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Ito's Formula for Continuous (N, d)-Processes

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Introduction

This paper was written for the purpose of studying local times of continuous vector valued multi-parameter processes by means of an appropriate stochastic calculus. As is well known from the theory of one-parameter semimartingales (cf. for example Azema, Yor [1]), Tanaka's formula links stochastic calculus and local times. It can be derived from Ito's formula by extending the latter to a larger class of functions. In this sense this paper is basic to the study of multi-parameter local times (see Imkeller [8,9]): several notions of stochastic measures and integrals are considered by means of which versions of Ito's formula are stated and proved.

§1 is devoted to an extension of the formula which was given by Allain [2] to $\mathbb{R}^d$-valued continuous processes within Allain's abstract framework. At the basis of Allain's calculus is the following observation: a one-parameter process is a semimartingale if and only if it it admits - roughly speaking - a reasonable stochastic calculus (i.e. it is an "L_0-integrator", see Bichteler [3], Metivier, Pellaumail [11], p. 155). Allain's (real valued) "p-semimartingales of order k" by definition give rise to such a calculus in L^p-sense. In this paper $\mathbb{R}^d$-valued p-integrable processes X are studied, $p \geqq 1$. If for $k \in \mathbb{N}$, some finite measures ρ_l, $1 \leqq |l| \leqq k$, $l \in \mathbb{N}_0^d$, $q \geqq 1, X$ is submitted to a so-called "$(\rho_l, 1 \leqq |l| \leqq k; q)^*$-domination" condition, which is more restrictive than Allain's conditions, but looks natural for example for processes with independent increments, the existence of the "l^{th} variations" $\mu_{X^{(l)}}$, $1 \leqq |l| \leqq k$ (p-stochastic measures which are direct generalizations of the "quadratic variation" of one-parameter semimartingales) can be established. A simple additional condition concerning the "fluctuation" of X assures that for k times continuously differentiable $f: \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives the formula

$$f(X_t) - f(0) = \sum_{1 \leqq |l| \leqq k} \frac{1}{l!} \int_{\Omega \times]0,t]} D^{(l)} f(X) \, d\mu_{X^{(l)}}, \qquad t \in [0,1]^N,$$

is valid (Theorem 1). $(\rho_l, 1 \leqq |l| \leqq k; q)^*$-domination is approximately a "domination property" of the variations

$$\|\int Y d\mu_{X^{(l)}}\|_p \leqq (\int |Y|^q d\rho_l)^{1/q}, \qquad Y \text{ previsible}, \quad 1 \leqq |l| \leqq k$$

(see Metivier, Pellaumail [11], p. 20). Contrary to the classical formula, Theorem 1 yields only an equality of random variables (t is fixed). For the case of the (N,d)-Wiener process W however, in §2 and §3 an improvement is made by using multi-parameter martingale theory. More precisely, by an appeal to the multi-parameter versions of Doob's maximal inequalities (Cairoli [5], Wong, Zakai [13]), the existence of continuous versions of the integral processes of $\mu_{W^{(l)}}$, $1 \leqq |l| \leqq 2N$, can be proved. This is accomplished in §2 by decomposing these processes into "martingales" (Theorem 2) which are deduced from W in the same way as are the integral processes $\int . dW$, $\int . dW dW$, $\int . d\lambda dW$, $\int . dW d\lambda$, $\int . d\lambda d\lambda$, well-known from 2-parameter theory (see for example Cairoli, Walsh [5], Wong, Zakai [13, 14, 15], Guyon, Prum [7], Merzbach [10]; for N-parameter results see Yor [16], Sanz [12]). One can imagine the decomposition of (the 2-stochastic measure) $\mu_{W^{(l)}}(\Omega \times]0,t])$ in the following way: for each partition $\mathscr{T}$ of $\{1, \ldots, N\}$ functions $\phi: \mathscr{T} \to \{0, 1, \ldots, d\}$ are taken to note whether $\mu_{W^{(l)}}$ varies in T-direction "linearly" with W^j ($\phi(T)=j$), $1 \leqq j \leqq d$, or "quadratically" with any W^j, $1 \leqq j \leqq d$ ($\phi(T)=0$), $T \in \mathscr{T}$. In the latter case the variation in T-direction is given by Lebesgue measure. The "order" of variation, i.e. $2|\mathscr{T}^0| + |\mathscr{T}^1|$, where $\mathscr{T}^0 = \{T: \phi(T)=0\}$, $\mathscr{T}^1 = \{T: \phi(T) \neq 0\}$, is equal to $|l|$. Thus one obtains components $\mu^{(\mathscr{T}, \phi)}$ of $\mu_{W^{(l)}}$ (2-stochastic measures as well) with integrals $I^{(\mathscr{T}, \phi)}$, whose integral processes $I_{\cdot}^{(\mathscr{T}, \phi)}$ turn out to be "martingales" possessing continuous versions in consequence of the maximal inequalities.

This result is applied in §3 to yield an improvement of the formula of Theorem 1:

$$f(W_\cdot) - f(0) = \sum_{(\mathscr{T}, \phi)} \frac{1}{2^{|\mathscr{T}^0|}} I_{\cdot}^{(\mathscr{T}, \phi)}([D^{(\mathscr{T}, \phi)} f(W)]^{\mathscr{T}}) \qquad \text{(Theorem 3).}$$

Here $D^{(\mathscr{T}, \phi)}$ is the differential operator which is obtained by applying $|\mathscr{T}^0|$ times the Laplacian and differentiating $|\{T: \phi(T)=j\}|$ times in direction j, $1 \leqq j \leqq d$. Its order is identical with the order of $(\mathscr{T}, \phi)$. For any process Y, $Y^{\mathscr{T}}$ is a "corner function" of Y (cf. Guyon, Prum [7]): $(s_T)_{T \in \mathscr{T}} \to Y(\sup_{T \in \mathscr{T}} s^T)$. See Guyon, Prum [7] for a comparable formula for a class of $(2,d)$-processes, Sanz [12] for $(N,1)$-Wiener process.

In §4 a modification of the formula of Theorem 3 is developed which is particularly useful in applications on local times (Imkeller [9]): it is desirable to have a formula in which – besides the term of highest order – all terms are of the lowest possible differentiation order. Like in applications of the classical formula of Green this is achieved by replacing integrals over intervals by integrals over their surfaces, thus reducing the order of differentiation of the integrands. For this purpose it is necessary to give a precise meaning to the notion of "iterated stochastic integration", to have a "stochastic Fubini's theorem" and to introduce stochastic integrals of W on surfaces of intervals. Ideas of Cairoli, Walsh [5] can be employed (see also Guyon, Prum [7], Dozzi [6]). In the resulting formula (Theorem 4), the above mentioned orders do not exceed N.

§ 0. Notations and Preliminaries

For a fixed number $N\in\mathbb{N}(=\{1,2,\ldots\}$, whereas $\mathbb{N}_0=\{0,1,\ldots\})$, the parameter set is $\mathbb{I}=[0,1]^N$. $\mathbb{I}$ is endowed with the usual partial ordering (i.e. coordinatewise linear ordering on $[0,1]$) "$\leqq$", with respect to which intervals are defined in the usual way. Let $\mathfrak{J}$ be the set of all intervals in $\mathbb{I}$ of the form $]s,t], s,t\in\mathbb{I}$, $]0,t]$ being denoted by R_t. The symbol $\hat{\mathbb{I}}^2$ is used for the for the set of all pairs $(s,t)\in\mathbb{I}^2$, $s\leqq t$. Vectors of "time points" $(s^i)_{i\in I}$ are denoted by $\mathfrak{s}$ whenever there is no ambiguity about the index set I. Projections of vectors (intervals) defined by subsets H of the index set are always provided with a subscript H. For example: if $U\in\Pi_N$ (set of all subsets of $\{1,\ldots,N\}$), $s\in\mathbb{I}$, $J\in\mathfrak{J}$, then s_U resp. J_U is the projection of s resp. J on the U-coordinates; if $\mathfrak{s}=(s^i)_{i\in I}\in\mathbb{I}^I$ and $K\subset I$, then $\mathfrak{s}_K=(s^i)_{i\in K}$. The set of all J_U is denoted by $\mathfrak{J}_U$. For any $m\in\mathbb{R}$, let $\underline{m}$ be the vector in $\mathbb{R}^N$, all of whose coordinates are equal to m. A "decomposition (partition) of $\mathbb{I}$ in $\mathfrak{J}$" is understood to be a decomposition of $]0,1]^N$ by intervals in $\mathfrak{J}$.

Given any function $f:\mathbb{I}\to\mathbb{R}$, any interval $J=]s,t]\in\mathfrak{J}$ and $T\in\Pi_N$, denote the "increment of f over the T-boundary of J" by

$$\Delta_J^T f=\sum_{S\subset T}(-1)^{|T|-|S|}f((s_{\bar{S}},t_S))$$

("$^{-}$" denotes the complement w.r.t. fixed reference sets). For $T=\{1,\ldots,N\}$ the superscript T may be omitted. Note that, in case f vanishes on $\partial\mathbb{R}_+^N\cap\mathbb{I}$, by setting $J^T=[(0_{\bar{T}},s_T),(s_{\bar{T}},t_T)]$,

$$\Delta_J^T f=\Delta_{J^T}f.$$

The "variation of f over $\mathbb{I}$" is defined by

$$v(f)=\sup\{\sum_{1\leqq i\leqq n}|\Delta_{J_i}f|:(J_i)_{1\leqq i\leqq n}\text{ is a decomposition of }\mathbb{I}\text{ in }\mathfrak{J},\ n\in\mathbb{N}\}.$$

Given two measurable spaces $(B,\mathfrak{B})$, $(C,\mathfrak{C})$, the space of all measurable functions from B to C is denoted by $\mathcal{M}(\mathfrak{B},\mathfrak{C})$. $(\Omega,\mathfrak{F},P)$, the basic probability space, is always assumed to be complete, $(\mathfrak{F}_t)_{t\in\mathbb{I}}$, the basic filtration (family of σ-fields, increasing with respect to the partial ordering on $\mathbb{I}$) to be augmented by the 0-sets of $\mathfrak{F}$. There are several relevant notions of "previsability" with respect to $(\mathfrak{F}_t)_{t\in\mathbb{I}}$: for $T\in\Pi_N$, $t\in\mathbb{I}$, let $\mathfrak{F}_t^T=\mathfrak{F}_{(t_T,\underline{1}_{\bar{T}})}$. Then, for $T\subset S\in\Pi_N$,

$$\mathfrak{R}_S^T=\{F\times J_S: J_S=]s_S,t_S]\in\mathfrak{J}_S,\ F\in\mathfrak{F}^T_{(s_S,\underline{1}_{\bar{S}})}\}$$

is called "set of T-previsible rectangles in $\mathbb{I}_S$". Let the algebra resp. σ-algebra generated by $\mathfrak{R}_S^T$ resp. the linear hull of characteristic functions of T-previsible rectangles in $\mathbb{I}_S$ be denoted by $\mathfrak{A}_S^T$ resp. $\mathfrak{P}_S^T$ resp. $\mathfrak{E}_S^T$. $\mathfrak{P}_S^T$ is called "σ-algebra of T-previsible sets in $\mathbb{I}_S$", $\mathfrak{E}_S^T$ "space of T-previsible elementary functions in $\mathbb{I}_S$". In case $\bar{T}=\emptyset$ ($\bar{S}=\emptyset$) the superscript (subscript) T (S) is omitted.

Let $d\in\mathbb{N}$. A stochastic process X with values in $\mathbb{R}^d$ is always understood to belong to $\mathcal{M}(\mathfrak{F}\times\mathfrak{B}(\mathbb{I}),\mathfrak{B}(\mathbb{R}^d))$. X is said to be "previsible", if $X\in\mathcal{M}(\mathfrak{P},\mathfrak{B}(\mathbb{R}^d))$, "adapted", if $X_t\in\mathcal{M}(\mathfrak{F}_t,\mathfrak{B}(\mathbb{R}^d))$ for each $t\in\mathbb{I}$. For $t\in\mathbb{I}$, the vector $(\sup_{s\leqq t} X_s^1,\ldots,\sup_{s\leqq t} X_s^d)$ is denoted by $\bar{X}_t$. All stochastic processes to be considered here are supposed to fulfil the following conditions

(0.1) $X_t=0$ for $t\in\partial\mathbb{R}_+^N\cap\mathbb{I}$,

(0.2) X is adapted and has continuous trajectories.

In particular, this is the case for the most important process to be studied here, W, the "(N,d)-Wiener process". W is an $\mathbb{R}^d$-valued Gaussian random field with mean zero and covariance function $E(W_t^i W_s^j)=\delta_{ij}\lambda^N(R_t\cap R_s)$, $1\leqq i, j\leqq d$, $s,t\in\mathbb{I}$. When dealing with W, the filtration $(\mathfrak{F}_t)_{t\in\mathbb{I}}$ is always assumed to be the natural filtration $(\sigma(W_s\colon s\leqq t))_{t\in\mathbb{I}}$, augmented by zero-sets.

In order to introduce the stochastic integrals belonging to W in §2 below, some special notation has to be established. Set

$$\sigma=\{\mathcal{S}\colon \mathcal{S}\in\Pi_N, S\neq\emptyset \text{ for all } S\in\mathcal{S}\},\qquad \tau=\{\mathcal{T}\in\sigma\colon T\cap S=\emptyset \text{ for all } S,T\in\mathcal{T}\}.$$

For $U\in\Pi_N$ denote by σ_U (τ_U) the subset of σ (τ) composed of those $\mathcal{S}$ such that $\bigcup_{S\in\mathcal{S}} S=U$ ("N" is used instead of "$\{1,\ldots,N\}$"). Further, e_j being the j^{th} unit vector in $\mathbb{R}^d$, let

$$\begin{aligned}
&\Phi_0=\{(\mathcal{S},\phi)\colon \mathcal{S}\in\sigma,\ \phi\colon\mathcal{S}\to\mathbb{N}_0^d\setminus\{0\}\},\\
&\Psi_0=\{(\mathcal{T},\phi)\colon (\mathcal{T},\phi)\in\Phi_0, \mathcal{T}\in\tau,\ \phi(S)=i\,e_j, i=1,2,\ 1\leqq j\leqq d,\ S\in\mathcal{T}\},\\
&\Psi=\{(\mathcal{T},\phi)\colon \mathcal{T}\in\tau, \phi\colon\mathcal{T}\to\{0,1,\ldots,d\}\},\\
&\Lambda=\{(\mathcal{T},\phi)\colon (\mathcal{T},\phi)\in\Psi,\ \phi(T)\neq 0, \text{ whenever } T\in\mathcal{T}, |T|=1\}.
\end{aligned}$$

For all of these sets, the subscript U may be added in the same sense as above. For example, $\Psi_{0,N}$ is obtained by replacing τ by τ_U in the definition of Ψ_U. One can consider the following relation "$<$" on τ (Ψ):

$\mathcal{S}<\mathcal{T}$ $((\mathcal{S},\psi)<(\mathcal{T},\phi))$, iff there is a one-to-one mapping $g\colon\mathcal{T}\to\mathcal{S}$ such that $g(T)\subset T$ for each $T\in\mathcal{T}$ (and $\psi(g(T))=\phi(T)$).

"$<$" turns out to be a partial ordering.

For an individual $(\mathcal{T},\phi)\in\Psi$, put

$$\mathcal{T}^0=\{T\in\mathcal{T}\colon\phi(T)=0\},\qquad \mathcal{T}_j^1=\{T\in\mathcal{T}\colon\phi(T)=j\},\quad 1\leqq j\leqq d,\qquad \mathcal{T}^1=\bigcup_{1\leqq j\leqq d}\mathcal{T}_j^1,$$

$$\underline{\mathcal{T}}^0=\bigcup_{T\in\mathcal{T}^0} T,\qquad \underline{\mathcal{T}}_j^1=\bigcup_{T\in\mathcal{T}_j^1} T,\qquad \underline{\mathcal{T}}^1=\bigcup_{T\in\mathcal{T}^1} T,\qquad \underline{\mathcal{T}}=\bigcup_{T\in\mathcal{T}} T.$$

The integer $m(\mathcal{T},\phi)=|\mathcal{T}^1|+2|\mathcal{T}^0|$, the "order of $(\mathcal{T},\phi)$", is seen to be equal to the order of the differential operator

$$D^{(\mathcal{T},\phi)}=D^{(|\mathcal{T}_1^1|,\ldots,|\mathcal{T}_d^1|)}\,\mathbb{D}^{|\mathcal{T}^0|},$$

which is defined on the space $C^{m(\mathcal{T},\phi)}(\mathbb{R}^d)$, $\mathbb{D}$ denoting the Laplace-operator. Note that $D^{(\mathcal{S},\psi)}=D^{(\mathcal{T},\phi)}$, whenever $(\mathcal{S},\psi)<(\mathcal{T},\phi)$. For $k\in\mathbb{N}$, $C_b^k(\mathbb{R}^d)\,(C_c^k(\mathbb{R}^d))$ is the subspace of $C^k(\mathbb{R}^d)$ consisting of bounded functions (with compact support). Finally, if ϕ is a mapping from a set A to $\mathbb{N}_0^d$, then

$$\phi!=\prod_{a\in A}\prod_{1\leqq j\leqq d}\phi_j(a)!,\qquad |\phi|=\Big(\sum_{a\in A}\phi_1(a),\ldots,\sum_{a\in A}\phi_d(a)\Big)\qquad(\in\mathbb{N}_0^d).$$

For any multiindex $k\in\mathbb{N}_0^d, x\in\mathbb{R}^d$

$$x^k=\prod_{1\leqq j\leqq d}x_j^{k_j}.$$

Occasionally, real valued functions are tacitly assumed to be trivially extended to larger domains.

