$ is contained in the ε-neighbourhood of suborbit $\mathscr{R}_u x$ for any u, in which case the bound $\tau_x(D_r(y)) < Cr$ is the best possible. As a preliminary step in the proof we'll show that it is not so.

Lemma 4. *For any $x \in S^{2n-1}$ there exist an element $R \in \mathbf{SO}(3)$ and a subgroup $\mathscr{R}_u \in \mathbf{SO}(3)$, for which*

$$\rho(x, \mathscr{R}_u R x) \geqq \frac{\pi}{8}.$$

For the proof we shall use two following lemmas.

Lemma 5.

$$R_v(\pi) R_u(\pi) = R_{[u,v]}(2r(u,v)).$$

Here $[u, v]$ denotes the vector product of u and v.

The proof is immediate.

Lemma 6. *Let $A \subset \mathbf{SO}(3)$ be*

i) *symmetric:*
for $R \in A, R^{-1} \in A$, and

ii) *transitive on S^2:*
for $u, v \in S^2$ there exist $R \in A$ with $Ru = v$.

Then $A^4 = \mathbf{SO}(3)$; in other words, the set of all four-products of elements of A coincides with $\mathbf{SO}(3)$ for any A.

Proof. Let $u \in S^2$. By (ii), there exist $v \in S^2$, $v \perp u$ such that $R_v(\pi) \in A$ (because it is the only way for A to be able to carry u to $(-u)$). Now, for any $w \in S^2$, $R_w(\pi) \in A^3$. To see this, let us find an element $R \in A$, which carries w to v: $Rw = v$. Then $R^{-1} R_v(\pi) R \in A^3$ by (i). But $R_w(\pi) = R^{-1} R_v(\pi) R$, because

$$R^{-1} R_v(\pi) R w = w, \qquad (R^{-1} R_v(\pi) R)^2 = \mathrm{id}, \qquad R^{-1} R_v(\pi) R \neq \mathrm{id}.$$

Let's denote $C = \{R_w(\pi), w \in S^2\}$. It is enough to show $CA = \mathbf{SO}(3)$. To see this, consider an arbitrary $x \in S^2$. Let us show that $R_x(\theta) \in CA$ for any $\theta \in [0, 2\pi]$. Again, let $y \in S^2$ be orthogonal to x with $R_y(\pi) \in A$.

Let $z = R_x(\theta/2) y$. Then $R_z(\pi) \in C$, and $R_x(\theta) = R_z(\pi) R_y(\pi)$ by Lemma 5, and the result follows.

Remark. In the preliminary version of this paper we have proved only, that $A^6 = \mathbf{SO}(3)$. The remark about $A^4 = \mathbf{SO}(3)$ is due to V. Aisenstat. He also pointed out, that the inclusion $A^2 \subset \mathbf{SO}(3)$ can be proper.

Proof of Lemma 4. Let

$$\lambda = \max_{R \in \mathbf{SO}(3),\, u \in S^2} \{\rho(x, \mathscr{R}_u R x)\}.$$

Put $A = \{R \in \mathbf{SO}(3), \rho(x, Rx) \leqq \lambda\}$. Then A is

i) *symmetric*, because $\rho(x, Rx) = \rho(x, R^{-1} x)$ due to the unitarity of T,

ii) *transitive on* S^2. To see it, let $u, v \in S^2$ and $Ru = v$, $R \in \mathbf{SO}(3)$. Then $R_v(\theta) R u = v$ for any $\theta \in [0, 2\pi]$. But $\rho(x, \mathscr{R}_v R x) \leqq \lambda$, hence there exist $\theta_0 \in [0, 2\pi]$ with $\rho(x, R_v(\theta_0) R x) \leqq \lambda$ which implies $R_v(\theta_0) R \in A$.

By Lemma 6, $A^4 = \mathbf{SO}(3)$, hence for any $R \in \mathbf{SO}(3)$

$$\rho(x, Rx) \leqq 4\lambda \tag{11}$$

as it follows from the triangle inequality.

