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A Method for Solving a Class of Recursive Stochastic Equations

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Summary. The existence and uniqueness of solutions of a special type of recursive stochastic equations is investigated. Such equations occur in many stochastic models in which the stochastic process describing the behaviour of the system will be generated by the so-called input.

For example, let the input be a stationary sequence $\{X_t\}_{t=-\infty}^{+\infty}$ of random variables with values in a measurable space M and let P be the distribution of this sequence. Consider an equation of the form $Z_{t+1}=f^*(X_t, Z_t)$. Assume that there is a system $\{A(t, \{x_i\}_{i=-\infty}^{+\infty}): t\in\Gamma, \{x_i\}_{i=-\infty}^{+\infty}\in M^{\Gamma}\}$ of subsets of the state space Z with the property

$$P(f^*(X_t, A(t, \{X_u\}_{u=-\infty}^{+\infty}))\subseteq A(t+1, \{X_u\}_{u=-\infty}^{+\infty}) \quad \textit{for all } t\in\Gamma,$$
$$0<|A(0, \{X_u\}_{u=-\infty}^{+\infty})|<\infty)=1.$$

Then, under some regularity conditions, there is a stationary solution of the given equation, i.e. there is a stationary sequence $\{Z_t\}_{t=-\infty}^{+\infty}$ for which the equation almost surely holds with respect to the common distribution of $(\{X_t\}_{t=-\infty}^{+\infty}, \{Z_t\}_{t=-\infty}^{+\infty})$. Analogous results are obtained in more general models.

1. Introduction

Recursive equations occur in many stochastic models in which the stochastic process describing the temporal behaviour of the system under consideration will be generated by the so-called input. For almost all realizations of the input the "future" of the realization of the generated process, given its present state, is determined by means of a measurable function, i.e. a recursive equation holds with probability 1. Most queueing systems work in such a way. There are some methods to construct queueing processes, i.e. to solve recursive equations for queueing systems, cf. e.g. [1, 4, 8]. However, all these concepts make use of the special form of these equations given by the type of the queueing system.

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In this paper a general method for the solution of a wide class of recursive stochastic equations will be introduced. This method is a generalization of a construction given in [6] where a stationary state distribution was constructed for queueing systems without delay. The same method may be used in inventory and other fields of stochastic modelling, cf. [7]. As a further application a generalization of Wald's Identity for dependent variables was provided in [5].

2. Basic Notations

Let $[M, \mathfrak{M}]$ be an arbitrary measurable space. A random element Φ of $[M, \mathfrak{M}]$ will be interpreted as the *input*. Let P be the distribution of Φ. For instance, Φ may be a sequence of random variables or a random marked point process, cf. [4]. Now we consider an abelian group of measurable transformations (*shifts*) Θ_t on $[M, \mathfrak{M}]$, where t is an element of an abelian group $[S, +]$. Assume that an order relation $\leqq$ is defined on S. Consider $S^+ = \{t \in S\colon\ 0 \leqq t\}$ and an arbitrary σ-field $\mathfrak{S}^+$ of subsets of S^+. In the following S will be interpreted as the *time axis*. For applications the cases $S = R$ (the set of real numbers) and $S = \Gamma$ (the set of integers) with the corresponding Borel σ-fields are most important.

The input Φ is called *stationary* if its distribution P is invariant with respect to all shifts Θ_t, $t \in S$. A stationary input is called *ergodic* if $P(B) = 0$ or $P(B) = 1$ holds for every invariant set $B \in \mathfrak{M}$, i.e. for every set B for which $\Theta_t \mathrm{B} = \mathrm{B}$ holds for all $t \in S$.

Furthermore, let a measurable state space $[Z, \mathfrak{Z}]$ be given. The space Z^S of functions from S into Z will be used as the *space of trajectories* of the stochastic process to be constructed. Consider the σ-field $\mathfrak{Z}^S$ generated by the cylinder sets on Z^S. Let $A \in \mathfrak{M}$ be an invariant set with $P(A) = 1$ and $\mathfrak{M}_A \underset{\mathrm{df}}{=} \{B \in \mathfrak{M}\colon\ B \subseteq A\}$ the restriction of $\mathfrak{M}$ to A. The stochastic process under investigation is generated by Φ in the following manner: If $\varphi \in A$ is the realization of the input and $z \in Z$ the state of the process at time 0, then the state at each time $t \geqq 0$ is determined by a measurable function $f\colon [A \times Z \times S^+, \mathfrak{M}_A \otimes \mathfrak{Z} \otimes \mathfrak{S}^+] \to [Z, \mathfrak{Z}]$. Assume that f fulfills the conditions $f(\varphi, z, 0) = z$ and

$$f(\varphi, z, t) = f(\Theta_u \varphi, f(\varphi, z, u), t - u) \tag{1}$$

for all $\varphi \in A$, $z \in Z$, and $0 \leqq u \leqq t$.

