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Transformations of the Brownian Motion on a Riemannian Symmetric Space

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1. Introduction

A famous result by Cameron-Martin [2] states that, for a d-dimensional Brownian motion $B(t)$ with $B(0)=0$ on a finite time interval $[0,T]$ and a continuous function a: $[0,T]\to\mathbf{R}^d$, the processes $(B(t);\ 0\leqq t\leqq T)$ and $(B(t)+a(t);\ 0\leqq t\leqq T)$ are *equivalent* in the sense that the laws on the path space $W^d=C([0,T]\to\mathbf{R}^d)$ are mutually absolutely continuous if and only if $a(0)=0$ and all components of a are absolutely continuous with square-integrable derivatives. A generalization of this result is given as follows: let $u=(u^\alpha_\beta(t))$ be an $SO(d)$-valued continuous function defined on $(0,T]$ ($SO(d)$ is the set of all orthogonal matrices of order d with determinant 1), $B(t)$ and $a=(a^\alpha(t))$ be as above and ask when two processes $(B(t);\ 0\leqq t\leqq T)$ and $(\eta(t);\ 0\leqq t\leqq T)$ defined by $\eta^\alpha(t)=\sum_{\beta=1}^{d} u^\alpha_\beta(t)B^\beta(t)+a^\alpha(t)$, $\alpha=1,\ldots,d$, are equivalent. As we shall see in Sect. 5, they are equivalent if and only if a satisfies the same condition as above and u satisfies that all components of u are absolutely continuous on $(0,T]$ and $\int_0^T t\{\dot{u}^\alpha_\beta(t)\}^2\,dt<\infty$, $\alpha,\beta=1,2,\ldots,d$.

Since $\mathbf{R}^d$ is a Riemannian symmetric space and $x\mapsto ux+a$, $u\in SO(d)$, $a\in\mathbf{R}^d$, is an isometric transformation of $\mathbf{R}^d$ it might be natural to generalize the problem to Brownian motions on Riemannian symmetric spaces in the following manner. Let M be a d-dimensional *Riemannian symmetric space* and take an arbitrary point $o\in M$ and fix it. The *Brownian motion* on M is the diffusion on M which is generated by $\frac{1}{2}\Delta$, Δ being the Laplace-Beltrami operator on M. Let $X=(X_t;0\leqq t\leqq T)$ be the Brownian motion on M with $X_0=o$ given on a finite time interval $[0,T]$ and let $g=(g_t)$ be a continuous map $(0,T]\to G$ where G is the connected component containing the identity e of the Lie group $I(M)$ formed of all isometries of M. We want to know when $X=(X_t;\ 0\leqq t\leqq T)$

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and $Y=(Y_t=g_tX_t,\ 0<t\leqq T,\ Y_0=o)$ are equivalent. The purpose of the present paper is to answer this question and our result is the following. Let $\mathfrak{g}$ be the Lie algebra of G an element of which, as usual, is identified with a left invariant vector field over G. Let $\mathfrak{g}=\mathfrak{m}+\mathfrak{k}$ be the usual direct sum decomposition (see Sect. 3 for the definition) and let $\{A_1,\ldots,A_d\}$ and $\{A_{d+1},\ldots,A_n\}$ be bases of $\mathfrak{m}$ and $\mathfrak{k}$ respectively ($n=\dim(G)$). We assume that

$$\lim_{t\to 0} g_t o = o \tag{A.1}$$

which is clearly necessary in order that X and Y are equivalent. We further assume that

$$t\mapsto g_t \quad \textit{is absolutely continuous on every closed interval of } (0,T]. \tag{A.2}$$

Under *(A.2)*, we can define the coefficient ζ_t^I of $\dot{g}_t\in T_{g_t}(G)$ with respect to the basis $A_I(g_t)=(A_I)_{g_t}\in T_{g_t}(G)$ for almost all $t\in(0,T]$, $I=1, 2, \ldots, n$, that is, we can write the differential equation for g_t as

$$dg_t=\sum_{I=1}^{n} \zeta_t^I A_I(g_t)\,dt \qquad 0<t\leqq T$$

and we assume furthermore that

$$\begin{aligned}&\text{(i)} \int_0^T (\zeta_t^\alpha)^2\,dt<\infty, \qquad \alpha=1,\ldots,d,\\ &\text{(ii)} \int_0^T t(\zeta_t^i)^2\,dt<\infty, \qquad i=d+1,\ldots,n.\end{aligned} \tag{A.3}$$

Our main result in this paper now can be stated as follows: *under the assumptions (A.1) (A.2) and (A.3), the processes X and Y are equivalent.* We can also obtain an explicit formula of the Radon-Nikodym derivative of the law of Y with respect to the law of X. We could not show that these conditions are also necessary for the equivalence of X and Y but, in view of the above result in the Euclidean case, they look like to be almost necessary.

Our method in obtaining the above result is as follows. First we construct our Brownian motion X on the Riemannian symmetric space M by constructing a left Brownian motion on the Lie group G and lift the equivalence problem for processes on M to that for the processes on G. Writing the stochastic equations for these processes on G with respect to the basis $\{A_I,\ I=1,2,\ldots,n\}$, of the left invariant vector fields, it is further reduced to the equivalence problem for semimartingales on the Euclidean space. Then we can appeal to known results for such problems, e.g., results by Kailath-Zakai [12] and Ershov [5]. In the above reductions of the problem, an important role is played by the Itô-formula for products and inverses of semimartingales on the Lie group G which will be discussed in Sect. 2. These formulas may be regarded as a special case of general formulas obtained by e.g. Kunita [14] and Bismut [1] for the composites of stochastic flows of diffeomorphisms: our formulas in the case of a Lie group can be given more explicitly, however.

2. Stochastic Differential Equations on Lie Groups

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, $(\mathscr{F}_t)_{t \geqq 0}$ be a right continuous increasing family of sub σ-fields of $\mathscr{F}$ such that $\mathscr{F}_0$ contains all P-null sets. Let $B=(B_t;\ 0 \leqq t < \infty)$ be a d-dimensional $(\mathscr{F}_t)$-Brownian motion starting at 0 (we follow the terminology in Ikeda-Watanabe [8]). Let M be a manifold. We will assume that all manifolds discussed in this paper are always C^∞, connected and σ-compact. Let $A_0, A_1, \ldots, A_d$ be C^∞ vector fields on M. We denote the totality of C^∞ vector fields on M by $\Gamma^\infty(T(M))$. Let us consider the following stochastic differential equation on M;

$$\begin{aligned} dX_t &= \sum_{\alpha=1}^{d} A_\alpha(X_t) \circ dB_t^\alpha + A_0(X_t)\,dt \\ X_0 &= x \in M. \end{aligned} \tag{2.1}$$

The meaning of the equation (2.1) is that $X=(X_t)$ is a continuous $(\mathscr{F}_t)$-adapted process such that for any C^∞-function f on M with compact support, it holds that

$$f(X_t) - f(X_0) = \sum_{\alpha=1}^{d} \int_0^t (A_\alpha f)(X_s) \circ dB_s^\alpha + \int_0^t (A_0 f)(X_s)\,ds \tag{2.2}$$

where the symbol $\circ$ means the Fisk-Stratonovich symmetric integral. It is well-known that (2.1) has a unique strong solution up to the explosion time.

Let M and M' be C^∞ manifolds and ϕ be a C^∞ mapping from M into M'. Suppose that $A_0, A_1, \ldots, A_d$ are in $\Gamma^\infty(T(M))$ and $A'_0, A'_1, \ldots, A'_d$ are in $\Gamma^\infty(T(M'))$ such that

$$(\phi_*)_x (A_\alpha)_x = (A'_\alpha)_{\phi(x)} \qquad \alpha = 0, 1, \ldots, d$$

for any $x \in M$ where $(\phi_*)_x$ is the differential of ϕ at x. Let $X=(X_t)$ and $Y=(Y_t)$ be the solutions of the following stochastic differential equations on M and M' respectively;

$$\begin{aligned} dX_t &= \sum_{\alpha=1}^{d} A_\alpha(X_t) \circ dB_t^\alpha + A_0(X_t)\,dt \\ X_0 &= x \in M, \end{aligned} \tag{2.3}$$

$$\begin{aligned} dY_t &= \sum_{\alpha=1}^{d} A'_\alpha(Y_t) \circ dB_t^\alpha + A'_0(Y_t)\,dt \\ Y_0 &= \phi(x) \in M'. \end{aligned} \tag{2.4}$$

Then the following lemma is easily obtained from the definition.

Lemma 2.1. *Suppose that ϕ is surjective or (X_t) has an infinite explosion time. Let τ_X and τ_Y be explosion times of (X_t) and (Y_t) respectively. Then $\tau_X \leqq \tau_Y$* a.e. *and $Y_t = \phi(X_t)$ for all t up to τ_X* a.e.

Let M be a manifold and G be a Lie group acting on M on the right. The action of an element g in G is denoted by $x \mapsto xg$ for $x \in M$. Let $\mathfrak{g}$ be a Lie algebra of G, i.e., the set of all left invariant vector fields on G. For each $A \in \mathfrak{g}$, the 1-parameter subgroup $\{\exp tA; t \in \mathbf{R}\}$ induces a vector field on M, which we denote by A^*. The mapping from $\mathfrak{g}$ into $\Gamma^\infty(T(M))$ which sends A into A^* is a Lie algebra homomorphism (see, e.g., Kobayashi-Nomizu [13], I.4.). Let us consider the following stochastic differential equations on G and M respectively;

$$\begin{aligned} dg_t &= \sum_{\alpha=1}^{d} A_\alpha(g_t) \circ dB_t^\alpha + A_0(g_t)\,dt \\ g_0 &= e \end{aligned} \tag{2.5}$$

$$\begin{aligned} dX_t &= \sum_{\alpha=1}^{d} A_\alpha^*(X_t) \circ dB_t^\alpha + A_0^*(X_t)\,dt \\ X_0 &= x \end{aligned} \tag{2.6}$$

where $A_0, A_1, \ldots, A_d$ are in $\mathfrak{g}$. Define the mapping ϕ: $G \to M$ by $\phi(g) = xg$. Since $(\phi_*)_g(A_\alpha)_g = (A_\alpha^*)_{\phi(g)}$ $\alpha = 0, 1, \ldots, d$, $g \in G$, and (g_t) is conservative (see, e.g., Mc-Kean [19]), the following corollary is an easy consequence from Lemma 2.1.

Corollary 2.1. *If* (g_t) *and* (X_t) *are solutions of* (2.5) *and* (2.6) *respectively, then* $X_t = xg_t$ *for all* $t \geqq 0$ a.e.

The solution of (2.5) is called the left Brownian motion on G (c.f. [19]). We will discuss the processes on the Lie group G in the remainder of this section. We will generalize (2.5) to the case that $A_0, A_1, \ldots, A_d$ are random and depend on the time t. More precisely, let $(\xi_{0,t})_{t \geqq 0}$ be $\mathfrak{g}$-valued process such that each component is in L_1^{loc} where L_1^{loc} is the set of all measurable and $(\mathscr{F}_t)$-adapted processes $(\Phi_t)_{t \geqq 0}$ such that

$$P\left[\int_0^t |\Phi_s|\,ds < \infty \text{ for all } t \geqq 0\right] = 1$$

and let $(\xi_{\alpha,t})_{t \geqq 0}$ $\alpha = 1, \ldots, d$ be $\mathfrak{g}$-valued continuous semimartingales. Here a continuous semimartingale is a process which can be represented as a sum of a continuous local martingale and a continuous process of bounded variation. $\mathfrak{g}$-valued continuous semimartingale is a $\mathfrak{g}$-valued continuous process such that each component is a continuous semimartingale. Since all the semimartingales which we discuss in this paper are continuous, we sometimes omit the adjective "continuous". We consider the following stochastic differential equation on G;

$$\begin{aligned} dg_t &= \sum_{\alpha=1}^{d} \xi_{\alpha,t}(g_t) \circ dB_t^\alpha + \xi_{0,t}(g_t)\,dt \\ g_0 &= g \in G. \end{aligned} \tag{2.7}$$

Lemma 2.2. *The stochastic differential equation* (2.7) *has a unique conservative solution.*

Proof. The proof is similar to that of McKean [19]. Let us consider the following stochastic differential equation on G;

$$\begin{aligned} dg_t &= \sum_{\alpha=1}^{r} A_\alpha(g_t) \circ dM_t^\alpha \\ g_0 &= e \end{aligned} \tag{2.8}$$

where $A_1, \ldots, A_r$ are in $\mathfrak{g}$ and $M^\alpha = (M_t^\alpha)$ $(\alpha = 1, \ldots, r)$ are semimartingales. First assume that G is a subgroup of $GL(m, \mathbf{R})$ for some integer m and hence $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(m, \mathbf{R})$, the Lie algebra of $GL(m, \mathbf{R})$. Since $GL(m, \mathbf{R})$ is an open submanifold of R^{m^2}, there exist the natural coordinates $(x_j^i;\ i,j = 1, \ldots, m)$. Hence each element A in $\mathfrak{g}$ corresponds to an $m \times m$ matrix $(A_j^i)_{i,j=1,\ldots,m}$ in the following manner;

$$A = \sum_{i,j,k=1}^{m} x_j^i A_k^j \frac{\partial}{\partial x_k^i}.$$

Then the Eq. (2.8) can be written in the component form as follows;

$$\begin{aligned} dg_{j,t}^i &= \sum_{\alpha=1}^{r} \sum_{k=1}^{m} g_{k,t}^i (A_\alpha)_j^k \, dM_t^\alpha + \tfrac{1}{2} \sum_{\alpha,\beta=1}^{r} \sum_{k,l=1}^{m} g_{k,t}^i (A_\alpha)_l^k (A_\beta)_j^l \, d\langle M^\alpha, M^\beta \rangle_t \\ g_{j,0}^i &= \delta_j^i \qquad i,j = 1, \ldots, m. \end{aligned} \tag{2.9}$$

Here, $(A_\alpha)_j^k$ $k,j = 1, \ldots, m$, are the components of a matrix corresponding to A_α, $\langle M^\alpha, M^\beta \rangle$ $\alpha, \beta = 1, \ldots, r$, are quadratic variational processes of M^α and M^β and δ_j^i is the Kronecker delta. It is well-known that (2.9) has a unique conservative solution (see, e.g., Ikeda-Watanabe [8], Chap. III, Th. 2.1.) and hence (2.8) has a unique conservative solution.