But the representation T has no invariant vectors, so

$$\int_{\mathbf{SO}(3)} R x \, d\chi(R) = 0.$$

In particular

$$\int_{\mathbf{SO}(3)} \operatorname{Re} \langle x, Rx \rangle \, d\chi(R) = 0,$$

which together with $\langle x, x \rangle = 1$ implies the existence of the element $R_0 \in \mathbf{SO}(3)$ with $\operatorname{Re}\langle x, R_0 x \rangle < 0$, or $\rho(x, R_0 x) > \frac{\pi}{2}$. Together with (11) it gives $\lambda \geqq \frac{\pi}{8}$ and the result follows.

Proof of the Theorem 3. In order to obtain the desired upper bound for $\tau_x(D_r(y))$ it is enough, according to (10), to get a lower bound on $P_r(\mathcal{O}(x))$. So we are left with the packing problem.

We consider three different cases according to the specific choice of the vector x.

Case 1. $\dim \mathcal{O}(x) = 2$. In this case we can suppose without loss of generality that the stationary group $H(x)$ contains the subgroup $\mathscr{R}_3 \subset \mathbf{SO}(3)$ (the rotations around Oz axis). (The inclusion $\mathscr{R}_3 \subset H(x)$ is not necessarily proper. The manifold $\mathcal{O}(x)$ is diffeomorphic to S^2 or $\mathbb{RP}^2$.)

Consider the mapping $\varphi: S^2 \to \mathcal{O}(x)$, which is given by the formula

$$\varphi(u) = Rx, \tag{12}$$

where $Re_3 = u$. This formula is well-defined, because for any R', $R' e_3 = u$ we have $R' = R R_3(\theta)$ for some θ, so $Rx = R'x$.

For $u, v \in S^2$ let

$$\tilde{\rho}(u, v) = \rho(\varphi(u), \varphi(v)).$$

The function $\tilde{\rho}$ has all the properties of the usual metric with the only possible exception that $\tilde{\rho}(u, v)$ can be equal to zero for $u \neq v$.

The metric $\tilde{\rho}$ is $\mathbf{SO}(3)$-invariant. To see it, let $u, v \in S^2$, and $R', R'' \in \mathbf{SO}(3)$ be such that $R' e_3 = u$, $R'' e_3 = v$. Then for $R \in \mathbf{SO}(3)$,

$$\tilde{\rho}(Ru, Rv) = \rho(RR'x, RR''x) = \rho(R'x, R''x) = \rho(u, v).$$

Hence the metric $\tilde{\rho}(u, v)$ depends actually only on $r(u, v)$. We need also the fact, that for any $u \in S^2$ there exist $v \in S^2$ with $\tilde{\rho}(u, v) > \frac{\pi}{2}$. This property follows from the fact that T has no invariant vectors, and was derived during the proof of Lemma 4.

Now we need the following:

Lemma 7. *Let the function $\tilde{\rho}(u, v)$ on $S^2 \times S^2$*

i) *depends only on $r(u, v)$,*
ii) *satisfies the triangle inequality,*
iii) *takes the values, greater than $\frac{\pi}{2}$.*

Then $\tilde{\rho}(u, v) \geqq \frac{1}{3} r(u, v)$, provided $r(u, v) \leqq \frac{\pi}{2}$.

Proof of Lemma 7. Suppose for some $u \neq v$, $\lambda > 0$

$$\tilde{\rho}(u, v) < \lambda r(u, v), \qquad r(u, v) \leqq \frac{\pi}{2}. \tag{13}$$

Then for any $v' \in S^2$ with $r(u, v') \leqq k r(u, v)$, $k = 2, 3, \ldots$

$$\tilde{\rho}(u, v') \leqq k \lambda r(u, v). \tag{14}$$

Indeed, from $r(u, v') \leqq k\, r(u, v)$, $r(u, v) \leqq \frac{\pi}{2}$ follows the existence of the sequence $v_1, \ldots, v_k \in S^2$ with

$$v_1 = v, \qquad v_k = v', \qquad r(v_i, v_{i+1}) = r(u, v), \qquad i = 1, 2, \ldots k-1.$$

Hence, (14) follows from (i), (ii).