In order to formally describe the relations between the input and the generated process we consider the product space $[M \times Z^S, \mathfrak{M} \otimes \mathfrak{Z}^S]$. Let π_1 and π_2 be the projections from $M \times Z^S$ into M and Z^S, respectively.

Definition 1. Let $\varphi \in A$ be fixed. Then the element $\tilde{\varphi} \in M \times Z^S$ is called an *extension* of φ iff. $\pi_1 \tilde{\varphi} = \varphi$ and

$$\pi_2 \tilde{\varphi}(t) = f(\Theta_u \pi_1 \tilde{\varphi}, \pi_2 \tilde{\varphi}(u), t - u) \textit{ hold for all } t, u \in S \textit{ with } u \leqq t. \tag{2}$$

That means, given the effect of the input at time u and the state at this time, the whole future of the realization of the process may be calculated by means of the function f. In this sense (2) is a recursive equation. This is especially clear in the case of $S=\Gamma$. Here, in view of (1), the function f is uniquely determined by $f^*(\varphi, z) \underset{\text{df}}{=} f(\varphi, z, 1)$, $\varphi \in A$, $z \in Z$. Then the condition

$$\pi_2 \tilde{\varphi}(t) = f^*(\Theta_{t-1} \pi_1 \tilde{\varphi}, \pi_2 \tilde{\varphi}(t-1)) \qquad \textit{for all } t \in \Gamma \tag{2'}$$

is equivalent to (2).

Definition 2. Consider the input Φ with the distribution P. A random element $\tilde{\Phi}$ of the space $[M \times Z^S, \mathfrak{M} \otimes \mathfrak{Z}^S]$ is called an *extension* of Φ iff. its distribution $\tilde{P}$ fulfills the following conditions:

$$\tilde{P}((.) \times Z^S) = P(.), \tag{3}$$

and there exists a set $C \in \mathfrak{M} \otimes \mathfrak{Z}^S$ with $P(C)=1$ and $C \subseteq \{\tilde{\varphi}: \tilde{\varphi} \textit{ satisfying } (2)\}$.

The latter condition means that Eq. (2) may almost surely hold. Thus an extension is a solution of a recursive stochastic equation of the form (2). In the theory of stochastic differential equations our type of solution is sometimes called a weak solution since there is no requirement about the existence of a measurable mapping $\varphi \mapsto \tilde{\varphi}$. The second component $\pi_2 \tilde{\Phi}$ of an extension $\tilde{\Phi}$ may be interpreted as a state process controlled by Φ. This was the reason to investigate extensions in special queueing models, cf. [6, 7]. In the following we will deal with the existence and uniqueness of extensions for a given input.

3. Existence of a Solution

The following definition will play a central role in our solution of the existence problem.

Definition 3. A family $\mathfrak{A} = \{A(t, \varphi)\colon t \in S, \varphi \in A\}$ will be called *potential state system* iff. the following conditions are fulfilled:

(A) $A(t, \varphi) \subseteq Z$ and $A(t, \varphi) \neq \emptyset$ hold for all $\varphi \in A$ and $t \in S$.

(B) The elements of $A(t, \varphi)$ are numbered for every $t \in S$ and $\varphi \in A$ with $|A(t, \varphi)| < \infty$:

$$A(t, \varphi) = \{z_1(t, \varphi), z_2(t, \varphi), \ldots, z_{|A(t, \varphi)|}(t, \varphi)\}.$$

(C) Let $i \in \Gamma$, $i \geqq 1$, and $t \in S$ be fixed. Then $\{\varphi: i \leqq |A(t, \varphi)| < \infty\} \in \mathfrak{M}_A$, and the mapping $\varphi \mapsto z_i(t, \varphi)$ defined on $\{\varphi: i \leqq |A(t, \varphi)| < \infty\}$ is measurable.

(D) $f(\Theta_u \varphi, A(u, \varphi), t-u) \subseteq A(t, \varphi)$ holds for all $\varphi \in A$ and $t, u \in S$ with $u \leqq t$.

Now suppose that there is a potential state system $\mathfrak{A}$. We define

$$B(v, \varphi) \underset{\text{df}}{=} \bigcap_{t \leqq v} f(\Theta_t \varphi, A(t, \varphi), v-t)$$

for all $\varphi \in A$ and $v \in S$.