Secondly we consider a general Lie group G. By Ado's theorem there exist an integer m and a Lie subgroup G' of $GL(m, \mathbf{R})$ such that the Lie algebra $\mathfrak{g}'$ of G' is isomorphic to $\mathfrak{g}$. Hence the universal covering groups of G and G' coincide and we denote it by $\tilde{G}$. Let $A'_1, \ldots, A'_r$ be in $\mathfrak{g}'$ which correspond to $A_1, \ldots, A_r$ in $\mathfrak{g}$ under the isomorphism between $\mathfrak{g}$ and $\mathfrak{g}'$. $\tilde{A}_1, \ldots, \tilde{A}_r$ in $\tilde{\mathfrak{g}}$ are similarly defined where $\tilde{\mathfrak{g}}$ is the Lie algebra of $\tilde{G}$. We consider the following stochastic differential equations on G' and $\tilde{G}$ respectively;

$$\begin{aligned} dg'_t &= \sum_{\alpha=1}^{r} A'_\alpha(g'_t) \circ dM_t^\alpha \\ g'_0 &= e \end{aligned} \tag{2.8$'$}$$

$$\begin{aligned} d\tilde{g}_t &= \sum_{\alpha=1}^{r} \tilde{A}_\alpha(\tilde{g}_t) \circ dM_t^\alpha \\ \tilde{g}_0 &= e. \end{aligned} \tag{2.8$''$}$$

The above result implies that there exists a unique conservative solution of (2.8)′ on G' and by lifting it we obtain a solution of (2.8)″. Then a solution of (2.8) on G is obtained by projecting the solution of (2.8)″. The uniqueness of (2.8) is easy to see. Thus (2.8) has a unique conservative solution. Note that if (g_t) is a solution of (2.8), then for g in G, (gg_t) is a solution of the same equation (2.8) with the initial condition, however, replaced by g.

Now we go back to the equation (2.7). Let $\{A_1, \dots, A_n\}$ be a basis of $\mathfrak{g}$ where n is the dimension of G. If we write

$$\xi_{\alpha,t} = \sum_{i=1}^{n} \xi^i_{\alpha,t} A_i \qquad \alpha = 0, 1, \dots, d,$$

then $(\xi^i_{0,t})$ is in L_1^{loc} and $(\xi^i_{\alpha,t})$ $\alpha = 1, \dots, d$ are semimartingales. Define semimartingales $M^i = (M^i_t)$ $i = 1, \dots, n$ by

$$M^i_t = \sum_{\alpha=1}^{d} \int_0^t \xi^i_{\alpha,s} \circ dB^\alpha_s + \int_0^t \xi^i_{0,s}\, ds.$$

Then (2.7) is equivalent to the following stochastic differential equation on G;

$$\begin{aligned} dg_t &= \sum_{i=1}^{n} A_i(g_t) \circ dM^i_t \\ g_0 &= g. \end{aligned} \tag{2.10}$$

Therefore we can conclude by the above argument that (2.7) has a unique conservative solution. □

Let (g_t) and (h_t) be G-valued continuous processes. If we set $k_t = g_t h_t$, then we have a new G-valued continuous process (k_t). Assume that (g_t) and (h_t) satisfy some stochastic differential equations. Then we can obtain the stochastic differential equation of (k_t) as follows which may be considered as a particular case of more general results due to Kunita ([14], Prop. 4.2) and Bismut ([1], Th. 2.3).

Proposition 2.1. *Let $(\xi_{0,t})$ and $(\eta_{0,t})$ be $\mathfrak{g}$-valued processes all of whose components are in L_1^{loc} and $(\xi_{\alpha,t})$, $(\eta_{\alpha,t})$ $\alpha = 1, \dots, d$ be $\mathfrak{g}$-valued semimartingales. Let (g_t) and (h_t) be the solution of the following stochastic differential equation on G respectively:*

$$\begin{aligned} dg_t &= \sum_{\alpha=1}^{d} \xi_{\alpha,t}(g_t) \circ dB^\alpha_t + \xi_{0,t}(g_t)\, dt \\ g_0 &= g \end{aligned} \tag{2.11}$$

$$\begin{aligned} dh_t &= \sum_{\alpha=1}^{d} \eta_{\alpha,t}(h_t) \circ dB^\alpha_t + \eta_{0,t}(h_t)\, dt \\ h_0 &= h, \end{aligned} \tag{2.12}$$

where g and h are in G. If we put $k_t = g_t h_t$, then the stochastic process (k_t) satisfies the following stochastic differential equation on G;

$$\begin{aligned} dk_t &= \sum_{\alpha=1}^{d} (\mathrm{Ad}(h_t^{-1}) \xi_{\alpha,t} + \eta_{\alpha,t})(k_t) \circ dB^\alpha_t \\ &\quad + (\mathrm{Ad}(h_t^{-1})\, \xi_{0,t} + \eta_{0,t})(k_t)\, dt \\ k_0 &= gh \end{aligned} \tag{2.13}$$

where Ad: $G \to GL(\mathfrak{g})$ *is the adjoint representation of* G. *Moreover, if* $\eta_{\alpha,t} = \sum_{i=1}^{n} \eta_{\alpha,t}^{i} A_i$ $\alpha = 1, \ldots, d$ *and* $\mathrm{Ad}(h_t^{-1}) A_j = \sum_{i=1}^{n} \mathrm{Ad}(h_t^{-1})_j^i A_i$ *by fixing a basis* $\{A_1, \ldots, A_n\}$ *in* $\mathfrak{g}$, *then* $(\mathrm{Ad}(h_t^{-1})_j^i)_{i,j=1,\ldots,n}$ *satisfies the following stochastic differential equation;*

$$\begin{aligned} d\,\mathrm{Ad}(h_t^{-1})_j^i &= \sum_{\alpha=1}^{d} \sum_{k,l=1}^{n} C_{kl}^i \,\mathrm{Ad}(h_t^{-1})_j^k \eta_{\alpha,t}^l \circ dB_t^\alpha \\ &\quad + \sum_{k,l=1}^{n} C_{kl}^i \,\mathrm{Ad}(h_t^{-1})_j^k \eta_{0,t}^l \, dt \qquad (2.14) \\ \mathrm{Ad}(h_0^{-1})_j^i &= \mathrm{Ad}(h^{-1})_j^i \end{aligned}$$

where C_{kl}^i*'s are the structure constants, i.e.,*

$$[A_k, A_l] = \sum_{i=1}^{n} C_{kl}^i A_i.$$

Proof. It is enough to prove the first assertion only locally. Let $(x^1, \ldots, x^n)$, $(y^1, \ldots, y^n)$ and $(z^1, \ldots, z^n)$ be local coordinates on neighborhoods of g, h and gh respectively. Define the mapping F^i by $F^i(x, y) = z^i(xy)$. By the Itô formula, we have

$$\begin{aligned} dF^i(g_t, h_t) &= \sum_{\alpha=1}^{d} \{((\xi_{\alpha,t})_x F^i)(g_t, h_t) + ((\eta_{\alpha,t})_y F^i)(g_t, h_t)\} \circ dB_t^\alpha \\ &\quad + \{((\xi_{0,t})_x F^i)(g_t, h_t) + ((\eta_{0,t})_y F^i)(g_t, h_t)\}\, dt \end{aligned}$$

where $(\xi_{\alpha,t})_x$ and $(\eta_{\alpha,t})_y$ stand for the differentials with respect to the first variable and the second variable respectively. But generally we have that, for $A \in \mathfrak{g}$,

$$\begin{aligned} A_x F^i(x, y) &= A_x(z^i(xy)) = \frac{d}{dt} z^i(x \exp tA\, y)|_{t=0} \\ &= A(z^i \circ R_y)(x) = ((R_{y*} A) z^i)(xy) = \mathrm{Ad}(y^{-1}) A z^i(xy) \end{aligned}$$

since $R_{y*} = \mathrm{Ad}(y^{-1})$ where R_y: $G \to G$ is the mapping defined by $R_y(g) = gy$. Similarly we have

$$A_y F^i(x, y) = A_y(z^i(xy)) = \frac{d}{dt} z^i(xy \exp tA)|_{t=0} = A z^i(xy).$$

Thus we have

$$\begin{aligned} dz^i(g_t h_t) &= \sum_{\alpha=1}^{d} (\mathrm{Ad}(h_t^{-1}) \xi_{\alpha,t} + \eta_{\alpha,t}) z^i(g_t h_t) \circ dB_t^\alpha \\ &\quad + (\mathrm{Ad}(h_t^{-1}) \xi_{0,t} + \eta_{0,t}) z^i(g_t h_t)\, dt \end{aligned}$$

and this proves (2.13).

To show the second assertion, note that G acts on $\mathfrak{g}$ on the right in the following manner; $Ag = \mathrm{Ad}(g^{-1})A$ for $A \in \mathfrak{g}$ and $g \in G$. From the remark after Lemma 2.1, there exists a Lie algebra homomorphism $A \mapsto A^*$ from $\mathfrak{g}$ into

$\Gamma^\infty(T(\mathfrak{g}))$ such that if we define $\sigma_X: G \to \mathfrak{g}$ for $X \in \mathfrak{g}$ by $\sigma_X(g) = Xg$, then $(\sigma_X)_* A = A^*$ for $A \in \mathfrak{g}$. Since $\{A_1, \ldots, A_n\}$ is a basis of $\mathfrak{g}$, we can represent $X \in \mathfrak{g}$ as $X = \sum_{i=1}^n X^i A_i$ and $(X^1, \ldots, X^n)$ forms a system of local coordinates in $\mathfrak{g}$. Since $\mathfrak{g}$ is a vector space, we can identify $T_X(\mathfrak{g})$ and $\mathfrak{g}$ for any $X \in \mathfrak{g}$. We denote this isomorphism from $\mathfrak{g}$ into $T_X(\mathfrak{g})$ by ι_X. Then we have for $X, Y \in \mathfrak{g}$

$$\begin{aligned}(Y^*)_X &= (\sigma_{X^*})_e Y_e = \frac{d}{dt}\sigma_X(\exp tY)|_{t=0} = \frac{d}{dt}\mathrm{Ad}(\exp(-tY))X|_{t=0} \\ &= \frac{d}{dt}\exp(-t\,\mathrm{ad}\,Y)X|_{t=0} = \iota_X(-\mathrm{ad}\,Y(X)) \\ &= \iota_X\left(\left[\sum_{i=1}^n X^i A_i, \sum_{j=1}^n Y^j A_j\right]\right) = \iota_X\left(\sum_{i,j,k=1}^n X^i Y^j C_{ij}^k A_k\right) \\ &= \sum_{i,j,k=1}^n X^i Y^j C_{ij}^k \left(\frac{\partial}{\partial X^k}\right)_X\end{aligned}$$

where ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$, i.e., $\mathrm{ad}\,X(Y) = [X, Y]$. Thus we have

$$(\eta_{\alpha,t}^*)_X = \sum_{i,j,k=1}^n X^i \eta_{\alpha,t}^j C_{ij}^k \left(\frac{\partial}{\partial X^k}\right)_X.$$

Since Corollary 2.1 also holds for more general stochastic differential equations of type (2.7), we can apply it to the Eq. (2.12) and thus obtain (2.14). □

The following corollary is easily obtained from the above proposition.

Corollary 2.2. *Let $(\xi_{0,t})$ be a $\mathfrak{g}$-valued process such that each component is in L_1^{loc} and $(\xi_{\alpha,t})$ $\alpha = 1, \ldots, d$ be $\mathfrak{g}$-valued semimartingales and let (g_t) be a solution of the following stochastic differential equation on G;*

$$\begin{aligned} dg_t &= \sum_{\alpha=1}^d \xi_{\alpha,t}(g_t) \circ dB_t^\alpha + \xi_{0,t}(g_t)\,dt \\ g_0 &= g. \end{aligned} \tag{2.15}$$

Then $(h_t = g_t^{-1})$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} dh_t &= -\sum_{\alpha=1}^d \mathrm{Ad}(g_t)\,\xi_{\alpha,t}(h_t) \circ dB_t^\alpha - \mathrm{Ad}(g_t)\,\xi_{0,t}(h_t)\,dt \\ h_0 &= g^{-1} \end{aligned} \tag{2.16}$$

Proof. Let (h_t) be a solution of (2.16). Then by Proposition 2.1, we have

$$\begin{aligned} d(h_t g_t) &= \sum_{\alpha=1}^d \{\mathrm{Ad}(g_t^{-1})(-\mathrm{Ad}(g_t)\xi_{\alpha,t})(h_t g_t) + \xi_{\alpha,t}(h_t g_t)\} \circ dB_t^\alpha \\ &\quad + \{\mathrm{Ad}(g_t^{-1})(-\mathrm{Ad}(g_t)\xi_{0,t})(h_t g_t) + \xi_{0,t}(h_t g_t)\}\,dt \\ &= 0 \end{aligned}$$

and $h_0 g_0 = e$. Hence $h_t g_t = e$ for all $t \geqq 0$ a.e. and this completes the proof. □

3. The Brownian Motion on the Riemannian Symmetric Space

Let M be a d-dimensional Riemannian manifold, $O(M)$ be the *orthonormal frame bundle* and let π: $O(M)\to M$ be the natural projection. Let ω be the *Riemannian connection form* on $O(M)$ (see, e.g., [13]) and θ be the *canonical* 1-*form* on $O(M)$, i.e., the $\mathbf{R}^d$-valued 1-form defined by $\theta_u(X)=u^{-1}(\pi_* X)$ for $u\in O(M)$ and $X\in T_u(O(M))$ where we regard an element $u\in O(M)$ as a linear isomorphism from $\mathbf{R}^d$ onto $T_{\pi(u)}(M)$ which preserves the inner product. We associate with each $\xi\in\mathbf{R}^d$ a vector fields $B(\xi)$ on $O(M)$ such that $\omega_u(B(\xi)_u)=0$ and $\theta_u(B(\xi)_u)=\xi$ for $u\in O(M)$. $B(\xi)$ is called a *standard horizontal vector field* corresponding to ξ. Using these notations, we can construct the Brownian motion on M by the following method of moving frames (c.f. [18]). Let (U_t) be the solution of the following stochastic differential equation on $O(M)$;

$$\begin{aligned} dU_t &= \sum_{\alpha=1}^{d} L_\alpha(U_t)\circ dB_t^\alpha \\ U_0 &= u \end{aligned} \tag{3.1}$$

where $L_\alpha=B(e_\alpha)$, $e_\alpha=(0,\dots,0,\overset{\alpha}{\breve{1}},0,\dots,0)$ and $u\in O(M)$. Then $(\pi(U_t))$ is a Brownian motion on M starting at $\pi(u)$.