§ 1. Ito's Formula for a Class of Vector-Valued Processes

Let a real number $p\geqq 1$ be fixed throughout this paragraph and suppose X fulfils the additional condition

$$(1.1)\qquad X_t^i\in L^p(\Omega,\mathfrak{F},P)\quad\text{for all } t\in\mathbb{I},\quad 1\leqq i\leqq d.$$

For a class of real-valued ($d=1$) processes (the so-called "p-semimartingales of order k_0") Allain [2] proved a transformation theorem (Ito's formula), employing the following simple method: take a sequence of partitions of parameter space whose mesh tends to zero; for each partition, apply Taylor's formula to get an approximation of Ito's formula; study the behaviour of the approximating finite sums as the partitions become finer; if the process is "good", these finite sums converge to stochastic integrals, thus giving the desired formula. In this paragraph Allain's result (and method of proof) will be extended to the $\mathbb{R}^d$-valued case.

To this end, let $k\in\mathbb{N}$, $f\in C^k(\mathbb{R}^d)$, $J\in\mathfrak{J}$ and $(J^{j,n}=]s^{j,n},t^{j,n}]$: $1\leqq j\leqq r(n))$, $n\in\mathbb{N}$, be a sequence of partitions of J in $\mathfrak{J}$ whose mesh goes to zero. For each $T\in\Pi_N$, $1\leqq j\leqq r(n)$, apply Taylor's formula to develop $f(X_{(s_{\bar T}^{j,n},t_T^{j,n})})$ at $X_{s^{j,n}}$. Sum over T (alternately) and j to obtain

$$(1.2)\qquad \Delta_J f(X)=\sum_{1\leqq|l|\leqq k}\frac{1}{l!}\sum_{1\leqq j\leqq r(n)}D^{(l)}f(X_{s^{j,n}})\,\Delta_{J^{j,n}}^{(l)}X+\sum_{1\leqq j\leqq r(n)}R(f,J^{j,n},k),$$

thereby setting

$$\Delta_K^{(l)}X=\sum_{\emptyset\neq T\in\Pi_N}(-1)^{N-|T|}(X_{(u_{\bar T},v_T)}-X_u)^l$$

(recall: l is multiindex; $x^l=\prod_{1\leqq i\leqq d}x_i^{l_i}$),

$$R(f,K,k)=\sum_{\emptyset \neq T\in \Pi_N}(-1)^{N-|T|}$$

$$\cdot \sum_{|l|=k}\frac{1}{l!}[D^{(l)}f(X_{(u_{\bar T},u_T+\theta_T(v_T-u_T))})-D^{(l)}f(X_u)]\,(X_{u_{\bar T},v_T)}-X_u)^l$$

for $K=]u,v]\in\mathfrak{J}$, $l\in\mathbb{N}_0^d$, suitable $\theta_T\in[0,1]$.

For an appropriate class of processes the looked-for formula will emerge, as n tends to infinity in (1.2). We will give a criterion for the existence of p-stochastic measures $\mu_{X^{(l)}}$, $1\leqq|l|\leqq k$, such that the integral $\int D^{(l)}f(X)\,d\mu_{X^{(l)}}$ is equal to the limit of the corresponding term on the right of (1.2). As will be seen below, this is essentially a disguised "dominated property" for the stochastic measures $\mu_{X^{(l)}}$, $1\leqq|l|\leqq k$ (cf. Metivier, Pellaumail [11], p. 20).

First note the following special representation of a previsible elementary function. Let R be a finite subset of $\mathbb{I}$. Consider the intervals in $\mathfrak{J}$ which originate from decomposing $\mathbb{I}$ by all hyperplanes which are parallel to the axes and go through at least one point of R. Enumerate them according to the succession of the points of intersection of the hyperplanes with each coordinate axis by N-multiindices and call this "*partition of* $\mathbb{I}$ *generated by* R". If $Y_0=\sum_{1\leqq i\leqq n}a_i 1_{F_i\times J_i}$, $J_i=]s^i,t^i]$, is a previsible elementary function, Q a finite subset of $\mathbb{I}$ and $(K^k\colon 1\leqq k\leqq r)$ the partition of $\mathbb{I}$ generated by $Q\cup\{s^i,t^i\colon 1\leqq i\leqq n\}$, set

$$\alpha_k=\sum_{1\leqq k\leqq r}a_i 1_{F_i}1_{\{J:J\subset K^k\}}(J_i),\qquad 1\leqq k\leqq r.$$

Then $Y_0=\sum_{1\leqq k\leqq r}\alpha_k 1_{K^k}$ is said to be a "$\mathbb{I}$-*representation of* Y_0 *subordinate to* Q" (in case $Q=\emptyset$: "$\mathbb{I}$-*representation of* Y_0"). Note that, in consequence of previsability, α_k is a $\mathfrak{F}_{u^k}$-measurable step function, if $K^k=]u^k,v^k]$, $1\leqq k\leqq r$.

Definition 1. Let $k\in\mathbb{N}$, $q\geqq 1$. For $l\in\mathbb{N}_0^d$, $1\leqq|l|\leqq k$, let ρ_l be a finite measure on $\mathfrak{P}$ (we recall that p is fixed throughout; see (1.1)). X is said to be "$(\rho_l, 1\leqq|l|\leqq k;\, q)^*$-dominated", if

i) for each finite subset R of $\mathbb{I}$ there is a finite subset Q of $\mathbb{I}$ containing R and generating a partition $(J^i\colon 1\leqq i\leqq r)$ of $\mathbb{I}$, such that for all $Y_0\in\mathfrak{E}$ which have an $\mathbb{I}$-representation $Y_0=\sum_{1\leqq i\leqq r}\alpha_i 1_{J^i}$, and all $l\in\mathbb{N}_0^d$, $1\leqq|l|\leqq k$, the inequality

$$\Big\|\sum_{1\leqq i\leqq r}\alpha_i\,\Delta_{J^i}^{(l)}X\Big\|_p\leqq\Big(\int|Y_0|^q\,d\rho_l\Big)^{1/q}$$

is valid,

ii) $\bar X^{l-m}\in L^q(\Omega\times\mathbb{I},\mathfrak{P},\rho_m)$, $l,m\in\mathbb{N}_0^d$, $m\leqq l$, $|l|\leqq k$.

"*-domination" is sufficient for the above mentioned p-stochastic measures to exist (i.e. vector measures on $\mathfrak{P}$ with values in $L^p(\Omega,\mathfrak{F},P)$, see Metivier, Pellaumail [11]). This fact is proved in

Proposition 1. *Let X be a $(\rho_l, 1\leqq|l|\leqq k;\, q)^*$-dominated process. Then, for $l\in\mathbb{N}_0^d$, $1\leqq|l|\leqq k$, there exist p-stochastic measures $\mu_{X^{(l)}}$ whose integrals exist on $L^q(\Omega\times\mathbb{I},\mathfrak{P},\rho_l)$, such that the following is true*

(1.3) *if $F\times J\in\mathfrak{R}$ and if $(J^{j,n}: 1\leqq j\leqq r(n))_{n\in\mathbb{N}}$ is a sequence of partitions of J (with intervals belonging to $\mathfrak{J}$) whose mesh goes to 0, then*

$$1_F \sum_{1\leqq j\leqq r(n)} \Delta^{(l)}_{J^{j,n}} X \to \mu_{X^{(l)}}(F\times J),$$

$$\text{(1.4)}\qquad \|\textstyle\int Y\,d\mu_{X^{(l)}}\|_p \leqq (\int |Y|^q\,d\rho_l)^{1/q} \quad \text{for } Y\in L^q(\Omega\times\mathrm{II},\mathfrak{P},\rho_l).$$

Remark. (1.4) signifies, that the p-stochastic measure $\mu_{X^{(l)}}$ is "(ρ_l,q)-dominated".

Proof. It is sufficient to show: if $Y_0=\sum_{\underline{1}\leqq i\leqq r(n)} \alpha_i^n 1_{J^{i,n}}$, $n\in\mathbb{N}$, is a sequence of II-representations of $Y_0\in\mathfrak{E}$, the mesh of $(J^{i,n}=]s^{i,n},t^{i,n}]: \underline{1}\leqq i\leqq r(n))_{n\in\mathbb{N}}$ going to zero, then $(\sum_{\underline{1}\leqq i\leqq r(n)} \alpha_i^n \Delta^{(l)}_{J^{i,n}} X)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega,\mathfrak{F},P)$ for all $1\leqq|l|\leqq k$. In view of (1.3), the limit of such a sequence is the only possible candidate for $\int Y_0\,d\mu_{X^{(l)}}$. Furthermore, it is uniquely determined. By density of $\mathfrak{E}$ in $L^q(\Omega\times\mathrm{II},\mathfrak{P},\rho_l)$, a familiar extension argument yields the assertion. Let first $J=]u,v]\in\mathfrak{J}$ and a partition $(K^i=]u^i,v^i]: \underline{1}\leqq i\leqq r)$ of II in $\mathfrak{J}$ be given. Apply (1.2) to the functions $f_l:\mathbb{R}^d\to\mathbb{R}$, $x\to x^l$, $1\leqq|l|\leqq k$. Hence

$$\Delta_J X^l = \sum_{0\neq m\leqq l} \binom{l}{m} \sum_{\underline{1}\leqq i\leqq r} X^{l-m}_{u^i} \Delta^{(m)}_{K^i} X.$$

In case $r=\underline{1}$ check, by substituting, that this formula can be "inverted":

$$\Delta^{(l)}_J X = \sum_{0\neq m\leqq l} \underline{(-1)^{l-m}} \binom{l}{m} X^{l-m}_u \Delta_J X^m.$$

Apply the latter to each K^i separately and combine. This gives

$$\text{(1.5)}\qquad \Delta^{(l)}_J X = \sum_{0\neq m\leqq l}\sum_{m\leqq j\leqq l} \underline{(-1)^{l-j}} \binom{l}{j}\binom{j}{m} \sum_{\underline{1}\leqq i\leqq r} X^{l-j}_u X^{j-m}_{u^i} \Delta^{(m)}_{K^i} X.$$

To prove that $(\sum_{\underline{1}\leqq i\leqq r(n)} \alpha_i^n \Delta^{(l)}_{J^{i,n}} X)_{n\in\mathbb{N}}$ is a Cauchy sequence, let now $h,n\in\mathbb{N}$. Put $R=\{s^{i,h},t^{i,h},s^{i',n},t^{i',n}: \underline{1}\leqq i\leqq r(h),\ \underline{1}\leqq i'\leqq r(n)\}$ and let Q be chosen according to i) of Definition 1. By $(K^i=]u^i,v^i]: \underline{1}\leqq i\leqq r)$ denote the partition of II generated by Q. For convenience set

$$Y^{l,j,g} = \sum_{\underline{1}\leqq i\leqq r(g)} \alpha_i^g X^{l-j}_{s^{i,g}} 1_{J^{i,g}}, \quad g\in\mathbb{N},\quad j,l\in\mathbb{N}_0^d,\ 0\neq j\leqq l.$$

Now make use of (1.5), replacing J by $J^{i,h}$ resp. $J^{i,n}$, to obtain

$$\sum_{\underline{1}\leqq i\leqq r(h)} \alpha_i^h \Delta^{(l)}_{J^{i,h}} X - \sum_{\underline{1}\leqq i\leqq r(n)} \alpha_i^n \Delta^{(l)}_{J^{i,n}} X$$

$$= \sum_{0\neq m\leqq l}\sum_{m\leqq j\leqq l} \underline{(-1)^{l-j}} \binom{l}{j}\binom{j}{m} \sum_{\underline{1}\leqq i\leqq r} (Y^{l,j,h}_{u^i} - Y^{l,j,n}_{u^i}) X^{j-m}_{u^i} \Delta^{(m)}_{K^i} X.$$

To estimate the right side of this equality, note that the inequality of Definition 1, i) can be easily extended to functions of the form $\sum_{\underline{1}\leqq i\leqq r} \zeta_i 1_{J^i}$, the step functions α_i being replaced by random variables ζ_i which have the same measurability properties and satisfy $\sum_{1\leqq i\leqq r} \zeta_i 1_{J^i} \in L^q(\Omega\times \mathrm{II}, \mathfrak{P}, \rho_l)$. Thus

$$(1.6)\quad \Big\| \sum_{1\leqq i\leqq r(h)} \alpha_i^h \Delta_{J^{i,h}}^{(l)} X - \sum_{1\leqq i\leqq r(n)} \alpha_i^n \Delta_{J^{i,n}}^{(l)} X\Big\|_p$$

$$\leqq \sum_{0\neq m\leqq l}\sum_{m\leqq j\leqq l} \binom{l}{j}\binom{j}{m}\Big(\int \sum_{1\leqq i\leqq r} |(Y_{u^i}^{l,j,h} - Y_{u^i}^{l,j,n}) X_{u^i}^{j-m}|^q 1_{K^i}\, d\rho_m\Big)^{1/q}$$

$$\leqq \sum_{0\neq m\leqq l}\sum_{m\leqq j\leqq l} \binom{l}{j}\binom{j}{m}\Big(\int [|Y^{l,j,h} - Y^{l,j,n}|\, \bar{X}^{j-m}]^q\, d\rho_m\Big)^{1/q}.$$

As for $g\in\mathbb{N}$, $j,l\in\mathbb{N}_0^d$, $j\leqq l$, we have $Y^{l,j,g}\leqq |Y_0|\,\bar{X}^{l-j}$, and as X has continuous trajectories, an appeal to ii) of Definition 1 and Lebesgue's dominated convergence theorem completes the proof. $\lrcorner$

Definition 2. Let X be a $(\rho_l, 1\leqq |l|\leqq k; q)^*$-dominated process. For $1\leqq |l|\leqq k$ the p-stochastic measure $\mu_{X^{(l)}}$ which exists according to Proposition 1 is called "l^{th} *variation of* X".

Go back to (1.2) for a moment. If X is *-dominated and the derivatives of f not too "big", the existence of the limit of the first term on the right is assured by Proposition 1. But the convergence of the rest term to zero must still be forced by an additional condition.

Theorem 1. *Let $k\in\mathbb{N}$, such that X is a $(\rho_l,\ 1\leqq |l|\leqq k; q)^*$-dominated process. For $J=]s,t]\in\mathfrak{J}$ let $(J^{i,n}=]s^{i,n},t^{i,n}]: 1\leqq i\leqq r(n))_{n\in\mathbb{N}}$ be the sequence of partitions of II generated by $Q_n=\{s,t\}\cup\left\{\frac{i}{n}: 0\leqq i\leqq n\right\}$. For $J\in\mathfrak{J}$, $T\in\Pi_N$, $l\in\mathbb{N}_0^d$ such that $|l|=k+1$, suppose that X satisfies*

$$(1.7)\qquad \lim_{n\to\infty} E\Big(\sum_{1\leqq i\leqq r(n)} |(X_{(s_T^{i,n}, t_T^{i,n})} - X_{s^{i,n}})^l|\Big)=0.$$

Then there is a unique number $k_0\leqq k$ such that $\mu_{X^{(l)}}\neq 0$ for at least one $l\in\mathbb{N}_0^d$, $|l|=k_0$ and $\mu_{X^{(l)}}=0$ for all $l\in\mathbb{N}_0^d$, $k_0<|l|\leqq k$. For each $f\in C^{k_0}(\mathbb{R}^d)$ satisfying

$$(1.8)\qquad D^{(l)}f(X)\in L^q(\Omega\times\mathrm{II},\mathfrak{P},\rho_l),\qquad 1\leqq |l|\leqq k_0,$$

and for each $J\in\mathfrak{J}$ the equation

$$\Delta_J f(X) = \sum_{1\leqq |l|\leqq k_0} \frac{1}{l!}\int_{\Omega\times J} D^{(l)} f(X)\, d\mu_{X^{(l)}}$$

is valid.

Remark. For $A\in\mathfrak{P}$, $1\leqq |l|\leqq k_0$ we adopt the notation "$\int_A Y d\mu_{X^{(l)}}$" instead of "$\int 1_A Y d\mu_{X^{(l)}}$".

Proof. A familiar approximation argument shows, that it is sufficient to consider $f\in C_b^{k+1}(\mathbb{R}^d)$. Let $J\in\mathfrak{J}$ and $(J^{i,n}: 1\leqq i\leqq r(n))_{n\in\mathbb{N}}$ be as above. Write down (1.2), with k in place of $k+1$, and simplify the k^{th} and $(k+1)^{\text{st}}$ terms to obtain

$$(1.9)\qquad \Delta_J f(X) = \sum_{1\leqq |l|\leqq k}\frac{1}{l!}\sum_{1\leqq i\leqq r(n)} D^{(l)} f(X_{s^{i,n}})\,\Delta_{J^{i,n}}^{(l)} X + \sum_{1\leqq i\leqq r(n)} S(f, J^{i,n}, k),$$

thereby setting

$$S(f,K,k)=\sum_{\emptyset\neq T\in\Pi_N}(-1)^{N-|T|}\sum_{|l|=k+1}\frac{1}{l!}D^{(l)}f(X_{(u_{\bar{T}},u_T+\theta_T(v_T-u_T))})(X_{(u_{\bar{T}},v_T)}-X_u)^l$$

for $K=]u,v]\in\mathfrak{I}$, suitable $\theta_T\in[0,1]$.

