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Hypoellipticity Theorems and Conditional Laws

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In this paper, we consider two kinds of second order differential equations: a heat equation with coefficients irregular with respect to the time variable and a stochastic partial differential equation as, for example, the Zakaï equation arising in the filtering theory. We prove the smoothness of the solutions of these equations under a condition of the Hörmander type. For this purpose, we adapt the proof by J.J. Kohn [11] of the classical Hörmander theorem [10]: in each case, we develop a symbolic calculus on a special class of pseudo-differential operators and obtain a priori inequalities in well fitted Sobolev spaces, which lead to the smoothness of the solutions as in the classical case.

As a direct application of these two theorems, we get regularity results for the conditional laws of filtering and smoothing theory under a global Hörmander condition. This gives a new proof of this smoothness result, which has been already obtained by J.M. Bismut-D. Michel [4], using the Malliavin calculus [3, 20, 21] under a local Hörmander condition.

H. Kunita [16, 17] has a similar theorem in the particular case where the coefficients of the Zakaï equation do not depend on the observation.

Let us remark that the idea is closely related to that of E. Pardoux [24, 26], N. Krylov-B. Rozovskii [12] and B. Rozovskii-A. Shimizu [27] who generalized regularity results for solutions of elliptic PDE (partial differential equations) to solutions of SPDE (stochastic partial differential equations).

The paper is structured as follows. In the first part, we establish the deterministic theorem, in the second part the probabilistic one (in these two parts, we follow the expository paper of M. Chaleyat-Maurel [5] on the paper of J.J. Kohn [11]). The last part is devoted to the applications.

We are grateful to J.M. Bismut for pointing out to us this approach of the problem and encouraging us during our investigations.

1. A Hörmander Theorem for a Class of Heat Equations

a) Notations

Let us denote by $C_{b,c}^{\infty} = C_{b,c}^{\infty}(\mathbb{R}^+ \times \mathbb{R}^n)$ the space of real valued functions f defined on $\mathbb{R}^+ \times \mathbb{R}^n$ such that:

a) f is $\mathscr{B}(\mathbb{R}^+)\otimes\mathscr{B}(\mathbb{R}^n)$ measurable.

b) for every $t>0$, $f(t,.)$ is smooth and all the derivatives w.r.t. x are bounded on compact subsets of $\mathbb{R}^{+*}\times\mathbb{R}^n$.

$V_0, \ldots, V_d$ are $d+1$ vector fields on $\mathbb{R}^+\times\mathbb{R}^n$ such that: $V_i(t,x)=V_i^j(t,x)\dfrac{\partial}{\partial x_j}$ and the V_i^j's are in $C_{b,c}^\infty$; c is an element of $C_{b,c}^\infty$. We call $\mathscr{L}(V_1, \ldots, V_d)$ the Lie algebra with coefficients in $C_{b,c}^\infty$ generated by $V_1, \ldots, V_d$ and, for $N\in\mathbb{N}$, $\mathscr{L}_N(V_1, \ldots, V_d)$ the sub-space of $\mathscr{L}(V_1, \ldots, V_d)$ generated by the brackets of $V_1, \ldots, V_d$ whose length is smaller than N.

The aim of this paragraph is to prove a hypoellipticity theorem for the operator $L=\dfrac{\partial}{\partial t}+V_0+\sum_{i=1}^{d} V_i^2+c$ acting on the following class of distributions: $\mathscr{G}$ is the class of Radon measures on $\mathbb{R}^+\times\mathbb{R}^n$ of the form $d\mu_t\times dt$ where $(\mu_t)_{t\in\mathbb{R}}$ is a family of Radon measures on $\mathbb{R}^n$ such that the map $t\to\mu_t$ from $\mathbb{R}^+$ to $\mathscr{M}(\mathbb{R}^n)$, the set of Radon measures on $\mathbb{R}^n$, is bounded on bounded sets of $\mathbb{R}^+$.

1.1. **Theorem.** *Assume that, for every compact set K of $\mathbb{R}^{+*}\times\mathbb{R}^n$, there exists $N\in\mathbb{N}$ such that: H: $\mathscr{L}_N(V_1, \ldots, V_d)(t,x)=\mathbb{R}^n$ for every $(t,x)\in K$.*

Let u be in $\mathscr{G}$, satisfying:

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{d} V_i^2 u+V_0 u+cu=f\in C_{b,c}^\infty.$$

Then, u has a density p and both p and $\dfrac{\partial p}{\partial t}$ are elements of $C_{b,c}^\infty$.

In the classical Hörmander theorem, the vector fields are supposed to be smooth; here we allow a measurable dependance with respect to t. Nevertheless, Kohn's method works because the only derivation w.r.t. t appearing in L is $\dfrac{\partial}{\partial t}$, i.e. a first order operator, and condition (H) only involves the "sum of squares" part of L.

So, the proof of Theorem 1.1 consists in integrating with respect to t the classical subelliptic inequalities (cf. [11]); this is justified since the assumptions on the V_i's and u imply that the estimates are uniform w.r.t. t.

b) A Class of Test Functions and Sobolev Norms

We define $\mathscr{S}_1$ as the class of real valued, measurable functions on $\mathbb{R}^+\times\mathbb{R}^n$, with compact support in $\mathbb{R}^{+*}\times\mathbb{R}^n$, smooth with respect to x for every $t>0$ and with all their derivatives w.r.t. x in $L^2(\mathbb{R}^+\times\mathbb{R}^n)$.

We take as a scalar product on $\mathscr{S}_1$ the scalar product on $L^2(\mathbb{R}^+\times\mathbb{R}^n)$ and we denote it by $(\,,\,)$.

We introduce now the class of pseudo-differential operators on which we shall work and the related Sobolev norms.

First, for v in $\mathscr{S}_1$, $\hat{v}(t,\xi)$ denotes the Fourier transform of $v(t,.)$ considered as a function on $\mathbb{R}^n$, and if Λ^α is the Bessel potential on $\mathbb{R}^n$ for $\alpha\in\mathbb{R}$, we still call Λ^α the operator acting on $\mathscr{S}_1$ in the following way:

$$\widehat{\Lambda^\alpha v}(t,\xi)=(1+|\xi|^2)^{\alpha/2}\,\hat{v}(t,\xi).$$

Then the associated α-Sobolev norm is:

$$\|v\|_\alpha^2=\int_{\mathbb{R}^+\times\mathbb{R}^n}|\Lambda^\alpha v(t,x)|^2\,dt\,dx.$$

We remark that if v is in $\mathscr{S}_1$: $\|v\|_\alpha<+\infty$, $\forall\alpha\in\mathbb{R}$. So v is in $L^2(\mathbb{R}^+, H^\alpha(\mathbb{R}^n))$.

We recall that, even if u is a smooth function on $\mathbb{R}^n$ with compact support, $\Lambda^\alpha u$ will not have compact support unless $\alpha\in\mathbb{N}$. In particular, Λ^α does not map $\mathscr{S}_1$ into itself. On the other hand, we have a decomposition: $\Lambda^\alpha=\Lambda'^\alpha+\Lambda''^\alpha$ (cf. [30]), where Λ'^α is a properly supported operator (so it sends $\mathscr{S}_1$ into itself) and Λ''^α is an operator of order $-\infty$ (so we shall ignore it in the a priori estimations). Then we consider the algebra $\mathscr{A}$ with coefficients in $C_{b,c}^\infty$ generated by the derivations $\frac{\partial}{\partial x_i}$, $i=1\dots n$ and the operators Λ'^α, $\alpha\in\mathbb{R}$. Now we state some lemmas dealing with the properties of $\mathscr{A}$.

1.2. **Lemma.** *$\mathscr{A}$ acts on $\mathscr{S}_1$.*

Proof. It suffices to prove that, if u is in $\mathscr{S}_1$, Pu is in $\mathscr{S}_1$ when $P=\Lambda'^\alpha$ or $P=\frac{\partial}{\partial x_i}$ or $P=b(t,x)$ with b in $C_{b,c}^\infty$ and this is immediate from the definitions.

1.3. Now, we recall the main definitions concerning a class of pseudo-differential operators that we need in the following (for more details, see [30]).

If U is a bounded open set in $\mathbb{R}^n$ and $m\in\mathbb{N}$, let $S_m(U,U)$ be the linear space of C^∞ functions a in $U\times U\times\mathbb{R}^n$ which have the following property: to every compact subset K of $U\times U\times\mathbb{R}^n$ and of n-tuples (α,β,γ), there is a constant $C_{\alpha,\beta,\gamma}(K)>0$ such that:

$$|D_\xi^\alpha D_x^\beta D_y^\gamma a(x,y,\xi)|\leqq C_{\alpha,\beta,\gamma}(K)(1+|\xi|)^{m-|\alpha|}.$$

To every a in $S^m(U,U)$, we associate the pseudo-differential operator A with amplitude a defined by the oscillating integral:

$$\forall u\in C_c^\infty(U),\qquad Au(x)=(2\pi)^{-1}\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{i(x-y)\xi}\,a(x,y,\xi)\,u(y)\,dy\,d\xi$$

(m is the order of A).

Let $\Psi_m(U)$ be the space of such operators. Every element of $\Psi_m(U)$ extends, by duality to a linear map on $\mathscr{E}'(U)$ the space of distributions with compact support in U. If K is a compact subset of U and φ a C^∞ function with support in K, we define a seminorm $p_{K,\varphi}$ on $\Psi_m(U)$ by:

$$p_{K,\varphi,\alpha}(A)=\sup_{\substack{u\in C_c^\infty(U)\\ \operatorname{sup}u\subset K}}\frac{\|\varphi Au\|_{H_c^{\alpha-m}}}{\|u\|_{H_c^\alpha}}.$$

We give now the special form of the amplitudes for the generating elements of $\mathscr{A}$: $(1+|\xi|^2)^{\alpha/2}$ is an amplitude for Λ^α; so, if φ is an element of $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which is identically equal to one in a neighborhood of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ and with compact support when multiplied by $g(x)$ or $g(y)$, $g \in C_c^\infty(\mathbb{R}^n)$; $\varphi(x, y)$ $(1+|\xi|^2)^{\alpha/2}$ is an amplitude for Λ'^α. Finally, if $A = b(t, x)\dfrac{\partial}{\partial x_i}$, $b(t, x)\xi_i$ is an amplitude for A.

1.4. **Lemma.** *Every element P of $\mathscr{A}$ of order m defines a map $t \to P_t$ from $\mathbb{R}^+$ to $\Psi_m(U)$ which sends bounded sets of $\mathbb{R}^{+*}$ into bounded sets of $\Psi_m(U)$.*

Proof. Writing the general form of an amplitude for P and using the properties of elements of $C_{b,c}^\infty$, we get the result.