Assume that there is a countable subgroup $\{t_j: j \in \Gamma\} \subseteq S$ with the following properties:

(i) $t_j = jt_1$ for all $j \in \Gamma$.

(ii) $t_j \leqq t_k$ for $j \leqq k$.

(iii) For every $t \in S$ there is a $j \in \Gamma$ with $t_j \leqq t$.

(The existence of such a sequence $\{t_j\}_{j=-\infty}^{+\infty}$ is trivial in the cases of $S=\Gamma$ and $S=R$ since $t_j=j$ suffices the conditions.) Then

$$B(v, \varphi) = \bigcap_{j=0}^{-\infty} f(\Theta_{t_j}\varphi, A(t_j, \varphi), v-t_j)$$

holds. Let

$$j(v, \varphi) \underset{\mathrm{df}}{=} \sup\{j \in \Gamma : f(\Theta_{t_j}\varphi, A(t_j, \varphi), v-t_j) = B(v, \varphi)\}.$$

The condition $j(v, \varphi) > -\infty$ is sufficient for $B(v, \varphi) \neq \emptyset$ since

$$|f(\Theta_{t_{j(v,\varphi)}}\varphi, A(t_{j(v,\varphi)}, \varphi), v - t_{j(v,\varphi)})| \geqq 1$$

holds because of (A). It follows from (1) and property (D) that

$$\begin{aligned} B(v, \varphi) &\subseteq f(\Theta_u\varphi, A(u, \varphi), v-u) \\ &\subseteq f(\Theta_t\varphi, A(t, \varphi), v-t) \subseteq A(v, \varphi) \end{aligned} \tag{4}$$

holds for $u \leqq t \leqq v$. From there we obtain

$$\begin{aligned} B(v, \varphi) &= \bigcap_{u \leqq t} f(\Theta_u\varphi, A(u, \varphi), v-u) \\ &= \bigcap_{u \leqq t} f(\Theta_t\varphi, f(\Theta_u\varphi, A(u, \varphi), t-u), v-t) \\ &= f(\Theta_t\varphi, B(t, \varphi), v-t) \end{aligned} \tag{5}$$

for all $t \leqq v$.

We denote the power of $B(v, \varphi)$ by $i_\varphi(v)$ and obtain

$$i_\varphi(v) \leqq i_\varphi(t) \qquad \text{for } t \leqq v \tag{6}$$

from (5). It follows from (4) that $j(v, \varphi) > -\infty$ and $1 \leqq i_\varphi(v) < \infty$ hold for all φ with $|A(v, \varphi)| < \infty$. Let $v \in S$ and $\varphi \in M$ with $|A(v, \varphi)| < \infty$ be fixed. We number the elements of $B(v, \varphi)$ in the same order as they occur in $A(v, \varphi)$ and denote them by $z^1(v, \varphi), \ldots, z^{i_\varphi(v)}(v, \varphi)$. Then the mappings

$$\varphi \in \{\varphi' : i \leqq i_{\varphi'}(v) \leqq |A(v, \varphi')| < \infty\} \mapsto z^i(v, \varphi),$$

$i \geqq 1$, $v \in S$, are measurable. For abbreviation, the notation

$$M_{\mathfrak{A}} \underset{\mathrm{df}}{=} \{\varphi : |A(t_j, \varphi)| < \infty \quad \textit{for all } j \textit{ with } t_j \leqq 0\}$$

will be used. Now we choose a $\varphi \in M_{\mathfrak{A}}$ and an element $z^i(0, \varphi) \in B(0, \varphi)$. Then it follows from (5) that there is an element $z \in B(t_{-1}, \varphi)$ with

$$f(\Theta_{t_{-1}}\varphi, z, t_1) = z^i(0, \varphi). \tag{7}$$

We choose that solution z of (7) which has the minimal number among the elements of $B(t_{-1}, \varphi)$ and denote it by $z(i, t_{-1}, \varphi)$. This construction may be continued recursively: Let $j \geqq 1$ and $z(i, t_{-j}, \varphi) \in B(t_{-j}, \varphi)$ be given. Then there is an element $z \in B(t_{-j-1}, \varphi)$ with

$$f(\Theta_{t_{-j-1}}\varphi, z, t_1) = z(i, t_{-j}, \varphi) \tag{8}$$

because of (5). The solution z of (8) with the minimal number will be denoted by $z(i, t_{-j-1}, \varphi)$. In this way a sequence $\{z(i, t_{-j}, \varphi)\}_{j=1}^{\infty}$ is defined for every $i \in \{1, \ldots, i_\varphi(0)\}$. We define

$$z(i, t, \varphi) \underset{\mathrm{df}}{=} \begin{cases} f(\Theta_{t_j}\varphi, z(i, t_j, \varphi), t - t_j) & \text{for } t_j < t \leqq t_{j+1}, j < 0, \\ f(\varphi, z^i(0, \varphi), t) & \text{for } t > 0. \end{cases} \tag{9}$$