If in particular M is a Riemannian symmetric space, a Brownian motion on M is constructed from a Brownian motion on a transformation Lie group of M as follows. Let G be an connected component containing e of the isometric transformation group $I(M)$ of M. It is well-known that the Lie group G acts on M on the left transitively. Take any points $o\in M$ and $u_0\in O(M)$ such that $\pi(u_0)=o$ and fix them. Hereafter we consider the Brownian motion on M starting at o. Let K be the isotropy subgroup of G at o and λ be the linear isotropy representation of K, i.e., λ is a homomorphism from K into $GL(T_o(M))$ defined by $\lambda(k)=(k_*)_o$. Then λ induces a homomorphism Λ: $K\to O(d)$ defined by $\Lambda(k)=u_0^{-1}\circ\lambda(k)\circ u_0$ for $k\in K$. Let s be the symmetry at o and σ be an involutive automorphism of G defined by $\sigma(g)=s\circ g\circ s$ for $g\in G$. We can easily see that $\sigma^2=\mathrm{id}_G$ where id_G is the identity mapping of G. Let σ_* be the differential of σ. Then σ_* induces an involutive automorphism of $\mathfrak{g}$ denoted by σ_* also. Since $\sigma_*^2=\mathrm{id}_{\mathfrak{g}}$, $\mathfrak{g}$, as a vector space, is a direct sum of eigenspaces for 1 and -1, denoted by $\mathfrak{k}$ and $\mathfrak{m}$ respectively. Then $A\in\mathfrak{g}$ can be represented uniquely as $A=A_1+A_2$ such that $A_1\in\mathfrak{m}$ and $A_2\in\mathfrak{k}$. We denote A_1 and A_2 by $A_\mathfrak{m}$ and $A_\mathfrak{k}$ respectively. It is known that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is invariant under $\mathrm{Ad}(K)$, i.e., $\mathrm{Ad}(k)\mathfrak{m}\subseteq\mathfrak{m}$ for any $k\in K$. This fact plays an important role later. Since G acts on M, G acts on $O(M)$ in the following manner; $gu=(g_*)_{\pi(u)}\circ u$ for $g\in G$ and $u\in O(M)$. This means that gu is a composite mapping of u which is an inner product preserving linear isomorphism from $\mathbf{R}^d$ onto $T_{\pi(u)}(M)$, and $(g_*)_{\pi(u)}$ which is an inner product preserving linear isomorphism from $T_{\pi(u)}(M)$ onto $T_{g\pi(u)}(M)$. Then G can be regarded as a closed submanifold of $O(M)$ by an inclusion map ι: $G\to O(M)$ defined by $\iota(g)=gu_0$. Since G acts on M and $O(M)$, 1-parameter subgroup $\{\exp tA;\ t\in\mathbf{R}\}$ for $A\in\mathfrak{g}$ defines a vector fields on M and $O(M)$, denoted by A^* and $\tilde{A}$ respectively. We define a linear map L: $\mathfrak{m}\to T_o(M)$ by $L(A)=(A^*)_o$ for $A\in\mathfrak{m}$. It is easy to see that L is a linear isomorphism. Hence there exists an inverse map of L and we denote it by l. For $\xi\in\mathbf{R}^d$ we define $A(\xi)$ in $\mathfrak{m}$ by $A(\xi)=l(u_0(\xi))$. Then it holds that $\imath_* A(\xi)=B(\xi)$ and $\mathrm{Ad}(k)A(\xi)=A(A(k)\xi)$ for $k\in K$.

Now we can construct a Brownian motion on M by using a Brownian motion on the Lie group G. The following proposition is announced in M.P. Malliavin, P. Malliavin [17],

Proposition 3.1. *Let (h_t) be a solution of the following stochastic differential equation on G;*

$$\begin{aligned} dh_t &= \sum_{\alpha=1}^{d} A(e_\alpha)(h_t)\circ dB_t^\alpha \\ h_0 &= e. \end{aligned} \tag{3.2}$$

Then $(X_t=h_t o)$ is a Brownian motion on M starting at o.

The proof is easily obtained from Lemma 2.1. It is obtained at the same time that a Brownian motion on M is conservative.

4. Main Theorems

We keep the same notations as before. In the following we consider processes only on the time interval $[0,T]$ for some fixed constant T, so for example, the Brownian motion (B_t) is defined for $t\in[0,T]$. Without loss of generality we can suppose that the Brownian motion (B_t) is canonically realized on the probability space $(W_0^d, \mathscr{B}(W_0^d), P^W)$, i.e., W_0^d is a set of all continuous paths w: $[0,T]\to\mathbf{R}^d$ such that $w_0=0$, $\mathscr{B}(W_0^d)$ is a Borel σ-field, P^W is the Wiener measure and $B_t(w)=w_t$. Set $\mathscr{B}_t(W_0^d)=\sigma(w_s;\ 0\leqq s\leqq t)$. Let $\overline{\mathscr{B}(W_0^d)}$ be a completion of $\mathscr{B}(W_0^d)$ with respect to P^W and let $\mathscr{N}\subseteqq\overline{\mathscr{B}(W_0^d)}$ be a set of all P^W-null sets. Set $\Omega=W_0^d$, $P=P^W$, $\mathscr{F}=\overline{\mathscr{B}(W_0^d)}$ and $\mathscr{F}_t=\mathscr{B}_t(W_0^d)\vee\mathscr{N}$ for $t\in[0,T]$. In the sequel we will consider on this specific probability space $(\Omega,\mathscr{F},P,(\mathscr{F}_t)_{0\leqq t\leqq T})$. We define a class of processes L_2^{loc} by

$$L_2^{\mathrm{loc}}=\left\{\Phi=(\Phi_t;\ 0\leqq t\leqq T)\,|\,\Phi \text{ is a real measurable } (\mathscr{F}_t)\text{-adapted process on } \Omega \text{ such that } \int_0^T \Phi_t^2\,dt<\infty \text{ a.e.}\right\}.$$

Now we give here a precise formulation of the theorem. Let $t\mapsto g_t$ be a G-valued continuous function on $(0,T]$. Instead of continuity of g_t at $t=0$ we assume the following condition;

$$\lim_{t\to 0} g_t o = o. \tag{A.1}$$

Let $W_o(M)$ be as in Sect. 1. We define the transformation I of $W_o(M)$ by

$$(Iw)_t=\begin{cases} o & \text{if } t=0 \\ g_t w_t & \text{if } 0<t\leqq T. \end{cases} \tag{4.1}$$

Then I is well-defined, i.e., Iw is in $W_o(M)$ by the assumption *(A.1)*. In fact, $\overline{\lim_{t\to 0}} d(g_t w_t, o) \leqq \lim d(g_t w_t, g_t o) + \lim_{t\to 0} d(g_t o, o) = \lim_{t\to 0} d(w_t, o) = 0$ where d is the Riemannian distance. Let (h_t) be a solution of the following stochastic differential equation on G;

$$\begin{aligned} dh_t &= \sum_{\alpha=1}^{d} A_\alpha(h_t) \circ dB_t^\alpha \\ h_0 &= e \end{aligned} \tag{4.2}$$

where $A_\alpha = A(e_\alpha) \in \mathfrak{m}$ for $\alpha = 1, \ldots, d$, were defined in the previous section. Set $X_t = h_t o$. Then $(X_t;\ 0 \leqq t \leqq T)$ is a Brownian motion on M starting at o by Proposition 3.1. We denote the law of (X_t) on $W_o(M)$ by μ. Let $\nu = \mu \circ I^{-1}$ be the induced measure of μ by the mapping I. Our main problem then is to study the condition on (g_t) so that two measures μ and ν on $W_o(M)$ are equivalent. Let $n = \dim(G)$ and $p = \dim(K)$. Then it holds that $n = d + p$. Take any basis $\{A_{d+1}, \ldots, A_{d+p}\}$ of $\mathfrak{k}$ and fix it through the argument. Then $\{A_1, \ldots, A_d, A_{d+1}, \ldots, A_{d+p}\}$ forms a basis of $\mathfrak{g}$. In the sequel we use the following conventions. Indices $I, J, K, L, \ldots$ run over $1, 2, \ldots, n = d + p$, indices $\alpha, \beta, \gamma, \delta, \ldots$ run over $1, 2, \ldots, d$, and indices $i, j, k, l, \ldots$ run over $d+1, d+2, \ldots, d+p$. Following the usual convention we omit the summation signs for repeated indices and we do so even if they appear in the top or in the bottom at the same time, e.g., $\xi^I A_I$ for $\sum_{I=1}^{n} \xi^I A_I$, $\xi^i A_i$ for $\sum_{i=d+1}^{n} \xi^i A_i$ and $\int_0^T \xi_t^\alpha dB_t^\alpha$ for $\sum_{\alpha=1}^{d} \int_0^T \xi_t^\alpha dB_t^\alpha$, etc.

Next we assume that (g_t) satisfies the following condition;

$$t \mapsto g_t \text{ is absolutely continuous on every closed interval of } (0, T]. \tag{A.2}$$

Under the condition *(A.2)* (g_t) satisfies the following differential equation on G;

$$dg_t = \zeta_t dt \qquad \text{for } 0 < t \leqq T \tag{4.3}$$

where $\zeta_t \in \mathfrak{g}$, $0 < t \leqq T$, is the unique left invariant vector field on G such that $\zeta_t(g_t) = \frac{dg_t}{dt} \in T_{g_t}(G)$. If we write $\zeta_t = \zeta_t^I A_I$ then clearly $\int_\varepsilon^T |\zeta_t^I| dt < \infty$ for $I = 1, \ldots, n$ and conversely, given such a system of functions ζ_t^I if we set $\zeta_t = \zeta_t^I A_I$ then any solution (g_t) of (4.3) defines a continuous curve g_t: $(0, T] \to G$ which satisfies *(A.2)*. Furthermore we assume that

$$\begin{aligned} &\text{(i)} \int_0^T (\zeta_t^\alpha)^2 dt < \infty \qquad \text{for } \alpha = 1, \ldots, d, \\ &\text{(ii)} \int_0^T t(\zeta_t^i)^2 dt < \infty \qquad \text{for } i = d+1, \ldots, n. \end{aligned} \tag{A.3}$$

Now we can state the main theorem.

Theorem 4.1. *Under the assumptions (A.1), (A.2) and (A.3), μ and $\nu=\mu\circ I^{-1}$ are equivalent.*

If μ and ν are equivalent, then the natural question arises: What is the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$? We will give the expression of $\frac{d\nu}{d\mu}$. Let $W_e(G)$ be the set of all continuous paths in G defined on the interval $[0,T]$ starting at e. Since (h_t) is a strong solution of (4.2), there exists a measurable and non-anticipative mapping Ξ: $W_0^d\to W_e(G)$ such that $(h.)=\Xi(B.)$. Let P^h be the induced measure of P^W by Ξ, i.e., $P^h=P^W\circ\Xi^{-1}$. Define the mapping $\tilde{p}$: $W_e(G)\to W_o(M)$ by $\tilde{p}(h.)_t=h_t o$. Then the measure μ is nothing but the induced measure of P^h by $\tilde{p}$. Notice that there exist inverse mappings of Ξ and $\tilde{p}$, denoted by D and H respectively; H: $W_o(M)\to W_e(G)$ is the "stochastic horizontal lift" defined by P. Malliavin [18], (see also [23]) and D: $W_e(G)\to W_0^d$ is the "stochastic development" defined by Ikeda-Manabe [7], i.e.,

$$D(h.)_t=(\int_{h[0,t]}\omega_1,\dots,\int_{h[0,t]}\omega_d)$$

where $\omega_1,\dots,\omega_d,\ \omega_{d+1},\dots,\omega_n$ are the dual 1-forms of the vector fields $A_1,\dots,A_n$ defined by $\omega_I(A_J)=\delta_{IJ}$ for $I,J=1,\dots,n$, and $\int_{h[0,t]}\omega_\alpha$ is the integral of the 1-form ω_α along the path $(h_s;0\leqq s\leqq t)$ (see, e.g., Ikeda-Manabe [7], Ikeda-Watanabe [8]). Namely we have that $H\circ\tilde{p}=\mathrm{id}_{W_e(G)}\,P^h$-a.e., $\tilde{p}\circ H=\mathrm{id}_{W_o(M)}\,\mu$-a.e., $D\circ\Xi=\mathrm{id}_{W_0^d}\,P^W$-a.e. and $\Xi\circ D=\mathrm{id}_{W_e(G)}\,P^h$-a.e. Hence we can represent the processes (B_t) and (h_t) as the non-anticipative functionals of the Brownian motion (X_t) on M, i.e., $(h.)=H(X.)$ and $(B.)=D\circ H(X.)$. Define a mapping $F=(F^1,\dots,F^d)$: $\mathfrak{g}\times G\to\mathbf{R}^d$ by $F(A,g)=(g\,u_0)^{-1}A^*_{go}$ which is often called a *scalarization function* of the vector field A^*. Then the Radon-Nikodym derivative is given by the following form;

Theorem 4.2. *Under the same assumptions of Theorem* 4.1,

$$\frac{d\nu}{d\mu}(X.)=\exp\left\{\int_0^T F^\alpha(\mathrm{Ad}(g_t)\zeta_t,h_t)\,dB_t^\alpha-\tfrac{1}{2}\int_0^T\|(\mathrm{Ad}(g_t)\zeta_t)^*_{X_t}\|^2\,dt\right\}\tag{4.4}$$

*where * stands for the homomorphism from* $\mathfrak{g}$ *into* $\Gamma^\infty(T(M))$ *defined in Sect.* 3 *and* $\|\cdot\|$ *is the norm on* $T(M)$ *defined by the Riemannian metric* $\langle\cdot,\cdot\rangle$.

If in particular, (g_t) *is defined on* $[0,T]$ *and smooth (at least* C^4 *class) then we can express the density as the functional of* (X_t) *more explicitly;*

$$\frac{d\nu}{d\mu}(X.)=\exp\left\{\int_0^T\omega_t\circ dX_t-\tfrac{1}{2}\int_0^T[\mathrm{div}(\mathrm{Ad}(g_t)\zeta_t)^*(X_t)+\|(\mathrm{Ad}(g_t)\zeta_t)^*_{X_t}\|^2]\,dt\right\}\tag{4.5}$$

where ω_t *is the* 1*-form defined by* $\omega_t(\cdot)=\langle(\mathrm{Ad}(g_t)\zeta_t)^*,\cdot\rangle$.

Remark. In the right hand side of (4.4), we regard (h_t) and (B_t) as the functionals of the Brownian motion (X_t) on M. In the right hand side of (4.5), $\int_0^T \omega_t \circ dX_t$ is the integral of the 1-form (ω_t). Note that (ω_t) depends on the time t and hence we need a slight generalization in defining $\int_0^T \omega_t \circ dX_t$. See [23] for details.