Let us point out that this property of $\mathscr{A}$ is the main tool in the generalization of Kohn's proof to our case. We have in particular:

1.5. **Lemma.** *If P is an element of $\mathscr{A}$ of order m:*

$$\|Pv\|_\alpha \leqq c(\alpha)\,\|v\|_{\alpha+m}, \qquad \forall\, v \in \mathscr{S}_1.$$

The form of the operator occurring in Theorem 1.1 leads us to introduce a sub-class $\mathscr{S}_2$ of $\mathscr{S}_1$:

$$\mathscr{S}_2 = \left\{u \in \mathscr{S}_1,\ \frac{\partial u}{\partial t} \in \mathscr{S}_1\right\}$$

and, for $v \in \mathscr{S}_2$, we define:

$$Lv = \frac{\partial v}{\partial t} + \sum_{i=1}^{d} V_i^2 v + V_0 v + c v.$$

If U is a bounded open set in $\mathbb{R}^{+*} \times \mathbb{R}^n$, $\mathscr{S}_2(U)$ is the set of elements of U whose support is contained in U.

c) Proof of the Theorem

In a first stage, we establish that L satisfies an energy inequality for every v in $\mathscr{S}_2$. Then we obtain a priori estimates for L in terms of the Sobolev norms defined in § b., under the hypothesis H. After, if u is a Radon measure satisfying the hypothesis of Theorem 1.1, we localize and regularize u and, applying the a priori estimates to the regularized functions, we show that all the α-norms of ξu ($\xi \in C_c^\infty(\mathbb{R}^{+*} \times \mathbb{R}^n)$) are finite, which implies the smoothness of u for almost every t. The last step is to get the result for every t.

1.6. **Proposition.** *If U is a bounded open set in $\mathbb{R}^{+*} \times \mathbb{R}^n$, there exists $C > 0$ such that, for every u in $\mathscr{S}_2(U)$:*

$$\sum_{i=1}^{d} \|V_i u\|_0^2 \leqq C(|(Lu, u)| + \|u\|_0^2).$$

Proof. Integrating by parts and noticing that $\left(\frac{\partial u}{\partial t}, u\right)=0$, we get the estimation in the same way as in the classical case.

From now on, we suppose assumption H is verified.

1.7. **Proposition.** *There exists* $\varepsilon>0$ *such that, for all* $s\in\mathbb{R}$ *and all* u *in* $\mathcal{S}_2(U)$:

$$\|u\|_{s+\varepsilon}\leqq C(s)(\|Lu\|_s+\|u\|_s).$$

Proof. We sketch the proof in case $s=0$.

Using the inequality: $\|u\|_\varepsilon\leqq\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_{\varepsilon-1}+\|u\|_0$ (valid for $\varepsilon\leqq1$), we see that we have to get estimations of $\left\|\frac{\partial u}{\partial x_i}\right\|_{\varepsilon-1}$, $1\leqq i\leqq n$, by means of $\|Lu\|_0$ and $\|u\|_0$. This is, at this stage, that we use assumption H.

Indeed, there exists $N\in\mathbb{N}$ such that, at each point (t,x) of U, every $\frac{\partial}{\partial x_i}$ is a linear combination of brackets of $V_1,\dots,V_d$ whose length is less than N:

$$\frac{\partial}{\partial x_i}=\sum a^i_{i_1\dots i_k}F_{i_1\dots i_k}$$

where $F_{i_1\dots i_k}=[V_{i_k},F_{i_1\dots i_{k-1}}]$ and $F_{i_1}=V_{i_1}$.

The coefficients $a^i_{i_1\dots i_k}$ being in $C^\infty_{b,c}$, they are bounded in U. So it remains to bound $\|F_{i_1\dots i_k}u\|_{\varepsilon-1}$ by $C(\|Lu\|_0+\|u\|_0)$ and it is obtained recursively by the symbolic calculus via the formula:

$$\|F_{i_1\dots i_k}u\|_{\varepsilon-1}\leqq C(\|F_{i_1\dots i_{k-1}}u\|_{2\varepsilon-1}+\|Lu\|_0+\|u\|_0)$$

which leads to:

$$\|F_{i_1\dots i_k}u\|_{\varepsilon-1}\leqq C\left(\sum_{i=1}^{d}\|V_iu\|_{2^{k-1}\varepsilon-1}+\|Lu\|_0+\|u\|_0\right).$$

If $2^{N-1}\varepsilon\leqq1$, we conclude by using energy inequality.

The estimate, for $s\neq0$, is proved, as in the classical case, by commuting L and Λ^s.

1.8. *Localization and Regularization.* Let ξ and ξ_1 be elements of $C^\infty_c(\mathbb{R}^+\times\mathbb{R}^n)$ with support in $\mathbb{R}^{+*}\times\mathbb{R}^n$, such that $\xi_1=1$ on $\operatorname{supp}\xi$.

If φ is in $C^\infty_c(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\varphi(x)\,dx=1$, we set: $\varphi_\delta(x)=\delta^{-n}\varphi(\delta^{-1}x)$. Then, if u is an element of $\mathcal{G}$ (cf. § a)) such that $Lu=f\in C^\infty_{b,c}$, we define:

$$S_\delta\xi u(t,x)=\xi u*\varphi_\delta=\int_{\mathbb{R}^n}\xi(t,y)\,u(t,y)\,\varphi_\delta(x-y)\,dy.$$

S_δ is an element of $\Psi_{-\infty}(\mathbb{R}^n)$ and, now, we work on the algebra of pseudodifferential operators generated by $\mathcal{A}$ and the S_δ, $0\leqq\delta\leqq1$.

In order to apply Proposition 1.7 to $S_\delta\xi u$, we prove first the following:

1.9. **Lemma.** *$S_\delta \xi u$ is in $\mathcal{S}_2$.*

Proof. It is clear that $S_\delta \xi u$ and $\frac{d}{dt}(S_\delta \xi u)$ have compact support in $\mathbb{R}^{+*} \times \mathbb{R}^n$ and are smooth w.r.t. x.

We only have to prove that all their derivatives w.r.t. x are in $L^2(\mathbb{R}^+ \times \mathbb{R}^n)$. We have:

$$\frac{d}{dt}(S_\delta \xi u) = S_\delta \left(\frac{\partial \xi}{\partial t} u\right) + S_\delta \xi \left(-\sum_{i=1}^{d} V_i^2 u - V_0 u - cu - f\right).$$

We must study expressions of the form: $D_x^\alpha(S_\delta \eta Wu)$ where η is in $C_c^\infty(\mathbb{R}^{+*} \times \mathbb{R}^n)$ with support in $\mathbb{R}^{+*} \times \mathbb{R}^n$ and W is a differential operator on $\mathbb{R}^n$ with coefficients in $C_{b,c}^\infty$. Denoting by W^* the adjoint of W in $C_c^\infty(\mathbb{R}^n)$, we have:

$$D_x^\alpha(S_\delta \eta Wu) = \int u(t, y) W^*(t, .) \eta(t, y) D^\alpha \varphi_\delta(x-y) \, dy.$$

To show that this is in $L^2(\mathbb{R}^+ \times \mathbb{R}^n)$, we only have to prove that it is bounded since it has compact support.

For this, we recall that u is in $\mathcal{G}$ so:

$$D_x^\alpha(S_\delta \eta Wu) = \int W^*(t, .) \eta(t, y) D^\alpha \varphi(x-y) \, d\mu_t(y)$$

and for (x, t) varying in a compact set, the set of functions $y \to W^*(t, .) \eta(t, y) D^\alpha \varphi(x-y)$ is bounded for the sup norm.

The mollified distribution satisfies the following estimate:

1.10. **Proposition.** *There exists $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ and all u in $\mathcal{G}$*

$$\|S_\delta \xi u\|_{s+\varepsilon} \leqq C(s) \{\|\xi_1 f\|_s + \|\xi_1 u\|_s\}.$$

Proof. We remark first that $\operatorname{supp} S_\delta \xi u$ is contained in a compact set independant of δ. As in the classical case, we apply Proposition 1.7 to $S_\delta \xi u$ and let L and $S_\delta \xi$ commute. Using Lemma 1.4, we can generalize Friedrich lemma (cf. [30]) and get:

$$\|[T, S_\delta \xi] u\|_s \leqq C \|\xi_1 u\|_s$$

where T is a first order element of $\mathcal{A}$ and C is independant of δ.

1.11. *End of the Proof.* We can see easily that, μ_t being a Radon measure on $\mathbb{R}^n$ belonging to $\mathcal{G}$, we have: $\|\xi_1 u\|_{-n} < +\infty$.

We deduce then from Proposition 1.10 that there exists a constant C independant of δ such that:

$$\|S_\delta \xi u\|_{-n+\varepsilon} \leqq C$$

which implies that $\|\xi u\|_{-n+\varepsilon} < +\infty$ using the following lemma:

1.12. **Lemma.** *Properties* (i) *and* (ii) *are equivalent:*

(i) *there exists C independant of δ such that $\|S_\delta \xi u\|_\alpha < C$,*

(ii) $\|\xi u\|_\alpha < +\infty$.

Iterating this argument, we can show that for any $\xi_0 \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, for any $s \in \mathbb{R}$: $\|\xi_0 u\|_s < +\infty$ and so, for almost every $t \in \mathbb{R}^{+*}$, $u_t (= \mu_t)$ is in $C^\infty(\mathbb{R}^n)$. To get this smoothness result for all $t \in \mathbb{R}^{+*}$, we shall integrate with respect to t the equation giving $\frac{\partial}{\partial t}(\xi u)$ and use the following lemma:

1.13. **Lemma.** *Let v be a Radon measure on $\mathbb{R}^m$ such that, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, there exists a constant $C_\alpha > 0$ such that:*

$$\left|\int_{\mathbb{R}^m} D^\alpha \varphi(x)\, v(dx)\right| \leqq C_\alpha \|\varphi\|_\infty, \qquad \forall \varphi \in C_b^\infty(\mathbb{R}^m).$$

Then, v is a smooth function.

Let us fix $t \in \mathbb{R}^+$. If φ is in $C_b^\infty(\mathbb{R}^n)$, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} D^\alpha \varphi(x)\, \xi(t,x)\, \mu_t(dx) \\
&= \int_0^t \int_{\mathbb{R}^n} (-1)^\alpha \varphi(x) D^\alpha \left\{ \xi f(s,x) + \left(\frac{\partial \xi}{\partial s}(s,x) - \xi(s,x) \sum_{i=1}^d V_i^2(s,x) \right.\right. \\
&\quad \left.\left. - \xi(s,x)\, V_0(s,x) - \xi(s,x)\, c(s,x) \right) \mu_s(x) \right\} dx\, ds.
\end{aligned}$$

Then we get:

$$\left|\int_{\mathbb{R}^n} D^\alpha \varphi(x)\, \xi(t,x)\, \mu_t(dx)\right| \leqq \{C \|\xi_1 u\|_{\alpha+2} + C'\} \|\varphi\|_\infty.$$

and, by Lemma 1.13, μ_t is a smooth function.