Now from (7), (8), (9), and (1) we get that $\psi_i(\varphi) \underset{\mathrm{df}}{=} (\varphi, z(i, .\,, \varphi))$ is an extension of φ for every $i \leqq i_\varphi(0)$. It follows from (C) and from our construction that the mappings $\varphi \mapsto \psi_i(\varphi)$, $i \geqq 1$, which are defined on $\{\varphi : \varphi \in M_{\mathfrak{A}},\ i \leqq i_\varphi(0)\}$, are measurable. The result of the construction may be summarized as follows:

Theorem 1. *Consider the input Φ with the distribution P and assume that there is a potential state system $\mathfrak{A}$ with $P(M_{\mathfrak{A}}) = 1$. Then $\psi_1(\Phi)$ is an extension of Φ.*

Consider a random element Ψ of $[M \times Z^S, \mathfrak{M} \otimes \mathfrak{Z}^S]$ with the distribution Q defined by

$$Q(\Psi \in (.)) \underset{\mathrm{df}}{=} \int \frac{1}{i_\varphi(0)} \sum_{i=1}^{i_\varphi(0)} \mathbb{1}\{\psi_i(\varphi) \in (.)\}\, P(\mathrm{d}\varphi). \tag{10}$$

Then Ψ is also an extension of Φ.

(The proof of Theorem 1 will be completed in Sect. 6.1.) In order to prove the existence of a solution of the given recursive equation it suffices to show the existence of a potential state system $\mathfrak{A}$ with $P(M_{\mathfrak{A}}) = 1$. The special extension Ψ of Φ will be of interest in the following chapters.

4. Existence of a Stationary Solution

Stationarity and ergodicity are defined for extensions in a natural way:

Definition 4. An extension $\tilde{\Phi}$ of the input Φ is called *stationary* if its distribution $\tilde{P}$ is invariant with respect to all transformations

$$T_t : \tilde{\varphi} \in M \times Z^S \mapsto T_t \tilde{\varphi} = (\Theta_t \pi_1 \tilde{\varphi}, \pi_2 \tilde{\varphi}((.) + t)), \quad t \in S.$$

A stationary extension $\tilde{\Phi}$ is called *ergodic* (metrically transitive) if $\tilde{P}(C) = 0$ or $\tilde{P}(C) = 1$ holds for all $C \in \mathfrak{M} \otimes \mathfrak{Z}^S$ for which $T_t C = C$ holds for all $t \in S$.

If the extension $\tilde{\Phi}$ is stationary, then the components $\pi_1 \tilde{\Phi}$ and $\pi_2 \tilde{\Phi}$ are stationary, too. Now our question is, whether there is a stationary extension $\tilde{\Phi}$ for a given stationary input Φ. For that reason the construction from chapter 3 will be considered under the additional assumption that Φ is stationary. Let $\mathfrak{A}$ be a potential state system with the property

(E) $A(t, \varphi) = A(0, \Theta_t \varphi)$ for all $\varphi \in A$, $t \in S$.

Then the construction becomes much simpler as the following Lemma shows:

Lemma. *Let* $\mathfrak{A}$ *be a potential state system with* (E) *and let* P *be the distribution of the stationary input* Φ *with* $P(M_{\mathfrak{A}})=1$. *Then*

$$1 \leqq i_\varphi(t) = i_{\Theta_t \varphi}(0) = i_\varphi(0) < \infty \tag{11}$$

holds for almost all φ *and all* $t \in S$.

(The proof of the Lemma is contained in Sect. 6.2.) For abbreviation, we denote the number $i_\varphi(0)$ by i_φ. It follows from (E) and the stationarity of Φ that the process $\{|A(t,\Phi)|, t \in \Gamma\}$ is stationary, too. Hence $P(|A(0,\Phi)|<\infty)=1$ is equivalent to

$$P(|A(t_j, \varphi)| < \infty \text{ for all } j) = 1.$$

That means, one can replace the assumption $P(M_{\mathfrak{A}})=1$ by $P(|A(0,\Phi)|<\infty)=1$.