5. Proofs of the Theorems

In this section we give the proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1: Firstly we shall prove that ν is absolutely continuous with respect to μ under the conditions in Theorem 4.1. Let $(\tilde{l}_t)$ be the solution of the following differential equation on G;

$$d\tilde{l}_t = (\zeta_t^i A_i)(\tilde{l}_t)\,dt = (\zeta_t)_{\mathfrak{k}}(\tilde{l}_t)\,dt \qquad 0<t\leqq T$$
$$\tilde{l}_T = e.$$

Note that $(\tilde{l}_t)$ is defined only for $0<t\leqq T$. From now on we omit the variable $\tilde{l}_t$ in the above equation for simplicity. Thus we write $d\tilde{l}_t = \zeta_t^i A_i\,dt$ in place of $d\tilde{l}_t = (\zeta_t^i A_i)(\tilde{l}_t)\,dt$. This convention will be used for all differential and stochastic differential equations considered in the future.

From Corollary 2.2, $(\tilde{l}_t^{-1})$ satisfies the following differential equation on G;

$$d\tilde{l}_t^{-1} = -\operatorname{Ad}(\tilde{l}_t)(\zeta_t)_{\mathfrak{k}}\,dt \qquad 0<t\leqq T$$
$$\tilde{l}_T^{-1} = e.$$

Set $\tilde{m}_t = g_t \tilde{l}_t^{-1}$, then from Proposition 2.1, we have

$$d\tilde{m}_t = \{\operatorname{Ad}(\tilde{l}_t)\zeta_t - \operatorname{Ad}(\tilde{l}_t)(\zeta_t)_{\mathfrak{k}}\}\,dt = \operatorname{Ad}(\tilde{l}_t)(\zeta_t)_{\mathfrak{m}}\,dt \qquad 0<t\leqq T.$$

We denote the components of $\operatorname{Ad}(\tilde{l}_t)$ with respect to the basis $\{A_1, \ldots, A_n\}$ by $\operatorname{Ad}(\tilde{l}_t)^I_J$, i.e., $\operatorname{Ad}(\tilde{l}_t)A_J = \operatorname{Ad}(\tilde{l}_t)^I_J A_I$. Hereafter we use this convention without mentioning. Since $\mathfrak{m}$ and $\mathfrak{k}$ are invariant under $\operatorname{Ad}(K)$, it holds that $\operatorname{Ad}(k)^\alpha_i = \operatorname{Ad}(k)^i_\alpha = 0$ for $k \in K$. Moreover $(\operatorname{Ad}(k)^\alpha_\beta)$ is in $O(d)$ because $(\operatorname{Ad}(k)^\alpha_\beta) = \Lambda(k)$. We use these facts frequently. Thus $(\tilde{m}_t)$ satisfies the following differential equation on G;

$$d\tilde{m}_t = \operatorname{Ad}(\tilde{l}_t)^\beta_\alpha \zeta_t^\alpha A_\beta\,dt = \operatorname{Ad}(\tilde{l}_t)(\zeta_t)_{\mathfrak{m}}\,dt \qquad 0<t\leqq T$$
$$\tilde{m}_T = g_T.$$

By the assumption *(A.3)* and the boundedness of $(\operatorname{Ad}(\tilde{l}_t)^\alpha_\beta)$, $(\operatorname{Ad}(\tilde{l}_t)^\beta_\alpha \zeta_t^\alpha)$ belongs to $L^2([0, T])$. Hence $(\tilde{m}_t)$ is defined on $[0, T]$. i.e., $\tilde{m}_0 = \lim_{t\to 0} \tilde{m}_t$ exists and it can be shown that $\tilde{m}_0 \in K$. In fact $\tilde{m}_0 o = \lim_{t\to 0} \tilde{m}_t o = \lim_{t\to 0} g_t \tilde{l}_t^{-1} o = \lim_{t\to 0} g_t o = o$ from the assumption *(A.1)*. Set $m_t = \tilde{m}_t \tilde{m}_0^{-1}$ and $l_t = \tilde{m}_0 \tilde{l}_t$, then $g_t = m_t l_t$ and (l_t), (l_t^{-1}) and (m_t) satisfy the following differential equations on G respectively;

$$\begin{aligned} dl_t &= (\zeta_t)_{\mathfrak{k}}\, dt \qquad 0<t\leqq T \\ l_T &= \tilde{m}_0 \in K, \end{aligned} \tag{5.1}$$

$$\begin{aligned} dl_t^{-1} &= -\operatorname{Ad}(l_t)(\zeta_t)_{\mathfrak{k}}\, dt \qquad 0<t\leqq T \\ l_T^{-1} &= \tilde{m}_0^{-1} \end{aligned} \tag{5.2}$$

$$\begin{aligned} dm_t &= \operatorname{Ad}(l_t)(\zeta_t)_{\mathfrak{m}}\, dt \\ m_0 &= e. \end{aligned} \tag{5.3}$$

Note that $l_t \in K$ for all $t \in (0, T]$.

From (4.2) (5.1) (5.2) and Proposition 2.1, we see that $(l_t h_t l_t^{-1})$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} d(l_t h_t l_t^{-1}) &= \operatorname{Ad}(l_t) A_\alpha \circ dB_t^\alpha + \{\operatorname{Ad}(l_t h_t^{-1})(\zeta_t)_{\mathfrak{k}} - \operatorname{Ad}(l_t)(\zeta_t)_{\mathfrak{k}}\}\, dt \\ &= \operatorname{Ad}(l_t)_\alpha^\beta A_\beta \circ dB_t^\alpha + \{\zeta_t^i(\operatorname{Ad}(h_t^{-1})_i^I - \delta_i^I) \operatorname{Ad}(l_t)_I^J A_J\}\, dt. \end{aligned}$$

Again by Proposition 2.1, $(\operatorname{Ad}(h_t^{-1})_J^I)$ satisfies the following stochastic differential equation;

$$\begin{aligned} d \operatorname{Ad}(h_t^{-1})_J^I &= C_{K\alpha}^I \operatorname{Ad}(h_t^{-1})_J^K\, dB_t^\alpha + \tfrac{1}{2} C_{K\alpha}^I C_{L\alpha}^K \operatorname{Ad}(h_t^{-1})_J^L\, dt \\ \operatorname{Ad}(h_0^{-1})_J^I &= \delta_J^I. \end{aligned} \tag{5.4}$$

Since this is a linear equation in $\operatorname{Ad}(h_t^{-1})_J^I$, there exists a constant c_1 such that $E[(\operatorname{Ad}(h_t^{-1})_J^I)^2] \leqq c_1$ for $I, J = 1, \ldots, n$ and $t \in [0, T]$. We have

$$\operatorname{Ad}(h_t^{-1})_J^I - \delta_J^I = \int_0^t C_{K\alpha}^I \operatorname{Ad}(h_s^{-1})_J^K\, dB_s^\alpha + \tfrac{1}{2} \int_0^t C_{K\alpha}^I C_{L\alpha}^K \operatorname{Ad}(h_s^{-1})_J^L\, ds$$

and hence

$$\begin{aligned} & E[(\operatorname{Ad}(h_t^{-1})_J^I - \delta_J^I)^2] \\ & \leqq 2E\left[\left\{\int_0^t C_{K\alpha}^I \operatorname{Ad}(h_s^{-1})_J^K\, dB_s^\alpha\right\}^2\right] + E\left[\left\{\int_0^t C_{K\alpha}^I C_{L\alpha}^K \operatorname{Ad}(h_s^{-1})_J^L\, ds\right\}^2\right] \\ & \leqq 2E\left[\int_0^t \sum_\alpha \{C_{K\alpha}^I \operatorname{Ad}(h_s^{-1})_J^K\}^2\, ds\right] + tE\left[\int_0^t \sum_L (C_{L\alpha}^K C_{K\alpha}^I)^2 \sum_M (\operatorname{Ad}(h_s^{-1})_J^M)^2\, ds\right] \\ & \leqq (2n^2 d c_1 c_2^2 + n^4 d^2 c_1 c_2^4 T)\, t \end{aligned}$$

where $c_2 = \max_{I,J,K} |C_{JK}^I|$. Setting $c_3 = 2n^2 d c_1 c_2^2 + n^4 d^2 c_1 c_2^4 T$, we have therefore

$$E[(\operatorname{Ad}(h_t^{-1})_J^I - \delta_J^I)^2] \leqq c_3 t. \tag{5.5}$$

Let $(\tilde{h}_t)$ be a solution of the following stochastic differential equation on G;

$$\begin{aligned} d\tilde{h}_t &= \operatorname{Ad}(l_t)_\alpha^\beta A_\beta \circ dB_t^\alpha + \zeta_t^i [\operatorname{Ad}(h_t^{-1})_i^I - \delta_i^I] \operatorname{Ad}(l_t)_I^J A_J\, dt \\ \tilde{h}_0 &= e. \end{aligned} \tag{5.6}$$

By the assumption *(A.3)* and (5.5), the coefficients on the right hand side of the above equation belong to L_2^{loc} and hence $(\tilde{h}_t)$ is defined on the whole interval $[0,T]$. Now it is easy to see that $(l_t h_t l_t^{-1} \tilde{h}_t^{-1})$ is constant in $t \in (0,T]$. Set $\eta = l_t h_t l_t^{-1} \tilde{h}_t^{-1}$. Since K is compact and $l_t \in K$, there exists a sequence $\{t_k\}_{k\in N}$ of $(0,T]$ such that $\lim_{k\to\infty} t_k = 0$ and $\lim_{k\to\infty} l_{t_k}$ exists, say λ. Then we have

$$\eta = \lim_{k\to\infty} l_{t_k} h_{t_k} l_{t_k}^{-1} \tilde{h}_{t_k}^{-1} = \lambda e \lambda^{-1} e = e$$

and hence $\tilde{h}_t = l_t h_t l_t^{-1}$.

By (5.3) and (5.6), $(m_t \tilde{h}_t)$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} d(m_t \tilde{h}_t) &= \mathrm{Ad}(l_t) A_\alpha \circ dB_t^\alpha + \{\mathrm{Ad}(l_t h_t^{-1}) \zeta_t - \mathrm{Ad}(l_t)(\zeta_t)_{\mathfrak{k}}\} dt \\ m_0 \tilde{h}_0 &= e. \end{aligned} \tag{5.7}$$

Let (k_t) be a solution of the following stochastic differential equation on G;

$$\begin{aligned} dk_t &= \mathrm{Ad}(l_t)(\mathrm{Ad}(h_t^{-1}) \zeta_t - \zeta_t)_{\mathfrak{k}} dt = \mathrm{Ad}(l_t)_i^j (\mathrm{Ad}(h_t^{-1})_I^i - \delta_I^i) \zeta_t^I A_j dt \\ k_0 &= e. \end{aligned}$$

Since the coefficients of the above equation belong to L_2^{loc} by *(A.2)* and (5.5), (k_t) is defined on $[0,T]$. By Corollary 2.2, (k_t^{-1}) satisfies the following differential equation on G;

$$\begin{aligned} dk_t^{-1} &= -\mathrm{Ad}(k_t l_t) \{(\mathrm{Ad}(h_t^{-1}) \zeta_t)_{\mathfrak{k}} - (\zeta_t)_{\mathfrak{k}}\} dt \\ k_0^{-1} &= e. \end{aligned} \tag{5.8}$$

Then from (5.7) and (5.8), we see that $(m_t \tilde{h}_t k_t^{-1})$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} d(m_t \tilde{h}_t k_t^{-1}) &= \mathrm{Ad}(k_t l_t) A_\alpha \circ dB_t^\alpha + \mathrm{Ad}(k_t l_t)(\mathrm{Ad}(h_t^{-1}) \zeta_t)_{\mathfrak{m}} dt \\ &= \mathrm{Ad}(k_t)_\beta^\alpha \,\mathrm{Ad}(l_t)_\gamma^\beta A_\alpha \circ dB_t^\gamma \\ &\quad + \mathrm{Ad}(k_t)_\beta^\alpha \,\mathrm{Ad}(l_t)_\gamma^\beta \,\mathrm{Ad}(h_t^{-1})_I^\gamma \zeta_t^I A_\alpha dt \\ m_0 \tilde{h}_0 k_0^{-1} &= e. \end{aligned} \tag{5.9}$$

Now if we prove the equivalence of the laws of (h_t) and $(m_t \tilde{h}_t k_t^{-1})$, then we can conclude the equivalence of μ and ν which are just laws of $(h_t o)$ and $(g_t h_t o)$ respectively. In fact since $k_t, l_t \in K$ and $g_t = m_t l_t$, it holds that $m_t \tilde{h}_t k_t^{-1} o = m_t l_t h_t l_t^{-1} k_t^{-1} o = g_t h_t o$. Define a d-dimensional continuous process $(Z_t) = (Z_t^1, \ldots, Z_t^d)$ by

$$Z_t^\alpha = \int_0^t \mathrm{Ad}(k_s)_\beta^\alpha \,\mathrm{Ad}(l_s)_\gamma^\beta \, dB_s^\gamma + \int_0^t \mathrm{Ad}(k_s)_\beta^\alpha \,\mathrm{Ad}(l_s)_\gamma^\beta \,\mathrm{Ad}(h_s^{-1})_I^\gamma \zeta_s^I ds. \tag{5.10}$$

Further $(\mathrm{Ad}(k_t)_J^I)$ and $(\mathrm{Ad}(l_t)_J^I)$ satisfy the following differential equations respectively from Proposition 2.1

$$\begin{aligned} d\,\mathrm{Ad}(k_t)^I_J &= C^K_{jJ}\,\mathrm{Ad}(k_t)^I_K\,\mathrm{Ad}(l_t)^j_i(\mathrm{Ad}(h_t^{-1})^i_L-\delta^i_L)\,\zeta^L_t\,dt \\ \mathrm{Ad}(k_0)^I_J &= \delta^I_J \end{aligned} \tag{5.11}$$

$$\begin{aligned} d\,\mathrm{Ad}(l_t)^I_J &= C^K_{iJ}\,\mathrm{Ad}(l_t)^I_K\,\zeta^i_t\,dt \\ \mathrm{Ad}(l_T)^I_J &= \mathrm{Ad}(\tilde{m}_0)^I_J. \end{aligned} \tag{5.12}$$

Then we can rewrite the Eq. (5.9) as follows;

$$\begin{aligned} d(m_t\,\tilde{h}_t\,k_t^{-1}) &= A_\alpha \circ dZ^\alpha_t \\ m_0\,\tilde{h}_0\,k_0^{-1} &= e. \end{aligned} \tag{5.13}$$

Here we used the fact that $\mathrm{Ad}(k_t)^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma \circ dB^\gamma_t = \mathrm{Ad}(k_t)^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma\,dB^\gamma_t$ but it is easily obtained from (5.11) and (5.12). Note that the stochastic differential Eqs. (4.2) and (5.13) have the unique strong solutions and hence, (h_t) and $(m_t\,\tilde{h}_t\,k_t^{-1})$ are non-anticipative functionals in the same form of (B_t) and (Z_t) respectively: $(h.)=\Phi(B.)$ and $(m.\tilde{h}.k.^{-1})=\Phi(Z.)$. Therefore the equivalence of the laws of (h_t) and $(m_t\,\tilde{h}_t\,k_t^{-1})$ follows at once from that of the laws of (B_t) and (Z_t). Thus it remains now only to show that the laws of (B_t) and (Z_t) are equivalent.