2. A Stochastic Hörmander Theorem

The celebrated "sum of squares" Hörmander theorem gives the hypoellipticity of the heat operator $\frac{\partial}{\partial t} + \sum_{i=1}^d V_i^2 + V_0$ whose coefficients do not depend on the variable t, under the condition: the Lie algebra generated by $V_1, \ldots, V_d$ and the brackets of $V_0, V_1, \ldots, V_d$ is $\mathbb{R}^n$ at each $x \in \mathbb{R}^n$. In the previous paragraph, we allowed a dependance on t for the V_i's and so the vector field V_0 can not occur in the assumption. In this section, we consider stochastic heat operators where the dependance on t of the coefficients is through a brownian motion β; explicitly: $V_i(t,x) = \bar{V}_i(x, \beta_t)$, $1 \leqq i \leqq d$ and

$$V_0(t,x) = \tilde{V}_0(x, \beta_t) + \sum_{i=1}^p \tilde{V}_i(x, \beta_t) \frac{d\beta_t^i}{dt},$$

where $(x,z) \to \bar{V}_i(x,z)$, $\tilde{V}_0(x,z)$, $\tilde{V}_i(x,z)$ are C^∞ functions in $\mathbb{R}^n \times \mathbb{R}^p$ $\left(\frac{d\beta^i}{dt}\right.$ being a white noise$\Big)$. As we shall see in Sect. 3, operators of this form arise naturally in filtering theory. We shall prove a hypoellipticity result for these operators under a Hörmander condition including V_0 in a certain sense. We can see what happens by computing formally the bracket:

$$\left[\frac{\partial}{\partial t}+V_0, V_i\right]=[\tilde{V}_0, \bar{V}_i]+\left[\tilde{V}_j+\frac{\partial}{\partial z_j}, \bar{V}_i\right]\frac{d\beta^j}{dt}.$$

a) Notations

Let us denote by $C_b^\infty = C_b^\infty(\mathbb{R}^n \times \mathbb{R}^p)$ the space of smooth functions on $\mathbb{R}^n \times \mathbb{R}^p$ such that all their derivatives are bounded.

$V_0, V_1, \ldots, V_m, \tilde{V}_1, \ldots, \tilde{V}_p$ are $m+p+1$ vector fields on $\mathbb{R}^n \times \mathbb{R}^p$ such that:

$$V_i(x,z)=\sum_{j=1}^{n} V_i^j(x,z)\frac{\partial}{\partial x_j}, \qquad 0 \leqq i \leqq m$$

$$\tilde{V}_i(x,z)=\sum_{j=1}^{n} \tilde{V}_i^j(x,z)\frac{\partial}{\partial x_j}, \qquad 1 \leqq i \leqq p,$$

where the V_i^j's and $\tilde{V}_i^j$'s are in C_b^∞; $f, g_1, \ldots, g_p$ are $p+1$ elements of C_b^∞.

We want to work on a special class of random distributions $\mathscr{G}$ which is defined in the following way: let $(\Omega, \mathscr{F}, P)$ be a probability space filtered by $(\mathscr{F}_t)_{t\in\mathbb{R}^+}$ and $(\beta_t)_{t\in\mathbb{R}^+}$ be a $\mathscr{F}_t$-brownian motion on Ω with values in $\mathbb{R}^p$. We shall denote by ω the generic element of Ω.

If X is a semi-martingale defined on $(\Omega, \mathscr{F}_t, P)$, δX denotes its Ito differential and dX its Stratonovitch differential (cf. [22]). We denote by $\delta_M X$ (resp. $\delta_D X$) the martingale part (resp. the bounded variation part) of δX.

$\mathscr{G}$ is the class of random variables on Ω with values in $\mathscr{M}(\mathbb{R}^+ \times \mathbb{R}^n)$ of the form $\mu_t(\omega, dx)\,dt$ such that:

i) $\mu(\cdot, dx)$ is $\mathscr{F}_t$-progressively measurable with values in $\mathscr{M}(\mathbb{R}^n)$.

ii) the map $t \to \mu_t$ sends bounded sets of $\mathbb{R}^{+*}$ into bounded sets of the space of random radon measures on $\mathbb{R}^n$ normed by:

$$\|\mu(\omega)\| = E(\|\mu\|^q_{\mathscr{M}(\mathbb{R}^n)})^{1/q} \quad \text{where} \quad q=\sup(n+1, 4).$$

On this class, we can state the following SPDE:

$$\int_{\mathbb{R}^n} \varphi(t,x)\,\mu_t(dx)+\int_0^t\left(\int_{\mathbb{R}^n}\left\{\mathscr{L}^*_{0,s}\varphi(s,x)-\frac{\partial\varphi}{\partial s}(s,x)\right\}\mu_s(dx)\right)ds$$

$$+\int_0^t\left(\int_{\mathbb{R}^n}\mathscr{L}^*_{i,s}\varphi(s,x)\,\mu_s(dx)\right)\delta\beta^i=\int_{\mathbb{R}^n}\left(\int_0^t h_0(x,\beta_s)\,ds+\int_0^t h_i(x,\beta_s)\,\delta\beta_s^i\right)\varphi(t,x)\,dx,$$

a.s. for every $\varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ where:

i) $\mathscr{L}_{0,s}=\left\{-\frac{1}{2}\left(\sum_{i=1}^{m} V_i^2+\sum_{i=1}^{p}\tilde{V}_i^2\right)+V_0+f\right\}(\cdot, \beta_s)$,

ii) $\mathscr{L}_{i,s}=(\tilde{V}_i+g_i)(\cdot, \beta_s)$, $1 \leqq i \leqq p$,

iii) $h_0, \ldots, h_p$ are in C_b^∞.

In the following, we shall write $h(t, x, \omega) = \int_0^t h_0(x, \beta_s)\, ds + \int_0^t h_i(x, \beta_s)\, \delta \beta_s^i$.

Let $\mathscr{R}$ be the Lie algebra, with coefficients in C_b^∞, generated by $V_1 \ldots V_m$ and the brackets of $V_1 \ldots V_m$, $\tilde{V}_1 + \frac{\partial}{\partial z_1}, \ldots, \tilde{V}_p + \frac{\partial}{\partial z_p}$, where, at least, one V_i appears.

If $N \in \mathbb{N}$, we denote by $\mathscr{R}_N$ the sub-space of $\mathscr{R}$ generated by the brackets of length smaller than N.

Our aim is to prove the following theorem.

2.1. **Theorem.** *Assume that, for every compact set K of $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that:*

$$\tilde{H}: \mathscr{R}_N(x, z) = \mathbb{R}^n \quad \textit{for every } (x, z) \in K \times \mathbb{R}^p.$$

Let $u = \mu_t\, dt$ in $\tilde{\mathscr{G}}$ satisfying equation (1). *Then,* a.s., *for every $t > 0$, μ_t admits a smooth density.*

b) Classes of Test Semi-Martingales

We define first a wide class of semi-martingales on which a stochastic Fubini theorem is valid.

2.2. **Definition.** $\tilde{\mathscr{S}}_0$ is the class of maps $u: \mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ such that there exist $p+1$ maps $u_0, u_1, \ldots, u_p$: $\mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ satisfying:

i) u_i is $\mathscr{F}_t$-progressively measurable.

ii) u_i is in $L^2(\mathbb{R}^+ \times \mathbb{R}^n \times \Omega)$.

iii) $u(t, x, \omega) = \int_0^t u_0(s, x, \omega)\, ds + \int_0^t u_i(s, x, \omega)\, \delta \beta_s^i$.

By a theorem of C. Stricker-M. Yor [28], this implies that u admits an $\mathscr{F}_t$-progressively measurable version.

Let us denote by $(,)$ the standard scalar product on $L^2(\mathbb{R}^n)$.

In the proof of Theorem 2.1, we shall often use the following stochastic Fubini theorem that we state without proof, the arguments being standard:

Let f: $\mathbb{R}^n \to \mathbb{R}$ be in $L^2(\mathbb{R}^n)$ and u in $\tilde{\mathscr{S}}_0$. Then:

$$\int_{\mathbb{R}^n} f(x)\, u(t, x, \omega)\, dx = \int_0^t \Big(\int_{\mathbb{R}^n} f(x)\, u_i(s, x, \omega)\, dx\Big)\, \delta \beta_s^i + \int_0^t \Big(\int_{\mathbb{R}^n} f(x)\, u_0(s, x, \omega)\, dx\Big)\, ds, \tag{2}$$

a.s. for every $t \in \mathbb{R}^+$.

2.3. **Definition.** $\tilde{\mathscr{S}}_1$ is the class of maps u: $\mathbb{R}^+ \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ such that:

i) u is $\mathscr{B}(\mathbb{R}^+) \times \mathscr{B}(\mathbb{R}^n) \times \mathscr{F}$ measurable.

ii) a.s., for every $t > 0$, $u(t, ., \omega)$ is in $C^\infty(\mathbb{R}^n)$.

iii) u has compact support in $\mathbb{R}^{+*} \times \mathbb{R}^n$ independent of ω.

iv) all the derivatives of u w.r.t. x are in $L^q(\mathbb{R}^+ \times \mathbb{R}^n \times \Omega)$ $(q = \sup(n+1, 4))$.

If U is a bounded open set in $\mathbb{R}^{+*} \times \mathbb{R}^n$, $\tilde{\mathscr{S}}_1(U)$ is the set of elements of $\tilde{\mathscr{S}}_1$ whose support is in U.

As in the deterministic case (cf. § 1), we define a class of pseudo-differential operators acting on $\tilde{\mathscr{S}}_1$. First, for u in $\tilde{\mathscr{S}}_1$, $\hat{u}(t, \xi, \omega)$ denotes the Fourier transform of $u(t, ., \omega)$ considered as a function on $\mathbb{R}^n$ and, if Λ^α is the Bessel potential on $\mathbb{R}^n$, for $\alpha \in \mathbb{R}$, we still call Λ^α the operator acting on $\tilde{\mathscr{S}}_1$ in the following way:

$$\widehat{\Lambda^\alpha u}(t, \xi, \omega) = (1+|\xi|^2)^{\alpha/2}\, \hat{u}(t, \xi, \omega).$$

As in [12] and [25], we introduce the partial Sobolev norms:

$$\|u\|_\alpha^2 = E\Big(\int_{\mathbb{R}^+} \int_{\mathbb{R}^n} |\Lambda^\alpha u(t, x, \omega)|^2\, dx\, dt\Big).$$

We remark that, if u is in $\tilde{\mathscr{S}}_1$, $\|u\|_\alpha < +\infty$, $\forall \alpha \in \mathbb{R}$; so u is in $L^2(\Omega \times \mathbb{R}^+, H^\alpha(\mathbb{R}^n))$.

At last, we consider the algebra $\mathscr{C}$ of pseudo-differential operators on $\mathbb{R}^n$ depending on a parameter $z \in \mathbb{R}^p$ generated by the derivations $\frac{\partial}{\partial x_i}$, $1 \leqq i \leqq n$ and the operators Λ'^α (associated to the Λ^α as in § 1b) with coefficients $a(x, z)$, a being in C_b^∞. Then, we define an algebra $\tilde{\mathscr{A}}$ of random pseudo-differential operators on $\mathbb{R}^+ \times \mathbb{R}^n$: an element of $\tilde{\mathscr{A}}$ is obtained from an element of $\mathscr{C}$ by replacing z by β_t; so the random character of $\tilde{\mathscr{A}}$ is introduced through the brownian motion β.

This algebra has the following properties.

