Z. Wahrscheinlichkeitstheorie verw. Gebiete
68, 163-189 (1984)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Asymptotic Behaviour of Solutions of Stochastic Differential Equations

G. Keller[1]*, G. Kersting[2]*, and U. Rösler[3]

[1] Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg, Federal Republic of Germany
[2] Fachbereich Mathematik, Universität Frankfurt, Robert-Mayer-Straße 10, D-6000 Frankfurt 1, Federal Republic of Germany
[3] Institut für mathematische Stochastik, Universität Göttingen, Lotzestraße 13, D-3400 Göttingen, Federal Republic of Germany

Summary. In this paper we study the asymptotic behaviour of the solution of the stochastic differential equation $dX_t = g(X_t)\,dt + \sigma(X_t)\,dW_t$, where σ and g are positive functions and W_t is a Wiener process. We clarify, under which conditions X_t may be approximated on $\{X_t \to \infty\}$ by means of a deterministic function. Further the question is treated, whether X_t converges in distribution on $\{X_t \to \infty\}$. We deal with the Ito-solution as well as the Stratonovitch-solution and compare both.

1. Introduction

In this paper we study the asymptotic behaviour of the solution X_t of the stochastic differential equation

$$dX_t = g(X_t)\,dt + \sigma_1(X_t)\,dW_t, \qquad X_0 \equiv 1, \tag{1}$$

$t \geqq 0$, where W_t is a standard Brownian motion. We shall analyse the behaviour of X_t conditioned on the event $\{X_t \to \infty, \text{ as } t \to \infty\}$. The basic assumption in the paper is that not only $\sigma_1(t)$ but also $g(t)$ is a strictly positive function. Furthermore we are only interested in situations, in which the event $\{X_t \to \infty\}$ takes place with positive probability and infinity will not be reached in finite time. The left boundary 0 of the state space will be assumed to be absorbing, if it is at all attainable. This last assumption, however, is insignificant for the results and the reader will have no difficulties to treat other cases.

One may view X_t as modelling a randomly disturbed growth process. In fact our starting point were certain Markovian growth models with discrete time, as Galton-Watson processes with state-dependent offspring distributions and other models, where the rate of divergence in general is no longer exponential.

* Partially supported by the SFB 123 "Stochastische Mathematische Modelle", Heidelberg

These processes often behave similar to the solution of (1), if g and σ_1 are properly chosen. Thus the results of this study should give some indication, which behaviour has to be expected in the discrete time situation. This case is much more complicated technically and our results on this will be published in a forthcoming paper.

Denote by $\mu(t)=\mu_t$ the solution of the deterministic differential equation, given by $\sigma_1 \equiv 0$. An interesting question to ask is, under which conditions X_t/μ_t converges to 1 on $\{X_t \to \infty\}$. It turns out that in this problem not only the order of magnitude of $\sigma_1(t)$, but also that of $g(t)$ plays a significant role. If one writes for a moment $X_t=\mu(t+\xi_t)$, then one might expect that ξ_t converges almost surely on $\{X_t \to \infty\}$, if the random irritation $\sigma_1(t)$ is small enough, and the limit ξ_∞ will be non-degenerate for strictly positive σ_1. This behaviour would entail $X_t/\mu_t \to 1$ only, if $\log \mu_t = o(t)$ or equivalently $g(t)=o(t)$. In situations similar to the Galton-Watson process μ_t grows at an exponential rate, and in fact X_t/μ_t converges to a non-degenerate limit. It is this influence of μ_t, which led us to postpone the treatment of the problem above. We first study a transformed process Y_t, whose behaviour does no longer depend on μ_t in the described manner.