Under the assumptions of the Lemma the extension Ψ is defined via Theorem 1. We obtain

Theorem 2. *Consider the stationary input* Φ *with the distribution* P *and assume that there is a potential state system* $\mathfrak{A}$ *with the property* (E) *and* $P(|A(0,\Phi)|<\infty)=1$. *Then*

$$Q(\Psi \in (.)) = \int \frac{1}{i_\varphi} \sum_{i=1}^{i_\varphi} \mathbb{1}\{\psi_i(\varphi) \in (.)\}\, P(\mathrm{d}\varphi) \tag{12}$$

defines a stationary extension Ψ *of* Φ.

(The proof is given in Sect. 6.3.) The extension $\psi_1(\Phi)$, however, is not stationary in general, cf. Corollary 2 in Chap. 5.

We consider the special case $S=\Gamma$. Then property (D) is equivalent to

(D′) $f^*(\Theta_t\varphi, A(t,\varphi)) \subseteq A(t+1,\varphi)$ for all $\varphi \in A$, $t \in \Gamma$.

Theorem 2 provides

Corollary 1. *Let* P *be the distribution of a strictly stationary sequence* $\Phi = \{X_t\}_{t=-\infty}^{+\infty}$ *with values in an arbitrary measurable space, and* $f^*: A \times Z \to Z$ *a measurable function. Assume that there is a system* $\mathfrak{A} = \{A(t,\varphi): t \in \Gamma, \varphi \in A\}$ *of non-empty subsets of* Z *with the properties* (B), (C), (D′), (E), *and* $P(|A(0,\Phi)|<\infty)=1$. *Then there is a stationary solution of the recursive equation*

$$Z_{t+1} = f^*(\Theta_t \Phi, Z_t), \qquad t \in \Gamma, \tag{13}$$

i.e. there is a distribution of the pair $(\Phi, \{Z_t\}_{t=-\infty}^{+\infty})$ *for which* (13) *is fulfilled almost surely, where* $\{Z_t\}_{t=-\infty}^{+\infty}$ *is a strictly stationary sequence.*

Via (12) the solution $\{Z_t\}_{t=-\infty}^{+\infty}$ is given by $Z_t = \pi_2 \Psi(t)$. Thus our problem has been solved constructively.

5. On the Uniqueness of the Solution

Consider the distribution P of a stationary input Φ and a potential state system $\mathfrak{A}$ with $P(|A(0,\Phi)|<\infty)=1$ and (E). Furthermore, let Φ be ergodic. The sets $\{\varphi: i_\varphi = i\}$, $i=1,2,\ldots$, are disjoint and invariant. Their union has probability 1. Hence there is an integer $i_P \geqq 1$ with $P(i_\Phi = i_P)=1$. Then the distribution Q of the constructed stationary extension Ψ (cf. Theorem 2) has the form

$$Q(\Psi \in (.)) = \frac{1}{i_P} \sum_{i=1}^{i_P} P(\psi_i(\Phi) \in (.)).$$

Now an additional condition on the potential state system $\mathfrak{A}$ will be introduced:

(F) For every stationary extension $\tilde{\Phi}$ of Φ

$$\tilde{P}(\tilde{\Phi} \in \{\psi_1(\pi_1 \tilde{\Phi}), \ldots, \psi_{i_P}(\pi_1 \tilde{\Phi})\}) = 1$$

holds. (Here $\tilde{P}$ denotes the distribution of $\tilde{\Phi}$.)

Definition 5. Let Φ be stationary. A potential state system $\mathfrak{A}$ is called *sufficient* iff. (E), (F), and $P(|A(0, \Phi)| < \infty) = 1$ hold.

The interpretation of this definition is as follows: If $\mathfrak{A}$ is a sufficient potential state system, then the construction from Chap. 3 almost surely yields all extensions of the realization φ of Φ. It especially follows that the number i_P is the same for all sufficient potential state systems. Hence i_P is characterized only by P if it exists. The constructed extension Ψ is not the only stationary extension in general, as examples in [6] show. In order to investigate the uniqueness problem, first a method from queueing theory (cf. [4]) will be used. Let $\mathfrak{A}$ be a sufficient potential state system.

Definition 6. Let $\varphi \in A$ be fixed. An element $t' \in S$ is called *construction point* if there is a $t \in S$ and a state $z' \in Z$ with $f(\Theta_t \varphi, A(t, \varphi), t'-t) = \{z'\}$.