2.4. **Lemma.** *$\tilde{\mathscr{A}}$ acts on $\tilde{\mathscr{S}}_1$.*

Proof. Let u be in $\tilde{\mathscr{S}}_1$ and P in $\tilde{\mathscr{A}}$. It suffices to prove that Pu is in $\tilde{\mathscr{S}}_1$ for $P = P_1 = \Lambda'^\alpha$ and $P = P_2 = a(x, \beta_t)\frac{\partial}{\partial x_i}$. Properties (i) and (ii) are clear. (iii) is valid because P_1 is properly supported and P_2 is a differential operator.

The definition of P_1 implies that P_1 satisfies condition (iv). The same is true for P_2 because a is in C_b^∞.

In the classical case, a pseudo-differential operator on $\mathbb{R}^n$ of order m continuously sends $H_{\mathrm{loc}}^{\alpha+m}(\mathbb{R}^n)$ into $H_{\mathrm{loc}}^\alpha(\mathbb{R}^n)$. We generalize this result to our case.

2.5. **Lemma.** *If P is an element of $\tilde{\mathscr{A}}$ of order m and U a bounded open set of $\mathbb{R}^+ \times \mathbb{R}^n$, there exist constants $C(\alpha)$, $\alpha \in \mathbb{R}$, such that:*

$$\|Pu\|_\alpha \leqq C(\alpha)\, \|u\|_{\alpha+m}, \quad \forall u \in \tilde{\mathscr{S}}_1(U). \tag{3}$$

Proof. It suffices to prove this inequality for the generators of $\tilde{\mathscr{A}}$ i.e. $\frac{\partial}{\partial x_i}$, $1 \leqq i \leqq n$ Λ'^α, $\alpha \in \mathbb{R}$ and the multiplication by $a(x, \beta_t)$ with a in C_b^∞. Since $\frac{\partial}{\partial x_i}$ and Λ'^α do not depend on the variable t, we obtain (3) by integrating w.r.t. t the classical inequality on $\mathbb{R}^n$. The same property is valid for $a(x, \beta_t)$ since the support of u is contained in a fixed compact set and a is in C_b^∞.

This inequality could also have been obtained as a consequence of a lemma similar to Lemma 1.4.

Finally we define the class on which we shall work.

2.6. **Definition.** $\tilde{\mathscr{S}}_2$ is the class of semi-martingales u in $\tilde{\mathscr{S}}_1 \cap \tilde{\mathscr{S}}_0$ with u_i, $i=0$ to p, also in $\tilde{\mathscr{S}}_1$. If U is a bounded open set of $\mathbb{R}^n$ we define: $\tilde{\mathscr{S}}_2(U) = \tilde{\mathscr{S}}_2 \cap \tilde{\mathscr{S}}_1(U)$.

2.7. **Lemma.** *$\tilde{\mathscr{A}}$ acts on $\tilde{\mathscr{S}}_2$.*

It suffices to prove that the generators of $\tilde{\mathscr{A}}$ acts on $\tilde{\mathscr{S}}_2$ i.e. that Λ'^{α}, $\frac{\partial}{\partial x_i}$ and the multiplication by $a(x, \beta_t)$ ($a \in C_b^\infty$) send $\tilde{\mathscr{S}}_2$ into itself. As we saw in §1.3., Λ'^{α} and $\frac{\partial}{\partial x_i}$ have amplitudes independent of t and we shall prove the property for them in the next lemma. For the multiplication by $a(x, \beta_t)$, we write Ito's formula for the product $a(x, \beta_t)\, u(t, x, \omega)$, $u \in \tilde{\mathscr{S}}_2$:

$$\begin{aligned} a(x, \beta_t)\, u(t, x, \omega) = & \int_0^t \left\{ \frac{\partial a}{\partial z_i}(x, \beta_s)\, u(s, x, \omega) + a(x, \beta_s)\, u_i(s, x, \omega) \right\} \delta \beta_s^i \\ & + \int_0^t \left\{ \frac{1}{2} \sum_{i=1}^{p} \frac{\partial^2 a}{\partial z_i^2}(x, \beta_s)\, u(s, x, \omega) \right. \\ & \left. + \sum_{i=1}^{p} \frac{\partial a}{\partial z_i}(x, \beta_s)\, u_i(s, x, \omega) + a(x, \beta_s)\, u_0(s, x, \omega) \right\} ds \end{aligned}$$

a being in C_b^∞ and u in $\tilde{\mathscr{S}}_1 \cap \tilde{\mathscr{S}}_0$, it is clear that the four properties of $\tilde{\mathscr{S}}_1$ are satisfied by $a(x, \beta_t)\, u$.

2.8. **Lemma.** *Let P be an element of $\tilde{\mathscr{A}}$ whose amplitude does not depend on t. Then, if u is in $\tilde{\mathscr{S}}_2$, we have:* a.s. *for every (t, x) in $\mathbb{R}^+ \times \mathbb{R}^n$:*

$$P u(t, x, \omega) = \int_0^t P u_i(s, x, \omega)\, \delta \beta_s^i + \int_0^t P u_0(s, x, \omega)\, ds. \tag{4}$$

Proof. Let us fix $x \in \mathbb{R}^n$. If P is an element of $\tilde{\mathscr{A}}$ satisfying the assumption of the lemma, it can be viewed as a pseudo-differential operator on $\mathbb{R}^n$ and it admits the representation

$$P u(t, x, \omega) = (2\pi)^{-n} \iint e^{i(x-y)\xi}\, b(x, y, \xi)\, u(t, y, \omega)\, dy\, d\xi$$

where b can be written:

$$b(x, y, \xi) = \sum_{r_i + s_i \leq m} \varphi_i(x, y)\, \xi^{r_i} (1 + |\xi|^2)^{s_i/2} \quad \text{(cf. §1.3).}$$

We first notice that we can suppose $b(., ., \xi)$ has compact support on $\mathbb{R}^n \times \mathbb{R}^n$ independant of ξ. Then, by a standard argument of integration by parts in oscillatory integrals, we can consider that $b(x, ., .)$, as well as $\frac{\partial b}{\partial x_i}$ and $\frac{\partial b}{\partial y_i}$, $1 \leq i \leq n$, is uniformally bounded in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ when x stays in a compact set.

To obtain (4) a.s. for every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$ we use the Kolmogorov theorem to show that a.s., the two members of (4) are continuous w.r.t. (t,x). We prove first that:

$$E(Pu(t',x',\omega)-Pu(t,x,\omega))^{4q}\leqq C(|t-t'|^q+|x-x'|^{4q}),$$

for $q=\sup(4,n+1)$. Indeed, if we define:

$$\Delta_{t',t}=Pu(t',x',\omega)-Pu(t,x',\omega)=(2\pi)^{-n}\iint e^{i(x'-y)\xi}b(x',y,\xi)$$
$$(u(t',y,\omega)-u(t,y,\omega))\,dy\,d\xi,$$

applying Schwarz inequality, we get:

$$E(\Delta_{t',t}^{4q})\leqq CE\Big(\int_{\mathbb{R}^n}(u(t',y,\omega)-u(t,y,\omega))^{4q}dy\Big)\leqq CE\left(\int_{\mathbb{R}^n}\left(\int_t^{t'}\sum_{i=0}^{p}u_i^2(s,y,\omega)\,ds\right)^{2q}dy\right)$$
$$\leqq C|t'-t|^qE\left(\int_{\mathbb{R}^+\times\mathbb{R}^n}\sum_{i=1}^{p}u_i^{4q}(s,y,\omega)\,ds\,dy\right)\leqq C|t'-t|^q$$

and here we have used property (iv) of $\tilde{\mathscr{S}}_1$.

Writing $\Delta_{x',x}=Pu(t,x',\omega)-Pu(t,x,\omega)$, we are led to estimate the oscillating integral:

$$\iint e^{i(x''-y)\xi}\left(i\xi_j b(x'',y,\xi)+\frac{\partial b}{\partial x_j}(x'',y,\xi)\right)u(t,y,\omega)\,dy\,d\xi$$

where x'' is in the segment joining x and x'.

First, we integrate by parts w.r.t. y to make ξ_j disappear. The estimation follows then from the fact that b, $\dfrac{\partial b}{\partial x_j}$ and $\dfrac{\partial b}{\partial y_j}$ were supposed to be in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$ and from property (4) of $\tilde{\mathscr{S}}_1$.

The same estimation can be obtained for the second member of (4).

2.9. *Definition of the Operator L.* For u and v in $\tilde{\mathscr{S}}_2$, let us define:

$$\langle Lu,v\rangle(t,\omega)=\int_{\mathbb{R}^n}dx\left(\int_0^t v(s,x,\omega)\,\delta u(s,x,\omega)\right.$$
$$\left.+\int_0^t v(s,x,\omega)\,\mathscr{L}_{0,s}u(s,x,\omega)\,ds+\int_0^t v(s,x,\omega)\,\mathscr{L}_{i,s}u(s,x,\omega)\,\delta\beta_s^i\right)$$

where the first integral on $[0,t]$ is a stochastic integral with respect to the semi-martingale u (cf. [22]). (It exists because v and u_i are in $L^q(\mathbb{R}^+\times\mathbb{R}^n\times\Omega)$).

From the Fubini Theorem (2), we know that $\langle Lu,v\rangle$ is a semi-martingale. $\langle Lu,v\rangle_M$ will be its martingale part, $\langle Lu,v\rangle_D$ its bounded variation part. Let us introduce furthermore:

$$L_i u(t,x,\omega)=u_i(t,x,\omega)+\mathscr{L}_{i,t}u(t,x,\omega),\qquad 1\leqq i\leqq p$$

which is a kind of martingale part of "Lu", and the associated quantities:

$$\|Lu\|_{\alpha,\mathcal{M}}^2 = \sum_{i=1}^{p} \{\|L_i u\|_\alpha^2 + |\langle L_i u, \mathcal{L}_i u\rangle_\alpha|\}$$

where $\langle\,,\rangle_\alpha$ is the scalar product associated to the norm $\|\;\|_\alpha$. We notice that, if u is in $\tilde{\mathcal{S}}_2$, $\|Lu\|_{\alpha,\mathcal{M}}$ is finite for all $\alpha\in\mathbb{R}$.

c) The A-Priori Inequalities

The organization of this paragraph is the same as in the deterministic case (cf. §1c).