Let

$$G(t)=\int_1^t \frac{ds}{g(s)}, \tag{2}$$

$$Y_t = G(X_t), \tag{3}$$

$t \geqq 0$. Note that $G(t)$ is the inverse function of $\mu(t)$, thus in the deterministic situation ($\sigma_1 \equiv 0$) $Y_t = t$. It turns out that (contrarily to the validity of $X_t \sim \mu_t$) $Y_t \sim t$ on $\{X_t \to \infty\}$ is true in practically all cases in which $P(X_t \to \infty)>0$ and explosions are excluded. (The slightly stronger assumptions that we need are given in Sect. 2.) Furthermore we show that Y_t has a representation $Y_t=\alpha(t) + Z_t$, where $\alpha(t)$ is the solution of a deterministic differential equation and Z_t either converges a.s. on $\{X_t \to \infty\}$ or else behaves asymptotically like a standard Brownian motion after a certain deterministic transformation of the time scale.

These results, which are developed in Sect. 3, clarify quite well the stochastic behaviour of X_t on $\{X_t \to \infty\}$. Still it seems to be desirable to derive asymptotic properties of X_t itself. From the results for Y_t it is difficult to see, what happens with the stochastic behaviour of X_t, if g varies, since then the transforming function G also varies. Especially in a statistical context, where g is unknown, it will be necessary to get results on the process X_t itself. Section 4 is devoted to this question. In Theorem 2 we give a necessary and sufficient condition, under which $X_t \sim \mu_t$ in probability on $\{X_t \to \infty\}$. It turns out that this property is also equivalent to the existence of real numbers β_t such that in probability $X_t \sim \beta_t$ on $\{X_t \to \infty\}$. Theorem 3 settles the question, under which conditions the law of X_t, properly normalized and restricted to $\{X_t \to \infty\}$, converges in distribution to a non-degenerate probability measure. Here three cases may arise: X_t may converge a.s., or X_t has asymptotically a normal or log-normal distribution. There is another interesting aspect of these results. All the described alternatives of behaviour can only occur, if $\sigma_1(t)$ does not grow too fast, as $t\to\infty$. If the rate of divergence of σ_1 exceeds a certain limit, then the stochastic behaviour of X_t changes drastically. In this case there is no longer any limiting distribution of X_t. Furthermore $X_t=o(\mu_t)$ in probability (Theorem 4). The growth condition, essential in this context, is a little bit involved. In Sect. 5 we shall explain it further and give examples. - Almost sure approximations to X_t can also be derived, but there are no longer neat conditions, which are as well necessary as sufficient. We discuss this question in Sect. 6. Here we content ourselves with a rather general result on the possibility of a.s. approximations to $\log X_t$.

Up to now we have not stressed that we always have been talking about the Ito-solution of the stochastic differential equation. As is well-known, there are situations, where it seems to be more appropriate to consider the so-called Stratonovitch-solution of (1) (compare [7], p. 348ff., or [1], Chap. 10). In Sect. 7 we show that it is possible to develop a similar theory for this type of solution. An interesting feature is that both types of solution show a different behaviour, if and only if $\sigma_1(t)$ exceeds the critical rate of growth, which we already were talking about.

The question, if there exists a deterministic function β_t such that $X_t\sim\beta_t$ a.s., has been treated before. Our results contain practically all results on this question, which Gihman and Skorohod have included in their book [5]. The reader will notice that our assumptions may be weakened in several respects. Let us point to one of them: Much of the theory remains valid, if σ_1 additionally depends on the time t, i.e. if one considers the equation

$$dX_t=g(X_t)\,dt+\sigma_1(t,X_t)\,dW_t.$$

2. Notations and Assumptions

In order to increase readability we list all the notations used in some place in the paper. We agree upon denoting the value of any function f at the point t by as well $f(t)$ as f_t. - Since $g(t)$ is positive, it is possible to rewrite (1) in the form

