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Necessary and Sufficient Qualitative Axioms for Conditional Probability

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1. Introduction

In a previous paper (Suppes and Zanotti, 1976) we gave simple necessary and sufficient qualitative axioms for the existence of a unique expectation function for the set of extended indicator functions. As we defined this set of functions earlier, it is the closure of the set of indicator functions of events under function addition. In the present paper we extend the same approach to conditional probability. One of the more troublesome aspects of the qualitative theory of conditional probability is that $A|B$ is not an object – in particular it is not a new event composed somehow from events A and B. Thus the qualitative theory rests on a quaternary relation $A|B \geqq C|D$, which is read: event A given event B is at least as probable as event C given event D. There have been a number of attempts to axiomatize this quaternary relation (Koopman, 1940a, 1940b; Aczél, 1961, 1966, p. 319; Luce, 1968; Domotor, 1969; Krantz et al., 1971; and Suppes, 1973). The only one of these axiomatizations to address the problem of giving necessary and sufficient conditions is the work of Domotor, which approaches the subject in the finite case in a style similar to that of Scott (1964).

By using indicator functions or, more generally, extended indicator functions, the difficulty of $A|B$ not being an object is eliminated, for $A^i|B$ is just the indicator function of the set A restricted to the set B, that is, $A^i|B$ is a partial function whose domain is B. In similar fashion if X is an extended indicator function, $X|A$ is that function restricted to the set A. The use of such partial functions requires care in formulating the algebra of functions in which we are interested, for functional addition $X|A + Y|B$ will not be well defined when $A \neq B$ but $A \cap B \neq \emptyset$. Thus, to be completely explicit we begin with a nonempty set Ω, the probability space, and an algebra $\mathscr{F}$ of events, that is, subsets of Ω, with it understood that $\mathscr{F}$ is closed under union and complementation. Next we extend this algebra to the algebra $\mathscr{F}^*$ of extended indicator functions, that is, the smallest semigroup (under function addition) containing the indicator functions of all events in $\mathscr{F}$. This latter algebra is now

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extended to include as well all partial functions on Ω that are extended indicator functions restricted to an event in $\mathscr{F}$. We call this algebra of partial extended indicator functions $\mathscr{R}\mathscr{F}^*$, or, if complete explicitness is needed, $\mathscr{R}\mathscr{F}^*(\Omega)$. From this definition it is clear that if $X|A$ and $Y|B$ are in $\mathscr{R}\mathscr{F}^*$, then

(i) If $A=B$, $X|A+Y|B$ is in $\mathscr{R}\mathscr{F}^*$.
(ii) If $A\cap B=\emptyset$, $X|A\cup Y|B$ is in $\mathscr{R}\mathscr{F}^*$.

In the more general setting of decision theory or expected utility theory there has been considerable discussion of the intuitive ability of a person to directly compare his preferences or expectations of two decision functions with different domains of restriction. Without reviewing this literature, we do want to state that we find no intuitive general difficulty in making such comparisons. Individual cases may present problems, but not necessarily because of different domains of definition. In fact, we believe comparisons of expectations under different conditions is a familiar aspect of ordinary experience. In the present setting the qualitative comparison of restricted expectations may be thought of as dealing only with beliefs and not utilities. The fundamental ordering relation is a weak ordering $\geqq$ of $\mathscr{R}\mathscr{F}^*$ with strict order $>$ and equivalence $\sim$ defined in the standard way.

The axioms we give are strong enough to prove that the probability measure constructed is unique when it is required to cover expectation of random variables. It is worth saying something more about this problem of uniqueness. The earlier papers mentioned have all concentrated on the existence of a probability distribution, but from the standpoint of a satisfactory theory it seems obvious for many different reasons that one wants a unique distribution. For example, if we go beyond properties of order and have uniqueness only up to a convex polyhedron of distributions, as is the case with Scott's axioms for finite probability spaces, we are not able to deal with a composite hypothesis in a natural way, because the addition of the probabilities is not meaningful.

2. Axioms

We incorporate our axioms in the usual form of a definition.