2.10. *The energy inequality for the V_i's.*

Proposition. *Let U be a bounded open set of $\mathbb{R}^{+*}\times\mathbb{R}^n$. There exists a constant C such that, for every u in $\tilde{\mathcal{S}}_2(U)$:*

$$\sum_{i=1}^{m} \|V_i u\|_0^2 \leqq C\{E(|\langle Lu, u\rangle_D(\infty,\omega)|) + \|u\|_0^2 + \|Lu\|_{0,\mathcal{M}}^2\}.$$

Proof.

$$\langle Lu, u\rangle(t,\omega) = \int_{\mathbb{R}^n} dx \left(\int_0^t u\,\delta u + \int_0^t u\,\mathcal{L}_0 u\,ds + \int_0^t u\,\mathcal{L}_i u\,\delta\beta^i\right). \tag{5}$$

Applying the Ito formula for large t (in the following, we always suppose t large) we get:

$$\int_0^t u\,\delta u = -\tfrac{1}{2}\sum_{i=1}^{p}\int_0^t u_i^2\,ds.$$

We remark that there exist functions $q_0 \dots q_m$, $\tilde{q}_1 \dots \tilde{q}_p$ such that:

$$V_i^* = -V_i + q_i,\quad 0\leqq i\leqq m;\qquad \tilde{V}_i^* = -\tilde{V}_i + \tilde{q}_i,\quad 1\leqq i\leqq p.$$

Taking the bounded variation part of each member of (5), we have only to compute

$$\int_{\mathbb{R}^n}\int_0^t u\,\mathcal{L}_0 u\,ds\,dx.$$

First:

$$\int_{\mathbb{R}^n}\int_0^t u(V_0 u + cu)\,ds\,dx = \int_{\mathbb{R}^n}\int_0^t u(\tfrac{1}{2}q_0 + c)u\,ds\,dx.$$

As q_0 and c are in C_b^∞, this term gives a contribution $O(\|u\|_0^2)$. On the other hand, as in [11]:

$$(V_i^2 u, u) = -(V_i u, V_i u) + \tfrac{1}{2}(q_i^2 u, u).$$

In the same way:

$$(\tilde{V}_i^2 u, u) = -(\tilde{V}_i u, \tilde{V}_i u) + \tfrac{1}{2}(\tilde{q}_i^2 u, u).$$

Finally:

$$\sum_{i=1}^{m} \int_{\mathbb{R}^n} dx \left(\int_0^t (V_i u)^2\, ds\right) = 2\langle Lu, u\rangle_D(t,\omega)$$
$$+\sum_{i=1}^{p} \int_{\mathbb{R}^n} dx \left(\int_0^t (u_i^2 - (\tilde{V}_i u)^2)\, ds\right) + O\left(\int_{\mathbb{R}^n} dx \left(\int_0^t u^2\, ds\right)\right).$$

Let us examine $u_i^2 - (\tilde{V}_i u)^2$.

$$u_i^2 - (\tilde{V}_i u)^2 = (L_i u - \tilde{V}_i u - g_i u)^2 - (\tilde{V}_i u)^2 = (L_i u - g_i u)(L_i u - 2\tilde{V}_i u - g_i u)$$
$$= (L_i u - g_i u)(L_i u - 2\mathscr{L}_i u + g_i u) = (L_i u)^2 - (g_i u)^2 - 2\mathscr{L}_i u (L_i u - g_i u).$$

Integrating w.r.t. (t,x) and noticing that $g_i \mathscr{L}_i$ is a first order differential operator, we get:

$$E\sum_{i=1}^{p} \left|\int_{\mathbb{R}^n} dx \left(\int_0^t (u_i^2 - (\tilde{V}_i u)^2)\, ds\right)\right| \leqq C\{\|Lu\|_{0,\mathscr{M}}^2 + \|u\|_0^2\}.$$

2.11. *A first Sobolev a priori inequality.*

Proposition. *Assume $\tilde{H}$. If U is a bounded open set of $\mathbb{R}^{+*} \times \mathbb{R}^n$, there exist $\varepsilon_0 > 0$ and a constant C such that, for $\varepsilon < \varepsilon_0$ and $u \in \tilde{\mathscr{S}}_2(U)$:*

$$\|u\|_\varepsilon^2 \leqq C\{E(|\langle Lu, u\rangle_D|) + \|u\|_0^2 + \|Lu\|_{0,\mathscr{M}}^2\}. \tag{6}$$

Proof. Let u be in $\tilde{\mathscr{S}}_2(U)$. Then, as in the classical case:

$$\|u\|_\alpha^2 = \|\Lambda^\alpha u\|_0^2 = \|u\|_{\alpha-1}^2 + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{\alpha-1}^2 \leqq \|u\|_0^2 + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{\alpha-1}^2, \quad \text{if } \alpha \leqq 1. \tag{7}$$

The assumption $\tilde{H}$ implies that each derivation $D_i = \dfrac{\partial}{\partial x_i}$ can be expressed as:

$$D_i = \sum \lambda_i^{i_1\ldots i_k, j_1\ldots j_k}(x,z)\, F_{i_1\ldots i_k}^{j_1\ldots j_k}(x,z), \quad k \leqq N \text{ at each } (x,z) \in \mathbb{R}^n \times \mathbb{R}^p \tag{8}$$

where the coefficients $\lambda_i^{i_1\ldots i_k, j_1\ldots j_k}$ are in C_b^∞ and the $F_{i_1\ldots i_k}^{j_1\ldots j_k}$ are defined by:

$$F_{i_1}^{j_1} = V_{i_1}$$
$$F_{i_1\ldots i_k}^{j_1\ldots j_k} = [V_{i_k}, F_{i_1\ldots i_{k-1}}^{j_1\ldots j_{k-1}}] \qquad \text{if } j_k = 1$$
$$= \left[\tilde{V}_{i_k} + \frac{\partial}{\partial z_{i_k}}, F_{i_1\ldots i_{k-1}}^{j_1\ldots j_{k-1}}\right] \qquad \text{if } j_k = 2,$$

Indeed, using Jacobi's identity, we can always suppose that $\mathscr{R}(x,z)$ is generated by brackets of this form. In addition, we deduce from the hypoellipticity condition $\tilde{H}$ that we can take $j_1 = 1$ in each bracket $F_{i_1\ldots i_k}^{j_1\ldots j_k}$.

For simplicity, we will write:

$$F_{i_1\ldots i_k}^{j_1\ldots j_k} = F^k; \qquad F_{i_1\ldots i_{k-1}}^{j_1\ldots j_{k-1}} = F^{k-1} = X, \qquad F^k = [Y, X].$$

We write (8) at the point (x, β_t). Then, applying Lemma 2.5 and using inequality (7), it remains to bound $\|F^k u\|_{\alpha-1}^2$ by the right side of (6) since the λ_i's are uniformly bounded. We point out that it is at this step that we need the global Hörmander hypothesis; indeed, this allows us not to care about the λ_i's and to get the estimations recursively by means of the brackets.

We define: $T^{2\alpha-1}=\Lambda^{2\alpha-2}\circ F^k$. Then:

$$\|F^k u\|_{\alpha-1}^2=E\big(\int_{\mathbb{R}^+}([Y,X]u,T^{2\alpha-1}u)\,dt\big).$$

When $Y=V_{i_k}$, we can integrate on $\mathbb{R}^+\times\Omega$ the estimates of the classical case and we obtain:

$$\begin{aligned}\|F^k u\|_{\alpha-1}^2&\leqq C\left\{\|F^{k-1}u\|_{2\alpha-1}^2+\|u\|_0^2+\sum_{i=1}^{p}\|V_i u\|_0^2\right\}\\&\leqq C\{\|F^{k-1}u\|_{2\alpha-1}^2+\|u\|_0^2+E(|\langle Lu,u\rangle_D|)+\|Lu\|_{0,\mathcal{M}}^2\}.\end{aligned}$$

When $Y=\tilde{V}_{i_k}+\dfrac{\partial}{\partial z_{i_k}}$, it is similar to the estimation for the first order term in Hörmander's theorem which is obtained by using the adjoint operator; here, the notion of adjoint for L has no meaning, so we replace it by the following lemma.

Lemma. *Let u and v be in $\tilde{\mathcal{S}}_2$. Then:*

$$\int_{\mathbb{R}^n}\left(\int_0^t v\left[\mathcal{L}_i+\frac{\partial}{\partial z_i},X\right]u\,\delta\beta^i\right)dx=\langle LXu,v\rangle_M(t,\omega)-\langle Lu,X^*v\rangle_M(t,\omega).\tag{9}$$

Proof. $\langle LXu,v\rangle_M=\int_{\mathbb{R}^n}\left(\int_0^t v\,\delta_M Xu+\int_0^t v\mathcal{L}_i Xu\,\delta\beta^i\right)dx$

$$\langle Lu,X^*v\rangle_M=\int_{\mathbb{R}^n}\left(\int_0^t X^*v\,\delta_M u+\int_0^t X^*v\mathcal{L}_i u\,\delta\beta^i\right)dx.$$

First:

$$\int_{\mathbb{R}^n}\left(\int_0^t v\mathcal{L}_i Xu\,\delta\beta^i\right)dx-\int_{\mathbb{R}^n}\left(\int_0^t X^*v\mathcal{L}_i u\,\delta\beta^i\right)dx=\int_{\mathbb{R}^n}\left(\int_0^t v[\mathcal{L}_i,X]u\,\delta\beta^i\right)dx.$$

On the other hand, applying Lemma 2.7, we get:

$$\int_0^t v\,\delta_M Xu=\int_0^t v\left(\frac{\partial X}{\partial z_j}u+Xu_j\right)\delta\beta^j$$

where $\dfrac{\partial X}{\partial z_j}=\sum_{i=1}^n\dfrac{\partial X_i}{\partial z_j}\dfrac{\partial}{\partial x_i}$ if $X=\sum_{i=1}^n X_i\dfrac{\partial}{\partial x_i}$.

Then:

$$\int_{\mathbb{R}^n}\left(\int_0^t v\,\delta_M Xu\right)dx-\int_{\mathbb{R}^n}\left(\int_0^t X^*v\,\delta_M u\right)dx=\int_{\mathbb{R}^n}\int_0^t v\left(\frac{\partial X}{\partial z_j}u\,\delta\beta^j\right)dx.\quad\square$$

Now, we express the second member of (9) as a stochastic integral with respect to β.

$$\langle LXu, v\rangle_M = \int_{\mathbb{R}^n} \left(\int_0^t v\,\delta_M X u + \int_0^t v \mathscr{L}_i X u\,\delta\beta^i\right) dx$$
$$= \int_{\mathbb{R}^n} \left(\int_0^t -Xu\,\delta_M v + \int_0^t X u \mathscr{L}_i^* v\,\delta\beta^i\right) dx$$
$$= -\langle Lv, Xu\rangle_M + \int_{\mathbb{R}^n} \left(\int_0^t X u(\mathscr{L}_i^* + \mathscr{L}_i) v\,\delta\beta^i\right) dx.$$

$$\langle LXu, v\rangle_M - \langle Lu, X^* v\rangle_M$$
$$= \int_0^t \left(\int_{\mathbb{R}^n} (-XuL_i v - X^* v L_i u + Xu(\mathscr{L}_i^* + \mathscr{L}_i)v)\,dx\right)\delta\beta^i.$$

So, (9) implies:

$$\int_{\mathbb{R}^n} v\left[\mathscr{L}_i + \frac{\partial}{\partial z_i}, X\right] u\,dx = \int_{\mathbb{R}^n} (-XuL_i v - X^* v L_i u + Xu(\mathscr{L}_i^* + \mathscr{L}_i)v)\,dx.$$

Taking $v = T^{2\alpha-1}u$, we get:

$$\|F^k u\|_{\alpha-1}^2 = E\Big(\int_{\mathbb{R}^n\times\mathbb{R}^+} (-XuL_{i_k}v - X^* v L_{i_k} u + Xu(\mathscr{L}_{i_k}^* + \mathscr{L}_{i_k})v - v[g_{i_k}, X]u)\,dx\,dt.$$

Let us examine each term separately:

a) $L_i v = v_i + \mathscr{L}_i v = \dfrac{\partial T^{2\alpha-1}}{\partial z_i} u + [\mathscr{L}_i, T^{2\alpha-1}]u + T^{2\alpha-1} L_i u.$

b) $X^* v = -Xv + (X^* + X)v = -T^{2\alpha-1}Xu - [X, T^{2\alpha-1}]u + (X^* + X)v.$

c) $\mathscr{L}_{i_k}^* + \mathscr{L}_{i_k}$ is of order 0.