$$dX_t=g(X_t)\,dt+g(X_t)\,\sigma(X_t)\,dW_t,\qquad X_0\equiv 1 \tag{4}$$

with a suitable positive function $\sigma(t)=\sigma_t$. X_t denotes the Ito-solution of (4) and $\mu(t)=\mu_t$ the solution of the deterministic equation ($\sigma\equiv 0$), $G(t)$ is its inverse. If f is any function on the positive numbers, $\tilde f$ denotes the function given by

$$\tilde f(t)=f(\mu_t).$$

Further, let for $t>0$

$$h(t)=\tfrac{1}{2}g'(t)\,\sigma^2(t), \tag{5}$$

$$\psi(t)=\int_1^t\frac{\sigma^2(s)}{g(s)}\,ds,\qquad\text{or}\quad\tilde\psi(t)=\int_0^t\tilde\sigma^2(s)\,ds, \tag{6}$$

$$\varphi(t)=\int_1^t\frac{h(s)}{g(s)}\,ds=\frac{1}{2}\int_1^t g'(s)\,\psi'(s)\,ds \tag{7}$$

or

$$\tilde{\varphi}(t)=\int_0^t \tilde{h}(s)\,ds.$$

The function α_t, $t\geqq 0$, denotes the solution of the differential equation

$$\alpha'(t)=1-\tilde{h}(\alpha_t). \tag{8}$$

The initial value α_0 is chosen so large that α_t becomes an increasing function tending to infinity. This is possible in view of assumption (A2) below. (A2) implies that $\alpha_t \sim t$, as $t\to\infty$. Finally let

$$v_t=\mu(\alpha_t) \tag{9}$$

and

$$Z_t=\int_0^t \sigma(X_s)\,dW_s=\int_0^t \tilde{\sigma}(Y_s)\,dW_s \tag{10}$$

with Y_t as in (3). The stochastic integral is taken in Ito's sense.

We now come to the main assumptions of the paper:

(A1) $g\colon \mathbb{R}^+\to\mathbb{R}^+$ is strictly positive and twice continuously differentiable, and

$$G(\infty)=\int_1^\infty \frac{ds}{g(s)}=\infty.$$

(A2) $h(t)\to 0$, as $t\to\infty$.

(A3) $\sigma\colon \mathbb{R}^+\to\mathbb{R}^+$ is strictly positive and continuously differentiable, and

$$\int_0^\infty t^{-2}\,\tilde{\sigma}^2(t)\,dt<\infty.$$

Additionally to these assumptions we require regular behaviour of several functions:

(A4) The functions g, g', $\tilde{\sigma}^2$ and $\tilde{h}$ are ultimately concave or convex. If $\psi(\infty)=\infty$, we require the same behaviour for the function $\tilde{h}\circ\tilde{\psi}^{-1}$.

The rest of the section contains a discussion of these assumptions. The requirement $G(\infty)=\infty$ means that μ_t does not reach infinity in finite time. The same follows in the random situation. This is most easily seen from the stochastic differential equation, which Y_t obeys. Because of Ito's transformation rule

$$dY_t=(1-\tilde{h}(Y_t))\,dt+\tilde{\sigma}(Y_t)\,dW_t, \qquad Y_0\equiv 0, \tag{11}$$

or

$$Y_t=t-\int_0^t \tilde{h}(Y_s)\,ds+Z_t. \tag{12}$$

The state space of Y_t is the interval $(G(0), G(\infty))$, and the right boundary $G(\infty)$ will be reached in finite time, if and only if $G(\infty)<\infty$ (compare also [5], p. 229ff.). Thus requiring (A1), we exclude the possibility of explosions.