Definition. Let Ω be a nonempty set, let $\mathscr{R}\mathscr{F}^*(\Omega)$ be an algebra of partial extended indicator functions, and let $\geqq$ be a binary relation on $\mathscr{R}\mathscr{F}^*$. Then the structure $(\Omega, \mathscr{R}\mathscr{F}^*, \geqq)$ is a partial qualitative expectation structure *if and only if the following axioms are satisfied for every X and Y in $\mathscr{F}^*$ and every A, B and C in $\mathscr{F}$ with $A, B>\emptyset$:*

Axiom 1. *The relation $\geqq$ is a weak ordering of $\mathscr{R}\mathscr{F}^*$;*

Axiom 2. $\Omega^i>\emptyset^i$;

Axiom 3. $\Omega^i|A\geqq C^i|B\geqq\emptyset^i|A$;

Axiom 4a. *If* $X_1|A \geqq Y_1|B$ *and* $X_2|A \geqq Y_2|B$ *then*
$$X_1|A + X_2|A \geqq Y_1|B + Y_2|B.$$

Axiom 4b. *If* $X_1|A \leqq Y_1|B$ *and* $X_1|A + X_2|A \geqq Y_1|B + Y_2|B$ *then*
$$X_2|A \geqq Y_2|B.$$

Axiom 5. *If* $A \subseteq B$ *then*
$$X|A \geqq Y|A \text{ iff } X \cdot A^i|B \geqq Y \cdot A^i|B;$$

Axiom 6 *(Archimedean). If* $X|A > Y|B$ *then for every* Z *in* $\mathscr{F}^*$ *there is a positive integer* n *such that*
$$nX|A \geqq nY|B + Z|B.$$

The axioms are simple in character and their relation to the axioms of Suppes and Zanotti (1976) is apparent. The first three axioms are very similar. Axiom 4, the axiom of addition, must be relativized to the restricted set. Notice that we have a different restriction on the two sides of the inequality. We have been unable to show whether or not it is possible to replace the two parts of Axiom 4 by the following weaker and more natural axiom. *If* $X_2|A \sim Y_2|B$, *then* $X_1|A \geqq Y_1|B$ *iff* $X_1|A + X_2|A \geqq Y_1|B + Y_2|B$.

The really new axiom is Axiom 5. In terms of events and numerical probability, this axiom corresponds to the following: *If* $A \subseteq B$, *then* $P(C|A) \geqq P(D|A)$ *iff* $P(C \cap A|B) \geqq P(D \cap A|B)$. Note that in the axiom itself, function multiplication replaces intersection of events. (Closure of $\mathscr{F}^*$ under function multiplication is easily proved.) This axiom does not seem to have previously been used in the literature. Axiom 6 is the familiar and necessary Archimedean axiom.

3. Representation Theorem

We now state and prove the main theorem of this paper. In the theorem we refer to a strictly agreeing expectation function on $\mathscr{R}\mathscr{F}^*(\Omega)$. From standard probability theory and conditional expected utility theory, it is evident that the properties of this expectation should be the following for $A, B > \emptyset$:

(i) $E(X|A) \geqq E(Y|B)$ iff $X|A \geqq Y|B$,

(ii) $E(X|A + Y|A) = E(X|A) + E(Y|A)$,

(iii) $E(X \cdot A^i|B) = E(X|A)\,E(A^i|B)$ if $A \subseteq B$,

(iv) $E(\emptyset^i|A) = 0$ and $E(\Omega^i|A) = 1$.

Using primarily (iii), it is then easy to prove the following property, which occurs in the earlier axiomatic literature mentioned above:
$$E(X|A \cup Y|B) = E(X|A)\,E(A^i|A \cup B) + E(Y|B)\,E(B^i|A \cup B),$$
for $A \cap B = \emptyset$.