So:

$$\|F^k u\|_{\alpha-1}^2 \leqq C\{\|F^{k-1}u\|_{2\alpha-1}^2 + \|u\|_0^2 + \|Lu\|_{0,\mathscr{M}}^2\}.$$

Inductively, we then obtain:

$$\|F^k u\|_{\alpha-1}^2 \leqq C\left\{\sum_{i=1}^p \|V_i u\|_{2^k\alpha-1} + \|u\|_0^2 + \|Lu\|_{0,\mathscr{M}}^2\right\}.$$

If $2^k\alpha \leqq 1$, $\|V_i u\|_{2^k\alpha-1}^2 \leqq \|V_i u\|_0^2\ \forall k \leqq N$, and we use the energy inequality to conclude.

2.12. *A second Sobolev a priori estimate.*

Proposition. *Assume $\tilde{H}$. If U is a bounded open set in $\mathbb{R}^{+*}\times\mathbb{R}^n$, there exist $\varepsilon_0 > 0$ and $C_\alpha > 0$ such that for $\varepsilon < \varepsilon_0$ and u in $\mathscr{S}_2(U)$:*

$$\|u\|_{\alpha+\varepsilon}^2 \leqq C_\alpha\{E(|\langle Lu, \Lambda^{2\alpha}u\rangle_D|) + \|u\|_\alpha^2 + \|Lu\|_{\alpha,\mathscr{M}}^2\}.$$

Proof. We apply Proposition 2.11 to $\Lambda'^\alpha u$ (in the following, we identify Λ^α and Λ'^α as their difference is of order $-\infty$ and gives a trivial contribution to the estimations).

$$\|\Lambda^\alpha u\|_\varepsilon^2 \leqq C\{E(|\langle L\Lambda^\alpha u, \Lambda^\alpha u\rangle_D|) + \|\Lambda^\alpha u\|_0^2 + \|L\Lambda^\alpha u\|_{0,\mathcal{M}}^2\}.$$

Let us first examine $\|L\Lambda^\alpha u\|_{0,\mathcal{M}}^2$.

$$\|L\Lambda^\alpha u\|_{0,\mathcal{M}}^2 = \sum_{i=1}^{p} \{\|L_i\Lambda^\alpha u\|_0^2 + |\langle L_i\Lambda^\alpha u, \mathcal{L}_i\Lambda^\alpha u\rangle_0|\}.$$

We calculate $L_i\Lambda^\alpha u$:

$$L_i\Lambda^\alpha u = \Lambda^\alpha u_i + \mathcal{L}_i\Lambda^\alpha u = \Lambda^\alpha L_i u + [\mathcal{L}_i, \Lambda^\alpha]u.$$

As $[\mathcal{L}_i, \Lambda^\alpha]$ is of order α, we have:

$$\|L\Lambda^\alpha u\|_{0,\mathcal{M}}^2 \leqq C\{\|Lu\|_{\alpha,\mathcal{M}}^2 + \|u\|_\alpha^2 + E(\int_{\mathbb{R}^+} ([\mathcal{L}_i, \Lambda^\alpha]u, \Lambda^\alpha \mathcal{L}_i u)\,dt)\}.$$

For the last term, in order to lower the degree of the operator acting on u, we use the adjoint of the operator in the same way as to prove that: $(V_i u, u) = \frac{1}{2}(q_i u, u)$. We define: $T = \mathcal{L}_i^* \Lambda^\alpha [\mathcal{L}_i, \Lambda^\alpha]$

$$\begin{aligned} T^* &= -[\mathcal{L}_i^*, \Lambda^\alpha]\Lambda^\alpha \mathcal{L}_i = -[\mathcal{L}_i, \Lambda^\alpha]\Lambda^\alpha \mathcal{L}_i^* \\ &\quad + \text{term of order } 2\alpha = -[\mathcal{L}_i, \Lambda^\alpha]\mathcal{L}_i^*\Lambda^\alpha - [\mathcal{L}_i, \Lambda^\alpha][\Lambda^\alpha, \mathcal{L}_i^*] \\ &\quad + \text{term of order } 2\alpha = -T - [[\mathcal{L}_i, \Lambda^\alpha], \mathcal{L}_i^*\Lambda^\alpha] + \text{term of order } 2\alpha. \end{aligned}$$

So $T + T^*$ is of order 2α and this allows us to estimate

$$E(\int_{\mathbb{R}^+} ([\mathcal{L}_i, \Lambda^\alpha]u, \Lambda^\alpha \mathcal{L}_i u)\,dt) \quad \text{by} \quad C\|u\|_\alpha^2.$$

In the same way:

$$\langle L\Lambda^\alpha u, \Lambda^\alpha u\rangle_D = \langle Lu, \Lambda^{2\alpha}u\rangle_D + \int_{\mathbb{R}^n} (\Lambda^\alpha u, [\mathcal{L}_0, \Lambda^\alpha]u)\,dt.$$

For the last term, we lower the degree as previously.

d) Regularization

In this paragraph, we shall regularize and localize the distribution u satisfying the hypothesis of Theorem 1.1 and apply the a priori inequalities to the regularized maps.

2.13. **Definition.** Let ξ, ξ_1 be in $C_c^\infty(\mathbb{R}^{+*} \times \mathbb{R}^n)$, such that $\xi_1 = 1$ on $\operatorname{supp}\xi$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a even function with integral one.

We define: $\varphi_\eta(x) = \eta^{-n}\varphi(\eta^{-1}x)$ and:

$$u_\eta(t, x, \omega) = \int_{\mathbb{R}^n} \varphi_\eta(x - y)\,\xi(t, y)\,\mu_t(dy) = S_\eta \xi u(t, x, \omega).$$

2.14. **Proposition.** *For every $\eta \in \mathbb{R}^{+*}$, u_η is in $\tilde{\mathcal{S}}_2$.*

Let us first show that u_η is in $\tilde{\mathcal{S}}_0 \cap \tilde{\mathcal{S}}_1$. i), ii), iii) of $\tilde{\mathcal{S}}_1$ are clear from the properties of $\tilde{\mathcal{G}}$, ξ and φ.

To study property iv), we have to estimate:

$$\int_{\mathbb{R}^n \times \mathbb{R}^+} E\Big(\int_{\mathbb{R}^n} D^\alpha \varphi_\eta(x-y)\, \xi(t,y)\, \mu_t(dy)\Big)^q\, dx\, dt$$
$$= \int_{\mathbb{R}^n \times \mathbb{R}^+} E(\langle \mu_t D^\alpha \varphi_\eta(x-.)\, \xi(t,.)\rangle)^q\, dx\, dt \leqq C \sup_{t\in K} E(\|\mu_t\|^q_{\mathcal{M}(\mathbb{R}^n)}),$$

where K is the projection on $\mathbb{R}^+$ of $\operatorname{supp}\xi$, and this is finite by property $\tilde{\mathscr{G}}$ ii).

The semi-martingale property ($\tilde{\mathscr{S}}_0$, iii)) comes from the equation satisfied by u. Using the fact that $\mathscr{L}_i$, $0 \leqq i \leqq p$, is in $\tilde{\mathscr{A}}$, we can prove easily that the $(u_\eta)_i$, $0 \leqq i \leqq p$ are in $\tilde{\mathscr{S}}_1$.

2.15. **Proposition.** *Let α be a real number. Then, there exists a constant $C(\alpha)$ independent of η such that, for all ε smaller than ε_0:*

$$\|u_\eta\|_{\alpha+\varepsilon} \leqq C(\alpha)\left\{\|\xi_1 u\|_\alpha + \|\xi h(.,\beta.)\|_\alpha + \sum_{i=1}^p \|\xi h_i(.,\beta.)\|_{\alpha+1}\right\}.$$

Proof. By Proposition 2.13, we can apply Proposition 2.12 to u_η and we get:

$$\|u_\eta\|^2_{\alpha+\varepsilon} \leqq C_\alpha\{E(|\langle Lu_\eta, \Lambda^{2\alpha} u_\eta\rangle_D|) + \|u_\eta\|^2_\alpha + \|Lu_\eta\|^2_{\alpha,\mathcal{M}}\}.$$

In order to estimate the second member of this inequality by an expression independent of η, we need to adapt some known properties of the Friedrich mollifiers to the present case.

Let us denote by $\Psi_m(\Omega, \mathbb{R}^n)$ the space of random pseudo-differential operators on $\mathbb{R}^n$. We shall work on the subalgebra of $\Psi_m(\Omega, \mathbb{R}^n)$ generated by $\tilde{\mathscr{A}}$ and the S_η's, $0 \leqq \eta \leqq 1$.

2.16. **Lemma.** i) *Let u be in $\tilde{\mathscr{G}}$. Then:*

$$\|S_\eta \xi u\|_\alpha \leqq \|\xi u\|_\alpha$$

where $\|\xi u\|_\alpha$ is defined as $\|v\|_\alpha$ when v is in $\tilde{\mathscr{S}}_2$.

ii) *Friedrich lemma: If A* (resp. *B*) *is an element of $\tilde{\mathscr{A}}$ of order m* (resp. *m'*), *then the brackets $[S_\eta, A]$* (resp. *$[[S_\eta, A], B]$*) *are in a bounded set of the space $\Psi_{m-1}(\Omega, \mathbb{R}^n)$* (resp. *$\Psi_{m+m'-2}(\Omega, \mathbb{R}^n)$*) *when η varies in $[0,1]$ and A* (resp. *B*) *in a bounded set of $\Psi_m(\Omega, \mathbb{R}^n)$* (resp. *$\Psi_{m'}(\Omega, \mathbb{R}^n)$*).

Proof. Inequality i) is obtained by integrating on $\Omega \times \mathbb{R}^+$ the classical one. To prove ii), we integrate the classical proof on $\Omega \times \mathbb{R}^+$ and use the particular form of the elements of $\tilde{\mathscr{A}}$ as in Lemma 2.8.