At first, the convergence of the l^{th} term on the right side of (1.9) will be established. To do this, let Y_0 be a previsible elementary function, and for $n\in\mathbb{N}$ let $Y_0=\sum_{1\leq i\leq q(n)}\alpha_i^n 1_{K^{i,n}}, K^{i,n}=]u^{i,n},v^{i,n}]$, be an $\mathbb{I}$-representation of Y_0 subordinate to $Q_n=\{s^{i,n},t^{i,n}: 1\leq i\leq r(n)\}$. The following inequality produces three terms which will be evaluated separately:

$$\begin{aligned}(1.10)\quad &\Big\|\sum_{1\leq i\leq r(n)}D^{(l)}f(X_{s^{i,n}})\Delta^{(l)}_{J^{i,n}}X-\int_{\Omega\times J}D^{(l)}f(X)\,d\mu_{X^{(l)}}\Big\|_p\\ &\leq\Big\|\sum_{1\leq i\leq r(n)}D^{(l)}f(X_{s^{i,n}})\Delta^{(l)}_{J^{i,n}}X-\sum_{1\leq i\leq q(n)}\alpha_i^n\Delta^{(l)}_{K^{i,n}}X\Big\|_p\\ &\quad+\Big\|\sum_{1\leq i\leq q(n)}\alpha_i^n\Delta^{(l)}_{K^{i,n}}X-\int_{\Omega\times J}Y_0\,d\mu_{X^{(l)}}\Big\|_p\\ &\quad+\Big\|\int_{\Omega\times J}Y_0\,d\mu_{X^{(l)}}-\int_{\Omega\times J}D^{(l)}f(X)\,d\mu_{X^{(l)}}\Big\|_p.\end{aligned}$$

For the first, argue as in the proof of (1.6). For $n\in\mathbb{N}$, $j,l\in\mathbb{N}_0^d$, $0\neq j\leq l$, set

$$Y^{l,j,n}=\sum_{1\leq i\leq r(n)}D^{(l)}f(X_{s^{i,n}})X^{l-j}_{s^{i,n}}1_{J^{i,n}},$$

$$\hat{Y}^{l,j,n}=\sum_{1\leq i\leq r(n)}D^{(l)}f(X)X^{l-j}_{s^{i,n}}1_{J^{i,n}},$$

$$Z^{l,j,n}=\sum_{1\leq i\leq q(n)}\alpha_i^n X^{l-j}_{u^{i,n}}1_{K^{i,n}}.$$

Deduce

$$\begin{aligned}(1.11)\quad &\Big\|\sum_{1\leq i\leq r(n)}D^{(l)}f(X_{s^{i,n}})\Delta^{(l)}_{J^{i,n}}X-\sum_{1\leq i\leq q(n)}\alpha_i^n\Delta^{(l)}_{K^{i,n}}X\Big\|_p\\ &\leq\sum_{0\neq m\leq l}\sum_{m\leq j\leq l}\binom{l}{j}\binom{j}{m}\Big(\int[|Y^{l,j,n}-\hat{Y}^{l,j,n}|\,\bar{X}^{j-m}]^q\,d\rho_m\Big)^{1/q}\\ &\quad+\sum_{0\neq m\leq l}\sum_{m\leq j\leq l}\binom{l}{j}\binom{j}{m}\Big(\int[|\hat{Y}^{l,j,n}-Z^{l,j,n}|\,\bar{X}^{j-m}]^q\,d\rho_m\Big)^{1/q}.\end{aligned}$$

By definition, the integrands of the 2nd term on the right side of (1.11) can be estimated by $|D^{(l)}f(X)-Y_0|\,\bar{X}^{l-m}$. Thus, the density of $\mathfrak{E}$ in $L^q(\Omega\times\mathbb{I},\mathfrak{P},\rho_m)$ and ii) of Definition 1 imply, that the third term on the right of (1.10) and the 2nd term on the right of (1.11) can be made arbitrarily small. For the first term on the right of (1.11), use dominated convergence and path continuity of $D^{(l)}f(X)$. Finally, Proposition 1 assures the convergence of the 2nd term on the right of (1.10).

So far we have shown that both sides of (1.9) converge in $L^p(\Omega,\mathfrak{F},P)$. Using (1.7) for identification, we see that the limit of the "rest term" of (1.9) must be zero. Now the asserted formula follows by setting

$$k_0=\max\{j\leq k: \mu_{X^{(l)}}\neq 0 \text{ for at least one } l,\ |l|=j,\ \mu_{X^{(l)}}=0 \text{ for } |l|>j\}.$$

The uniqueness of k_0 is a consequence of (1.7) and the uniqueness of $\mu_{X^{(l)}}$ on $\mathfrak{P}$. $\lrcorner$

Definition 3. Let $k \in \mathbb{N}$, X be a $(\rho_l,\ 1 \leqq |l| \leqq k;\ q)^*$-dominated process satisfying (1.7). The number k_0 which exists according to Theorem 1 under these hypotheses, is called "*I-order of X*".

Remark. If X is a $(\rho_l,\ 1 \leqq |l| \leqq k;\ q)^*$-dominated process, then in consequence of Proposition 1 and (1.2) applied to $f(x) = x^l$ there exist p-stochastic measures μ_{X^l}, $1 \leqq l \leqq k$, determined by $\mu_{X^l}(F \times J) = 1_F \Delta_J X^l$ for $F \times J \in \mathfrak{R}$. A real-valued process X for which μ_{X^l}, $1 \leqq l \leqq k$, exists as a p-stochastic measure, is called "p-semimartingale of order k" (cf. Allain [2]). In Definition 3 the letter "I" is added in order not to conflict with this notion of "order" (for example, (1, 1)-Wiener process is a 2-semimartingale of order ∞, but of I-order $k_0 = 2$, k_0 indicating the maximum order of differentiation involved in Ito's formula).

§ 2. Existence and Decomposition of the 2-Stochastic Measures $\mu_{W^{(l)}}$

In this paragraph we show, that the (N, d)-Wiener process is *-dominated and that $\mu_{W^{(l)}}$ exists as a 2-stochastic measure for all $l \in \mathbb{N}_0^d$. As a consequence of this fact, Theorem 1 yields a transformation theorem for W. But, compared to the classical Ito's formula for (1, 1)-Wiener process, it has a considerable disadvantage: while the classical formula equates stochastic processes, Theorem 1 merely gives an equation of random variables (J is kept fix!). The reason is this: in the definition of Wiener integral martingale methods are used in a crucial way; path continuity is proved via the powerful Doob's inequality. By applying the tools of multi-parameter martingale theory in this and the following paragraph, the transformation theorem for W is improved to be an equation of process (Theorem 3). The most important step in this direction is the decomposition of the "processes" $(\mu_{W^{(l)}}(\Omega \times R_t))_{t \in \mathrm{II}}$ into "martingales" (Theorem 2). We first recall the well-known notions of multi-parameter martingales (see for example Cairoli, Walsh [5], Wong, Zakai [13], Merzbach [10]). The following generalization of the famous $(F4)$-condition of Cairoli, Walsh [5] is assumed to be satisfied by all processes that are considered in this paragraph:

(2.1) for each bounded $\alpha \in \mathscr{M}(\underline{\mathfrak{F}_1}, \mathfrak{B}(\mathbb{R}))$, each $t \in \mathrm{II}$ and all $S, T \in \Pi_N$, $S \subset T$

$$E(\alpha \mid \mathfrak{F}_t^T) = E(E(\alpha \mid \mathfrak{F}_t^S) \mid \mathfrak{F}_t^{T \setminus S}).$$

In particular, (2.1) is fulfilled by the filtration of W.

Let $S \in \Pi_N$. A real valued adapted stochastic process M such that M_t is integrable for all $t \in \Pi$ is called "*weak S-(sub-)martingale*", if $E(\Delta_J^S M \mid \mathfrak{F}_u) \overset{(\geqq)}{=} 0$ for $J =]u, v] \in \mathfrak{J}$, "*S-(sub-)martingale*", if $E(M_{(u_{\bar{S}}, v_S)} \mid \mathfrak{F}_u) \overset{(\geqq)}{=} 0$ for $(u, v) \in \hat{\mathrm{II}}^2$, "*strong S-(sub-)martingale*", if $E(\Delta_J M \mid \bigvee_{i \in S} \mathfrak{F}_u^{(i)}) \overset{(\geqq)}{=} 0$ for $J =]u, v] \in \mathfrak{J}$. (Weak, strong) $\{1, \ldots, \mathrm{N}\}$-martingales are simply called (weak, strong) martingales.

Remark. Of course, the hierarchy of the different notions of "martingale" is as indicated by the words "weak" and "strong" and is in general strict. More precisely, we have the following relations. Let $S, S_1, S_2 \in \Pi_N$, $S_1 \subset S_2$, be given. By (2.1),

$$E(\Delta_J^S M \mid \mathfrak{F}_u)=0 \quad \text{iff } E(\Delta_J M \mid \mathfrak{F}_u^S)=0 \quad \text{for } J=]u,v]\in\mathfrak{J}.$$

Therefore, weak S_1-martingales are weak S_2-martingales. Evidently, (strong) S_2-martingales are (strong) S_1-martingales. Furthermore, strong S-martingales are S-martingales and S-martingales are weak S-martingales. Consequently, M is an S-martingale iff M is an $\{i\}$-martingale for all $i\in S$.

Let now M be an S-martingale. Wishing to establish a maximal inequality of the Doob-Cairoli type, one has to keep in mind, that M is a (one-parameter) martingale in every direction i for $i\in S$, but nothing can be said about its behaviour in the $\bar{S}$-directions. If M is right continuous, $\sup_{t\in\Pi} M_t$ can be estimated by $\sup_{t_S\in\Pi_S} v(M_{(.,t_S)})$. M is said to be a "*proper S-martingale*", if $v(M_{(.,\underline{1}_S)})$ is integrable. If M is a proper S-martingale, the generalization of an observation made by Wong, Zakai [13] shows that $(v(M_{(.,t_S)}))_{t_S\in\Pi_S}$ is an $|S|$-parameter positive submartingale. Thus Cairoli's [4] inequality can be applied to yield the following result.

Proposition 2. (*inequalities of Doob, Cairoli, Wong, Zakai*): *Let $S\in\Pi_N$, $p>1$. Then there are constants c_1, c_2, c_3 such that for every right continuous proper S-martingale M and all $\lambda\geqq 0$*

i) $\lambda P(\sup_{t\in\Pi}|M_t|\geqq\lambda)\leqq c_1\, E(v(M_{(.,\underline{1}_S)})\log^+(v(M_{(.,\underline{1}_S)}))^{1\vee(|S|-1)})+c_2$,

ii) $E(\sup_{t\in\Pi}|M_t|^p)\leqq c_3\, E(v(M_{(.,\underline{1}_S)})^p)$.

The proof of Proposition 2 will be omitted, as it can obtained by direct generalization of the ideas of Wong, Zakai [13], p. 574.

We are now ready to demonstrate, that W is a *-dominated process, and thereby decompose its variations $\mu_{W^{(l)}}$ as indicated above. For $F\times J\in\mathfrak{R}$ consider the approximations $1_F\Delta_J^{(l)}W$ of $\mu_{W^{(l)}}(F\times J)$. We will write them as a sum of expressions which, on one hand, prove to be "natural" for deriving the "domination inequality" (i) of Definition 1) and, on the other hand, are approximations of stochastic integrals with martingale properties connected with the Wiener process. Therefore, the following two lemmas give condition i) of Definition 1 and a basis for the desired martingale representation of $(\mu_{W^{(l)}}(\Omega\times R_t))_{t\in\Pi}$.

Lemma 1. *For $(\mathcal{T},\phi)\in\Phi_{0,N}$, $J\in\mathfrak{J}$ set $\Delta_J^{(\mathcal{T},\phi)}W=\prod_{T\in\mathcal{T}}(\Delta_{J^T}W)^{\phi(T)}$. Then, for $l\in\mathbb{N}_0^d\setminus\{0\}$, $J=]s,t]\in\mathfrak{J}$*

$$\Delta_J^{(l)}W=\sum_{(\mathcal{T},\phi)\in\Phi_{0,N},\,|\phi|=l}\frac{l!}{\phi!}\Delta_J^{(\mathcal{T},\phi)}W.$$

Proof. For $T\in\Pi_N$, observe that $R_{(s_T,t_T)}\setminus R_s=\bigcup_{\emptyset\neq S\subset T}J^S$ and put

$$A=\{\psi:\psi:\Pi_N\setminus\{\emptyset\}\to\mathbb{N}_0^d\},\qquad V_\psi:=\bigcup\{S:\psi(S)\neq 0\}\quad\text{for }\psi\in A.$$

Then, by additivity of $\Delta_. W^j$, $1 \leqq j \leqq d$, and the polynomial theorem

$$\Delta_J^{(l)} W = \sum_{\emptyset \neq T \in \Pi_N} (-1)^{N-|T|} \prod_{1 \leqq j \leqq d} \Big(\sum_{\emptyset \neq S \subset T} \Delta_{JS} W^j \Big)^{l_j}$$

$$= \sum_{\psi \in A, |\psi| = l} \frac{l!}{\psi!} \Big(\sum_{V_\psi \subset T \in \Pi_N} (-1)^{N-|T|} \Big) \prod_{\emptyset \neq S \in \Pi_N} (\Delta_{JS} W)^{\psi(S)}.$$

Now note that $\sum_{V_\psi \subset T \in \Pi_N} (-1)^{N-|T|}$ is 0 resp. 1, if $V_\psi \neq \{1, \ldots, N\}$ resp. $= \{1, \ldots, N\}$ and, for $\psi \in A$, set $\mathcal{T} = \{S \colon \psi(S) \neq 0\}$, $\phi = \psi|_{\mathcal{T}}$. The asserted formula follows. ┘

Lemma 2. *Let $(\mathcal{T}, \phi) \in \Phi_{0,N}$. Then there is a constant $c_\phi \in \mathbb{R}$ and for each finite subset R of $\mathbb{I}$ there is a number $n_R \in \mathbb{N}$, such that for all $n \geqq n_R$, all $Y_0 \in \mathfrak{E}$ having an $\mathbb{I}$-representation $Y_0 = \sum_{\underline{1} \leqq i \leqq r(n)} \alpha_i^n 1_{J^{i,n}}$ (recall that i and $r(n)$ are $\mathbb{N}^N$-valued indices) with respect to the partition $(J^{i,n} \colon \underline{1} \leqq i \leqq r(n))$ generated by $Q_n = R \cup \left\{ \frac{i}{n} \colon \underline{0} \leqq i \leqq \underline{n} \right\}$, the following inequalities hold*

$$\Big\| \sum_{\underline{1} \leqq i \leqq r(n)} \alpha_i^n \Delta_{J^{i,n}}^{(\mathcal{T}, \phi)} W \Big\|_2 \leqq \sqrt{\frac{1}{n}}\, c_\phi \Big(\int |Y_0|^2 d(P \times \lambda^N)\Big)^{1/2}, \qquad \text{if } (\mathcal{T}, \phi) \notin \Psi_{0,N},$$

$$\Big\| \sum_{\underline{1} \leqq i \leqq r(n)} \alpha_i^n \Delta_{J^{i,n}}^{(\mathcal{T}, \phi)} W - \sum_{\underline{1} \leqq i \leqq r(n)} \alpha_i^n \prod_{T \in \mathcal{T}^1} (\Delta_{(J^{i,n})^T} W)^{\phi(T)} \prod_{T \in \mathcal{T}^0} \lambda^N((J^{i,n})^T) \Big\|_2$$

$$\leqq \sqrt{\frac{1}{n}}\, c_\phi \Big(\int |Y_0|^2 d(P \times \lambda^N)\Big)^{1/2}, \qquad \text{if } (\mathcal{T}, \phi) \in \Psi_{0,N}, \qquad \mathcal{T}^1 = \{T \in \mathcal{T} \colon |\phi(T)| = 1\},$$

$$\mathcal{T}^0 = \{T \in \mathcal{T} \colon |\phi(T)| = 2\}.$$

Proof. To derive the first inequality, set

$$\mathcal{S} = \{T \in \mathcal{T} \colon \phi_j(T) \text{ is odd for at least one } j,\ 1 \leqq j \leqq d\} \qquad \underline{\mathcal{S}} = \bigcup_{T \in S} T.$$

Write the integrand of its left side as a double sum over i, k. From the fact, that W has independent increments with zero odd moments, and the independence of W^j, $1 \leqq j \leqq d$, conclude that there is no contribution from terms such that $i_{\underline{\mathcal{S}}} \neq k_{\underline{\mathcal{S}}}$. Thus

$$(2.2) \quad \Big\| \sum_{\underline{1} \leqq i \leqq r(n)} \alpha_i^n \Delta_{J^{i,n}}^{(\mathcal{T}, \phi)} W \Big\|_2^2 = \sum_{\underline{1}_{\underline{\mathcal{S}}} \leqq i_{\underline{\mathcal{S}}} \leqq r(n)_{\underline{\mathcal{S}}}} E\Big(\Big(\sum_{\underline{1}_{\underline{\mathcal{S}}} \leqq i_{\underline{\mathcal{S}}} \leqq r(n)_{\underline{\mathcal{S}}}} \alpha_i^n \Delta_{J^{i,n}}^{(\mathcal{T}, \phi)} W \Big)^2\Big)$$

$$\leqq \sum_{\underline{1} \leqq i \leqq r(n)} E\big((\alpha_i^n)^2 (\Delta_{J^{i,n}}^{(\mathcal{T}, \phi)} W)^2\big) \prod_{k \in \underline{\mathcal{S}}} r(n)_k \qquad \text{(Cauchy-Schwartz)}$$

$$= c_{1,\phi} \sum_{\underline{1} \leqq i \leqq r(n)} E\big((\alpha_i^n)^2\big) \prod_{1 \leqq j \leqq d} \prod_{T \in \mathcal{T}} \lambda^N((J^{i,n})^T)^{\phi_j(T)} \prod_{k \in \underline{\mathcal{S}}} r(n)_k$$

$$\leqq c_{1,\phi} \int |Y_0|^2 d(P \times \lambda^N) \Big(\frac{1}{n}\Big)^{\sum_{T \in \mathcal{T}} |T| |\phi(T)| - N} (n + 1 + |R|)^{|\underline{\mathcal{S}}|} \ (Y_0 \text{ is previsible})$$

$$\Big(\lambda^N((J^{i,n})^T) \leqq \Big(\frac{1}{n}\Big)^{|T|},\ r(n)_k \leqq n + 1 + |R|\Big),$$

putting $c_{1,\phi} = \sup_{1 \leqq j \leqq d} \sup_{T \in \mathcal{T}} E((W_{\underline{1}}^1)^{2\phi_j(T)})$.