Now, let k satisfy

$$(k-1)\, r(u, v) < \pi \leqq k r(u, v). \tag{15}$$

Then $\{v' \in S^2, r(u, v') \leqq k r(u, v)\} = S^2$, hence by (14)

$$k \lambda r(u, v) \geqq \sup_{u, v \in S^2} \tilde{\rho}(u, v) \geqq \frac{\pi}{2}.$$

With the help of (15) we have then

$$\lambda \geqq \frac{\pi/2}{k r(u, v)} = \frac{\pi/2}{(k-1)\, r(u, v) + r(u, v)} > \frac{\pi/2}{\pi + \pi/2} = \frac{1}{3},$$

and Lemma 7 follows.

Hence, in Case 1,

$$P_r(\mathcal{O}(x)) \geqq \tfrac{1}{8} P_{3r}(S^2),$$

which, in turn, is bounded from below by $C_1 r^{-2}$ for some $C_1 > 0$ for all $r < r_1$ with some $r_1 > 0$.

Case 2. In Case 1 we have considered the vector x, for which there exist some $u \in S^2$ with $\mathcal{R}_u x = x$. Now consider those $x-s$, for which for some $u \in S^2$, $\operatorname{diam}\{\mathcal{R}_u x\} \leqq a$, where a is some small constant, the concrete value of which we'll specify later, at the end of the last Case 3.

The first step in treating the Case 2 is to show the existence of $z \in S^{2n-1}$, for which

$$\rho(x, z) \leqq 2a \tag{16}$$

and $\dim \mathcal{O}(z) = 2$.

Indeed, let

$$z' = (2\pi)^{-1} \int_0^{2\pi} R_u(\theta)\, x\, d\theta; \qquad z = z^1/\|z^1\|$$

(because of $\|z^1\| \geqq 1-a$, the last definition makes sense). It's clear, that $R_u(\theta)\, z \equiv z$ for any θ, so $\dim \mathcal{O}(z) = 2$. Now, $\rho(x, R_u(\theta)\, x) \leqq a$ implies

$$\|x - R_u(\theta)\, x\| \leqq 2\sin(a/2).$$

Because of the convexity of the ball

$$B = \{y, \|x-y\| \leqq 2\sin(a/2)\},$$

we have $\|x - z^1\| \leqq 2\sin(a/2)$. But the ball B is seen from the origin at the angle $\left(\arcsin 2\sin\frac{a}{2}\right) < (\arcsin \sin 2a) = 2a$ for $a < \frac{\pi}{4}$, and (16) follows.

Now we'll construct a two-dimensional piece of $\mathcal{O}(x)$ (which is uniformly big).

Without loss of generality let's suppose $u = e_3$. Then, as it follows from Case 1, for any $s \in [0, \pi/6]$ there exist $\theta_s \in [0, \pi/2]$ with $\rho(z, R_1(\theta_s)\, z) = s$. Hence, for any $r \in [2a, \pi/6 - 2a]$ there exist $\theta_r \in [0, \pi/2]$ with $\rho(z, R_1(\theta_r)\, x) = r$.

But then $\rho(z, R_1(\theta_r)\, z) \geqq r - 2a$, and with the help of metric $\tilde{\rho}$ of Case 1,

$$\operatorname{diam}\{\mathcal{R}_3 R_1(\theta_r)\, z\} \geqq r - 2a \quad \text{which implies} \quad \operatorname{diam}\{\mathcal{R}_3 R_1(\theta_r)\, x\} \geqq r - 6a.$$

Moreover, for any

$$r_1, r_2 \in [2a, \pi/6 - 2a], \qquad \varphi, \psi \in [0, 2\pi]$$

$$\rho(R_3(\varphi)\, R_1(\theta_{r_1})\, x,\ R_3(\psi)\, R_1(\theta_{r_2})\, x) \geqq |r_1 - r_2|$$

as it follows from triangle inequality.