From the first part of the lemma, we get first that:

$$\|u_\eta\|_\alpha \leqq \|\xi u\|_\alpha \leqq C\|\xi_1 u\|_\alpha.$$

Now, we examine $\|Lu_\eta\|^2_{\alpha,\mathcal{M}}$.

$$\|Lu_\eta\|^2_{\alpha,\mathcal{M}} = \sum_{i=1}^p \{\|L_i u_\eta\|^2_\alpha + |\langle L_i u_\eta, \mathscr{L}_i u_\eta\rangle_\alpha|\}.$$

In order to compute $L_i u_\eta$, we put down the equation satisfied by u_η:

$$u_\eta(t,x,\omega)=\int_0^t\left\{-S_\eta\xi\mathscr{L}_{0,s}u+S_\eta\frac{\partial\xi}{\partial s}u\right\}ds-\int_0^t S_\eta\xi\mathscr{L}_{i,s}u\,\delta\beta_s^i$$
$$+\int_{\mathbb{R}^n}h(t,y,\omega)\,\varphi_\eta(x-y)\,\xi(t,y)\,dy.$$

So: $(u_\eta)_i=-S_\eta\xi\mathscr{L}_i u+S_\eta\xi h_i$ and $L_i u_\eta=[\mathscr{L}_i,S_\eta\xi]\,u+S_\eta\xi h_i$. Using the formula:

$$[\mathscr{L}_i,S_\eta\xi]=[\mathscr{L}_i,S_\eta]\,\xi+S_\eta[\mathscr{L}_i,\xi],$$

we obtain:

$$\|L_i u_\eta\|_\alpha^2\leqq C\{\|\xi_1 u\|_\alpha^2+\|\xi h\|_\alpha^2\}$$

where C is independant of η by the second part of Lemma 2.16. On the other hand:

$$\langle L_i u_\eta,\mathscr{L}_i u_\eta\rangle_\alpha=\langle\Lambda^\alpha\{[\mathscr{L}_i,S_\eta\xi]\,u+S_\eta\xi h_i\},\Lambda^\alpha\mathscr{L}_i S_\eta\xi u\rangle_0$$

$$\langle\Lambda^\alpha S_\eta\xi h_i,\Lambda^\alpha\mathscr{L}_i S_\eta\xi u\rangle_0=\langle\Lambda^\alpha S_\eta\xi h_i,[\Lambda^\alpha,\mathscr{L}_i]\,S_\eta\xi u\rangle_0+\langle\mathscr{L}_i^*\Lambda^\alpha S_\eta\xi h_i,\Lambda^\alpha S_\eta\xi u\rangle_0.$$

So: $|\langle\Lambda^\alpha S_\eta\xi h_i,\Lambda^\alpha\mathscr{L}_i S_\eta\xi u\rangle_0|\leqq C\{\|\xi u\|_\alpha^2+\|\xi h_i\|_{\alpha+1}^2\}$.

It remains to estimate: $\langle\Lambda^\alpha[\mathscr{L}_i,S_\eta\xi]\,u,\Lambda^\alpha\mathscr{L}_i S_\eta\xi u\rangle_0$ which can be written:

$$\langle T_1\Lambda^\alpha\xi_1 u,\Lambda^\alpha\xi_1 u\rangle_0+\langle T_2\Lambda^\alpha\xi u,\Lambda^\alpha\xi u\rangle_0+O(\|\xi_1 u\|_\alpha^2)$$

where:

$$T_1=(\mathscr{L}_i S_\eta\xi)^*S_\eta[\mathscr{L}_i,\xi]$$
$$T_2=(\mathscr{L}_i S_\eta)^*[\mathscr{L}_i,S_\eta].$$

This can be easily seen by commuting Λ^α with $[\mathscr{L}_i,S_\eta]$, $S_\eta[\mathscr{L}_i,\xi]$ and $\mathscr{L}_i S_\eta$. We compute now $T_j+T_j^*$, $j=1,2$.

$$T_1+T_1^*=[\xi,\mathscr{L}_i^*+\mathscr{L}_i]\,S_\eta\mathscr{L}_i S_\eta\xi+[\mathscr{L}_i,\xi]\,S_\eta(\mathscr{L}_i+\mathscr{L}_i^*)S_\eta\xi-[\mathscr{L}_i,\xi]\,S_\eta[\mathscr{L}_i^*,S_\eta\xi]$$
$$-[\mathscr{L}_i,\xi]\,S_\eta[S_\eta,\xi]\,\mathscr{L}_i^*-[[\mathscr{L}_i,\xi]\,S_\eta,(\mathscr{L}_i S_\eta)^*]-(\mathscr{L}_i S_\eta)^*[[\mathscr{L}_i,\xi],S_\eta].$$

Using the second part of Lemma 2.16, we see that $T_1+T_1^*$ is in a bounded set of $\Psi_0(\Omega,\mathbb{R}^n)$ when η varies in $[0,1]$ and so:

$$|\langle T_1\Lambda^\alpha\xi_1 u,\Lambda^\alpha\xi_1 u\rangle_0|=|\tfrac{1}{2}\langle(T_1+T_1^*)\Lambda^\alpha\xi_1 u,\Lambda^\alpha\xi_1 u)\rangle_0|\leqq C\,\|\xi_1 u\|_\alpha.$$

Using the same argument, we get the same estimate for $\langle T_2\Lambda^\alpha\xi u,\Lambda^\alpha\xi u\rangle_0$. The last term to examine is: $E(|\langle Lu_\eta,\Lambda^{2\alpha}u_\eta\rangle_D|)$ which is equal to:

$$\left\langle\Lambda^{2\alpha}u_\eta,[\mathscr{L}_0,S_\eta\xi]\,u+S_\eta\frac{\partial\xi}{\partial t}u+S_\eta\left(h_0\xi+h\frac{\partial\xi}{\partial t}\right)\right\rangle_0$$

and it leads to the same estimates as before.

e) End of the Proof of Theorem 2.1

Let U and V be two bounded open sets in $\mathbb{R}^{+*}\times\mathbb{R}^n$ such that $\bar{U}\subset V$. U_0, $U_1\ldots U_k\ldots$ is a decreasing sequence of open sets satisfying:

$$U\subset\bar{U}\subset\ldots\subset U_k\subset\bar{U}_k\subset U_{k-1}\subset\ldots\subset U_0=V;$$

We choose a fitted sequence of smooth functions ξ_k on $\mathbb{R}^{+*} \times \mathbb{R}^n$ such that:

i) $\xi_k = 1$ on U_k.

ii) $\operatorname{supp} \xi_k \subset U_{k-1}$.

Let us show that, for $\xi \in C_c^\infty(\mathbb{R}^{+*} \times \mathbb{R}^n)$ with support in U: $\|\xi u\|_\alpha < +\infty$, $\forall \alpha \in \mathbb{R}$. First, u being a Radon measure on $\mathbb{R}^n$ satisfying property $\tilde{\mathscr{G}}$ ii), $\|\xi_k u\|_{-n} < +\infty$, $\forall k \in \mathbb{N}$.

Then, applying Proposition 2.14 with $\xi = \xi_2$, we get that $\|S_\eta \xi_2 u\|_{-n+\varepsilon} \leqq C$, C independant of η. We deduce from this and a stochastic version of Lemma 1.10 that: $\|\xi_2 u\|_{-n+\varepsilon} < +\infty$. So, inductively, we have:

$$\|\xi_k u\|_{-n+(k-1)\varepsilon} < +\infty \quad \text{and so:} \quad \|\xi u\|_\alpha < +\infty, \quad \forall \alpha \in \mathbb{R}.$$

Now, to deduce from that the smoothness of ξu w.r.t. x, we cannot conclude directly as we saw in §1 and so we use the same method as in the proof of Theorem 1.1.

If φ is in $C_b^\infty(\mathbb{R}^n)$, we have:

$$\begin{aligned} \langle \xi \mu_t, D^\alpha \varphi \rangle = & -\int_0^t \left\{ \langle \xi \mu_s, \mathscr{L}_{0,s}^* D^\alpha \varphi \rangle + \left\langle \frac{\partial \xi}{\partial s} \mu_s, D^\alpha \varphi \right\rangle \right\} ds \\ & -\int_0^t \langle \xi \mu_s, \mathscr{L}_{i,s}^* D^\alpha \varphi \rangle \, \delta \beta_s^i + \int_0^t \int_{\mathbb{R}^n} \xi h D^\alpha \varphi \, dx \, dt. \end{aligned}$$

Integrating by parts and using the fact that the coefficients of the $\mathscr{L}_i$'s are in C_b^∞, we get:

$$E\Big(\sup_{\substack{t \leqq T \\ \|\varphi\|_\infty = 1}} |\langle \xi \mu_t, D^\alpha \varphi \rangle| \Big) \leqq C \|\xi u\|_{\alpha+2} + C' < +\infty$$

and, so, a.s. all the derivatives of $\xi \mu_t$, $t \leqq T$, are bounded measures which proves, by Lemma 1.13, that $\xi \mu_t$, for $t \leqq T$, has a smooth density.

3. Applications

a) Background Material

We denote by $\mathscr{W}$ (resp. $\tilde{\mathscr{W}}$) the space $\mathscr{C}(\mathbb{R}^+, \mathbb{R}^m)$ (resp. $\mathscr{C}(\mathbb{R}^+, \mathbb{R}^p)$). A point of $\mathscr{W}$ (resp. $\tilde{\mathscr{W}}$) is denoted by w (resp. $\tilde{w}$). Let Q (resp. $\tilde{Q}$) be the Wiener measure on $\mathscr{W}$ (resp. $\tilde{\mathscr{W}}$) with $Q(w_0 = 0) = 1$ (resp. $\tilde{Q}(\tilde{w}_0 = 0) = 1$). If X is a stochastic process on $\mathscr{W} \times \tilde{\mathscr{W}}$, $\mathscr{B}_t^X$ is the σ-field $\mathscr{B}(X_s, s \leqq t)$. $X_0, X_1, \ldots, X_m, \tilde{X}_1, \ldots, \tilde{X}_p$, are $m+p+1$ vector fields defined on $\mathbb{R}^n \times \mathbb{R}^p$, such that:

$$X_i(x,z) = \sum_{j=1}^n X_i^j(x,z) \frac{\partial}{\partial x_j}, \qquad 0 \leqq i \leqq m$$

$$\tilde{X}_i(x,z) = \sum_{j=1}^n \tilde{X}_i^j(x,z) \frac{\partial}{\partial x_j}, \qquad 1 \leqq i \leqq p$$

$l_1 \ldots l_p$ are p functions defined on $\mathbb{R}^n \times \mathbb{R}^p$.

We assume that the components of the vector fields and the l_i's are in C_b^∞ (cf. 2.a).

Let (x_t, z_t) be the solution of the stochastic differential equation on $(\mathscr{W} \times \tilde{\mathscr{W}}, \mathscr{B}_t^{w,\tilde{w}}, Q \times \tilde{Q})$:

$$\begin{aligned} dx_t &= X_0(x_t, z_t)\,dt + X_i(x_t, z_t)\,dw_t^i + \tilde{X}_i(x_t, z_t)(d\tilde{w}_t^i + l^i(x_t, z_t)\,dt) \\ dz_t &= d\tilde{w}_t + l(x_t, z_t)\,dt \end{aligned} \tag{10}$$

x_0, z_0 fixed.

We let the l_i's appear in the first equation in order to simplify the Girsanov transformation.