Note that $(\mathrm{Ad}(k_t)^\alpha_\beta)$ and $(\mathrm{Ad}(l_t)^\alpha_\beta)$ are in $O(d)$. Hence the martingale part of (Z_t) is a d-dimensional Brownian motion and the integrand of the second term of (5.10) belongs to L^{loc}_2 by the assumption *(A.3)* and (5.5). Now by the following theorem of Shepp and Kadota, we can conclude that the law of (Z_t) is absolutely continuous with respect to the Wiener measure P^W.

Theorem 5.1 (Shepp-Kadota [10]). *Let $\xi=(\xi_t; 0\leqq t\leqq T)$ be a d-dimensional continuous process given by $\xi^\alpha_t = B^\alpha_t + \int_0^t \Phi^\alpha_s\,ds$ $(\alpha=1,\dots,d)$ such that (Φ^α_t) is in L^{loc}_2 for $\alpha=1,\dots,d$. Then the law of (ξ_t) is absolutely continuous with respect to the Wiener measure.*

Secondly we shall prove that μ is absolutely continuous with respect to ν. First we will show that $(\mathrm{Ad}(h_t^{-1})^I_J)$ and $(\mathrm{Ad}(k_t)^\alpha_\beta)$ are non-anticipative functionals of (Z_t). For this we show that $(\mathrm{Ad}(h_t^{-1})^I_J)$ satisfies the following stochastic differential equation with respect to dZ_t;

$$\begin{aligned} d\,\mathrm{Ad}(h_t^{-1})^I_J &= C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J\,\mathrm{Ad}(l_t^{-1})^\alpha_\beta\,\mathrm{Ad}(k_t^{-1})^\beta_\gamma \circ dZ^\gamma_t \\ &\quad - C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J\,\mathrm{Ad}(h_t)^\alpha_L\,\zeta^L_t\,dt \\ \mathrm{Ad}(h_0^{-1})^I_J &= \delta^I_J. \end{aligned} \tag{5.14}$$

In fact, the right hand side equals to

$$\begin{aligned} & C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J\,\mathrm{Ad}(l_t^{-1})^\alpha_\beta\,\mathrm{Ad}(k_t^{-1})^\beta_\gamma \circ \{\mathrm{Ad}(k_t)^\gamma_\delta\,\mathrm{Ad}(l_t)^\delta_\varepsilon\,dB^\varepsilon_t\} \\ &\quad + \{C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J\,\mathrm{Ad}(l_t^{-1})^\alpha_\beta\,\mathrm{Ad}(k_t^{-1})^\beta_\gamma\,\mathrm{Ad}(k_t)^\gamma_\delta\,\mathrm{Ad}(l_t)^\delta_\varepsilon\,\mathrm{Ad}(h_t^{-1})^\varepsilon_L\,\zeta^L_t \\ &\quad - C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J\,\mathrm{Ad}(h_t^{-1})^\alpha_L\,\zeta^L_t\}\,dt \\ &= C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1})^K_J \circ dB^\alpha_t = d\,\mathrm{Ad}(h_t^{-1})^I_J \end{aligned}$$

by (5.4). Here we used the fact that $(\mathrm{Ad}(l_t^{-1})^\alpha_\beta)$ and $(\mathrm{Ad}(k_t^{-1})^\alpha_\beta)$ are the inverse matrices of $(\mathrm{Ad}(l_t)^\alpha_\beta)$ and $(\mathrm{Ad}(k_t)^\alpha_\beta)$ respectively which follows from the fact that l_t and k_t are in K.

On the other hand, since $(\mathrm{Ad}(k)^{\alpha}_{\beta})$ is in $O(d)$ for $k \in K$, $\mathrm{Ad}(l_t^{-1})^{\alpha}_{\beta} = \mathrm{Ad}(l_t)^{\beta}_{\alpha}$ and $\mathrm{Ad}(k_t^{-1})^{\alpha}_{\beta} = \mathrm{Ad}(k_t)^{\beta}_{\alpha}$. Combining this with (5.11) and (5.14), we can see that $(\mathrm{Ad}(l_t)^I_J)$, $(\mathrm{Ad}(k_t)^I_J)$ and $(\mathrm{Ad}(h_t^{-1})^I_J)$ satisfy a system of stochastic differential equations with respect to (Z_t). Note that $\mathrm{Ad}(l_t)$ is uniquely determined as a non-random function. But it is not evident that $(\mathrm{Ad}(k_t)^I_J)$ and $(\mathrm{Ad}(h_t^{-1})^I_J)$ are obtained as strong solutions of these equations because ζ^i_t are singular at $t=0$. This difficulty is overcomed by a general result given in the next section (see Theorem 6.1) and therefore, as strong solutions of these equations, $(\mathrm{Ad}(h_t^{-1})^I_J)$ and $(\mathrm{Ad}(k_t)^{\alpha}_{\beta})$ are non-anticipative functionals of (Z_t). Now we can appeal to the following theorem due to Kailath-Zakai and Ershov.

Theorem 5.2 (Kailath-Zakai [12], Ershov [5]). *Let $(\xi_t;\ 0 \leqq t \leqq T)$ be a d-dimensional continuous process satisfying the following stochastic integral equation;*

$$\xi^{\alpha}_t = B^{\alpha}_t + \int_0^t \Phi^{\alpha}_s(\xi.)\,ds \qquad \alpha = 1, \dots, d,$$

where Φ^{α}_t is $\mathscr{F}_t$-measurable function from W^d_0 into $\mathbf{R}$. Assume further that both $(\Phi^{\alpha}_t(\xi.))$ and $(\Phi^{\alpha}_t(B.))$ belong to L^{loc}_2 $\alpha = 1, \dots, d$. Then the law P^{ξ} of (ξ_t) and the Wiener measure P^W are equivalent. Moreover the Radon-Nikodym derivative is given by

$$\frac{dP^{\xi}}{dP^W}(B.) = \exp\left\{\int_0^T \Phi^{\alpha}_s(B.)\,dB^{\alpha}_s - \tfrac{1}{2}\sum_{\alpha=1}^{d}\int_0^T \Phi^{\alpha}_s(B.)^2\,ds\right\}.$$

What remains to be shown is the part "$(\Phi^{\alpha}_t(B.)) \in L^{\mathrm{loc}}_2$" in the above theorem and this can be obtained by repeating a similar discussion as above. First (m_t) be a solution of (5.3). Then (m_t^{-1}) satisfies the following differential equation on G;

$$\begin{aligned} dm_t^{-1} &= -\mathrm{Ad}(m_t\, l_t)(\zeta_t)_{\mathfrak{m}}\, dt \\ m_0^{-1} &= e. \end{aligned}$$

Hence $(m_t^{-1} h_t)$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} d(m_t^{-1} h_t) &= A_{\alpha} \circ dB^{\alpha}_t - \mathrm{Ad}(h_t^{-1} m_t\, l_t)(\zeta_t)_{\mathfrak{m}}\, dt \\ &= A_{\alpha} \circ dB^{\alpha}_t - \mathrm{Ad}(h_t^{-1} m_t)^I_J\, \mathrm{Ad}(l_t)^J_{\alpha}\, \zeta^{\alpha}_t\, A_I\, dt \qquad (5.15) \\ m_0^{-1} h_0 &= e. \end{aligned}$$

Similarly $(l_t^{-1} m_t^{-1} h_t\, l_t)$ satisfies the following stochastic differential equation on G;

$$\begin{aligned} d(l_t^{-1} m_t^{-1} h_t\, l_t) &= \mathrm{Ad}(l_t^{-1})\, A_{\alpha} \circ dB^{\alpha}_t + \{(\zeta_t)_{\mathfrak{k}} - \mathrm{Ad}(l_t^{-1} h_t^{-1} m_t\, l_t)\, \zeta_t\}\, dt \\ &= \mathrm{Ad}(l_t^{-1})^{\beta}_{\alpha}\, A_{\beta} \circ dB^{\alpha}_t \\ &\quad + \{\mathrm{Ad}(l_t^{-1})^I_J (\delta^J_i - \mathrm{Ad}(h_t^{-1} m_t)^J_i)\, \mathrm{Ad}(l_t)^i_j\, \zeta^j_t\, A_I \qquad (5.16) \\ &\quad - \mathrm{Ad}(l_t^{-1})^I_J\, \mathrm{Ad}(h_t^{-1} m_t)^J_{\alpha}\, \mathrm{Ad}(l_t)^{\alpha}_{\beta}\, \zeta^{\beta}_t\, A_I\}\, dt \\ &0 < t \leqq T. \end{aligned}$$

By (5.15) and Proposition 2.1, $(\mathrm{Ad}(h_t^{-1} m_t)^I_J)$ satisfies the following stochastic differential equation;

$$d\,\mathrm{Ad}(h_t^{-1}m_t)^I_J = C^I_{K\alpha}\,\mathrm{Ad}(h_t^{-1}m_t)^K_J \circ dB^\alpha_t - C^I_{KL}\,\mathrm{Ad}(h_t^{-1}m_t)^K_J\,\mathrm{Ad}(h_t^{-1}m_t)^L_M\,\mathrm{Ad}(l_t)^M_\alpha\,\zeta^\alpha_t\,dt$$

$$\mathrm{Ad}(h_0^{-1}m_0)^I_J = \delta^I_J.$$

Hence

$$\mathrm{Ad}(h_t^{-1}m_t)^I_J - \delta^I_J = \int_0^t C^I_{K\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\,dB^\alpha_s + \int_0^t \{\tfrac{1}{2}C^I_{K\alpha}C^K_{M\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^M_J - C^I_{KL}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\,\mathrm{Ad}(h_s^{-1}m_s)^L_M\,\mathrm{Ad}(l_s)^M_\alpha\,\zeta^\alpha_s\}\,ds. \tag{5.17}$$

For any $N>0$, define a stopping time $\tau=\tau_N$ by

$$\tau = \begin{cases} \inf\{t\in[0,T] \mid \max_{I,J}|\mathrm{Ad}(h_t^{-1}m_t)^I_J| \geqq N\} \\ T \quad \text{if } \{\ \} = \phi. \end{cases}$$

We will show that $E[(\mathrm{Ad}(h_{t\wedge\tau}^{-1}m_{t\wedge\tau})^I_J - \delta^I_J)^2] \leqq c_4 t$ for some constant c_4 which may depend on N. Set $c_5 = \max_{I,J}\sup_{k\in K}\{|\mathrm{Ad}(k)^I_J|\}$. Then from (5.17) we have

$$\begin{aligned}
&E[(\mathrm{Ad}(h_{t\wedge\tau}^{-1}m_{t\wedge\tau})^I_J - \delta^I_J)^2] \\
&\leqq 3E\left[\left\{\int_0^{t\wedge\tau} C^I_{K\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\,dB^\alpha_s\right\}^2\right] + 3E\left[\left\{\int_0^{t\wedge\tau}\tfrac{1}{2}C^I_{K\alpha}C^K_{M\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^M_J\,ds\right\}^2\right] \\
&\quad + 3E\left[\left\{\int_0^{t\wedge\tau} C^I_{KL}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\,\mathrm{Ad}(h_s^{-1}m_s)^L_M\,\mathrm{Ad}(l_s)^M_\alpha\,\zeta^\alpha_s\,ds\right\}^2\right] \\
&\leqq 3E\left[\int_0^{t\wedge\tau}\sum_\alpha\{C^I_{K\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\}^2\,ds\right] \\
&\quad + 3tE\left[\int_0^{t\wedge\tau}\{\tfrac{1}{2}C^I_{K\alpha}C^K_{M\alpha}\,\mathrm{Ad}(h_s^{-1}m_s)^M_J\}^2\,ds\right] \\
&\quad + 3tE\left[\int_0^{t\wedge\tau} C^I_{KL}\,\mathrm{Ad}(h_s^{-1}m_s)^K_J\,\mathrm{Ad}(h_s^{-1}m_s)^L_M\,\mathrm{Ad}(l_s)^M_\alpha\,\zeta^\alpha_s\}^2\,ds\right] \\
&\leqq 3tn^4dN^2c_2^2 + \tfrac{3}{4}tTn^4d^2N^2c_2^4 + 3tTn^6dN^4c_5^2\sum_\alpha\int_0^T(\zeta^\alpha_t)^2\,dt \\
&\leqq c_4 t
\end{aligned}$$

where $c_4 = 3n^4dN^2c_2^2 + \tfrac{3}{4}Tn^4d^2N^2c_2^4 + 3Tn^6dN^4c_5^2\sum_\alpha\int_0^T(\zeta^\alpha_t)^2\,dt$. From this and *(A.3)* we have

$$\begin{aligned}
&E\left[\int_0^{\tau_N}\{\mathrm{Ad}(l_t^{-1})^I_J(\delta^J_i - \mathrm{Ad}(h_t^{-1}m_t)^J_i)\,\mathrm{Ad}(l_t)^i_j\,\zeta^j_t\}^2\,dt\right] \\
&\leqq \sum_{J,i,j} c_5^4\int_0^T E[(\delta^J_i - \mathrm{Ad}(h_{t\wedge\tau}^{-1}m_{t\wedge\tau})^J_i)^2](\zeta^j_t)^2\,dt \\
&\leqq n^2c_5^4c_4\sum_i\int_0^T t(\zeta^i_t)^2\,dt < \infty.
\end{aligned}$$

This shows that $\int_0^T \{\mathrm{Ad}(l_t^{-1})_J^I(\delta_i^J - \mathrm{Ad}(h_t^{-1} m_t)_i^J)\,\mathrm{Ad}(l_t)_j^i \zeta_t^j\}^2\,dt < \infty$ a.e. on a set $\{\tau_N = T\}$. But since $P^W\{\tau_N = T\} \uparrow 1$ as $N \to \infty$, we have

$$P^W\left[\int_0^T \{\mathrm{Ad}(l_t^{-1})_J^I(\delta_i^J - \mathrm{Ad}(h_t^{-1} m_t)_i^J)\,\mathrm{Ad}(l_t)_j^i \zeta_t^j\}^2\,dt < \infty\right] = 1.$$