Now from (2.2) it becomes clear, that the first inequality is a consequence of

$$(2.3) \quad \sum_{T \in \mathcal{T}} |T| |\phi(T)| - N - |\underline{\mathcal{S}}| \geqq 1.$$

But since $(\mathcal{T},\phi)\notin\Psi_{0,N}$, there is at least one $T\in\mathcal{S}$ with $|\phi(T)|>1$ or $T\in\bar{\mathcal{S}}$ with $|\phi(T)|>2$. This implies $\sum_{T\in\mathcal{T}}|T||\phi(T)|-N-|\underline{\mathcal{S}}|>\sum_{T\in\mathcal{S}}|T|+2\sum_{T\in\bar{\mathcal{S}}}|T|-N-|\underline{\mathcal{S}}|\geq 0$, whence (2.3).

In a similar way, the second inequality is proved. It is trivially true in case $\mathcal{T}^0=\emptyset$. Assume $\mathcal{T}^0\neq\emptyset$ and enumerate its elements by $T_1,\dots,T_q$. For convenience, omit the index n in the following estimation. By telescoping

$$\begin{aligned}(2.4)\quad &\Big\|\sum_{1\leq i\leq r}\alpha_i\Delta_{J^i}^{(\mathcal{T},\phi)}W-\sum_{1\leq i\leq r}\alpha_i\prod_{T\in\mathcal{T}^1}(\Delta_{(J^i)^T}W)^{\phi(T)}\prod_{T\in\mathcal{T}^0}\lambda^N((J^i)^T)\Big\|_2\\ &\leq\sum_{1\leq k\leq q}\Big\|\sum_{1\leq i\leq r}\alpha_i\prod_{T\in\mathcal{T}^1}(\Delta_{(J^i)^T}W)^{\phi(T)}\prod_{1\leq j\leq k-1}(\Delta_{(J^i)^{T_j}}W)^{\phi(T_j)}\\ &\quad\cdot[(\Delta_{(J^i)^{T_k}}W)^{\phi(T_k)}-\lambda^N((J^i)^{T_k})]\prod_{k+1\leq j\leq q}\lambda^N((J^i)^{T_j})\Big\|_2.\end{aligned}$$

Fix k, $1\leq k\leq q$, and replace $\mathcal{S}$ resp. $\underline{\mathcal{S}}$ by $\mathcal{T}^1\cup\{T_k\}$ resp. $\bigcup_{T\in\mathcal{T}^1}T\cup T_k$ in the above arguments. Observe that $[(\Delta_{(J^i)^{T_k}}W)^{\phi(T_k)}-\lambda^N((J^i)^{T_k})]$ has mean zero and variance $c\lambda^N((J^i)^{T_k})^2$, where c is a constant not depending on n. This gives the desired conclusion for the k^{th} term on the right of (2.4). k being arbitrary, the proof is complete. $\lrcorner$

Combined with Lemma 1, the inequalities of Lemma 2 signify, that $\mu_{W^{(t)}}$ gets no contribution from $(\mathcal{T},\phi)$-terms such that $(\mathcal{T},\phi)\notin\Psi_{0,N}$, whereas for $(\mathcal{T},\phi)\in\Psi_{0,N}$ there is a contribution which is asymptotically equivalent to an "elementary stochastic integral" of Wiener process on a space containing the set

$$\mathfrak{D}=\Big\{\sum_{1\leq i\leq n}a_i1_{F_i}1_{\prod_{T\in\mathcal{T}}(J_i)^T}:Y_0=\sum_{1\leq i\leq n}a_i1_{F_i\times J_i}\in\mathfrak{E}\Big\},$$

where it is defined by

$$\sum_{1\leq i\leq n}a_i1_{F_i}1_{\prod_{T\in\mathcal{T}}(J_i)^T}\to\sum_{1\leq i\leq n}a_i1_{F_i}\prod_{T\in\mathcal{T}^1}(\Delta_{(J_i)^T}W)^{\phi(T)}\prod_{T\in\mathcal{T}^0}\lambda^N((J_i)^T).$$

It will now be discussed how to define those integrals, how to extend them and what are the martingale properties of the corresponding integral processes. As will be shown in §4, they can be obtained by "iterated stochastic integration". With this in mind, we will define integrals not only for $\mathcal{T}\in\tau_N$, but for arbitrary $\mathcal{T}\in\tau$. Note that the σ-field generated by $\mathfrak{D}$ on $\Omega\times\mathrm{II}^{\mathcal{T}}$, contains all sets of the form $F\times\prod_{T\in\mathcal{T}}J^T$, $F\times J\in\mathfrak{R}$. But the latter class is not closed for relative complements. This fact makes plausible, that we may start with the following definition.

Definition 4. Let $\mathcal{T}\in\tau$.

1. $\mathrm{II}_{\mathcal{T}}=\{\mathfrak{s}=(s^T)_{T\in\mathcal{T}}\in\mathrm{II}^{\mathcal{T}}:\ s_T^T>s_T^S$ for $S,T\in\mathcal{T}$, $S\neq T\}$ is called "*set of $\mathcal{T}$-ordered points*", $\mathfrak{R}_{\mathcal{T}}:=\{F\times\prod_{T\in\mathcal{T}}A^T:\ A^T=]s^T,t^T]\in\mathfrak{I}$, $s_T^T\geq t_T^S$ for $S,T\in\mathcal{T}$, $S\neq T$, $F\in\mathfrak{F}^{\mathcal{T}}_{\bigvee_{T\in\mathcal{T}}s^T}\}$ "*set of $\mathcal{T}$-previsible rectangles*" ($\subset\Omega\times\mathrm{II}_{\mathcal{T}}$). Let $\mathfrak{A}_{\mathcal{T}}$ denote the ring generated by $\mathfrak{R}_{\mathcal{T}}$, $\mathfrak{P}_{\mathcal{T}}$ the σ-algebra generated by $\mathfrak{R}_{\mathcal{T}}$ ("*σ-algebra of $\mathcal{T}$-previsible sets*"). $\mathfrak{E}_{\mathcal{T}}$, the linear hull of characteristic functions belonging to $\mathfrak{R}_{\mathcal{T}}$, is called "*set of $\mathcal{T}$-previsible elementary functions*".

2. Let $Y_0 = \sum_{1 \leq i \leq n} a_i 1_{F_i \times \prod_{T \in \mathscr{T}} A_i^T}$ be a $\mathscr{T}$-previsible elementary function, $A_i^T =]s^{T,i}, t^{T,i}]$, and for $Q \subset \mathrm{II}$ let $(K^j\colon \underline{1} \leq j \leq r)$ be the partition of II generated by

$Q \cup \{s^{T,i}, t^{T,i}\colon 1 \leq i \leq n, T \in \mathscr{T}\}$, $\quad \alpha_{k} = \sum_{1 \leq i \leq n} a_i 1_{F_i} \prod_{T \in \mathscr{T}} 1_{\{K\colon K \subset A_i^T\}}(K^{k^T})$

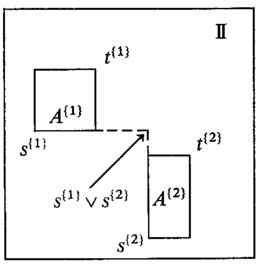
for $k = (k^T)_{T \in \mathscr{T}}$, $\underline{1} \leq k^T \leq r$, $T \in \mathscr{T}$. The representation $Y_0 = \sum_{\underline{1} \leq k^T \leq r} \alpha_{k} 1_{\prod_{T \in \mathscr{T}} K^{k^T}}$ is said to be a "$\mathrm{II}_{\mathscr{T}}$-*representation*" of Y_0 "*subordinate to* Q" (in case $Q = \emptyset$ "$\mathrm{II}_{\mathscr{T}}$-*representation*" of Y_0).

3. Let $\mathscr{S} \in \tau$ be such that $\mathscr{T}$ is a "refinement" of $\mathscr{S}$, i.e. each $S \in \mathscr{S}$ is the union over $\mathscr{T}_S = \{T \in \mathscr{T}\colon T \subset S\}$. Further let $Y \in \mathscr{M}(\mathfrak{P}_{\mathscr{S}}, \mathfrak{B}(\mathbb{R}))$. Set

$$Y^{\mathscr{T}}\colon \Omega \times \mathrm{II}_{\mathscr{T}} \to \mathbb{R},\ (\omega, s) \to Y(\omega, (\sup_{T \in \mathscr{T}_S} s^T)_{S \in \mathscr{S}}).$$

The process $Y^{\mathscr{T}}$ is called "$\mathscr{T}$-*corner function of* Y".
For $A \in \mathfrak{P}_{\mathscr{S}}$ let $A^{\mathscr{T}}$ be the set defined by $1_{A^{\mathscr{T}}} = (1_A)^{\mathscr{T}}$.

Example. Let $N = 2$. For $\mathscr{T} = \{\{1, 2\}\}$, we have $\mathrm{II}_{\mathscr{T}} = \mathrm{II}$, $\mathfrak{R}_{\mathscr{T}} = \mathfrak{R}$, $\mathfrak{P}_{\mathscr{T}} = \mathfrak{P}$, $\mathfrak{E}_{\mathscr{T}} = \mathfrak{E}$. For $\mathscr{T} = \{\{1\}, \{2\}\}$, $\mathrm{II}_{\mathscr{T}}$ consists of pairs of time points which are "incomparable" w.r.t. "$\leqq$". The following sketch may help to visualize $\{\{1\}, \{2\}\}$-previsible rectangles.



The $\{\{1\}, \{2\}\}$-corner function of $Y \in \mathscr{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ is just $Y(\omega, s^{\{1\}}, s^{\{2\}}) = Y(\omega, s^{\{1\}} \vee s^{\{2\}})$.

Remarks. 1. Let $Y_0 = \sum_{1 \leq k^T \leq r} \alpha_{k} 1_{\prod_{T \in \mathscr{T}} K^{k^T}}$ be an $\mathrm{II}_{\mathscr{T}}$-representation of $Y_0 \in \mathfrak{E}$, such that $K^j =]u^j, v^j]$, $\underline{1} \leq j \leq r$. Then, for every k, α_{k} is a $\mathfrak{F}^{\mathscr{T}}_{\bigvee_{T \in \mathscr{T}} u^{k^T}}$-measurable step function which vanishes if $\prod_{T \in \mathscr{T}} K^{k^T} \not\subset \mathrm{II}_{\mathscr{T}}$.

2. $\mathscr{T}$-corner functions are $\mathfrak{P}_{\mathscr{T}}$-measurable. If $\mathscr{T}$ is a refinement of $\mathscr{S}$, $Y \in \mathscr{M}(\mathfrak{P}^{\mathscr{T}}, \mathfrak{B}(\mathbb{R}))$, then obviously $Y^{\mathscr{T}} = (Y^{\mathscr{S}})^{\mathscr{T}}$. If $F \times J \in \mathfrak{R}$ and $(J^{j,n}\colon 1 \leq j \leq r(n))_{n \in \mathbb{N}}$ is a sequence of partitions of II in $\mathfrak{J}$ whose mesh goes to zero, then

$$\bigcup_{1 \leq j \leq r(n)} F \times \prod_{T \in \mathscr{T}} (J^{j,n})^T \to (F \times J)^{\mathscr{T}}.$$

(This gives the relation between $\mathfrak{D}$ (cf. p. 547) and the corner functions.)

3. The linear hull of $\{1_F 1_{\prod_{T \in \mathscr{T}} J^T}\colon F \times J \in \mathfrak{R}\}$ is identical with $\mathfrak{E}_{\mathscr{T}}$.

For $(\mathscr{T},\phi)\in\Psi_0$ the elementary integral whose approximation appears in the 2[nd] inequality of Lemma 2, can be introduced on $\mathfrak{E}_{\mathscr{T}}$. Note that it does not depend on the information provided by ϕ on $\mathscr{T}^0$ (cf. Lemma 2). Therefore we may use Ψ instead of Ψ_0.

Definition 5. Let $(\mathscr{T},\phi)\in\Psi$.

1. The linear mapping $I_0^{(\mathscr{T},\phi)}\colon \mathfrak{E}_{\mathscr{T}}\to L^2(\Omega,\mathfrak{F},P)$,

$$\sum_{1\leq i\leq n} a_i 1_{F_i\times\prod_{T\in\mathscr{T}} A_i^T}\to\sum_{1\leq i\leq n} a_i 1_{F_i}\prod_{T\in\mathscr{T}^1}\Delta_{A_i^T}W^{\phi(T)}\prod_{T\in\mathscr{T}^0}\lambda^N(A_i^T),$$

is called "*elementary* $(\mathscr{T},\phi)$-*integral*". Let $\mu_0^{(\mathscr{T},\phi)}$ be the restriction of $I_0^{(\mathscr{T},\phi)}$ to characteristic functions of sets in $\mathfrak{R}_{\mathscr{T}}$.

2. The linear mappings $I_{0,.}^{(\mathscr{T},\phi)}\colon \mathfrak{E}_{\mathscr{T}}\to L^2(\Omega\times\mathrm{II},\mathfrak{P},P\times\lambda^N)$, $Y_0\to I_0^{(\mathscr{T},\phi)}(Y_0(1_{\Omega\times R_t})^{\mathscr{T}})$, is called "*elementary* $(\mathscr{T},\phi)$-*integral process*".

Example. In case $N=2$, Definition 5 just gives the elementary versions of the well-known stochastic integrals of Wong and Zakai for the Wiener sheet. More precisely, the integrals $\int .dW$ and $\int .dW dW$, necessary for the description of "martingales" which are measurable w.r.t. the Wiener filtration (see Wong, Zakai [15], p. 118), are covered by "$\phi=1$":

$$I_0^{(\{\{1,2\}\},1)}\quad\text{corresponds to } \int .dW,\qquad I_0^{(\{\{1\},\{2\}\},1)}\quad\text{to } \int .dW dW.$$

In order to obtain a fully developed stochastic calculus, Wong and Zakai [13], Sect. 3, p. 574, studied another type of "iterated" stochastic integrals, the "mixed are integrals". For the Wiener sheet, they are recovered by taking

$\phi_1(\{1\})=1,\ \phi_1(\{2\})=0,\ \phi_2(\{1\})=0,\ \phi_2(\{2\})=1$:

$$I_0^{(\{\{1\},\{2\}\},\phi_1)}\quad\text{corresponds to } \int .du\,dW,\quad I_0^{(\{\{1\},\{2\}\},\phi_2)}\quad\text{to } \int .dW du.$$

Finally, "$\phi=0$" yields integrals w.r.t. Lebesgue measure:

$$I_0^{(\{\{1,2\}\},0)}\quad\text{corresponds to } \int .du,\quad I_0^{(\{\{1\},\{2\}\},0)}\quad\text{to } \int .du\,dv\quad\text{or}\quad\int .u_1 u_2\,du.$$

See Wong, Zakai [14] for a stochastic calculus with the above integrals. For extensions of Wong's and Zakai's notion of "iterated" stochastic integration see for example Cairoli, Walsh [5], Guyon, Prum [7], Merzbach [10], Yor [16], Sanz [12].

Remark. For $Y_0\in\mathfrak{E}_{\mathscr{T}}$ the process $I_{0,.}^{(\mathscr{T},\phi)}(Y_0)$ has continuous trajectories. By the following lemma, $I_0^{(\mathscr{T},\phi)}$ is "dominated". This makes an easy extension procedure possible.