From the definition it follows that construction points exist a.s. at most in the case of $i_P = 1$. Conversely, in the case of $i_P = 1$ all $t' \in S$ are construction points for almost all φ since $t = t_{j(t', \varphi)}$ (cf. Chap. 3) and $z' = z(1, t', \varphi)$ fulfill the conditions of Definition 6. The concept of construction points is due to Borovkov [2, 8], Franken and Kalähne [3]. These authors defined special kinds of construction points in queueing systems. Now we have seen that the existence of construction points is equivalent to $i_P = 1$ in our general context.

Consider an additional proposition:

(G) For every $z \in Z$ it holds that

$$P(\textit{There is a } t_z \in S \textit{ with } f(\Phi, z, t_z) \in A(t_z, \Phi)) = 1.$$

The following theorem is a generalization of a theorem by Franken and Kalähne [3] for queueing system without delay, cf. [4].

Theorem 3. *Let P be the distribution of a stationary ergodic input Φ and assume that there is a sufficient potential state system $\mathfrak{A}$ and that $i_P = 1$ holds. Then the following statements are valid:*

(a) *Ψ is the only stationary extension of Φ.*

(b) *Ψ is defined on the probability space $[M, \mathfrak{M}, P]$.*

(c) *Let* (G) *be fulfilled and let X be an arbitrary random element of* $[M \times Z^S, \mathfrak{M} \otimes \mathfrak{Z}^S]$ *with the marginal distribution P of the first component. Then*

$$V(f(\pi_1 X, \pi_2 X(0), t) \in (.)) \xrightarrow[t \to \infty]{\mathrm{Var}} Q(\pi_2 \Psi(0) \in (.))$$

holds. (Here V denotes the distribution of X and $\xrightarrow{\mathrm{Var}}$ *stands for the convergence in variation.)*

(The proof is contained in Sect. 6.4.) Of course, the condition $i_P = 1$ is not necessary for the uniqueness of the solution, cf. [6]. In the general case we obtain

Theorem 4. *Consider the stationary ergodic input Φ with the distribution P and assume that there is a sufficient potential state system $\mathfrak{A}$. Then Ψ is the only stationary extension of Φ if and only if Ψ is ergodic.*

The proof of this theorem is formally the same as in the special case considered in [6]. (We omit it here and refer to [6].) Unfortunately, the ergodicity of Ψ is hardly to check already in simple cases.

Corollary 2. *Let the assumptions of Theorem 4 be fulfilled and let Ψ be the only stationary extension of Φ. Then Ψ is defined on $[M, \mathfrak{M}, P]$ if and only if $i_P = 1$ holds.*

(The proof is given in Sect. 6.5.) Corollary 2 shows that the assumption $i_P = 1$ (i.e. the existence of construction points) is sufficiently general if one is only interested in extensions defined on $[M, \mathfrak{M}, P]$. Finally we get

Corollary 3. *Let the assumptions of Theorem 4 and* (G) *be fulfilled and let Ψ be the only stationary extension of Φ. Then*

$$P\left(T^{-1} \int_0^T \mathbb{1}\{f(\Phi, z, t) \in B\}\, \mathrm{d}t \xrightarrow[T \to \infty]{} \mathrm{Q}(\pi_2 \Psi(0) \in \mathrm{B})\right) = 1$$

holds for all $z \in Z$ and all $B \in \mathfrak{Z}$.

This result immediately follows from the individual ergodic theorem.

6. Proofs

6.1. The proof of Theorem 1

It remains to show that Ψ is an extension of Φ. In fact,

$$\begin{aligned} Q(\pi_1 \Psi \in (.)) &= \int \frac{1}{i_\varphi(0)} \sum_{i=1}^{i_\varphi(0)} \mathbb{1}\{\pi_1 \psi_i(\varphi) \in (.)\}\, P(\mathrm{d}\varphi) \\ &= \int \frac{1}{i_\varphi(0)} \sum_{i=1}^{i_\varphi(0)} \mathbb{1}\{\varphi \in (.)\}\, P(\mathrm{d}\varphi) \\ &= P(\Phi \in (.)) \end{aligned}$$

holds. Consider the set

$$C=\{\tilde{\varphi}\colon \pi_1\tilde{\varphi}\in M_{\mathfrak{A}},\ \tilde{\varphi}\in\{\psi_1(\pi_1\tilde{\varphi}),\ldots,\psi_{i_{\pi_1\tilde{\varphi}}(0)}(\pi_1\tilde{\varphi})\}\}.$$