Theorem. *Let* Ω *be a nonempty set, let* $\mathscr{F}$ *be an algebra of sets on* Ω, *and let* $\geqq$ *be a binary relation on* $\mathscr{F} \times \mathscr{F}$. *Then a necessary and sufficient condition that* *there is a strictly agreeing conditional probability measure on $\mathscr{F} \times \mathscr{F}$ is that there is an extension $\geqq^*$ of $\geqq$ from $\mathscr{F} \times \mathscr{F}$ to $\mathscr{R}\mathscr{F}^*(\Omega)$ such that the structure $(\Omega, \mathscr{R}\mathscr{F}^*(\Omega), \geqq^*)$ is a partial qualitative expectation structure. Moreover, if $(\Omega, \mathscr{R}\mathscr{F}^*(\Omega), \geqq^*)$ is a partial qualitative expectation structure, then there is a unique strictly agreeing expectation function on $\mathscr{R}\mathscr{F}^*(\Omega)$ and this expectation generates a unique strictly agreeing conditional probability measure on $\mathscr{F} \times \mathscr{F}$.*

Proof. For every $X|A$, with $A > \emptyset$, we define the set

$$S(X|A) = \left\{\frac{m}{n}: m\Omega^i|A \geqq nX|A\right\}.$$

(We note that it is easy to prove from the axioms that $\Omega^i \sim \Omega^i|A$, and thus for general purposes we can write: $m\Omega^i \geqq nX|A$.) Given this definition, on the basis of the reduction by Suppes and Zanotti (1976) of Axioms 1–4 and 6 to a known necessary and sufficient condition for extensive measurement (Krantz et al., 1971, Chap. 3), we know first that the greatest lower bound of $S(X|A)$ exists, and following the proof in Krantz et al. we use this to define the expectation of X given A:

$$E(X|A) = \text{g.l.b.}\left\{\frac{m}{n}: m\Omega^i \geqq nX|A\right\}. \tag{1}$$

It then follows from these earlier results that the function E (for fixed A) is unique and:

$$E(X|A) \geqq E(Y|A) \text{ iff } X|A \geqq Y|A. \tag{2}$$

$$E(X|A + Y|A) = E(X|A) + E(Y|A). \tag{3}$$

$$E(\emptyset^i|A) = 0 \text{ and } E(\Omega^i|A) = 1. \tag{4}$$

The crucial step is now to extend the results to the relation between given events A and B.

We first prove the preservation of order by the expectation function. For the first half of the proof, assume

$$X|A \geqq Y|B, \tag{5}$$

and suppose, on the contrary, that

$$E(Y|B) > E(X|A). \tag{6}$$

Then there must exist natural numbers m and n such that

$$E(Y|B) > \frac{m}{n} > E(X|A), \tag{7}$$

and so from the definition of the function E, we have

$$m\Omega^i < nY|B, \tag{8}$$

and

$$m\Omega^i \geqq nX|A, \tag{9}$$

whence

$$nY|B > nX|A, \tag{10}$$

but from (5) and Axiom 4a we have by a simple induction

$$nX|A \geqq nY|B, \tag{11}$$

which contradicts (10), and thus the supposition (6) is false.

Assume now

$$E(X|A) \geqq E(Y|B), \tag{12}$$

and suppose

$$Y|B > X|A. \tag{13}$$

Now if $E(X|A) > E(Y|B)$, by the kind of argument just given we can show at once that

$$X|A > Y|B, \tag{14}$$

which contradicts (13). On the other hand, if

$$E(X|A) = E(Y|B), \tag{15}$$

then we can argue as follows. By virtue of (13) and Axiom 6, there is an n such that

$$nY|B \geqq (n+1)X|A, \tag{16}$$

whence by the earlier argument

$$E(nY|B) \geqq E((n+1)X|A), \tag{17}$$

and by (3)

$$nE(Y|B) \geqq (n+1)E(X|A), \tag{18}$$

and so by (15) and (18)