Lemma 3. *Let* $(\mathscr{T},\phi)\in\Psi$, $Y_0\in\mathfrak{E}_{\mathscr{T}}$. *Then*

$$\|I_0^{(\mathscr{T},\phi)}(Y_0)\|_2^2=E\Big(\int_{\mathrm{II}^{\mathscr{T}^1}}\Big(\int_{\mathrm{II}^{\mathscr{T}^0}}Y_0(.,\sigma)\,d\sigma_{\mathscr{T}^0}\Big)^2 d\sigma_{\mathscr{T}^1}\Big).$$

Proof. The following arguments are similar to those used to prove Lemma 2. Let an II-representation $Y_0=\sum_{1\leq k^T\leq r}\alpha_k 1_{\prod_{T\in\mathscr{T}}K^{k^T}}$ of Y_0 be given. Insert this representation on the left side of the asserted equality to write it as a double sum over $\mathscr{k},\mathscr{j}\in(\mathbb{N}^N)^{\mathscr{T}}$. Observe that there is no contribution if $\mathscr{k}_{\mathscr{T}^1}\neq\mathscr{j}_{\mathscr{T}^1}$, since W has independent increments with mean zero. Therefore

$$\begin{aligned}\|I_0^{(\mathscr{T},\phi)}(Y_0)\|_2^2 &= \sum_{1\leq k^T\leq r,\, T\in\mathscr{T}^1} E\Big(\Big(\sum_{1\leq k^T\leq r,\, T\in\mathscr{T}^0} \alpha_{\mathscr{k}} \prod_{T\in\mathscr{T}^0} \lambda^N(K^{k^T})\Big)^2 \prod_{T\in\mathscr{T}^1} (\Delta_{K^{k^T}} W^{\phi(T)})^2\Big)\\ &= \sum_{1\leq k^T\leq r,\, T\in\mathscr{T}^1} E\Big(\Big(\sum_{1\leq k^T\leq r,\, T\in\mathscr{T}^0} \alpha_{\mathscr{k}} \prod_{T\in\mathscr{T}^0} \lambda^N(K^{k^T})\Big)^2 \prod_{T\in\mathscr{T}^1} \lambda^N(K^{k^T})\Big)\\ &\qquad\qquad (Y_0\in\mathscr{M}(\mathfrak{P}_{\mathscr{T}},\mathfrak{B}(\mathbb{R})))\\ &= E\Big(\int_{\mathbb{I}^{\mathscr{T}^1}} \Big(\int_{\mathbb{I}^{\mathscr{T}^0}} Y_0(\,.\,,\mathscr{s})\,d\mathscr{s}_{\mathscr{T}^0}\Big)^2 d\mathscr{s}_{\mathscr{T}^1}\Big). \quad \lrcorner\end{aligned}$$

Lemma 3 makes clear, how a "natural" domain of extension of $I_0^{(\mathscr{T},\phi)}$ must look like.

Definition 6. Let $(\mathscr{T},\phi)\in\Psi$. For $Y\in\mathscr{M}(\mathfrak{P}_{\mathscr{T}},\mathfrak{B}(\mathbb{R}))$ set

$$\|Y\|_{(\mathscr{T},\phi)}=\Big[E\Big(\int_{\mathbb{I}^{\mathscr{T}^1}}\Big(\int_{\mathbb{I}^{\mathscr{T}^0}}|Y|(\,.\,,\mathscr{s})\,d\mathscr{s}_{\mathscr{T}^0}\Big)^2 d\mathscr{s}_{\mathscr{T}^1}\Big)\Big]^{1/2},$$

$$L_{(\mathscr{T},\phi)}=\{Y\colon Y\in\mathscr{M}(\mathfrak{P}_{\mathscr{T}},\mathfrak{B}(\mathbb{R})),\ \|Y\|_{(\mathscr{T},\phi)}<\infty\}.$$

Remark. For $(\mathscr{T},\phi)\in\Psi$, $L_{(\mathscr{T},\phi)}$ is a Banach space with respect to the norm $\|\,.\,\|_{(\mathscr{T},\phi)}$, in which $\mathscr{E}_{\mathscr{T}}$ is dense. For $Y\in\mathscr{M}(\mathfrak{P}_{\mathscr{T}},\mathfrak{B}(\mathbb{R}))$, the inequality $\|Y\|_{(\mathscr{T},\phi)}\leqq\|Y\|_2$ is valid.

Proposition 3. *Let $(\mathscr{T},\phi)\in\Psi$. Then $\mu_0^{(\mathscr{T},\phi)}$ can be extended to $\mathfrak{P}_{\mathscr{T}}$ such that, denoting the extension by $\mu^{(\mathscr{T},\phi)}$, $\mu^{(\mathscr{T},\phi)}$ is a 2-stochastic measure. $I_0^{(\mathscr{T},\phi)}$ can be extended to $L_{(\mathscr{T},\phi)}$, such that its extension $I^{(\mathscr{T},\phi)}$ (the "$(\mathscr{T},\phi)$-integral") is a 2-stochastic integral satisfying*

$$\|I^{(\mathscr{T},\phi)}(Y)\|_2\leqq\|Y\|_{(\mathscr{T},\phi)}\quad \textit{for}\ \ Y\in L_{(\mathscr{T},\phi)}.$$

Proof. By Lemma 3,

$$\|I_0^{(\mathscr{T},\phi)}(Y_0)\|_2\leqq\|Y_0\|_{(\mathscr{T},\phi)}\quad \text{for}\ \ Y_0\in\mathfrak{E}_{\mathscr{T}}.$$

$\mathfrak{E}_{\mathscr{T}}$ being dense in $L_{(\mathscr{T},\phi)}$, the proposition follows. $\lrcorner$

The extension of elementary integral processes is based upon the following martingale property.

Lemma 4. *Let $(\mathscr{T},\phi)\in\Psi_N$, $Y_0\in\mathfrak{E}_{\mathscr{T}}$. Then $I_{0,.}^{(\mathscr{T},\phi)}(Y_0)$ is a continuous $\underline{\mathscr{T}}^1$-martingale.*

Proof. Evidently, $I_{0,t}^{(\mathscr{T},\phi)}(Y_0)$ is $\mathfrak{F}_t$-measurable and integrable for all $t\in\mathbb{I}$. By the remark to Definition 3, Remark 3 to Definition 4 and by linearity of $I_0^{(\mathscr{T},\phi)}$ it is sufficient to prove

$$E\big(I_0^{(\mathscr{T},\phi)}(1_F\, 1_{\prod_{T\in\mathscr{T}} K^T}(1_{\Omega\times J})^{\mathscr{T}}\,|\,\mathfrak{F}_{u\vee s}\big)=0, \tag{2.5}$$

$$F\times K\in\mathfrak{R},\quad K=]s,t],\quad J=]u,v]^{\{i\}},\quad i\in S\in\mathscr{T}^1.$$

Observe that

$$1_F 1_{\prod\limits_{T\in\mathcal{T}} K^T}(1_{\Omega\times J})^{\mathcal{T}} = 1_F 1_{\prod\limits_{T\in\mathcal{T}} K_T^T\cap J_T\times K_{\bar{T}}^T},\ K_T^T\cap J_T\times K_{\bar{T}}^T\subset]0,u\vee s] \quad \text{for } T\neq S,$$

$$K_S^S\cap J_S\times K_{\bar{S}}^S\cap]0,u\vee s]=\emptyset \quad \text{by definition of } J \text{ and } S.$$

Since W has independent increments with mean zero, (2.5) follows. $\lrcorner$

Remark. An analogon of Lemma 4 can be proved for arbitrary $(\mathcal{T},\phi)\in\Psi$: $I_{0,(\cdot,\underline{1}_{\mathcal{T}})}^{(\mathcal{T},\phi)}(Y_0)$ is an $\underline{\mathcal{T}}$-parameter $\underline{\mathcal{T}}^1$-martingale, $Y_0\in\mathfrak{E}_{\mathcal{T}}$. But since Y_0 is adapted only with respect to the filtration $(\mathfrak{F}_t^{\mathcal{T}})_{t\in\mathrm{II}}$, nothing can be said about the dependence of the elementary integral process on the $\bar{\mathcal{T}}$-coordinates. With the $\bar{\mathcal{T}}$-coordinates fixed, the following extension procedure could therefore also be carried out in this case.

We might try to extend the elementary integral process to $L_{(\mathcal{T},\phi)}$. But, in general, $L_{(\mathcal{T},\phi)}$ may be too large to produce continuous $\underline{\mathcal{T}}^1$-martingales. Since it is sufficient for the purposes of the following, we restrict our attention to $L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$ (although we could take a larger "intermediate" space). For $Z\in\mathcal{M}(\mathfrak{P},\mathfrak{B}(\mathbb{R}))$ set $\|Z\|_{(2,\infty)}=\|\sup\limits_{t\in\mathrm{II}}|Z_t|\|_2$ and

$$L^{(2,\infty)}(\Omega\times\mathrm{II},\mathfrak{P},P)=\{Z\colon Z\in\mathcal{M}(\mathfrak{P},\mathfrak{B}(\mathbb{R})),\ \|Z\|_{(2,\infty)}<\infty\}.$$

Note that $\mathrm{L}^{(2,\infty)}(\Omega\times\mathrm{II},\mathfrak{P},P)$ is a Banach space with respect to the norm $\|\cdot\|_{(2,\infty)}$. Lemma 4 does not tell all the truth about the elementary integral processes. As the proof of Proposition 4 shows, $I_{0,\cdot}^{(\mathcal{T},\phi)}(Y_0)$ is a proper $\underline{\mathcal{T}}^1$-martingale. Moreover, thanks to Proposition 2, $I_{0,\cdot}^{(\mathcal{T},\phi)}$ maps $\mathfrak{E}_{\mathcal{T}}$ into $L^{(2,\infty)}(\Omega\times\mathrm{II},\mathfrak{P},P)$. This important fact will be exploited in

Proposition 4. *Let $(\mathcal{T},\phi)\in\Psi_N$. The linear mapping $I_{0,\cdot}^{(\mathcal{T},\phi)}\colon\mathfrak{E}_{\mathcal{T}}\to L^{(2,\infty)}(\Omega\times\mathrm{II},\mathfrak{P},P)$ can be extended to $L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$. The extension, denoted by $I_{\cdot}^{(\mathcal{T},\phi)}$ (the "$(\mathcal{T},\phi)$-integral process") is linear, continuous and satisfies the following conditions*

i) $\|I_{\cdot}^{(\mathcal{T},\phi)}(Y)\|_{(2,\infty)}\leqq c\|Y\|_2$, $Y\in L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$,
c being a universal constant,

ii) $I_t^{(\mathcal{T},\phi)}(Y)=I_{\cdot}^{(\mathcal{T},\phi)}(Y(1_{\Omega\times R_t})^{\mathcal{T}})$, $t\in\mathrm{II}$, $Y\in L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$.

For each $Y\in L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$, $I_{\cdot}^{(\mathcal{T},\phi)}(Y)$ is a continuous proper $\underline{\mathcal{T}}^1$-martingale.

Proof. We will show

$$(2.6)\qquad E(v(I_{0,(\cdot,\underline{1}_{\mathcal{T}^1})}^{(\mathcal{T},\phi)}(Y_0))^2)\leqq\|Y_0\|_2^2,\qquad Y_0\in\mathfrak{E}_{\mathcal{T}}.$$

Once this is done, Lemma 4 and Proposition 2 yield a constant c, such that

$$(2.7)\qquad \|I_{0,\cdot}^{(\mathcal{T},\phi)}(Y_0)\|_{(2,\infty)}^2\leqq cE(v(I_{0,(\cdot,\underline{1}_{\mathcal{T}^1})}^{(\mathcal{T},\phi)}(Y_0))^2),\qquad Y_0\in\mathfrak{E}_{\mathcal{T}}.$$

Now combine (2.6) and (2.7) to obtain i) for $\mathcal{T}$-elementary functions. Use density of $\mathfrak{E}_{\mathcal{T}}$ in $L^2(\Omega\times\mathrm{II}_{\mathcal{T}},\mathfrak{P}_{\mathcal{T}},P\times(\lambda^N)^{|\mathcal{T}|})$, familiar extension arguments and the definition of $\|\cdot\|_{(2,\infty)}$ to finish the proof.

To prove (2.6), set $U := \mathscr{T}^0$ and let $(J_U^i =]s_U^i, t_U^i]: 1 \leqq i \leqq n)$ be a partition of $\mathbb{I}_U$ in $\mathfrak{I}$, $Y_0 = \sum_{\underline{1} \leqq k^T \leqq r} \alpha_{\ell} \prod_{T \in \mathscr{T}} 1_{K^{k^T}}$ and $\mathbb{I}_{\mathscr{T}}$-representation of $Y_0 \in \mathfrak{E}_{\mathscr{T}}$ subordinate to $Q = \{(u_U^i, \underline{1}_{\bar{U}}), (v_U^i, \underline{1}_{\bar{U}}): 1 \leqq i \leqq n\}$. Linearity of the elementary integral and of the operation "$Y \to Y^{\mathscr{T}}$" together give the following equation, valid for $\underline{1}_U \leqq k_U \leqq r_U$

$$
\begin{aligned}
(2.8) \quad \Delta_{K_U^{k_U}} I_{0,(.,\underline{1}_{\bar{U}})}^{(\mathscr{T},\phi)}(Y_0) &= I_0^{(\mathscr{T},\phi)}(Y_0 (1_{\Omega \times K_U^{k_U} \times (R_{\underline{1}})_{\bar{U}}})^{\mathscr{T}}) \\
&= I_0^{(\mathscr{T},\phi)}(Y_0 1_\Omega \prod_{T \in \mathscr{T}^0} 1_{(K_T^{k_U} \times \mathbb{I}_{\bar{T}})} \prod_{T \in \mathscr{T}^1} 1_{\mathbb{I}}) \\
&= \sum_{\underline{1} \leqq k^T \leqq r,\, k_T^T = k_T \text{ for } T \in \mathscr{T}^0} \alpha_{\ell} \prod_{T \in \mathscr{T}^0} \lambda^N(K^{k^T}) \prod_{T \in \mathscr{T}^1} \Delta_{K^{k^T}} W^{\phi(T)}.
\end{aligned}
$$

Sum (2.8) over k_U to obtain

$$
\begin{aligned}
& E\Big(\Big(\sum_{\underline{1}_U \leqq k_U \leqq r_U} |\Delta_{K_U^{k_U}} I_{0,(.,\underline{1}_{\bar{U}})}^{(\mathscr{T},\phi)}(Y_0)|\Big)^2\Big) \\
&= E\Big(\Big(\sum_{\substack{\underline{1}_T \leqq k_T^T \leqq r_T \\ T \in \mathscr{T}^0}} \Big| \sum_{\substack{\underline{1} \leqq k^T \leqq r \\ T \in \mathscr{T}^1}} \sum_{\substack{\underline{1}_{\bar{T}} \leqq k_{\bar{T}}^T \leqq r_{\bar{T}} \\ T \in \mathscr{T}^0}} \alpha_{\ell} \prod_{T \in \mathscr{T}^0} \lambda^{|\bar{T}|}(K_{\bar{T}}^{k^T}) \prod_{T \in \mathscr{T}^1} \Delta_{K^{k^T}} W^{\phi(T)}\Big| \\
&\qquad \cdot \prod_{T \in \mathscr{T}^0} \lambda^{|T|}(K_T^{k^T})\Big)^2\Big) \\
&\leqq \sum_{\substack{\underline{1}_T \leqq k_T^T \leqq r_T \\ T \in \mathscr{T}^0}} E\Big(\Big(\sum_{\substack{\underline{1} \leqq k^T \leqq r \\ T \in \mathscr{T}^1}} \sum_{\substack{\underline{1}_{\bar{T}} \leqq k_{\bar{T}}^T \leqq r_{\bar{T}} \\ T \in \mathscr{T}^0}} \alpha_{\ell} \prod_{T \in \mathscr{T}^0} \lambda^{|\bar{T}|}(K_{\bar{T}}^{k^T}) \prod_{T \in \mathscr{T}^1} \Delta_{K^{k^T}} W^{\phi(T)}\Big)^2\Big) \\
&\qquad \cdot \prod_{T \in \mathscr{T}^0} \lambda^{|T|}(K_T^{k^T}) \qquad \text{(Cauchy-Schwartz)} \\
&\leqq \int |Y_0|^2 \, d(P \times (\lambda^N)^{|\mathscr{T}|}) = \|Y_0\|_2^2
\end{aligned}
$$

(cf. proof of Lemma 3, remark to Definition 6).
Since the partition $(K_U^k U: \underline{1}_U \leqq k_U \leqq r_U)$ is finer than $(J_U^i: 1 \leqq i \leqq n)$, (2.6) follows. ⌟

Equipped with the $(\mathscr{T}, \phi)$-integrals of Proposition 3, we now apply the results of Lemma 2, to get *-domination for W and to decompose its variations.

Theorem 2. 1. *For each* $l \in \mathbb{N}_0^d \setminus \{0\}$ *there is a constant* c_l, *such that for all* $k \in \mathbb{N}$ W *is a* $(c_l P \times \lambda^N, 1 \leqq |l| \leqq k; 2)$**-dominated process.*

2. *For each* $l \in \mathbb{N}_0^d \setminus \{0\}$

$$
\mu_{W^{(l)}}(A) = \sum_{\substack{(\mathscr{T},\phi) \in \Psi_N, m(\mathscr{T},\phi) = |l| \\ l - |\mathscr{T}^1| \in 2\mathbb{N}_0^d}} \frac{1}{2^{|\mathscr{T}^0|}} \frac{l! \, |\mathscr{T}^0|!}{\left(\frac{l - |\mathscr{T}^1|}{2}\right)!} \mu^{(\mathscr{T},\phi)}(A^{\mathscr{T}}), \qquad A \in \mathfrak{P},
$$

$$
\int Y \, d\mu_{W^{(l)}} = \sum_{\substack{(\mathscr{T},\phi) \in \Psi_N, m(\mathscr{T},\phi) = |l| \\ l - |\mathscr{T}^1| \in 2\mathbb{N}_0^d}} \frac{1}{2^{|\mathscr{T}^0|}} \frac{l! \, |\mathscr{T}^0|!}{\left(\frac{l - |\mathscr{T}^1|}{2}\right)!} I^{(\mathscr{T},\phi)}(Y^{\mathscr{T}}),
$$

$$
Y \in L^2(\Omega \times \mathbb{I}, \mathfrak{P}, P \times \lambda^N).
$$

Remark. Since for $(\mathscr{T}, \phi) \in \Psi_N$ we have $m(\mathscr{T}, \phi) \leqq 2N$, Theorem 2 particularly says, that $\mu_{W^{(l)}} = 0$ for $|l| > 2N$.