For all $\tilde{\varphi}\in C$

$$\psi_1(\pi_1\tilde{\varphi}),\ldots,\psi_{i_{\pi_1\tilde{\varphi}}(0)}(\pi_1\tilde{\varphi})$$

are extensions of $\pi_1\tilde{\varphi}$. Hence it follows

$$\begin{aligned}&\sum_{i=1}^{i_\varphi(0)} \mathbb{1}\{\psi_i(\varphi)\in\{\psi_1(\pi_1\psi_i(\varphi)),\ldots,\psi_{i_{\pi_1\psi_i(\varphi)}(0)}(\pi_1\psi_i(\varphi))\}\}\\&\quad=\sum_{i=1}^{i_\varphi(0)} \mathbb{1}\{\psi_i(\varphi)\in\{\psi_1(\varphi),\ldots,\psi_{i_\varphi(0)}(\varphi)\}\}=i_\varphi(0)\end{aligned}$$

for all $\varphi\in M_{\mathfrak{A}}$, and thus $Q(C)=1$. The proof of Theorem 1 is completed.

6.2. *The proof of the Lemma*

It follows from (E) that $B(t,\varphi)=B(0,\Theta_t\varphi)$ holds for all $\varphi\in A$ and $t\in S$, and thus

$$i_\varphi(t)=i_{\Theta_t\varphi}(0)$$

is valid for all $\varphi\in M_{\mathfrak{A}}$. That means, the process $\{i_\Phi(t),\ t\in S\}$ is stationary. For every $\varphi\in M_{\mathfrak{A}}$ the function $i_\varphi(t)$ is non-increasing in t. Hence the realizations of the process $\{i_\Phi(t),\ t\in S\}$ are almost surely constant:

$$i_\varphi(t)=i_\varphi(0),\qquad t\in S.$$

The Lemma is proved.

6.3. *The proof of Theorem 2*

Via Theorem 1 Eq. (12) defines an extension of Φ. It remains to show that Ψ is stationary.

For almost all φ the sets $B(t,\varphi)$, $t\in S$, have the power i_φ (by the Lemma). Then the solution of Eq. (7) resp. (8) is uniquely determined, cf. Chap. 3. Thus for all fixed t and almost all φ the following statement holds: For every $i\in\{1,\ldots,i_\varphi\}$ there is exactly one number $j(i,t,\varphi)\in\{1,\ldots,i_\varphi\}$ with

$$z(i,(\cdot)+t,\varphi)=z(j(i,t,\varphi),\cdot,\Theta_t\varphi).$$

From there we get

$$\begin{aligned}&\sum_{i=1}^{i_\varphi} \mathbb{1}\{(\Theta_t\varphi, z(i,(\cdot)+t,\varphi))\in B\}\\&\quad=\sum_{i=1}^{i_\varphi} \mathbb{1}\{(\Theta_t\varphi, z(i,\cdot,\Theta_t\varphi))\in B\}\end{aligned}$$

for all $t\in S$, $B\in\mathfrak{M}\otimes\mathfrak{Z}^S$, and almost all φ.

Now the stationarity of Ψ can easily by proved:

$$\begin{aligned}
&Q((\Theta_t\pi_1\Psi, \pi_2\Psi((.)+t))\in B)\\
&=\int \frac{1}{i_\varphi}\sum_{i=1}^{i_\varphi} \mathbb{1}\{(\Theta_t\varphi, z(i,(.)+t,\varphi))\in B\} P(\mathrm{d}\varphi)\\
&=\int \frac{1}{i_\varphi}\sum_{i=1}^{i_\varphi} \mathbb{1}\{(\Theta_t\varphi, z(i,.,\Theta_t\varphi))\in B\} P(\mathrm{d}\varphi)\\
&=\int \frac{1}{i_{\Theta_t\varphi}}\sum_{i=1}^{i_{\Theta_t\varphi}} \mathbb{1}\{(\Theta_t\varphi, z(i,.,\Theta_t\varphi))\in B\} P(\mathrm{d}\varphi)\\
&=\frac{1}{\mathrm{i}_\varphi}\sum_{i=1}^{i_\varphi} \mathbb{1}\{(\varphi, z(i,.,\varphi))\in B\} P(\mathrm{d}\varphi)=Q(\Psi\in B)
\end{aligned}$$

holds for all $t\in S$, q.e.d.