Thus all the coefficients of the Eq. (5.16) belong to L_2^{loc}. Hence we can consider the following stochastic differential equation on G;

$$\begin{aligned} dn_t &= \{(\zeta_t)_{\mathfrak{k}} - [\mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{k}}\}\,dt \\ &= [\mathrm{Ad}(l_t^{-1})_j^i(\delta_k^j - \mathrm{Ad}(h_t^{-1} m_t)_k^j)\,\mathrm{Ad}(l_t)_l^k \zeta_t^l A_i \\ &\quad - \mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)_\alpha^i \zeta_t^\alpha A_i]\,dt \\ n_0 &= e. \end{aligned}$$

Then (n_t^{-1}) satisfies the following stochastic differential equation on G;

$$\begin{aligned} dn_t^{-1} &= \{\mathrm{Ad}(n_t)[\mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{k}} - \mathrm{Ad}(n_t)(\zeta_t)_{\mathfrak{k}}\}\,dt \\ n_0^{-1} &= e. \end{aligned} \tag{5.18}$$

Hence by (5.16), (5.18) we have

$$\begin{aligned} &d(l_t^{-1} m_t^{-1} h_t l_t n_t^{-1}) \\ &= \mathrm{Ad}(n_t l_t^{-1}) A_\alpha \circ dB_t^\alpha - \mathrm{Ad}(n_t)[\mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{m}}\,dt \\ &= \mathrm{Ad}(n_t l_t^{-1})_\alpha^\beta A_\beta \circ dB_t^\alpha + \{\mathrm{Ad}(n_t)_\beta^\alpha\,\mathrm{Ad}(l_t^{-1})_\gamma^\beta(\delta_i^\gamma - \mathrm{Ad}(h_t^{-1} m_t)_i^\gamma)\,\mathrm{Ad}(l_t)_j^i \zeta_t^j A_\alpha \\ &\quad - \mathrm{Ad}(n_t)_\beta^\alpha\,\mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)_\gamma^\beta \zeta_t^\gamma A_\alpha\}\,dt \qquad 0 < t \leqq T. \end{aligned}$$

It can be shown that the coefficients of the above equation belong to L_2^{loc} by a similar method and hence we can consider the following stochastic differential equation on G;

$$\begin{aligned} dp_t &= \mathrm{Ad}(n_t l_t^{-1}) A_\alpha \circ dB_t^\alpha - \mathrm{Ad}(n_t)[\mathrm{Ad}(l_t^{-1} h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{m}}\,dt \\ p_0 &= e. \end{aligned} \tag{5.19}$$

Then we can prove that $p_t = l_t^{-1} m_t^{-1} h_t l_t n_t^{-1}$ for $0 < t \leqq T$ a.e. Note that $\mathrm{Ad}(p_t^{-1}) = \mathrm{Ad}(n_t l_t^{-1} h_t^{-1} m_t l_t)$. Hence by (5.19) and Proposition 2.1, $(\mathrm{Ad}(p_t^{-1})_J^I)$ satisfies the following stochastic differential equation;

$$\begin{aligned} d\,\mathrm{Ad}(p_t^{-1})_J^I &= C_{K\alpha}^I\,\mathrm{Ad}(p_t^{-1})_J^K\,\mathrm{Ad}(n_t l_t^{-1})_\beta^\alpha \circ dB_t^\beta \\ &\quad - C_{K\alpha}^I\,\mathrm{Ad}(p_t^{-1})_J^K\,\mathrm{Ad}(p_t^{-1})_L^\alpha \zeta_t^L\,dt \\ \mathrm{Ad}(p_0^{-1})_J^I &= \delta_J^I. \end{aligned} \tag{5.20}$$

Also from (5.1), (5.2) and (5.18), we see that $(l_t n_t l_t^{-1})$ satisfies the following stochastic differential equation on G;

$$d(l_t n_t l_t^{-1}) = \{\mathrm{Ad}(l_t n_t^{-1})(\zeta_t)_{\mathfrak{k}} - [\mathrm{Ad}(h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{k}}\}\,dt$$
$$= \{\mathrm{Ad}(l_t)^i_j(\mathrm{Ad}(n_t^{-1})^j_k - \delta^j_k)\zeta^k_t A_i$$
$$-(\mathrm{Ad}(h_t^{-1} m_t)^i_I - \delta^i_I)\,\mathrm{Ad}(l_t)^I_J \zeta^J_t A_i\}\,dt \qquad 0<t\leqq T.$$

We can show that the coefficients of the above equation belong to L_2^{loc} as before. Hence if (q_t) is a solution of the following stochastic differential equation on G;

$$dq_t = \{\mathrm{Ad}(l_t n_t^{-1})(\zeta_t)_{\mathfrak{k}} - [\mathrm{Ad}(h_t^{-1} m_t l_t)\zeta_t]_{\mathfrak{k}}\}\,dt$$
$$= \{\mathrm{Ad}(l_t n_t^{-1})^i_j \zeta^j_t - \mathrm{Ad}(h_t^{-1} m_t l_t)^i_I \zeta^I_t\} A_i\,dt \tag{5.21}$$
$$q_0 = e$$

then it holds that $q_t = l_t n_t l_t^{-1}$ for $0<t\leqq T$, a.e. By (5.21) and Proposition 2.1, $(\mathrm{Ad}(q_t^{-1})^I_J)$ satisfies the following stochastic differential equation;

$$d\,\mathrm{Ad}(q_t^{-1})^I_J = C^I_{Ki}\,\mathrm{Ad}(q_t^{-1})^K_J\{\mathrm{Ad}(l_t n_t^{-1})^i_j \zeta^j_t - \mathrm{Ad}(h_t^{-1} m_t l_t)^i_L \zeta^L_t\}\,dt$$
$$\mathrm{Ad}(q_0^{-1})^I_J = \delta^I_J. \tag{5.22}$$

On the other hand, since $\mathrm{Ad}(g)$: $\mathfrak{g}\to\mathfrak{g}$ is a Lie algebra homomorphism for $g\in G$, it holds that

$$\mathrm{Ad}(g)[A_I, \mathrm{Ad}(g^{-1})A_J] = [\mathrm{Ad}(g)A_I, A_J] = [\mathrm{Ad}(g)^K_I A_K, A_J]$$
$$= \mathrm{Ad}(g)^K_I C^M_{KJ} A_M.$$

But the left hand side equals to

$$\mathrm{Ad}(g)[A_I, \mathrm{Ad}(g^{-1})^K_J A_K] = \mathrm{Ad}(g)\,\mathrm{Ad}(g^{-1})^K_J C^L_{IK} A_L$$
$$= \mathrm{Ad}(g)^M_L\,\mathrm{Ad}(g^{-1})^K_J C^L_{IK} A_M.$$

Thus we have $C^M_{KJ}\,\mathrm{Ad}(g)^K_I = C^L_{IK}\,\mathrm{Ad}(g)^M_L\,\mathrm{Ad}(g^{-1})^K_J$. By noting this and $q_t = l_t n_t l_t^{-1}\in K$, $p_t^{-1} = n_t l_t^{-1} h_t^{-1} m_t l_t$, we can rewrite (5.22) as follows;

$$d\,\mathrm{Ad}(q_t^{-1})^I_J$$
$$= C^K_{JL}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(q_t)^L_i\{\mathrm{Ad}(l_t n_t^{-1})^i_j \zeta^j_t - \mathrm{Ad}(h_t^{-1} m_t l_t)^i_M \zeta^M_t\}\,dt$$
$$= \{C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(q_t l_t n_t^{-1})^i_j \zeta^j_t - C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K$$
$$\times\,\mathrm{Ad}(q_t h_t^{-1} m_t l_t)^i_M \zeta^M_t\}\,dt$$
$$= \{C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t n_t l_t^{-1}\, l_t n_t^{-1})^i_j \zeta^j_t$$
$$- C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t n_t l_t^{-1} h_t^{-1} m_t l_t)^i_M \zeta^M_t\}\,dt$$
$$= \{C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t)^i_j \zeta^j_t - C^K_{Ji}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t)^i_j\,\mathrm{Ad}(p_t^{-1})^j_L \zeta^L_t\}\,dt$$
$$= C^K_{iJ}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t)^i_j\{\mathrm{Ad}(p_t^{-1})^j_L - \delta^j_L\}\zeta^L_t\,dt.$$

In the last line we used the fact $C^K_{iJ} = -C^K_{Ji}$. Thus $(\mathrm{Ad}(q_t^{-1})^I_J)$ satisfies the following differential equation;

$$d\,\mathrm{Ad}(q_t^{-1})^I_J = C^K_{iJ}\,\mathrm{Ad}(q_t^{-1})^I_K\,\mathrm{Ad}(l_t)^i_j\{\mathrm{Ad}(p_t^{-1})^j_L - \delta^j_L\}\zeta^L_t\,dt$$
$$\mathrm{Ad}(q_0^{-1})^I_J = \delta^I_J. \tag{5.23}$$

Also we can rewrite (5.20) as follows;

$$\begin{aligned} d\,\mathrm{Ad}(p_t^{-1})^I_J &= C^I_{K\alpha}\,\mathrm{Ad}(p_t^{-1})^K_J\,\mathrm{Ad}(l_t^{-1})^\alpha_\beta\,\mathrm{Ad}(q_t)^\beta_\gamma\circ dB^\gamma_t \\ &\quad - C^I_{K\alpha}\,\mathrm{Ad}(p_t^{-1})^K_J\,\mathrm{Ad}(p_t^{-1})^\alpha_L\,\zeta^L_t\,dt \\ \mathrm{Ad}(p_0^{-1})^I_J &= \delta^I_J \end{aligned} \tag{5.24}$$

Comparing the equations (5.11), (5.14) and (5.23), (5.24), we see that $(\mathrm{Ad}(h_t^{-1})^I_J)$, $(\mathrm{Ad}(k_t)^I_J)$ and (Z_t) correspond to $(\mathrm{Ad}(p_t^{-1})^I_J)$, $(\mathrm{Ad}(q_t^{-1})^I_J)$ and (B_t) respectively. Hence $(\mathrm{Ad}(p_t^{-1})^I_J)$ and $(\mathrm{Ad}(q_t^{-1})^I_J)$ are represented as non-anticipative functionals of (B_t) in the same way as $(\mathrm{Ad}(h_t^{-1})^I_J)$ and $(\mathrm{Ad}(k_t)^I_J)$ are represented as non-anticipative functionals of (Z_t). In the above argument we saw that $(\mathrm{Ad}(q_t^{-1})^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma\,\mathrm{Ad}(p_t^{-1})^\gamma_I\,\zeta^I_t)$ belongs to L_2^{loc} which corresponds to the part "$(\Phi^\alpha_t(B.))\in L_2^{\mathrm{loc}}$" in Theorem 5.2. Thus the proof of the equivalence of the laws (Z_t) and (B_t) is now completed.

Proof of Theorem 4.2: We will obtain the Radon-Nikodym derivative. We denote the law of (Z_t) by P^Z. Then from Theorem 5.2, we have

$$\begin{aligned} \frac{dP^Z}{dP^W}(B.) = \exp\Bigg\{ &\int_0^T \mathrm{Ad}(q_t^{-1})^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma\,\mathrm{Ad}(p_t^{-1})^\gamma_I\,\zeta^I_t\,dB^\alpha_t \\ &-\tfrac{1}{2}\int_0^T \sum_\alpha (\mathrm{Ad}(q_t^{-1})^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma\,\mathrm{Ad}(p_t^{-1})^\gamma_I\,\zeta^I_t)^2\,dt \Bigg\}. \end{aligned} \tag{5.25}$$

On the other hand, it holds that $\mathrm{Ad}(q_t^{-1})^\alpha_\beta\,\mathrm{Ad}(l_t)^\beta_\gamma\,\mathrm{Ad}(p_t^{-1})^\gamma_I = \mathrm{Ad}(q_t^{-1}\,l_t p_t^{-1})^\alpha_I$ $= \mathrm{Ad}(l_t\,n_t^{-1}\,l_t^{-1}\,l_t\,n_t\,l_t^{-1}\,h_t^{-1}\,m_t\,l_t)^\alpha_I = \mathrm{Ad}(h_t^{-1}\,g_t)^\alpha_I$. We defined the function F: $\mathfrak{g}\times G\to\mathbf{R}^d$ by $F(X,g)=(g\,u_0)^{-1}X^*_{go}$ for $X\in\mathfrak{g}$ and $g\in G$. Let L: $\mathfrak{m}\to T_o(M)$ be as in Sect. 3 and σ_{go} be the mapping from G into M defined by $\sigma_{go}(h)=hgo$. Define the mapping p: $G\to M$ by $p(g)=go$. Then it holds that $g^{-1}\circ\sigma_{go}=p\circ A_{g^{-1}}$ where $A_{g^{-1}}$ is the automorphism of G defined by $A_{g^{-1}}(h)=g^{-1}hg$. Note that $(A_{g^{-1}})_*=\mathrm{Ad}(g^{-1})$, $p_*|_{\mathfrak{m}}=L$ and $u_0^{-1}\circ L(A_\alpha)=e_\alpha$. Then if we write $X=X^I A_I$ for $X\in\mathfrak{g}$, we have

$$\begin{aligned} F(X,g) &= (g u_0)^{-1}X^*_{go} = u_0^{-1}\circ(g^{-1})_*\circ(\sigma_{go})_* X_e = u_0^{-1}\circ p_*(\mathrm{Ad}(g^{-1})X)_e \\ &= u_0^{-1}\circ p_*([\mathrm{Ad}(g^{-1})X]_{\mathfrak{m}})_e = u_0^{-1}\circ L(\mathrm{Ad}(g^{-1})^\alpha_I X^I A_\alpha) \\ &= \mathrm{Ad}(g^{-1})^\alpha_I X^I e_\alpha. \end{aligned}$$

From this it follows that

$$F^\alpha(\mathrm{Ad}(g_t)\zeta_t, h_t) = \mathrm{Ad}(h_t^{-1})^\alpha_I\,\mathrm{Ad}(g_t)^I_J\,\zeta^J_t = \mathrm{Ad}(h_t^{-1}\,g_t)^\alpha_I\,\zeta^I_t.$$

Since $g u_0$: $\mathbf{R}^d\to T_{go}(M)$ preserves the inner product we have $\sum_\alpha F^\alpha(X,g)^2$ $=\|X^*_{go}\|^2$. Therefore (4.4) is now obtained from (5.25).