Finally, we have

$$P_r(\mathcal{O}(x)) \geqq P_r^{\#}(\Delta),$$

where by Δ we mean any (plane) triangle with base $(\pi/6 - 8a)$ and height $(\pi/6 - 8a)$, and by $P_r^{\#}(\Delta)$ we mean the number of points of the square lattice on the plane with spacing $2r$ inside Δ, provided that one of the axes of the lattice is parallel to the base of Δ.

Case 3. Consider now those x, for which $\operatorname{diam}\{\mathcal{R}_u x\} > a$ for each $u \in S^2$. (The constant a is the same as in Case 2.) According to Lemma 4 there exist $u, v \in S^2$,

$\theta_0 \in [0, 2\pi]$ with $\rho(x, \{\mathscr{R}_u R_v(\theta_0) x\}) \geqq \pi/8$. The function of θ, $\rho(x, \{\mathscr{R}_u R_v(\theta) x\})$, is continuous, being zero at $\theta = 0$. For any θ, diam $\{\mathscr{R}_u R_v(\theta) x\} \geqq a$.

The triangle inequality implies

$$\rho(\{\mathscr{R}_u R_v(\theta_1) x\}, \{\mathscr{R}_u R_v(\theta_2) x\}) \geqq |\rho(x, \{\mathscr{R}_u R_v(\theta_1) x\}) - \rho(x, \{\mathscr{R}_u R_v(\theta_2) x\})|.$$

Hence we have

$$P_r(\mathscr{O}(x)) \geqq P_r^{\#}(\square),$$

where $\square$ stands for the plane rectangle with sides a, $\pi/8$, and $P_r^{\#}(\square)$ stands for the number of sites in $\square$ of square plane lattice with spacing $2r$, provided the axes of the lattice are parallel to the sides of $\square$.

To complete the proof of the theorem we have only to choose a. To do it, observe that $P_r^{\#}(\Delta) \sim r^{-2} S(\Delta)$, $P_r^{\#}(\square) \sim r^{-2} S(\square)$, where S means area. But $S(\Delta) = \frac{1}{2}(\pi/6 - 8a)^2$, $S(\square) = \frac{a\pi}{8}$, so taking $a = 1/50$ we'll have $S(\Delta)$, $S(\square) > 0$. The proof of the theorem is thus finished.

5. The Case of Arbitrary Groups

Let $\mathbf{G}$ be any compact connected group. Consider the number

$$k(\mathbf{G}) = \min\{\dim M,\ M \text{ is a manifold, } \mathbf{G} \text{ acts transitively on } M\}.$$

It is clear, that $k(\mathbf{G})$ is defined by the above formula, and $1 \leqq k(\mathbf{G}) < \infty$. For example, k(Abelean group) $= 1$

$$k(\mathbf{SO}(n)) = n - 1.$$

We conjecture here the following

Hypothesis. *Let $T: \mathbf{G} \to U(n)$ be any irreducible representation and τ be $(\mathbf{G}, T)$-invariant probability measure on $S^{2n-1} \subset \mathbb{C}^n$. Then there exist $C = C(\mathbf{G})$, $r = r(\mathbf{G})$ such that*

$$\tau(\mathscr{D}_r(x)) \leqq C r^{k(\mathbf{G})}, \quad \text{for all } T, \quad x \in S^{2n-1}, \quad r \leqq r(\mathbf{G}).$$

The validity of this hypothesis would imply the following.

Theorem 4. *Theorem 1 remains valid after replacing the estimation* (4) *by the improved one:*

$$\sup_{g \in G} |q_n(g) - 1| \leqq \|p_i - 1\|_{L^2} \|p_j - 1\|_{L^2} \prod_{\substack{s=1 \\ s \neq i, j}}^{n} (1 - \tilde{C}(\mathbf{G}) S_{k(\mathbf{G})}(\mu_s))$$

for some $\tilde{C}(\mathbf{G}) > 0$.

Acknowledgement. The author would like to thank S.I. Gelfand for many helpful discussions.

References

1. Major, P., Shlosman, S.B.: A local limit theorem for the convolution of probability measures on a compact connected group. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 137–148 (1979)

Received January 30, 1980