Proof. 1. Lemmata 1–3 and the easy inequality

$$\|Y^{\mathscr{T}}\|_{(\mathscr{T},\phi)} \leqq \|Y\|_2, \qquad Y \in L^2(\Omega \times \mathrm{II}, \mathfrak{P}, P \times \lambda^N),$$

yield constants $c_l, l \in \mathbb{N}_0^d \setminus \{0\}$, such that condition i) of Definition 1 is satisfied for $\rho_l = c_l P \times \lambda^N$, $q=2$. As W^j is a strong martingale, and all moments of $W_{\underline{1}}^j$ exist, $1 \leqq j \leqq d$, condition ii) of Definition 1 can be derived from Proposition 2.

2. By definition of all appearing integrals and by linearity, it is enough to prove the first of the asserted equalities for $A = F \times J \in \mathfrak{R}$. Let $(J^{j,n}: 1 \leqq j \leqq r(n))$, $n \in \mathbb{N}$, be a sequence of partitions of J in $\mathfrak{J}$ whose mesh goes to zero, and define the "projection"

$$\Gamma: \Psi_{0,N} \to \Psi_N, (\mathscr{T},\phi) \to (\mathscr{T},\psi), \quad \text{where}$$

$$\psi: \mathscr{T} \to \{0,1,\ldots,d\}, \quad T \to \begin{cases} 0, & \text{if } |\phi(T)| = 2, \\ j, & \text{if } \phi(T) = e_j, \ 1 \leqq j \leqq d. \end{cases}$$

Then

$$\begin{aligned}
(2.9) \quad \mu_{W^{(l)}}(F \times J) &= (L^2-) \lim_{n\to\infty} 1_F \sum_{1 \leqq j \leqq r(n)} \Delta_{J^{j,n}}^{(l)} W \qquad ((1.3)) \\
&= \sum_{(\mathscr{T},\phi)\in\Psi_{0,N},\, |\phi|=l} \frac{l!}{\phi!} (L^2-) \lim_{n\to\infty} 1_F \sum_{1 \leqq j \leqq r(n)} \Delta_{J^{j,n}}^{(\mathscr{T},\phi)} W \qquad \text{(Lemmata 1, 2)} \\
&= \sum_{(\mathscr{T},\phi)\in\Psi_{0,N},\, |\phi|=l} \frac{l!}{\phi!} (L^2-) \lim_{n\to\infty} \mu^{\Gamma((\mathscr{T},\phi))}\Big(\bigcup_{1 \leqq j \leqq r(n)} F \times \prod_{T\in\mathscr{T}} (J^{j,n})^T\Big) \\
&\qquad \text{(Lemma 2, Definition 5)} \\
&= \sum_{(\mathscr{T},\psi)\in\Psi_N} \Big[\sum_{\Gamma((\mathscr{T},\phi))=(\mathscr{T},\psi)} \frac{l!}{\phi!}\Big] \mu^{(\mathscr{T},\psi)}((F \times J)^{\mathscr{T}}). \\
&\qquad \text{(Remark 2 to Definition 4)}
\end{aligned}$$

Now, by definition of Γ and $\Psi_{0,N}$, $\phi! = 2^{|\mathscr{T}^0|}$ for $(\mathscr{T},\phi) \in \Gamma^{-1}((\mathscr{T},\psi))$, and

$$|\{(\mathscr{T},\phi): (\mathscr{T},\phi) \in \Gamma^{-1}((\mathscr{T},\psi)), |\phi| = l\}| = \frac{|\mathscr{T}^0|!}{\left(\frac{l - |\mathscr{T}_\cdot^1|}{2}\right)!} 1_{\{m(\mathscr{T},\psi) = |l|,\, l - |\mathscr{T}_\cdot^1| \in 2\mathbb{N}_0^d\}},$$

where, of course, "$\mathscr{T}^0$" and "$\mathscr{T}^1$" are with respect to ψ. Inserting this into (2.9) yields the desired equality. $\lrcorner$

Corollary. *Let $l \in 2\mathbb{N}_0^d$, $|l| = 2N$. Then for $F \times J \in \mathfrak{R}$*

$$\mu_{W^{(l)}}(F \times J) = \frac{1}{2^N} \frac{l!\,N!}{(\frac{l}{2})!} 1_F \int_J \prod_{1 \leqq i \leqq N} u_i^{N-1}\, du.$$

Proof. Put $\mathscr{S} = \{\{i\}: 1 \leqq i \leqq N\}$, $\psi = 0$. $(\mathscr{S},\psi)$ is the only element of Ψ_N satisfying $m(\mathscr{S},\psi) = 2N$, since for $(\mathscr{T},\phi) \in \Psi_N$ such that $\mathscr{T}^1 \neq \emptyset$

$$m(\mathscr{T},\phi) < 2(|\mathscr{T}^0| + |\mathscr{T}^1|) = 2N.$$

But

$$\mu^{(\mathscr{S},\psi)}((\Omega \times R_s)^{\mathscr{S}}) = (\lambda^N)^{|\mathscr{S}|}\Big(\prod_{T\in\mathscr{S}} R_s \cap \mathrm{II}_{\mathscr{S}}\Big), \qquad s \in \mathrm{II},$$

whence by definition of $\mathrm{II}_{\mathscr{S}}$ and Theorem 2 the asserted equality follows for $J=R_s$, $s\in\mathrm{II}$. Use $\mu^{(\mathscr{S},\psi)}((F\times J)^{\mathscr{S}})=1_F\,\mu^{(\mathscr{S},\psi)}((\Omega\times J)^{\mathscr{S}})$ and additivity of $\mu^{(\mathscr{S},\psi)}$ to complete the proof. $\lrcorner$

§ 3. Ito's formula for (N, d)-Wiener Process

(N,d)-Wiener process satisfies condition (1.7) of Theorem 1, as will be shown below. Therefore, by Theorem 1, we get a ("weak") version of Ito's formula. Using the decomposition of Theorem 2 and the extension of Proposition 4, we obtain another ("strong") version which is an equation of processes.

Theorem 3. *Let* $f\in C^{2N}(\mathbb{R}^d)$.

1. *For each* $l\in\mathbb{N}_0^d$, $|l|\leqq 2N$, *suppose that* $D^{(l)}f(W)\in L^2(\Omega\times\mathrm{II},\mathfrak{P},P\times\lambda^N)$. *Then*

$$\Delta_J f(W)=\sum_{1\leqq|l|\leqq 2N}\frac{1}{l!}\int_{\Omega\times J} D^{(l)}f(W)\,d\mu_{W^{(l)}},\qquad J\in\mathfrak{J}.$$

2. *For each* $(\mathscr{T},\phi)\in\Psi_N$ *suppose* $D^{(\mathscr{T},\phi)}f(W)\in L^2(\Omega\times\mathrm{II},\mathfrak{P},P\times\lambda^N)$. *Then*

$$\Delta_{R_\cdot} f(W)=\sum_{(\mathscr{T},\phi)\in\Psi_N}\frac{1}{2^{|\mathscr{T}^0|}}\,I_\cdot^{(\mathscr{T},\phi)}((D^{(\mathscr{T},\phi)}f(W))^{\mathscr{T}}).$$

Proof. 1. To verify (1.7), let $J=]s,t]\in\mathfrak{J}$, $T\in\Pi_N$, $l\in\mathbb{N}_0^d$ such that $|l|\geqq 2N+1$ and $(J^{j,n}=]s^{j,n},t^{j,n}]$: $1\leqq j\leqq r(n))$ be the partition of II generated by $Q_n=\{s,t\}\cup\left\{\frac{i}{n}: 0\leqq i\leqq n\right\}$, $n\in\mathbb{N}$. Since for $u,v\in\mathrm{II}$, $u\leqq v$, $p\geqq 1$, $1\leqq j\leqq d$ the p^{th} moment of $W_v^j-W_u^j$ is a constant multiple of $[\lambda^N(R_v)-\lambda^N(R_u)]^{p/2}$, and since by choice of the partition $[\lambda^N(R_{(s_T^{j,n},t_T^{j,n})})-\lambda^N(R_{s^{j,n}})]\leqq\frac{N}{n}$, $r(n)\leqq n+2$, there is a constant c_1, such that

$$E\Big(\sum_{1\leqq j\leqq r(n)}|(W_{(s_T^{j,n},t_T^{j,n})}-W_{s^{j,n}})^l|\Big)\leqq c_1\sum_{1\leqq j\leqq r(n)}\left(\frac{N}{n}\right)^{|l|/2}\leqq c_1(n+2)^N\left(\frac{N}{n}\right)^{|l|/2}.$$

As $|l|\geqq 2N+1$, this implies (1.7). Use this together with Theorem 2.1 to verify the hypotheses of Theorem 1.

2. By Proposition 4, the processes $I_\cdot^{(\mathscr{T},\phi)}(D^{(\mathscr{T},\phi)}f(W)^{\mathscr{T}})$ are continuous. This fact and ii) of Proposition 4 assure that we are done once we have checked

$$\Delta_J f(W)=\sum_{(\mathscr{T},\phi)\in\Psi_N}\frac{1}{2^{|\mathscr{T}^0|}}\,I^{(\mathscr{T},\phi)}((1_{\Omega\times J}D^{(\mathscr{T},\phi)}f(W))^{\mathscr{T}}),\qquad J\in\mathfrak{J}. \tag{3.1}$$

The following equation is a consequence of Theorem 2.2 and linearity:

$$\begin{aligned}\Delta_J f(W)=&\sum_{1\leqq|l|\leqq 2N}\frac{1}{l!}\sum_{\substack{(\mathscr{T},\phi)\in\Psi_N,\, m(\mathscr{T},\phi)=|l|\\ l-|\mathscr{T}_\cdot^1|\in 2\mathbb{N}_0^d}}\frac{1}{2^{|\mathscr{T}^0|}}\frac{l!\,|\mathscr{T}^0|!}{\left(\frac{l-|\mathscr{T}_\cdot^1|}{2}\right)!}\\&\cdot I^{(\mathscr{T},\phi)}((1_{\Omega\times J}D^{(\mathscr{T},\phi)}f(W))^{\mathscr{T}})\\=&\sum_{(\mathscr{T},\phi)\in\Psi_N}\frac{1}{2^{|\mathscr{T}^0|}}I^{(\mathscr{T},\phi)}\left(\left[1_{\Omega\times J}\left(\sum_{\substack{m(\mathscr{T},\phi)=|l|\\ l-|\mathscr{T}_\cdot^1|\in 2\mathbb{N}_0^d}}\frac{|\mathscr{T}^0|!}{\left(\frac{l-|\mathscr{T}_\cdot^1|}{2}\right)!}D^{(l)}\right)f(W)\right]^{\mathscr{T}}\right).\end{aligned}$$

Since

$$\sum_{\substack{m(\mathcal{T},\phi)=|l|\\ l-|\mathcal{T}_{\cdot}^{1}|\in 2\mathbb{N}_0^d}} \frac{|\mathcal{T}^0|!}{\left(\frac{l-|\mathcal{T}_{\cdot}^{1}|}{2}\right)!} D^{(l)} = \sum_{k\in\mathbb{N}_0^d,\, |k|=|\mathcal{T}^0|} \frac{|\mathcal{T}^0|!}{k!} D^{(2k)} D^{(|\mathcal{T}_{\cdot}^1|)}$$
$$= \mathbb{D}^{|\mathcal{T}^0|} D^{(|\mathcal{T}_{\cdot}^1|)} = D^{(\mathcal{T},\phi)} \quad \text{for all } (\mathcal{T},\phi)\in\Psi_N,$$

(3.1) follows. ┘

Corollary 1. *Let* $\mathcal{S}=\{\{i\}\colon 1\leqq i\leqq N\}$, $\psi=0$. *In the formula of Theorem* 3.2 *the term belonging to* $(\mathcal{S},\psi)$ *(the term of highest order) is given by*

$$\frac{1}{2^{|\mathcal{S}^0|}} I_{\cdot}^{(\mathcal{S},\psi)}((D^{(\mathcal{S},\psi)} f(W))^{\mathcal{S}}) = \frac{1}{2^N} \int_{R_{\cdot}} \mathbb{D}^N f(W)_u \prod_{1\leqq i\leqq N} u_i^{N-1}\, du.$$

Proof. Look at the proof of the corollary of Theorem 2 and observe $\mathbb{D}^N = D^{(\mathcal{S},\psi)}$. ┘

Corollary 2. (N,d)-*Wiener process has I-order* $2N$.

Proof. Use Theorem 3 and the corollary of Theorem 2. ┘

§4. Iterated Stochastic Integration; Modification of Ito's Formula

In a forthcoming paper, the stochastic calculus developed here will be used to compute local times (especially for (N,d)-Wiener process). For the needs of this application, a modification of Ito's formula (Theorem 3) will now be derived. The idea is, that a "transformation formula" is available, in which - besides the term of highest order (cf. Corollary 1 to Theorem 3) - all terms are of the lowest possible differentiation order. It is established by "partial stochastic integration", a method whose classical counterpart may be found in applications of "Green's formula". The most important tools are a notion of "iterated stochastic integration" and a stochastic version of Fubini's theorem. We have to consider Wiener process on the affine submanifolds

$$\{(t_U, t_{\bar{U}})\colon t_U\in\mathbb{I}_U\}, \qquad t_{\bar{U}}\in\mathbb{I}_{\bar{U}}, \qquad U\in\Pi_N.$$

Definition 7. Let $U\in\Pi_N$.

1. Let $\mathcal{T}\in\tau_U$. $\mathbb{I}_{\mathcal{T}}^U=\{s=(s^T)_{T\in\mathcal{T}}\in\mathbb{I}_U^{\mathcal{T}}\colon s_T^T>s_T^S$ for $S,T\in\mathcal{T}$, $S\neq T\}$ is called "*set of* $\mathcal{T}$*-ordered points in* $\mathbb{I}_U$",

$$\mathfrak{R}_{\mathcal{T}}^U=\{F\times\prod_{T\in\mathcal{T}} A^T\colon A^T=]s^T,t^T]\in\mathfrak{J}_U,\, s_T^T>t_T^S$$
$$\text{for } S,T\in\mathcal{T},\, S\neq T,\, F\in\mathfrak{F}^U_{\bigvee_{T\in\mathcal{T}}(s^T,1_{\bar{U}})}\}$$

"*set of* $(\mathcal{T},U)$*-previsible rectangles*". Analogously to Definition 4, using $\mathbb{I}_{\mathcal{T}}^U$ resp. $\mathfrak{R}_{\mathcal{T}}^U$ instead of $\mathbb{I}_{\mathcal{T}}$ resp. $\mathfrak{R}_{\mathcal{T}}$, $\mathfrak{A}_{\mathcal{T}}^U$, $\mathfrak{P}_{\mathcal{T}}^U$ ("σ*-algebra of* $(\mathcal{T},U)$*-previsible sets*"), $\mathfrak{E}_{\mathcal{T}}^U$ ("*set of* $(\mathcal{T},U)$*-previsible elementary functions*"), "$\mathbb{I}_{\mathcal{T}}^U$*-representations*" and "$(\mathcal{T},U)$*-corner functions*" are defined.

2. Let $(\mathcal{T}, \phi) \in \Psi_U$, $t \in \mathrm{II}_{\bar{U}}$ and put

$$B_t \colon \mathcal{M}(\mathfrak{P}^U_{\mathcal{T}}, \mathfrak{B}(\mathbb{R})) \to \mathcal{M}(\mathfrak{P}_{\mathcal{T}}, \mathfrak{B}(\mathbb{R})),\ Y \to Z,$$

where $Z(\cdot, (s^T)_{T\in\mathcal{T}}) = Y(\cdot, (s^T_U)_{T\in\mathcal{T}}) \prod_{T\in\mathcal{T}} 1_{(R_t)_{\bar{U}}}(s^T_{\bar{U}})$.

Then

$$\begin{aligned}
\|Y\|^t_{(\mathcal{T},\phi)} &= \|B_t(Y)\|_{(\mathcal{T},\phi)},\ Y \in \mathcal{M}(\mathfrak{P}^U_{\mathcal{T}}, \mathfrak{B}(\mathbb{R})),\\
L^t_{(\mathcal{T},\phi)} &= \{Y\colon Y \in \mathcal{M}(\mathfrak{P}^U_{\mathcal{T}}, \mathfrak{B}(\mathbb{R})),\ \|Y\|^t_{(\mathcal{T},\phi)} < \infty\},\\
I^{(\mathcal{T},\phi,t)} &\colon L^t_{(\mathcal{T},\phi)} \to L^2(\Omega, \mathfrak{P}, P),\ Y \to I^{(\mathcal{T},\phi)}(B_t(Y)).
\end{aligned}$$

(The meaning of "$\mu^{(\mathcal{T},\phi,t)}$", "$I_0^{(\mathcal{T},\phi,t)}$", "$\mu_0^{(\mathcal{T},\phi,t)}$", "$I_{0,\cdot}^{(\mathcal{T},\phi,t)}$" is obvious.)