6.4. The proof of Theorem 3

It follows from Theorem 2 that $\Psi=\psi_1(\Phi)$ is a stationary extension defined on $[M, \mathfrak{M}, P]$. Thus (b) is shown, and (a) immediately follows from property (F). It remains to prove (c).

$$t_j - t_{j(t_j,\varphi)} < \infty \tag{14}$$

holds for all $j\in\Gamma$ and almost all φ. By means of property (E) we obtain

$$\begin{aligned}
t_{j+k}-t_{j(t_{j+k},\varphi)} &= t_j + kt_1 - t_{j(t_j+kt_1,\varphi)}\\
&= t_j - t_{j(t_j,\Theta_{kt_1}\varphi)}
\end{aligned}$$

for all j, $k\in\Gamma$ and almost all φ. Hence the sequence

$$\{t_j - t_{j(t_j,\Phi)}\}_{j=-\infty}^{+\infty}$$

is stationary. Furthermore, it is ergodic. Consider the events

$$A_n \underset{\mathrm{df}}{=} \{\varphi\colon \textit{There is a } j\in\Gamma \textit{ with } t_j - t_{j(t_j,\varphi)} < n\}, \quad n\geqq 1.$$

It follows from (14) that there is a number n with $P(A_n)>0$. Since A_n is invariant with respect to all transformations $\varphi\mapsto\Theta_{kt_1}\varphi$, $k\in\Gamma$, the equation $P(A_n)=1$ holds. Now

$$P(|\{j\colon j>0, t_j - t_{j(t_j,\Phi)} < n\}|=\infty)=1 \tag{15}$$

follows. Let $z\in Z$ be arbitrary but fixed. From (G), (D), and (15) we obtain that the following statement holds for almost all φ: There is a $t\in S$ with

$$f(\varphi, z, t_{j(t,\varphi)})\in A(t_{j(t,\varphi)}, \varphi)$$

and hence $f(\varphi, z, t)=\pi_2\psi_1(\varphi)(t)$. For all $t'\geqq t$ the functions $f(\varphi, z, t')$ and $\pi_2\psi_1(\varphi)(t')$ coincide. From there one can obtain (c) by standard calculations, cf. e.g. [4].

6.5. The proof of Corollary 2

In the case of $i_P=1$ the extension Ψ is defined on $[M, \mathfrak{M}, P]$ by means of Theorem 3. Now let $i_P>1$. Suppose there were a measurable mapping $h: M \to M \times Z^S$ with $T_t h(\varphi)=h(\Theta_t \varphi)$ for all $t \in S$ and P-almost-all φ and $h(\Phi)$ were an extension of Φ. Let $\varphi \in M_{\mathfrak{A}}$ be arbitrary but fixed. Then, by means of (F), there is a number $i_h \in \{1, \ldots, i_P\}$ with $h(\varphi)=\psi_{i_h}(\varphi)$. Now we number the constructed extensions in a new order:

$$\psi_1^*(\varphi) \underset{\mathrm{df}}{=} \psi_{i_h}(\varphi)=h(\varphi),$$

$$\psi_i^*(\varphi) \underset{\mathrm{df}}{=} \begin{cases} \psi_{i-1}(\varphi) & \text{for } 1<i \leqq i_h, \\ \psi_i(\varphi) & \text{for } i>i_h. \end{cases}$$

Then

$$Q(\Psi \in (.))=\frac{1}{i_P} \sum_{i=1}^{i_P} P(\psi_i^*(\Phi) \in (.))$$

holds, and $\psi_1^*(\Phi)$ is a stationary extension. In the case of $i_P>1$ this contradicts the ergodicity of Ψ. Thus there is no measurable mapping h with the above mentioned properties. Corollary 2 is proved.

References

1. Borovkov, A.A.: Ergodic and Stability Theorems for One Class of Stochastic Equations and Their Applications. (In Russian). Teor. Verojatnost. i Primenen. **XXIII**, 241–262 (1978)
2. Borovkov, A.A.: Continuity Theorems for Multi-server Systems without Delay. (In Russian). Teor. Verojatnost. i Primenen. **XVII**, 458–468 (1972)
3. Franken, P., Kalähne, U.: Existence, Uniqueness and Continuity of Stationary Distributions for Queueing Systems without Delay. Math. Nachr. **86**, 97–115 (1978)
4. Franken, P., König, D., Arndt, U., Schmidt, V.: Queues and Point Processes. Akademie-Verlag, Berlin, and J. Wiley, New York 1981
5. Franken, P., Lisek, B.: On Wald's Identity for Dependent Variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **60**, 143–150 (1982)
6. Lisek, B.: Construction of Stationary State Distributions for Loss Systems. Math. Operationsforsch. Statist., Ser. Statistics, **10**, no. 4, 561–581 (1979)
7. Lisek, B., Lisek, M.: On Qualitative Properties of Inventory Models with Discrete Demand. Elektron. Informationsverarb. Kybernetik **16**, 425–432 (1980)
8. Loynes, R.M.: The Stability of a Queue with Non-independent Inter-arrival and Service Times. Proc. Cambridge Philos. Soc. **58**, 497–520 (1962)

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