$$E(Y|B) \leqq 0, \tag{19}$$

but from (2)–(4) it follows easily that

$$E(Y|B) \geqq 0, \tag{20}$$

whence

$$E(Y|B) = 0, \tag{21}$$

but then, using again (2)–(4), we obtain

$$Y|B \sim \emptyset^i|B, \tag{22}$$

and by virtue of Axiom 3

$$X|A \geqq \emptyset^i|B, \tag{23}$$

whence from (22) and (23) by transitivity

$$X|A \geqq Y|B, \tag{24}$$

contradicting (13). We have thus now shown that

$$E(X|A) \geqq E(Y|B) \text{ iff } X|A \geqq Y|B. \tag{25}$$

Finally, we need to prove that for $A > 0$ and $A \subseteq B$

$$E(X \cdot A^i|B) = E(X|A)\,E(A^i|B). \tag{26}$$

We first note that by putting $m\Omega^i$ for X and nX for Y in Axiom 5, we obtain

$$m\Omega^i \geqq nX|A \text{ iff } mA^i|B \geqq nX \cdot A^i|B. \tag{27}$$

It follows directly from (27) that

$$\left\{\frac{m}{n}: m\Omega^i \geqq nX|A\right\} = \left\{\frac{m}{n}: mA^i|B \geqq nX \cdot A^i|B\right\}, \tag{28}$$

whence their greatest lower bounds are the same, and we have

$$E(X|A) = E'_{A^i|B}(X \cdot A^i|B), \tag{29}$$

where E' is the measurement function that has $A^i|B$ as a unit, that is,

$$E'_{A^i|B}(A^i|B) = 1.$$

As is familiar in the theory of extensive measurement, there exists a positive real number c such that for every X

$$cE'_{A^i|B}(X \cdot A^i|B) = E(X \cdot A^i|B). \tag{30}$$

Now by (29) and taking $X = \Omega^i$

$$cE(\Omega^i|A) = E(\Omega^i \cdot A^i|B),$$

but $E(\Omega^i|A) = 1$, so

$$c = E(\Omega^i \cdot A^i|B) = E(A^i|B). \tag{31}$$

Combining (29), (30) and (31) we obtain (26) as desired.

The uniqueness of the expectation function follows from (4) and the earlier results (Suppes and Zanotti, 1976) about unconditional probability.

For $A > \emptyset$, we then define for every B in $\mathscr{F}$,

$$P(B|A) = E(B^i|A),$$

and it is trivial to show the function P is a conditional probability measure on $\mathscr{F}$, which establishes the sufficiency of the axioms. The necessity of each of the axioms is easily checked.

References

Aczél, J.: Über die Begründung der Additions- und Multiplikationsformeln von bedingten Wahrscheinlichkeiten. Magyar Tudományos Akadémia Matematikai Kutató Int. Közleményei **6**, 110–122 (1961)

Aczél, J.: Lectures on functional equations and their applications. New York: Academic Press 1966

Domotor, Z.: Probabilistic relational structures and their applications (Tech. Rep. 144). Stanford, Calif.: Stanford University, Institute for Mathematical Studies in the Social Sciences 1969

Koopman, B.O.: The axioms and algebra of intuitive probability. Ann. Math. **41**, 269–292 (1940a)

Koopman, B.O.: The bases of probability. Bull. Amer. Math. Soc. **46**, 763–774 (1940b)

Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A.: Foundations of Measurement (Vol. I). New York: Academic Press 1971

Luce, R.D.: On the numerical representation of qualitative conditional probability. Ann. Math. Statis. **39**, 481–491 (1968)

Roberts, F.S., Luce, R.D.: Axiomatic thermodynamics and extensive measurement. Synthese **18**, 311–326 (1968)

Scott, D.: Measurement structures and linear inequalities. Journal of Mathematical Psychology **1**, 233–247 (1964)

Suppes, P.: New foundations of objective probability: Axioms for propensities. In P. Suppes, L. Henkin, G.C. Moisil, A. Joja (Eds.), Logic, Methodology, and Philosophy of Science IV: Proceedings of the Fourth International Congress for Logic, Methodology and Philosophy of Science, Bucharest, 1971, pp. 515–529. Amsterdam: North-Holland 1973

Suppes, P., Zanotti, M.: Necessary and sufficient conditions for existence of a unique measure strictly agreeing with a qualitative probability ordering. Journal of Philosophical Logic **5**, 431–438 (1976)

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