Remark. As in Lemma 4, it can be verified that $I_{0,\cdot}^{(\mathcal{T},\phi,t)}(Y_0)$, $Y_0 \in \mathfrak{E}^U_{\mathcal{T}}$, has martingale properties. The extension of Proposition 4 could be considered. But since this is irrelevant for the sequel, it will be omitted.

The following lemma on "L^0-extension" (cf. Cairoli, Walsh [5], p. 132), on one hand, is an important tool in the proof of the "stochastic Fubini's theorem" and, on the other hand, yields an extension of $I_{0,\cdot}^{(\mathcal{T},\phi)}$ resp. $I_0^{(\mathcal{T},\phi,\cdot)}$ to $L_{(\mathcal{T},\phi)}$ resp. $L^1_{(\mathcal{T},\phi)}$, $(\mathcal{T}, \phi) \in \Psi_U$.

Lemma 5. *Let $(\mathcal{T}, \phi) \in \Psi$, $(\Lambda, \mathfrak{G}, \nu)$ a finite measure space. Further, let Y, $Y_n \in \mathcal{M}(\mathfrak{P}_T \times \mathfrak{G}, \mathfrak{B}(\mathbb{R}))$, $Z_n \in \mathcal{M}(\mathfrak{F} \times \mathfrak{G}, \mathfrak{B}(\mathbb{R}))$, $n \in \mathbb{N}$, be such, that*

$$\lim_{n\to\infty} \int_\Lambda \|Y_n(\cdot, x) - Y(\cdot, x)\|^2_{(\mathcal{T},\phi)}\, d\nu(x) = 0,$$

$$Z_n(\cdot, x) = I^{(\mathcal{T},\phi)}(Y_n(\cdot, x)) \text{ for } \nu\text{-a.e. } x \in \Lambda.$$

Then there exists $Z \in \mathcal{M}(\mathfrak{F} \times \mathfrak{G}, \mathfrak{B}(\mathbb{R}))$ satisfying

$$\begin{aligned}
&\|Z_n(\cdot, x) - Z(\cdot, x)\|_2 \leqq \|Y_n(\cdot, x) - Y(\cdot, x)\|_{(\mathcal{T},\phi)} \text{ for } \nu\text{-a.e. } x \in \Lambda,\\
&Z(\cdot, x) = I^{(\mathcal{T},\phi)}(Y(\cdot, x)) \quad \text{for } \nu\text{-a.e. } x \in \Lambda.
\end{aligned}$$

In particular, $Z_n \to Z$ in measure $(P \times \nu)$.

Proof. By hypothesis and Proposition 3, $(Z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure $(P \times \nu)$ and

$$\|Z_n(\cdot, x) - Z_m(\cdot, x)\|_2 \leqq \|Y_n(\cdot, x) - Y_m(\cdot, x)\|_{(\mathcal{T},\phi)} \text{ for } n, m \in \mathbb{N},\ \nu\text{-a.e. } x \in \Lambda.$$

The limit Z of $(Z_n)_{n\in\mathbb{N}}$ has the desired properties. ⌟

Corollary. *Let $U \in \Pi_N$, $(\mathcal{T}, \phi) \in \Psi_U$.*

1. *Let ρ be a finite measure on $\mathfrak{B}(\mathrm{II})$. Then for each $Y \in L_{(\mathcal{T},\phi)}$ there exists $X^{(\mathcal{T},\phi)} \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ satisfying*

$$X_t^{(\mathcal{T},\phi)} = I^{(\mathcal{T},\phi)}(Y(1_{\Omega\times R_t})^{\mathcal{T}}) \quad \textit{for } \rho\text{-a.e. } t \in \mathrm{II}.$$

2. *Let ρ on $\mathfrak{B}(\mathrm{II})$ be the product of finite measures ρ_i, $1 \leqq i \leqq N$. Then for each $Y \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ such that*

$$\int_{\mathrm{II}_{\bar{U}}} [\|Y(\cdot, t_{\bar{U}})^{\mathcal{T}}\|^{t_{\bar{U}}}_{(\mathcal{T},\phi)}]^2\, d\rho_{\bar{U}}(t_{\bar{U}}) < \infty, \tag{4.1}$$

there exists $Z^{(\mathcal{T},\phi)} \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ *satisfying*

$$Z_t^{(\mathcal{T},\phi)} = I^{(\mathcal{T},\phi,t_{\bar{U}})}((1_{\Omega\times(R_t)_U} Y(.,t_{\bar{U}}))^{\mathcal{T}}) \quad \text{for } \rho\text{-a.e. } t \in \mathbb{I}.$$

Proof. 1. Observe that $\mathfrak{E}_{\mathcal{T}}$ is dense in $L_{(\mathcal{T},\phi)}$ and

$$\|(Y-Z)(1_{\Omega\times R_t})^{\mathcal{T}}\|_{(\mathcal{T},\phi)} \leqq \|Y-Z\|_{(\mathcal{T},\phi)}, \; Y, Z \in L_{(\mathcal{T},\phi)}.$$

Apply Lemma 5 with $(\Lambda, \mathfrak{G}, \nu) = (\mathbb{I}, \mathfrak{B}(\mathbb{I}), \rho)$.

2. $\mathfrak{E}$ is dense in the set of previsible functions satisfying (4.1). Furthermore,

$$\int_{\mathbb{I}} [\|[(Y(.,t_{\bar{U}}) - Z(.,t_{\bar{U}}))\, 1_{\Omega\times(R_t)_U}]^{\mathcal{T}}\|_{(\mathcal{T},\phi)}^{t_{\bar{U}}}]^2\, d\rho(t)$$
$$\leqq \rho_U(\mathbb{I}_U) \int_{\mathbb{I}_{\bar{U}}} [\|(Y(.,t_{\bar{U}}) - Z(.,t_{\bar{U}}))^{\mathcal{T}}\|_{(\mathcal{T},\phi)}^{t_{\bar{U}}}]^2\, d\rho_{\bar{U}}(t_{\bar{U}}), \qquad Y, Z \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R})).$$

Since for $Y \in \mathfrak{E}$ the assertion is true, Lemma 5 with $(\Lambda, \mathfrak{G}, \nu) = (\mathbb{I}, \mathfrak{B}(\mathbb{I}), \rho)$ can be applied. ┘

Proposition 5 ("stochastic Fubini's theorem"). *Let* $U, V \in \Pi_N$, $U \cap V = \emptyset$, $(\mathcal{T}, \phi) \in \Psi_U$, $(\mathcal{S}, \psi) \in \Psi_V$, $\mathcal{U} := \mathcal{T} \cup \mathcal{S}$, $\chi := \phi \cup \psi$, $Y \in \mathcal{M}(\mathfrak{P}_{\mathcal{U}}, \mathfrak{B}(\mathbb{R}))$ *such that*

$$\int_{\mathbb{I}^{\mathcal{T}}} \|Y(.,.,\mathfrak{s}_{\mathcal{T}})\|_{(\mathcal{S},\psi)}^2\, d\mathfrak{s}_{\mathcal{T}} < \infty.$$

Let ρ be a finite measure on $\mathfrak{B}(\mathbb{I})$. Then:

i) *for* $(\lambda^N)^{|\mathcal{T}|}$-a.e. $\mathfrak{s}_{\mathcal{T}}$, $\|Y(.,.,\mathfrak{s}_{\mathcal{T}})\|_{(\mathcal{S},\psi)}$ *is finite,*

ii) *there exists* $Z^{(\mathcal{S},\psi)} \in \mathcal{M}(\mathfrak{P}_{\mathcal{T}} \times \mathfrak{B}(\mathbb{I}), \mathfrak{B}(\mathbb{R}))$ *satisfying*

$$Z^{(\mathcal{S},\psi)}(.,\mathfrak{s}_{\mathcal{T}},.) \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R})) \quad \text{for all } \mathfrak{s}_{\mathcal{T}},$$
$$Z^{(\mathcal{S},\psi)}(.,\mathfrak{s}_{\mathcal{T}},t) = I^{(\mathcal{S},\psi)}((Y(1_{\Omega\times R_t})^{\mathcal{U}}(.,.,\mathfrak{s}_{\mathcal{T}})) \quad \text{for } (\lambda^N)^{|\mathcal{T}|} \times \rho\text{-a.e. } (\mathfrak{s}_{\mathcal{T}}, t) \in \mathbb{I}_{\mathcal{T}} \times \mathbb{I},$$

iii) $\int_{\mathbb{I}} \|Z^{(\mathcal{S},\psi)}(.,.,t)\|_{(\mathcal{T},\psi)}^2\, d\rho(t) < \infty$,

iv) *there exists* $X^{(\mathcal{U},\chi)} \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$, *such that*

$$X_t^{(\mathcal{U},\chi)} = I^{(\mathcal{T},\phi)}(Z^{(\mathcal{S},\psi)}(.,.,t)) = I^{(\mathcal{U},\chi)}(Y(1_{\Omega\times R_t})^{\mathcal{U}}) \quad \text{for } \rho\text{-a.e. } t \in \mathbb{I}.$$

Proof. i) is evident. For $Y_0 \in \mathfrak{E}_{\mathcal{U}}$ the assertion follows easily from the definition of the elementary integral. Let $Y \in \mathcal{M}(\mathfrak{P}_{\mathcal{U}}, \mathfrak{B}(\mathfrak{R}))$ be given, satisfying the above finiteness hypothesis, $(Y_0^n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{U}$-previsible elementary functions such that

$$(4.2) \qquad \int_{\mathbb{I}} \int_{\mathbb{I}^{\mathcal{T}}} \|(Y - Y_0^n)(1_{\Omega\times R_t})^{\mathcal{U}}(.,.,\mathfrak{s}_{\mathcal{T}})\|_{(\mathcal{T},\phi)}^2\, d\mathfrak{s}_{\mathcal{T}}\, d\rho(t) \to 0 \qquad (n \to \infty).$$

This sequence owes its existence to the density of $\mathfrak{E}_{\mathcal{U}}$ in $L_{(\mathcal{U},\chi)}$ (cf. proof of corollary of Lemma 5). For $n \in \mathbb{N}$, $\mathfrak{s}_{\mathcal{T}} \in \mathbb{I}_{\mathcal{T}}$, $t \in \mathbb{I}$ let

$$Z_n^{(\mathcal{S},\psi)}(.,\mathfrak{s}_{\mathcal{T}},t) = I^{(\mathcal{S},\psi)}(Y_0^n(1_{\Omega\times R_t})^{\mathcal{U}}(.,.,\mathfrak{s}_{\mathcal{T}})).$$

Put $\Lambda = \mathbb{I}_{\mathcal{T}} \times \mathbb{I}$, $\nu = (\lambda^N)^{|\mathcal{T}|}$. Then, by (4.2), the functions $Y(1_{\Omega\times R_.})^{\mathcal{U}}$, $Y_0^n(1_{\Omega\times R_.})^{\mathcal{U}}$ and $Z_n^{(\mathcal{S},\psi)}$ fulfil the conditions of Lemma 5. Hence there exists $Z^{(\mathcal{S},\psi)} \in \mathcal{M}(\mathfrak{P}_{\mathcal{T}} \times \mathfrak{B}(\mathbb{I}), \mathfrak{B}(\mathbb{R}))$, such that ii) and iii) are valid, in particular

(4.3) $$\int_{\mathrm{II}} \|Z_n^{(\mathscr{S},\psi)}(.,.,t) - Z^{(\mathscr{S},\psi)}(.,.,t)\|^2_{(\mathscr{T},\phi)}\, d\rho(t)$$
$$\leqq \int_{\mathrm{II}} \int_{\mathrm{II}^{\mathscr{T}}} \|(Y - Y_0^n)(1_{\Omega\times R_t})^{\mathscr{U}}(.,.,\mathfrak{s}_{\mathscr{T}})\|^2_{(\mathscr{T},\phi)}\, d\mathfrak{s}_{\mathscr{T}}\, d\rho(t).$$

Let $X_n^{(\mathscr{U},\chi)}$ correspond to $Z_n^{(\mathscr{S},\psi)}$ and satisfy iv), $n\in\mathbb{N}$. Set $\Lambda = \mathrm{II}$, $\nu = \rho$. This time $Z^{(\mathscr{S},\psi)}$, $Z_n^{(\mathscr{S},\psi)}$ and $X_n^{(\mathscr{U},\chi)}$, according to (4.2), (4.3), fulfil the conditions of Lemma 5. Consequently there exists $X^{(\mathscr{U},\chi)}\in\mathscr{M}(\mathfrak{P},\mathfrak{B}(\mathbb{R}))$ which has the desired property iv). ┘

Corollary ("stochastic Fubini's theorem for corner functions"). *Let* $U, V\in\Pi_N$, $U\cap V=\emptyset$, $(\mathscr{T},\phi)\in\Psi_U$, $(\mathscr{S},\psi)\in\Psi_V$, $\mathscr{U}=\mathscr{T}\cup\mathscr{S}$, $\chi=\phi\cup\psi$, $Y\in\mathscr{M}(\mathfrak{P}^{U\cup V},\mathfrak{B}(\mathbb{R}))$ *such that*
$$\int_{\mathrm{II}^{\mathscr{T}}} [\|(Y^{\{U\}\cup\mathscr{S}}\|^2_{(\mathscr{S},\psi)}]^{\mathscr{T}}\, d\mathfrak{s}_{\mathscr{T}} < \infty.$$

Let ρ be a finite measure on $\mathfrak{B}(\mathrm{II})$. Then:

i) *for* λ^N-a.e. $s^U\in\mathrm{II}$, $\|(Y^{\{U\}\cup\mathscr{S}}(.,.,s^U)\|_{(\mathscr{S},\psi)}$ *is finite,*

ii) *there exists* $Z^{(\mathscr{S},\psi)}\in\mathscr{M}(\mathfrak{P}^U\times\mathfrak{B}(\mathrm{II}),\mathfrak{B}(\mathbb{R}))$ *satisfying*

$$Z^{(\mathscr{S},\psi)}(.,s^U,.)\in\mathscr{M}(\mathfrak{P}^{U\cup V},\mathfrak{B}(\mathbb{R})) \quad \textit{for each } s^U\in\mathrm{II},$$
$$Z^{(\mathscr{S},\psi)}(.,s^U,t) = I^{(\mathscr{S},\psi)}([(Y1_{\Omega\times R_t})^{\{U\}\cup\mathscr{S}}](.,.,s^U)) \quad \textit{for } \lambda^N\times\rho\text{-a.e. } (s^U,t)\in\mathrm{II}^2,$$

iii) $\int_{\mathrm{II}} \|(Z^{(\mathscr{S},\psi)}(.,.,t))^{\mathscr{T}}\|^2_{(\mathscr{T},\phi)}\, d\rho(t) < \infty$,

iv) *there exists* $X^{(\mathscr{U},\chi)}\in\mathscr{M}(\mathfrak{P}^{U\cup V},\mathfrak{B}(\mathbb{R}))$ *such that*

$$X_t^{(\mathscr{U},\chi)} = I^{(\mathscr{T},\phi)}((Z^{(\mathscr{S},\psi)}(.,.,t))^{\mathscr{T}}) = I^{(\mathscr{U},\chi)}((Y1_{\Omega\times R_t})^{\mathscr{U}}) \quad \text{for } \rho\text{-a.e. } t\in\mathrm{II}.$$

Proof. According to Remark 2 after Definition 4
$$[\|Y^{\{U\}\cup\mathscr{S}}\|_{(\mathscr{S},\psi)}]^{\mathscr{T}}(.,.,\mathfrak{s}_{\mathscr{T}}) = \|Y^{\mathscr{U}}(.,.,\mathfrak{s}_{\mathscr{T}})\|_{(\mathscr{S},\psi)}.$$

Therefore, by hypothesis,
$$\int_{\mathrm{II}^{\mathscr{T}}} \|Y^{\mathscr{U}}(.,.,\mathfrak{s}_{\mathscr{T}})\|^2_{(\mathscr{S},\psi)}\, d\mathfrak{s}_{\mathscr{T}} < \infty.$$

Apply Proposition 5 to $Y^{\mathscr{U}}$. ┘

Now observe that, by scaling, Wiener process on the affine submanifolds
$$\{(t_U, t_{\bar U}) : t_U\in\mathrm{II}_U\}, \quad t_{\bar U}\in\mathrm{II}_{\bar U}, \quad U\in\Pi_N,$$
can be transformed into $|U|$-parameter Wiener processes. This basic fact is exploited in the following lemma which, as a link between stochastic integration over intervals and stochastic integration over their surfaces, could be considered to be an elementary "stochastic Green's formula" (see Dozzi [6], Guyon, Prum [7], Cairoli, Walsh [5]).