If we set $y_t = (x_t, z_t)$, Eq. (10) can be written:

$$dy_t = Y_0(y_t)\,dt + Y_i(y_t)\,dw^i + \tilde{Y}_i(y_t)(d\tilde{w}^i + l^i(y_t)\,dt), \tag{11}$$

$$y_0 \text{ fixed}$$

where: $Y_i = X_i$ and $\tilde{Y}_i = \tilde{X}_i + \dfrac{\partial}{\partial z_i}$, or, in matricial notations:

$$Y_i = \begin{pmatrix} X_i \\ 0 \end{pmatrix}, \quad 0 \leqq i \leqq m; \qquad \tilde{Y}_i = \begin{pmatrix} \tilde{X}_i \\ \partial/\partial z_i \end{pmatrix} \quad 1 \leqq i \leqq p.$$

In the following paragraphs, we show that, under appropriate hypoellipticity assumptions, the conditional laws $\pi_{t,T}$ defined by $\pi_{t,T} f = E(f(x_t)|\mathscr{B}_T^z)$, where f is a bounded measurable function, have a C^∞ density. We treat first the filtering case $(t = T)$ and deduce from it the smoothing case $(t < T)$.

We introduce now the Lie algebras which are involved in our two hypoellipticity hypothesis.

We call $\mathscr{L}(X_1, \ldots, X_m)$ the Lie algebra with coefficients in C_b^∞ generated by $X_1, \ldots, X_m$ and, for $N \in \mathbb{N}$, $\mathscr{L}_N(X_1, \ldots, X_m)$ is the subspace of $\mathscr{L}(X_1, \ldots, X_m)$ generated by the brackets of $X_1, \ldots, X_m$ whose length is smaller than N.

Let $\mathscr{R}$ be the Lie algebra, with coefficients in C_b^∞ generated by $X_1, \ldots, X_m$ and the brackets of $X_1, \ldots, X_m$, $\tilde{X}_1 + \dfrac{\partial}{\partial z_1}, \ldots, \tilde{X}_p + \dfrac{\partial}{\partial z_p}$ where, at least, one X_i appears; we denote by $\mathscr{R}_N$ the subspace of $\mathscr{R}$ generated by the brackets whose length is smaller than N.

We obtain our regularity result by applying Theorem 1.1 under the restricted Hörmander condition:

H_1: for every compact set K of $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that:

$$\mathscr{L}_N(X_1, \ldots, X_m)(x, z) = \mathbb{R}^n \quad \text{at each point} \quad (x, z) \in K \times \mathbb{R}^p.$$

And then we obtain the same regularity result by applying Theorem 2.1 under the extended Hörmander condition:

H_2: for every compact set K of $\mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that:

$$\mathscr{R}_N(x, z) = \mathbb{R}^n \quad \text{at each point} \quad (x, z) \in K \times \mathbb{R}^p.$$

In two of the following regularity results, we need an other assumption:

H_3: the projection on $\mathbb{R}^n$ of the supports of $\tilde{X}_1 \ldots \tilde{X}_m$, $l_1 \ldots l_p$ is bounded.

b) Filtering (cf. [4])

Let us examine first two particular cases (without interest in the filtering theory).

i) $l_i=0$, $\tilde{X}_i=0$, $1 \leqq i \leqq p$.

Thanks to a theorem of C. Doléans-Dade [7], one can fix z a.s. in Eq. (11) and solve it on $\mathscr{W}$. The problem amounts then to proving the regularity of the law of a diffusion whose coefficients do not depend regularly on time. We get the answer by applying Theorem 1.1 under assumption H_1.

ii) $l_i=0$, $\tilde{X}_i \neq 0$, $1 \leqq i \leqq p$.

One cannot fix any more $\tilde{w}$ in Eq. (11). Nevertheless we shall see that all is going as if the conditional law were the law of the diffusion associated to the "operator" $\frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0 + \tilde{X}_i \frac{d\tilde{w}^i}{dt}$, the drift splitting up into $(p+1)$ independant components $X_0, \tilde{X}_1 \dots \tilde{X}_p$.

In the general case ($l \neq 0$), we define a new probability measure Q_0 on $\mathscr{W} \times \tilde{\mathscr{W}}$ such that:

$$\left.\frac{dQ_0}{dQ \otimes d\tilde{Q}}\right|_{\mathscr{B}_t^{w,\tilde{w}}} = L_t^{-1},$$

where

$$L_t = \exp\left(\int_0^t l_i(y_s)\,\delta\tilde{w}_s^i + \frac{1}{2}\int_0^t \sum_{i=1}^{p} l_i^2(y_s)\,ds\right).$$

As usually, we define the unnormalized filter:

$\rho_t f = E_0(f(x_t)L_t | \mathscr{B}_t^z)$ which is related to π_t by the Kushner formula [18]:

$$\pi_t f = \frac{\rho_t f}{\rho_t 1}$$

and ρ_t verifies the Zakaï equation: if f is in $\mathscr{C}_b^2(\mathbb{R}^n)$, we have:

$$\rho_t f = \rho_0 f + \int_0^t \rho_s C f\,ds + \int_0^t \rho_s B_i f\,\delta z_s^i \tag{12}$$

where

$$B_i = \tilde{X}_i + l_i, \quad C = \frac{1}{2}\left(\sum_{i=1}^{m} X_i^2 + \sum_{i=1}^{p} \tilde{X}_i^2\right) + X_0 + l_i \tilde{X}^i + \sum_{i=1}^{p} \frac{\partial \tilde{X}_i}{\partial z_i}$$

or equivalently:

$$\rho_t f = \rho_0 f + \int_0^t \rho_s A f\,ds + \int_0^t \rho_s B_i f\,dz_s^i \tag{13}$$

where

$$A = \frac{1}{2}\sum_{i=1}^{m} X_i^2 + X_0 - \frac{1}{2}\tilde{X}_i l^i - \frac{1}{2}\sum_{i=1}^{p} l_i^2 - \frac{1}{2}\sum_{i=1}^{p} \frac{\partial l^i}{\partial z_i}.$$

By Girsanov theorem, z is a brownian motion under Q_0.

We can now state the regularity result for the filter.

3.1. **Theorem.** *Under condition* H_2, a.s., *for every* $t>0$, π_t *has a* C^∞ *density.*

Proof. From the Kushner formula, we see that it is equivalent to prove that ρ_t has a C^∞ density. To get this result, we apply Theorem 2.1 to Eq. (12), setting:

$$\begin{aligned}
&V_0 = X_0 + l_i \tilde{X}^i + \sum_{i=1}^{p} \frac{\partial \tilde{X}_i}{\partial z_i},\\
&V_i = X_i, \quad 1 \leqq i \leqq m; \qquad \tilde{V}_i = \tilde{X}_i, \quad 1 \leqq i \leqq p,\\
&f = 0; \quad g_i = l_i, \quad 1 \leqq i \leqq p; \qquad h_i = 0, \quad 0 \leqq i \leqq p.
\end{aligned}$$

$(\Omega, P) = (\mathscr{W} \times \tilde{\mathscr{W}}, Q_0)$; $\beta = z$.

We only have to prove that ρ_t is in $\tilde{\mathscr{G}}$ and this is straightforward.

An other way to get this result without using stochastic partial differential equations is to work on an ordinary partial differential equation derived from the Zakaï equation (cf. [4]). From now on, we suppose H_3 verified.

Let us introduce the flow $(\psi_t, v_t)(z, .)$ associated to the system (cf. [1, 2, 13, 14]):

$$\begin{aligned}
&d y_t = Y_0(y_t)\, dt + \tilde{Y}_i(y_t)\, dz_t^i\\
&d h_t = l_i(y_t)\, \delta z_t^i - \tfrac{1}{2} \left(\sum_{i=1}^{p} l_i^2(y_t) \right) dt\\
&y_0 \text{ fixed}\\
&h_0 = 0.
\end{aligned} \tag{14}$$

Applying the generalized Ito's formula ([2, 14]), we get:

$$\operatorname{Log} L_t = v_t(z, \bar{y}_t) - \int_0^t \frac{\partial v_s}{\partial y}(z, \bar{y}_s)\, d\bar{y}_s$$

where $\bar{y}_t = \psi_t^{-1}(z, y_t)$, y_t being the solution of (11):

This allows to fix z in L_t and to associate to ρ_t an operator v_t defined in the following way:

Let g be any bounded measurable function on $\mathbb{R}^n$; we set:

$$v_t g = E_0^{\mathscr{W}} \left(g(\bar{x}_t) \exp \left(-\int_0^t \frac{\partial v_s}{\partial y}(z, \bar{x}_s, z_0)\, d\bar{y}_s \right) \right)$$

where $E_0^{\mathscr{W}}$ is the expectation w.r.t. the trace of Q_0 on $\mathscr{W}$ and $\bar{x}$ is the projection of $\bar{y}$ on $\mathbb{R}^n$.

v_t and ρ_t are then linked by the relation:

$$\rho_t f = v_t(\exp v_t(z, ., z_0)\, f \circ p_1 \psi_t)$$

where p_1 is the projection on $\mathbb{R}^n$, and this shows the equivalence between the smoothness of ρ_t and that of v_t. The main interest of v_t is that for a.e. z, it satisfies the following P.D.E.:

$$\frac{\partial v_t}{\partial t} - \frac{1}{2} \sum_{i=1}^{m} (\psi_t^{*-1} Y_i)^{02} v_t - \left(\sum_{i=1}^{m} ((\psi_t^{*-1} Y_i) v) \psi_t^{*-1} Y_i \right)^0 v_t$$
$$+ \sum_{i=1}^{m} (\tfrac{1}{2}(\psi_t^{*-1} Y_i v)^2 - (\psi_t^{*-1} Y_i)^2 v) v_t = 0 \tag{15}$$

where ^{0} denotes the adjoint.

The regularity of v_t under H_1 is a direct consequence of Theorem 1.1; indeed, assumption H is easily deduced from assumption H_1 and the properties of the flow ψ.

c) Smoothing

If H_3 is verified the result of regularity for the smoothing problem can be obtained directly from the filtering case by the following method: first, we have, for every bounded measurable function f:

$$E(f(x_t)|\mathscr{B}_T^z) = \frac{E_0(f(x_t) L_T | \mathscr{B}_T^z)}{E_0(L_T|\mathscr{B}_T^z)} = \frac{E_0(f(x_t) L_t E_0(L_{T-t} | \mathscr{B}_t^w \vee \mathscr{B}_T^z) | \mathscr{B}_T^z)}{E_0(L_T|\mathscr{B}_T^z)}.$$

Now, by a result of J.M. Bismut-D. Michel [4] (cf. Theorem 3.10 and the proof of Theorem 3.12), there exists a function $u\colon \tilde{\mathscr{W}} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that:

$$E_0(L_{T-t} | \mathscr{B}_t^w \vee \mathscr{B}_T^z) = u(z, t, T-t, \bar{x}_t).$$

So, the conditional law $\pi_{t,T}$ admits the density $x \to p_t^z(x) u(z, t, T - t\, p_1 \psi_t^{-1}(z, x, z_0))$ where p_t^z is the density of the conditional law π_t. We, then, deduce the regularity of $\pi_{t,T}$, under assumption H_2, from Theorem 3.1 above and Theorem 3.10 of [4].

d) Remark

We point out that assumptions H_1 and H_2 are global conditions and so are much stronger than in [4] where the condition on the Lie algebra was only supposed to be verified at the starting point of the process y.

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