Next we will consider the case that (g_t) is smooth with respect to t. Let ω_t be the 1-form defined in Theorem 4.2. For $g\in G$, let $(g u_0)^*$: $T^*_{go}(M)\to(\mathbf{R}^d)^*$ be a dual operator of $g u_0$: $\mathbf{R}^d\to T_{go}(M)$. Then for $\xi\in\mathbf{R}^d$, it holds that

$$\begin{aligned} ((g u_0)^*\omega_t, \xi) &= (\omega_t, (g u_0)\xi) = \langle [\mathrm{Ad}(g_t)\zeta_t]^*, (g u_0)\xi\rangle \\ &= \langle (g u_0)^{-1}[\mathrm{Ad}(g_t)\zeta_t]^*, \xi\rangle = \langle F(\mathrm{Ad}(g_t)\zeta_t, g_t), \xi\rangle \end{aligned}$$

where (,) is the natural bilinear form between the vector space and its dual space. $(gu_0)^*\omega_t$ is called a *scalarization* of ω_t and we denote its components by $(\omega_t)_\alpha(gu_0)$ $\alpha=1,\dots,d$. Hence we have $(\omega_t)_\alpha(gu_0)=F^\alpha(\mathrm{Ad}(g_t)\zeta_t, g)$. The integral of the 1-form along the path (X_t) is represented by the scalarization, i.e., the following equality holds (see, e.g., [7], [8], [23])

$$\begin{aligned}\int_0^t \omega_s \circ dX_s &= \int_0^t (\omega_s)_\alpha(h_s u_0)\, dB_s^\alpha + \tfrac{1}{2}\int_0^t (\delta\omega_s)(X_s)\, ds \\ &= \int_0^t F^\alpha(\mathrm{Ad}(g_s)\zeta_s, h_s)\, dB_s^\alpha + \tfrac{1}{2}\int_0^t \mathrm{div}[(\mathrm{Ad}(g_s)\zeta_s)^*](X_s)\, ds\end{aligned}$$

where δ is the dual operator of the exterior derivative d. Thus (4.5) is obtained and this completes the proof.

Remark. We have studied only the sufficient condition and it is left open whether *(A.1)*, *(A.2)* and *(A.3)* are the necessary conditions for equivalence of μ and ν. The condition *(A.1)* is evidently necessary in order that our problem is well-posed. In the Euclidean case we can actually prove that *(A.2)* and *(A.3)* are necessary as we shall see.

Let $(B_t;\ 0\leqq t\leqq T)$ be the d-dimensional Brownian motion starting at 0. Let a continuous mapping $t\mapsto u(t)$ from $(0,T]$ into $SO(d)$ and a continuous mapping $t\mapsto a(t)$ from $[0,T]$ into $\mathbf{R}^d$ such that $a(0)=0$ be given. Define the d-dimensional continuous process $\xi=(\xi_t;\ 0\leqq t\leqq T)$ as follows;

$$\xi_t^\alpha = u_\beta^\alpha(t) B_t^\beta + a^\alpha(t) \qquad \alpha=1,\dots,d$$

where $(u_\beta^\alpha(t))$ and $(a^\alpha(t))$ are the components of $u(t)$ and $a(t)$ respectively. Suppose that the laws of (B_t) and (ξ_t) are equivalent. Since (ξ_t) is a Gaussian process, we can appeal to the following theorem due to Shepp. Here we need the necessary condition part of his theorem which may be stated as follows:

Theorem 5.3. (Shepp [19]) *Let $\eta=(\eta_t;\ 0\leqq t\leqq T)$ be a d-dimensional Gaussian process with mean $(m(t);\ 0\leqq t\leqq T)$ and the covariance $(R(s,t);\ 0\leqq s,\ t\leqq T)$;*

$$\begin{aligned} m^\alpha(t) &= E[\eta_t^\alpha] \qquad \alpha=1,\dots,d, \\ R^{\alpha\beta}(s,t) &= E[(\eta_s^\alpha - m^\alpha(s))(\eta_t^\beta - m^\beta(t))] \qquad \alpha,\beta=1,\dots,d.\end{aligned}$$

Suppose that the law of $\eta=(\eta_t)$ and the Wiener measure are equivalent. Then $(m(t))$ and $(R(s,t))$ satisfy the following conditions:

(i) *for every α there exists a function $k^\alpha\in L^2([0,T])$ for which*

$$m^\alpha(t)=\int_0^t k^\alpha(s)\, ds,$$

(ii) *for every α, β there exists a function $K^{\alpha\beta}\in L^2([0,T]\times[0,T])$ for which*

$$R^{\alpha\beta}(s,t)=\delta^{\alpha\beta}(s\wedge t)+\int_0^s\int_0^t K^{\alpha\beta}(u,v)\, du\, dv.$$

In our case, the mean vector $(m(t))$ and the covariance matrix $(R(s,t))$ of (ξ_t) are given by

$$m^\alpha(t)=a^\alpha(t), \tag{5.26}$$

$$R^{\alpha\beta}(s,t)=u^\alpha_\gamma(s)\,u^\beta_\gamma(t)(s\wedge t). \tag{5.27}$$

From the condition (i) of Theorem 5.3 and (5.26), we have that $(a^\alpha(t))$ is absolutely continuous and $\int_0^T \dot a^\alpha(t)^2\,dt<\infty$. By the condition (ii) of Theorem 5.3 and (5.27), there exists a function $K^{\alpha\beta}\in L^2([0,T]\times[0,T])$ such that

$$u^\alpha_\gamma(s)\,u^\beta_\gamma(t)\,(s\wedge t)=\delta^{\alpha\beta}(s\wedge t)+\int_0^s\int_0^t K^{\alpha\beta}(u,v)\,du\,dv. \tag{5.28}$$

Hence it is easy to see that $(u^\alpha_\beta(t))$ is absolutely continuous on $(0,T]$. From now on we will consider on the set $s<t$. By differentiating the both hands of (5.28) with respect to s and t successively we have

$$K^{\alpha\beta}(s,t)=(\dot u^\alpha_\gamma(s)s+u^\alpha_\gamma(s))\,\dot u^\beta_\gamma(t).$$

Hence

$$\begin{aligned}\sum_{\alpha,\beta}\int_0^T dt\int_0^t K^{\alpha\beta}(s,t)^2\,ds&=\sum_{\alpha,\beta}\int_0^T dt\int_0^t \{(\dot u^\alpha_\gamma(s)s+u^\alpha_\gamma(s))\,\dot u^\beta_\gamma(t)\}^2\,ds\\&=\int_0^T dt\int_0^t \{\dot u^\alpha_\gamma(s)\,\dot u^\alpha_\eta(s)s^2+(\dot u^\alpha_\gamma(s)\,u^\alpha_\eta(s)+u^\alpha_\gamma(s)\,\dot u^\alpha_\eta(s))s\\&\quad+u^\alpha_\gamma(s)\,u^\alpha_\eta(s)\}\,\dot u^\beta_\gamma(t)\,\dot u^\beta_\eta(t)\,ds.\end{aligned}$$

By noting that $u^\alpha_\gamma(s)\,u^\alpha_\eta(s)=\delta_{\gamma\eta}$ and $\dot u^\alpha_\gamma(s)\,u^\alpha_\eta(s)+u^\alpha_\gamma(s)\,\dot u^\alpha_\eta(s)=0$, we have

$$\begin{aligned}\sum_{\alpha,\beta}\int_0^T dt\int_0^t K^{\alpha\beta}(s,t)^2\,ds&=\int_0^T dt\int_0^t \{\dot u^\alpha_\gamma(s)\,\dot u^\alpha_\eta(s)\,\dot u^\beta_\gamma(t)\dot u^\beta_\eta(t)\,s^2+\dot u^\beta_\gamma(t)\,\dot u^\beta_\eta(t)\,\delta_{\gamma\eta}\}\,ds\\&=\sum_{\alpha,\beta}\int_0^T dt\int_0^t \{\dot u^\alpha_\gamma(s)\,\dot u^\beta_\gamma(t)s\}^2\,ds+\sum_{\beta,\gamma}\int_0^T t\,\dot u^\beta_\gamma(t)^2\,dt.\end{aligned}$$

Since $K^{\alpha\beta}\in L^2([0,T]\times[0,T])$, we have $\int_0^T t\,\dot u^\beta_\gamma(t)^2\,dt<\infty$. Thus we obtained the necessity of *(A.2)* and *(A.3)* and this proves that *(A.1) (A.2) (A.3)* are necessary and sufficient conditions in the Euclidean case.

6. Some Remarks on the Stochastic Differential Equation with the Singularity

We give here the proof of the fact which was reserved in the previous section. Let the indices $\alpha,\beta,\ldots,i,j,\ldots$ and $I,J,\ldots$ be as before. Let $a_\alpha=(a^I_\alpha)$, $b_\alpha=(b^I_\alpha)$ and $c_{Ii}=(c^J_{Ii})$ be $\mathbf{R}^n$-valued continuous functions defined on $\mathbf{R}^n$. Assume that a_α, b_α and c_{Ii} satisfy the following condition; for any $N>0$, there exists a positive constant $L=L_N$ such that for any $x, y\in\mathbf{R}^n$, $|x|\leqq N$, $|y|\leqq N$,

$$\sum_\alpha |a_\alpha(x)-a_\alpha(y)|^2+\sum_\alpha |b_\alpha(x)-b_\alpha(y)|^2+\sum_{I,i} |c_{Ii}(x)-c_{Ii}(y)|^2\leqq L_N|x-y|^2 \quad (6.1)$$

where $|\cdot|$ is the n-dimensional Euclidean norm, i.e., $|x|^2=\sum_I (x^I)^2$. Let $\eta_t=(\eta_t^I)$ be an $\mathbf{R}^d$-valued function defined on $t\in[0, T]$ satisfying

$$\int_0^T (\eta_t^\alpha)^2\, dt<\infty \qquad \alpha=1, \dots, d, \quad (6.2)$$

$$\int_0^T t(\eta_t^i)^2\, dt<\infty \qquad i=d+1, \dots, n. \quad (6.3)$$

We consider the following stochastic differential equation;

$$\begin{aligned} dX_t^I &= a_\alpha^I(X_t)\, dB_t^\alpha+\{b_\alpha^I(X_t)\eta_t^\alpha+c_{Ji}^I(X_t)(X_t^J-x_0^J)\eta_t^i\}\, dt \\ X_0^I &= x_0^I \end{aligned} \quad (6.4)$$

where $x_0=(x_0^I)$ is a fixed point in $\mathbf{R}^n$.

First we establish the uniqueness of the solution of (6.4). We prepare some lemmas which are the extensions of Gronwall's inequality.

Lemma 6.1. *Let ϕ and ψ be non-negative measurable functions on $[0, T]$ such that $\int_0^T \phi(t)\, dt<\infty$ and $\int_0^T t\psi(t)\, dt<\infty$. Let $\{u_n(t)\}_{n=0,1,\dots}$ be a sequence of non-negative measurable functions on $[0, T]$. Suppose that there exist constants a and c such that*

$$u_0(t)\leqq at \quad \text{and}$$

$$u_{n+1}(t)\leqq ct+\int_0^t u_n(s)\,\phi(s)\, ds+t\int_0^t u_n(s)\,\psi(s)\, ds \qquad n=0, 1, \dots$$

Then if we set $\rho(t)=\int_0^t (\phi(s)+s\psi(s))\, ds$, it holds that

$$u_n(t)\leqq ct\sum_{m=0}^{n-1}\frac{1}{m!}\rho(t)^m+at\frac{1}{n!}\rho(t)^n, \qquad n=0, 1, \dots$$

Proof. Define a sequence of functions $\{I_n(t)\}_{n=0,1,\dots}$ inductively by

$$I_0(t)=1 \quad \text{and}$$

$$I_{n+1}(t)=\int_0^t (\phi(s)+s\psi(s))\, I_n(s)\, ds \qquad n=0, 1, \dots$$

Then it is clear that $I_n(t)=\frac{1}{n!}\rho(t)^n$. Hence it suffices to show that

$$u_n(t)\leqq ct\sum_{m=0}^{n-1} I_m(t)+at\, I_n(t) \qquad n=0, 1, \dots \quad (6.5)$$

We prove it by induction on n. It is clear when $n=0$. Assume (6.5) for n. Then

$$\begin{aligned} u_{n+1}(t) &\leqq ct + \int_0^t \left\{ cs \sum_{m=0}^{n-1} I_m(s) + as\, I_n(s) \right\} \phi(s)\, ds \\ &\quad + t \int_0^t \left\{ cs \sum_{m=0}^{n-1} I_m(s) + as\, I_n(s) \right\} \psi(s)\, ds \\ &\leqq ct + ct \sum_{m=0}^{n-1} \int_0^t I_m(s) \{\phi(s) + s\psi(s)\}\, ds \\ &\quad + at \int_0^t I_n(s) \{\phi(s) + s\psi(s)\}\, ds \\ &= ct \sum_{m=0}^{n} I_m(t) + at\, I_{n+1}(t). \end{aligned}$$

This completes the proof. □

Lemma 6.2. *Let ϕ and ψ be as in Lemma 6.1 and u be a non-negative measurable function on $[0, T]$. Suppose that there exist constants a and c such that*

$$u(t) \leqq at \quad \text{and}$$

$$u(t) \leqq ct + \int_0^t u(s)\, \phi(s)\, ds + t \int_0^t u(s)\, \psi(s)\, ds.$$

Then it holds that

$$u(t) \leqq ct \exp \rho(t) \tag{6.6}$$

where $\rho(t) = \int_0^t (\phi(s) + s\psi(s))\, ds$. In particular, if $c=0$ then $u(t)=0$ for $t \in [0, T]$.

Proof. From Lemma 6.1, we have

$$u(t) \leqq ct \sum_{m=0}^{n-1} \frac{1}{m!} \rho(t) + at \frac{1}{n!} \rho(t) \leqq ct \exp \rho(t) + at \frac{1}{n!} \rho(t).$$

By letting $n \to \infty$ we have (6.6). □

Now we go back to the uniqueness of the solution.