Lemma 6. *For* $U\in\Pi_N$ *let* $(\mathscr{U},\chi)\in\Psi_U$. *Suppose that* $f\in C^{2|U|}(\mathbb{R}^d)$ *satisfies*
$$f(W),\, D^{(\mathscr{S},\psi)}f(W)\in L^2(\Omega\times\mathrm{II},\mathfrak{P},P\times\lambda^N), \quad (\mathscr{S},\psi)\in\Psi_U.$$

Let ρ on $\mathfrak{B}(\mathbb{I})$ be the product of finite measure ρ_i, $1 \leqq i \leqq N$, such that

(4.4) $$\int_{\mathbb{I}^{\mathscr{T}}} [\| f(W_{(.,t_{\mathscr{T}})})^{\mathscr{T}} \|_{(\mathscr{T},\phi)}^{t_{\mathscr{T}}}]^2 \, d\rho_{\mathscr{T}}(t_{\mathscr{T}}) < \infty, (\mathscr{U}, \chi) < (\mathscr{T}, \phi) \in \Psi.$$

Then, for $(\mathscr{U}, \chi) < (\mathscr{T}, \phi) \in \Psi$, there exist $X^{(\mathscr{T},\phi)} \in \mathscr{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$, such that

$$X_t^{(\mathscr{T},\phi)} = I^{(\mathscr{T},\phi,t_{\mathscr{T}})}([1_{\Omega \times (R_t)_{\mathscr{T}}} f(W_{(.,t_{\mathscr{T}})})]^{\mathscr{T}}) \quad \textit{for } \rho\text{-a.e. } t \in \mathbb{I},$$

$$\sum_{(\mathscr{U},\chi) < (\mathscr{T},\phi) \in \Psi} (-1)^{|\mathscr{T}|-|U|} X_t^{(\mathscr{T},\phi)} = \sum_{(\mathscr{S},\psi) \in \Psi_{\bar{U}}} \frac{1}{2^{|\mathscr{S}^0|}} I_t^{(\mathscr{S} \cup \mathscr{U}, \psi \cup \chi)} ((D^{(\mathscr{S},\psi)} f(W))^{\mathscr{S} \cup \mathscr{U}})$$

for ρ-a.e. $t \in \mathbb{I}$.

Remark. According to the hypothesis, (4.4) is satisfied for $\rho = \lambda^N$.

Proof. In consequence of the hypothesis, by a (classical) Fubini argument, for λ^N-a.e. $s^U \in \mathbb{I}$, we have

(4.5) $$\|[D^{(\mathscr{S},\psi)} f(W)^{\{U\} \cup \mathscr{S}}](.,.,s^U)\|_2 \quad \text{is finite, } (\mathscr{S},\psi) \in \Psi_{\bar{U}}.$$

Pick $s^U \in \mathbb{I}$ satisfying (4.5). Note that

$$X_{t_{\bar{U}}} = (s_U^U)^{-1/2} W_{(s_U^U, t_{\bar{U}})}$$

is a $|\bar{U}|$-parameter Wiener process. $g: R^d \to \mathbb{R}$, $x \to f((s_U^U)^{1/2} x)$, has the differentiability properties of f and

$$D^{(\mathscr{S},\psi)} g(x) = (s_U^U)^{1/2 m(\mathscr{S},\psi)} D^{(\mathscr{S},\psi)} f(y), \qquad y = (s_U^U)^{1/2} x, \qquad (\mathscr{S},\psi) \in \Psi_{\bar{U}}.$$

Let the $(\mathscr{S},\psi)$-stochastic integrals belonging to X be denoted by the letter "J". Then, by definition,

$$I^{(\mathscr{S},\psi)}(B_{s_U^U}(Y)) = I^{(\mathscr{S},\psi,s_U^U)}(Y) = (s_U^U)^{1/2 m(\mathscr{S},\psi)} J^{(\mathscr{S},\psi)}(Y), \qquad Y \in L_{(\mathscr{S},\psi)}^{s_U^U}, \ (\mathscr{S},\psi) \in \Psi_{\bar{U}}.$$

Use this to translate Ito's formula for X (Theorem 3.2) into the language of W:

(4.6) $$\begin{aligned} \Delta_{]s^U,t]}^{\bar{U}} f(W) &= \Delta_{]s^U,t]_{\bar{U}}} g(X) \\ &= \sum_{(\mathscr{S},\psi) \in \Psi_{\bar{U}}} \frac{1}{2^{|\mathscr{S}^0|}} J^{(\mathscr{S},\psi)}((1_{\Omega \times]s^U,t]_{\bar{U}}} D^{(\mathscr{S},\psi)} g(X))^{\mathscr{S}}) \\ &= \sum_{(\mathscr{S},\psi) \in \Psi_{\bar{U}}} \frac{1}{2^{|\mathscr{S}^0|}} I^{(\mathscr{S},\psi)}([(1_{\Omega \times R_t} D^{(\mathscr{S},\psi)} f(W))^{\{U\} \cup \mathscr{S}}](.,.,s^U)). \end{aligned}$$

Next observe that for $U \subset T \in \Pi_N$ the set $\mathbb{I}_{\mathscr{U}}$ can be written as the $(\lambda^N)^{|\mathscr{U}|}$-a.s. (pairwise disjoint) union of $\mathbb{I}_{\mathscr{T}}$, $\mathscr{U} < \mathscr{T}$, $\mathscr{T} \in \tau_T$. Thus, using (4.4), we obtain for ρ-a.e. $t \in \mathbb{I}$

(4.7) $$\begin{aligned} & I^{(\mathscr{U},\chi)}((1_{\Omega \times R_t} \Delta_{].,t]}^{\bar{U}} f(W))^{\mathscr{U}}) \\ &= I^{(\mathscr{U},\chi)}([1_{\Omega \times R_t} \sum_{U \subset T \in \Pi_N} (-1)^{|T|-|U|} f(W_{(.,t_{\bar{T}})})]^{\mathscr{U}}) \\ &= \sum_{U \subset T \in \Pi_N} (-1)^{|T|-|U|} \sum_{(\mathscr{U},\chi) < (\mathscr{T},\phi) \in \Psi_T} I^{(\mathscr{T},\phi,t_{\bar{T}})}([1_{\Omega \times (R_t)_T} f(W_{(.,t_{\bar{T}})})]^{\mathscr{T}}). \end{aligned}$$

Now combine (4.6) and (4.7) in the following way to complete the proof: from the corollary of Lemma 5, conclude, that there are $X^{(\mathcal{T},\phi)} \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$, such that

$$X_t^{(\mathcal{T},\phi)} = I^{(\mathcal{T},\phi,t_{\underline{\mathcal{T}}})}([1_{\Omega \times (R_t)_{\underline{\mathcal{T}}}} f(W_{(.,t_{\underline{\mathcal{T}}})})]^{\mathcal{T}})$$
$$\text{for } \rho\text{-a.e. } t \in \mathrm{II}, (\mathcal{U}, \chi) < (\mathcal{T}, \phi) \in \Psi;$$

finally apply Fubini's theorem for $\mathcal{S} \cup \mathcal{U}$-corner functions to (4.6) and compare. ⌟

Lemma 6 is the corner stone in the proof of the following modification of Theorem 3.

Theorem 4. *Let $f \in C^{2N}(\mathbb{R}^d)$ be such, that*

$$D^{(\mathcal{T},\phi)} f(W) \in L^2(\Omega \times \mathrm{II}, \mathfrak{P}, P \times \lambda^N), \qquad (\mathcal{T}, \phi) \in \Psi_N.$$

Let ρ on $\mathfrak{B}(\mathrm{II})$ be a product of finite measures ρ_i, $1 \leqq i \leqq N$, such that

$$(4.8) \qquad \int_{\mathrm{II}_{\underline{\mathcal{T}}}} [\| D^{(\mathcal{T},\phi)} f(W_{(.,t_{\underline{\mathcal{T}}})})^{\mathcal{T}} \|_{(\mathcal{T},\phi)}^{t_{\underline{\mathcal{T}}}}]^2 \, d\rho_{\underline{\mathcal{T}}}(t_{\underline{\mathcal{T}}}) < \infty, \qquad (\mathcal{T}, \phi) \in \Psi.$$

For $(\mathcal{T}, \phi) \in \Lambda$ define

$$\alpha_{(\mathcal{T},\phi)} = \prod_{T \in \mathcal{T}^0} (|T|-1)(-1)^{|\underline{\mathcal{T}}|-1} \sum_{0 \leqq i \leqq |\mathcal{T}|} \sum_{i \leqq k \leqq |\mathcal{T}|} (-1)^{|\mathcal{T}|-i} \binom{k}{i} i^{|\mathcal{T}|}.$$

Then:

for each $(\mathcal{T}, \phi) \in \Lambda$ there exists $X^{(\mathcal{T},\phi)} \in \mathcal{M}(\mathfrak{F} \times \mathfrak{B}(\hat{\mathrm{II}}^2), \mathfrak{B}(\mathbb{R}))$ such that

i) $X^{(\mathcal{T},\phi)}(.,s,.) \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$, $s \in \mathrm{II}$,

ii) $$X^{(\mathcal{T},\phi)}(.,s,t) = \Delta_{]s,t]_{\underline{\mathcal{T}}}} I^{(\mathcal{T},\phi,.)}([1_{\Omega \times]s,t]_{\underline{\mathcal{T}}}} D^{(\mathcal{T},\phi)} f(W_{(.,.)})]^{\mathcal{T}})$$
$$\textit{for } \rho \times \rho\text{-a.e. } (s,t) \in \hat{\mathrm{II}}^2,$$

iii) $$\Delta_{]s,t]} f(W) = \sum_{(\mathcal{T},\phi) \in \Lambda} \alpha_{(\mathcal{T},\phi)} \frac{1}{2^{|\mathcal{T}^0|}} X^{(\mathcal{T},\phi)}(.,s,t) + \frac{1}{2^N} \int_{]s,t]} \mathbb{D}^N f(W)_u \prod_{1 \leqq i \leqq N} u_i^{N-1} \, du$$
$$\textit{for } \rho \times \rho\text{-a.e. } (s,t) \in \hat{\mathrm{II}}^2.$$

Remarks. 1. According to the hypothesis, (4.8) is satisfied for $\rho = \lambda^N$.

2. For $(\mathcal{T}, \phi) \in \Lambda$ we have $m(\mathcal{T}, \phi) \leqq N$. This means that the orders of differentiation corresponding to $X^{(\mathcal{T},\phi)}$ in iii) do not exceed N.

3. According to ii) and the definition of $I^{(\mathcal{T},\phi,t)}$, $X^{(\mathcal{T},\phi)}$ may be considered as a stochastic integral on the surface of $]s,t]_{\underline{\mathcal{T}}}$.

Proof. As ρ is a product measure, a (classical) Fubini argument shows, that the assertion follows, once we have established: for each $(\mathcal{T}, \phi) \in \Lambda$ there exists $X^{(\mathcal{T},\phi)} \in \mathcal{M}(\mathfrak{P}, \mathfrak{B}(\mathbb{R}))$ such that ii) and iii) are valid with $s = 0$. Regarding Lemma 6, it is enough to prove

$$(4.9)\quad \Delta_{R_t} f(W) = \sum_{(\mathcal{T},\phi)\in\Lambda} \alpha_{(\mathcal{T},\phi)} \frac{1}{2^{|\mathcal{T}^0|}} I^{(\mathcal{T},\phi,t_{\underline{\mathcal{T}}})}([1_{\Omega\times(R_t)_{\underline{\mathcal{T}}}} D^{(\mathcal{T},\phi)} f(W_{(.,t_{\underline{\mathcal{T}}})})]^{\mathcal{T}})$$
$$+\frac{1}{2^N}\int_{R_t} \mathbb{D}^N f(W)_u \prod_{1\le i\le N} u_i^{N-1}\,du \qquad \text{for } \rho\text{-a.e. } t\in\mathbb{I}.$$

Pick $t\in\mathbb{I}$ such that $\|D^{(\mathcal{T},\phi)} f(W_{(.,t_{\underline{\mathcal{T}}})})^{\mathcal{T}}\|^{t_{\underline{\mathcal{T}}}}_{(\mathcal{T},\phi)}<\infty$ for all $(\mathcal{T},\phi)\in\Lambda$. According to (4.8), this is true for ρ-a.e. $t\in\mathbb{I}$.

For $\mathcal{T}\in\tau$, $k\in\mathbb{N}_0$ put

$$\beta_{\mathcal{T},k}=|\{\mathcal{S}\colon \mathcal{S} \text{ is a partition of } \mathcal{T}, |\mathcal{S}|=k+1\}|.$$

Further, let

$$h\colon \Psi_N\to\Lambda, (\mathcal{T},\phi)\to(\mathcal{S},\psi), \qquad \text{where } \mathcal{S}=\{T\in\mathcal{T}: |\phi(T)|\neq 0, \text{ if } |T|=1\},\ \psi=\phi|_{\mathcal{S}}.$$

Then, by induction on k,

$$(4.10)\quad \frac{1}{2^N}\int_{R_t} \mathbb{D}^N f(W)_u \prod_{1\le i\le N} u_i^{N-1}\,du$$
$$=\Delta_{R_t} f(W) + \sum_{(\mathcal{U},\chi)\in\Lambda} \frac{1}{2^{|\mathcal{U}^0|}}\Big(\sum_{0\le j\le k-1}(-1)^{j-1}\beta_{\mathcal{U},j}\Big)$$
$$\cdot \sum_{(\mathcal{U},\chi)<(\mathcal{T},\phi)\in\Lambda} (-1)^{|\underline{\mathcal{T}}|-|\underline{\mathcal{U}}|} I^{(\mathcal{T},\phi,t_{\underline{\mathcal{T}}})}([1_{\Omega\times(R_t)_{\underline{\mathcal{T}}}} D^{(\mathcal{T},\phi)} f(W_{(.,t_{\underline{\mathcal{T}}})})]^{\mathcal{T}})$$
$$+(-1)^{k-1}\sum_{(\mathcal{V},\eta)\in\Psi_N} \frac{1}{2^{|\mathcal{V}^0|}}\beta_{h_1(\mathcal{V},\eta),k} I^{(\mathcal{V},\eta)}((1_{\Omega\times R_t} D^{(\mathcal{V},\eta)} f(W))^{\mathcal{V}}).$$

To argue "$k\to k+1$", note that h is one-to-one and fix $(\mathcal{U},\chi)\in\Lambda\setminus\Lambda_N$. Apply Lemma 6 with $\rho=\varepsilon_{\{t\}}$ to get

$$\beta_{\mathcal{U},k}\frac{1}{2^{h_1^{-1}(\mathcal{U},\chi)}} I^{h^{-1}(\mathcal{U},\chi)}([1_{\Omega\times R_t} D^{h^{-1}(\mathcal{U},\chi)} f(W)]^{h_1^{-1}(\mathcal{U},\chi)})$$
$$=\beta_{\mathcal{U},k}\frac{1}{2^{|\mathcal{U}^0|}}\sum_{(\mathcal{U},\chi)<(\mathcal{T},\phi)\in\Lambda}(-1)^{|\underline{\mathcal{T}}|-|\underline{\mathcal{U}}|} I^{(\mathcal{T},\phi,t_{\underline{\mathcal{T}}})}([1_{\Omega\times(R_t)_{\underline{\mathcal{T}}}} D^{(\mathcal{T},\phi)} f(W_{(.,t_{\underline{\mathcal{T}}})})]^{\mathcal{T}})$$
$$-\sum_{(\mathcal{V},\eta)\in\Psi_N,\, h(\mathcal{V},\eta)\subsetneqq(\mathcal{U},\chi)} \beta_{\mathcal{U},k}\frac{1}{2^{|\mathcal{V}^0|}} I^{(\mathcal{V},\eta)}((1_{\Omega\times R_t} D^{(\mathcal{V},\eta)} f(W))^{\mathcal{V}}).$$

Now sum this equation over $(\mathcal{U},\chi)\in\Lambda$ and observe

$$\sum_{h(\mathcal{V},\eta)\subsetneqq(\mathcal{U},\chi)} \beta_{\mathcal{U},k}=\beta_{h_1(\mathcal{V},\eta),k+1}.$$

Since for $k\ge N-1$ the last term on the right side of (4.10) vanishes, we are left with the following assertion

$$(4.11)\qquad \alpha_{(\mathcal{T},\phi)}=-\sum_{\Lambda\ni(\mathcal{U},\chi)<(\mathcal{T},\phi)}(-1)^{|\underline{\mathcal{T}}|-|\underline{\mathcal{U}}|}\Big(\sum_{0\le j\le|\mathcal{T}|}(-1)^{j-1}\beta_{\mathcal{T},j}\Big),$$
$$(\mathcal{T},\phi)\in\Lambda, \qquad \beta_{\mathcal{T},j}=\beta_{\mathcal{U},j}, \text{ if } \mathcal{U}<\mathcal{T}).$$

On one hand, by the general addition theorem and the polynomial theorem, we have for $0 \leqq j \leqq |\mathscr{T}|$

$$\beta_{\mathscr{T},j} = \sum_{l \in \mathbb{N}^{j+1}, |l| = |\mathscr{T}|} \frac{|\mathscr{T}|!}{l!} = \sum_{S \in \Pi_{j+1}} (-1)^{|S|} \sum_{l \in \mathbb{N}_0^{j+1-|S|}, |l| = |\mathscr{T}|} \frac{|\mathscr{T}|!}{l!}$$
$$= \sum_{0 \leqq i \leqq j+1} (-1)^i \binom{j+1}{i} i^{|\mathscr{T}|}.$$

On the other hand, by the binomial theorem,

$$\sum_{A \ni (\mathscr{U}, \chi) < (\mathscr{T}, \phi)} (-1)^{|\mathscr{T}| - |\mathscr{U}|}$$
$$= \prod_{T \in \mathscr{T}^0} \Big(\sum_{S \subset T, |S| \geqq 2} (-1)^{|T| - |S|} \Big) \prod_{T \in \mathscr{T}^1} \Big(\sum_{S \subset T, |S| \geqq 1} (-1)^{|T| - |S|} \Big)$$
$$= (-1)^{|\mathscr{Z}| - |\mathscr{T}|} \prod_{T \in \mathscr{T}^0} (|T| - 1).$$

Hence (4.11) follows. $\lrcorner$

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