Theorem 6.1. *Suppose that the conditions* (6.1), (6.2), (6.3) *are satisfied and let $X = (X_t)$ and $Y = (Y_t)$ be any tow solutions of* (6.4). *Assume further that both X and Y satisfy the following conditions: for any $N > 0$, there exists a constant $c = c_N$ such that*

$$\begin{aligned} E[|X_t - x_0|^2 1_{\{t \leqq \tau_N\}}] &\leqq ct \\ E[|Y_t - x_0|^2 1_{\{t \leqq \sigma_N\}}] &\leqq ct \end{aligned} \tag{6.7}$$

where τ_N and σ_N are exit times of X and Y from $B(x_0; N) = \{x \in \mathbf{R}^n \mid |x - x_0| \leqq N\}$ respectively. Then $P^W[X_t = Y_t$ for all $t \in [0, T]] = 1$.

Proof. For any $N > 0$, take L and c satisfying (6.1) and (6.7) respectively. Set $\kappa = \tau_N \wedge \sigma_N$. Since X and Y are solutions of (6.4), we have

$$X^I_{t\wedge\kappa}-Y^I_{t\wedge\kappa}=\int_0^{t\wedge\kappa}\{a^I_\alpha(X_s)-a^I_\alpha(Y_s)\}\,dB^\alpha_s+\int_0^{t\wedge\kappa}\{b^I_\alpha(X_s)-b^I_\alpha(Y_s)\}\eta^\alpha_s\,ds$$
$$+\int_0^{t\wedge\kappa}\{c^I_{Ji}(X_s)-c^I_{Ji}(Y_s)\}(X^J_s-x^J_0)\eta^i_s\,ds$$
$$+\int_0^{t\wedge\kappa}c^I_{Ji}(Y_s)(X^J_s-Y^J_s)\eta^i_s\,ds.$$

Hence by Schwarz's inequality we have

$$(X^I_{t\wedge\kappa}-Y^I_{t\wedge\kappa})^2\leqq 4\left\{\int_0^{t\wedge\kappa}(a^I_\alpha(X_s)-a^I_\alpha(Y_s))\,dB^\alpha_s\right\}^2$$
$$+4t\int_0^{t\wedge\kappa}\sum_\alpha(b^I_\alpha(X_s)-b^I_\alpha(Y_s))^2\sum_\beta(\eta^\beta_s)^2\,ds$$
$$+4t\int_0^{t\wedge\kappa}\sum_{J,i}(c^I_{Ji}(X_s)-c^I_{Ji}(Y_s))^2\sum_K(X^K_s-x^K_0)^2\sum_j(\eta^j_s)^2\,ds$$
$$+4t\int_0^{t\wedge\kappa}\sum_{J,i}c^I_{Ji}(Y_s)^2\sum_K(X^K_s-Y^K_s)^2\sum_j(\eta^j_s)^2\,ds.$$

Since c_{Ii}'s are continuous, $M=\sup_{x\in B(x_0;N)}\sum_{I,i}|c_{Ii}(x)|^2<\infty$. Hence from (6.1) we have

$$E[|X_{t\wedge\kappa}-Y_{t\wedge\kappa}|^2]$$
$$\leqq 4E\left[\int_0^{t\wedge\kappa}\sum_\alpha|a_\alpha(X_s)-a_\alpha(Y_s)|^2\,ds\right]$$
$$+4tE\left[\int_0^{t\wedge\kappa}\sum_\alpha|b_\alpha(X_s)-b_\alpha(Y_s)|^2\sum_\beta(\eta^\beta_s)^2\,ds\right]$$
$$+4tE\left[\int_0^{t\wedge\kappa}\sum_{J,i}|c_{Ji}(X_s)-c_{Ji}(Y_s)|^2\,|X_s-x_0|^2\sum_j(\eta^j_s)^2\,ds\right]$$
$$+4tE\left[\int_0^{t\wedge\kappa}\sum_{J,i}|c_{Ji}(Y_s)|^2\,|X_s-Y_s|^2\sum_j(\eta^j_s)^2\,ds\right]$$
$$\leqq 4LE\left[\int_0^{t\wedge\kappa}|X_s-Y_s|^2\,ds\right]+4tLE\left[\int_0^{t\wedge\kappa}|X_s-Y_s|^2\sum_\alpha(\eta^\alpha_s)^2\,ds\right]$$
$$+4tLN^2E\left[\int_0^{t\wedge\kappa}|X_s-Y_s|^2\sum_i(\eta^i_s)^2\,ds\right]+4tME\left[\int_0^{t\wedge\kappa}|X_s-Y_s|^2\sum_j(\eta^j_s)^2\,ds\right]$$
$$\leqq 4L\int_0^t E[|X_s-Y_s|^2\,1_{\{s\leqq\kappa\}}]\,ds+4TL\int_0^t E[|X_s-Y_s|^2\,1_{\{s\leqq\kappa\}}]\sum_\alpha(\eta^\alpha_s)^2\,ds$$
$$+4t(LN^2+M)\int_0^t E[|X_s-Y_s|^2\,1_{\{s\leqq\kappa\}}]\sum_i(\eta^i_s)^2\,ds.$$

If we set $u(s)=E[|X_s-Y_s|^2\,1_{\{s\leqq\kappa\}}]$, then we have

$$u(t) \leqq \int_0^t u(s)\left\{4L + 4TL\sum_\alpha (\eta_s^\alpha)^2\right\} ds + t\int_0^t u(s)\, 4(LN^2+M)\sum_i (\eta_s^i)^2\, ds.$$

From (6.7) it is easy to see that $u(t) \leqq 2ct$ and, by Lemma 6.2, we can conclude that $u(t)=0$. Hence we have $\tau_N = \sigma_N$ a.e., and $P^W[X_t = Y_t \text{ for } t \leqq \tau_N] = 1$. By letting $N \to \infty$, we obtain a desired conclusion.

Remark. Without assuming (6.7), the uniqueness may fail. For example, let us consider the following ordinary differential equation on the interval $[0, \frac{1}{2}]$;

$$\begin{aligned} dX_t &= -(X_t/t\log t)\, dt \\ X_0 &= 0 \end{aligned} \tag{6.8}$$

Since $\int_0^{1/2} \{t/(t\log t)^2\}\, dt < \infty$, (6.2) and (6.3) hold. But for any constant c, $X_t = c/\log t$ is a solution of (6.8) and hence the uniqueness does not hold. Note also that $X_t = 0$ is a unique solution which satisfies (6.7).

Next we will discuss the existence.

Theorem 6.2. *Let η be as in Theorem 6.3. Suppose that a_α, b_α and c_{Ii} satisfy the following conditions; there exist positive constants L and K such that*

$$\begin{aligned} &\sum_\alpha |a_\alpha(x) - a_\alpha(y)|^2 + \sum_\alpha |b_\alpha(x) - b_\alpha(y)|^2 \\ &\qquad + \sum_i |c_{Ii}(x)(x^I - x_0^I) - c_{Ii}(y)(y^I - x_0^I)|^2 \leqq L|x-y|^2 \end{aligned} \tag{6.9}$$

for $x, y \in \mathbf{R}^n$ and

$$\sum_\alpha |a_\alpha(x)|^2 + \sum_\alpha |b_\alpha(x)|^2 \leqq K(1 + |x - x_0|^2) \tag{6.10}$$

$$\sum_{I,i} |c_{Ii}(x)|^2 \leqq K \tag{6.11}$$

for $x \in \mathbf{R}^n$. Then there exists a solution of (6.4) *such that $E[|X_t - x_0|^2] \leqq mt$, $0 \leqq t \leqq T$ for some positive constant m.*

Proof. We construct a solution by the method of successive approximations. We define a sequence $\{X_r = (X_r(t))\}_{r=0,1,\dots}$ of n-dimensional continuous processes inductively by

$$\begin{aligned} &X_0^I(t) = x_0^I \quad \text{and} \\ &X_{r+1}^I(t) = x_0 + \int_0^t a_\alpha^I(X_r(t))\, dB_t^\alpha + \int_0^t \{b_\alpha^I(X_r(t))\eta_t^\alpha \\ &\qquad + c_{Ji}^I(X_r(t))(X_r^J(t) - x_0^J)\eta_t^i\}\, dt \qquad r = 0, 1, \dots \end{aligned} \tag{6.12}$$

By Doob-Kolmogorov's inequality we have

$$
\begin{aligned}
&E[\sup_{0\leqq s\leqq t}|X_{r+1}(s)-x_0|^2]\\
&\leqq 3\sum_I E\left[\sup_{0\leqq s\leqq t}\left\{\int_0^s a_\alpha^I(X_r(v))\,dB_v^\alpha\right\}^2\right]+3\sum_I E\left[\sup_{0\leqq s\leqq t}\left\{\int_0^s b_\alpha^I(X_r(v))\,\eta_v^\alpha\,dv\right\}^2\right]\\
&\quad+3\sum_I E\left[\sup_{0\leqq s\leqq t}\left\{\int_0^s c_{Ji}^I(X_r(v))(X_r^J(v)-x_0^J)\eta_v^i\,dv\right\}^2\right]\\
&\leqq 12E\left[\int_0^t\sum_\alpha |a_\alpha(X_r(s))|^2\,ds\right]+3tE\left[\int_0^t\sum_\alpha |b_\alpha(X_r(s))|^2\sum_\beta(\eta_s^\beta)^2\,ds\right]\\
&\quad+3tE\left[\int_0^t\sum_{I,i}|c_{Ii}(X_r(s))|^2\,|X_r(s)-x_0|^2\sum_j(\eta_s^j)^2\,ds\right]\\
&\leqq 12E\left[\int_0^t K(1+|X_r(s)-x_0|^2)\,ds\right]+3tE\left[\int_0^t K(1+|X_r(s)-x_0|^2)\sum_\alpha(\eta_s^\alpha)^2\,ds\right]\\
&\quad+3tE\left[\int_0^t K|X_r(s)-x_0|^2\sum_i(\eta_s^i)^2\,ds\right].
\end{aligned}
$$

If we set $u_r(t)=E[\sup_{0\leqq s\leqq t}|X_r(s)-x_0|^2]$, then we have

$$
\begin{aligned}
u_{r+1}(t)\leqq &\left(12K+3K\int_0^T\sum_\alpha(\eta_s^\alpha)^2\,ds\right)t+\int_0^t u_r(s)\left\{12K+3TK\sum_\alpha(\eta_s^\alpha)^2\right\}ds\\
&+t\int_0^t u_r(s)\,3K\sum_i(\eta_s^i)^2\,ds.
\end{aligned}
$$

Note also that $u_0(t)=0$. Then from Lemma 6.1, we have

$$
u_r(t)\leqq ct\sum_{k=0}^{r-1}\frac{1}{k!}\rho(t)^k\leqq ct\exp\rho(t) \tag{6.13}
$$

where

$$
c=12K+3K\int_0^T\sum_\alpha(\eta_s^\alpha)^2\,ds,
$$

$$
\rho(t)=\int_0^t\{12K+3TK\sum_\alpha(\eta_s^\alpha)^2+3Ks\sum_i(\eta_s^i)^2\}\,ds.
$$

This implies that (6.12) is well-defined. We also have

$$
\begin{aligned}
&E[\sup_{0\leqq s\leqq t}|X_{r+1}(s)-X_r(s)|^2]\\
&\leqq 12E\left[\int_0^t\sum_\alpha|a_\alpha(X_r(s))-a_\alpha(X_{r-1}(s))|^2\,ds\right]\\
&\quad+3tE\left[\int_0^t\sum_\alpha|b_\alpha(X_r(s))-b_\alpha(X_{r-1}(s))|^2\sum_\beta(\eta_s^\beta)^2\,ds\right]\\
&\quad+3tE\left[\int_0^t\sum_i|c_{Ii}(X_r(s))(X_r^I(s)-x_0^I)\right.\\
&\quad\left.-c_{Ii}(X_{r-1}(s))(X_{r-1}^I(s)-x_0^I)|^2\sum_j(\eta_s^j)^2\,ds\right]
\end{aligned}
$$

$$\leq 12L\int_0^t E[|X_r(s)-X_{r-1}(s)|^2]\,ds$$

$$+3TL\int_0^t E[|X_r(s)-X_{r-1}(s)|^2]\sum_\alpha (\eta_s^\alpha)^2\,ds$$

$$+3tL\int_0^t E[|X_r(s)-X_{r-1}(s)|^2]\sum_i (\eta_s^i)^2\,ds.$$

If we set $v_r(t)=E[\sup_{0\leq s\leq t}|X_{r+1}(s)-X_r(s)|^2]$ then we have

$$v_r(t)\leq\int_0^t v_{r-1}(s)\{12L+3TL\sum_\alpha(\eta_s^\alpha)^2\}\,ds+t\int_0^t v_{r-1}(s)\,3L\sum_i(\eta_s^i)^2\,ds.$$

From (6.13), it holds that $v_0(t)\leq ct$. Then from Lemma 6.1, we have $v_r(t)\leq ct\,\xi(t)^r/r!$ where

$$\xi(t)=\int_0^t\{12L+3TL\sum_\alpha(\eta_s^\alpha)^2+3Ls\sum_i(\eta_s^i)^2\}\,ds.$$

Hence we have

$$P^W[\sup_{0\leq t\leq T}|X_{r+1}(t)-X_r(t)|>1/2^r]\leq 2^{2r}c\,T\xi(T)^r/r!.$$

By Borel-Cantelli's lemma we see that $(X_r(t))$ converges uniformly on $[0,T]$ a.e. Set $X_t=\lim_{r\to\infty}X_r(t)$. By letting $r\to\infty$ in (6.13), we have

$$E[\sup_{0\leq s\leq t}|X_s-x_0|^2]\leq ct\exp\rho(t).$$

On the other hand, since $E[\sup_{0\leq s\leq t}|X_{r+1}(s)-X_r(s)|^2]\leq ct\,\xi(t)^r/r!$, we have for $p,q\in\mathbf{N}$, $p<q$

$$E[\sup_{0\leq s\leq t}|X_q(t)-X_p(t)|^2]\leq ct\left(\sum_{k=p}^{q-1}\sqrt{\xi(t)^k/k!}\right)^2.$$

By letting $q\to\infty$, we have

$$E[\sup_{0\leq s\leq t}|X_t-X_p(t)|^2]\leq ct\left(\sum_{k=p}^{\infty}\sqrt{\xi(t)^k/k!}\right)^2.$$

Now it is easy to see that (X_t) is a solution of (6.4) and this completes the proof. □

If we assume only (6.1), (6.2) and (6.3), then the global solution may not exists. But we can show, by a truncation argument, that there exists a unique solution satisfying (6.7) up to the explosion time. Note that this solution is non anticipative functional of the Brownian motion (B_t): in fact this is clear from the above construction by successive approximations. If we replace the Brownian motion by a semimartingale (Z_t) such that the law of (Z_t) is absolutely continuous with respect to the Wiener measure, then the same result holds.

Moreover both functionals on W_0^d coincide. This fact is exactly what we needed in the previous section.

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