The meaning of (A2) becomes clear by an inspection of (12), the assumption ensures that $\int_0^t \tilde{h}(s)\,ds = o(t)$ a.s. on $\{X_t \to \infty\} = \{Y_t \to \infty\}$. For readers who may wonder, if (A2) is already enough for proving $Y_t = t + o(t)$ on $\{X_t \to \infty\}$, we give a counter-example. Let $X_t = n^{-1}(W_1^2(t) + \ldots + W_n^2(t))$, where $W_1(t), \ldots, W_n(t)$ are standard Wiener processes, independent of each other, and $n \geqq 3$ is fixed. Then

$$dX_t = dt + \frac{2\sqrt{t}}{n}\,dW_t$$

with some other Brownian motion W_t (compare [7], p. 175; X_t is a so-called Bessel process). $P(X_t \to \infty) = 1$, since the n-dimensional standard Wiener process is transient for $n \geqq 3$. Further $h \equiv 0$ and $X_t = Y_t$. Now the distribution of $t^{-1} Y_t$ is independent of t, namely up to a scaling factor a χ^2-distribution. This example shows that we can expect $Y_t \sim t$ on $\{X_t \to \infty\}$ only, if $\tilde{\sigma}(t) = o(\sqrt{t})$. (A3) is just a slight strengthening of this requirement. In the next section we show that (A2) and (A3) entail $Y_t \sim t$ a.s. on $\{X_t \to \infty\}$.

We discuss now the relation between the conditions (A2), (A3) and the property that $P(X_t \to \infty) > 0$. Let us assume for reasons of convenience that for some real c, d $h(t) \to c$ and $t^{-1}\tilde{\sigma}^2(t) \to d$, as $t \to \infty$. Now $X_t \to \infty$ with positive probability, iff

$$\int_0^\infty \exp\left\{-2\int_0^t (1 - \tilde{h}(s))\,\tilde{\sigma}(s)^{-2}\,ds\right\} dt < \infty$$

([5], p. 119). Therefore $P(X_t \to \infty) > 0$ implies $c \leqq 1$ and $d \leqq 2(1-c)$. On the other hand our assumptions $c = d = 0$ yield convergence to infinity with positive probability. Thus (A2) and (A3) restrict the order of $\tilde{\sigma}^2(t)$ in a slightly stronger way than it would be necessary, if one only wants to guarantee $P(X_t \to \infty) > 0$. In the limiting domain the behaviour of Y_t changes as indicated by the example just given.

(A2) and (A3) are also related to each other. Many smooth functions $g(t)$ have the property that the derivative $g'(t)$ is of order $t^{-1} g(t)$. If $G(t)$ behaves in the same way, its derivative $1/g(t)$ is of order $t^{-1} G(t)$. In this situation (A2) is equivalent to $\sigma^2(t) = o(G(t))$ or $\tilde{\sigma}^2(t) = o(t)$ and thus a consequence of (A3). These considerations can be verified for $g(t) = t^\alpha$, if $\alpha < 1$ and $\alpho \neq 0$. ($\alpha > 1$ is excluded because of (A1). Compare also Lemma 1 below.)

We finish with some comments on the regularity assumptions, formulated in (A4). They entail the following consequences, which will be used freely in the sequel. Either g is ultimately decreasing. Then it will be convex, $g'(t) \leqq 0$ and $g''(t) \geqq 0$ ultimately. Otherwise g eventually increases. If it is additionally concave, $0 \leqq g'(t) \leqq t^{-1} g(t)$ for large t. In the convex case $g'(t) \geqq t^{-1} g(t)/2$ ultimately. A similar distinction is valid for $\tilde{\sigma}^2(t)$. Because of (A3) $\tilde{\sigma}^2$ either is ultimately decreasing and convex, or it is increasing and concave. In both cases

$$\tilde{\sigma}^2(t) = o(t). \tag{13}$$

This and the convexity properties of $\tilde{\sigma}^2$ have further consequences, which will be useful later. If t is large enough

$$\tilde{\psi}(c\,t) \leqq c^2\,\tilde{\psi}(t) \quad \text{for all } c \geqq 1, \tag{14}$$

$$t\,\tilde{\sigma}^2(t) \leqq 2\tilde{\psi}(t), \tag{15}$$

$$\tilde{\psi}(t) = o(t^2), \tag{16}$$

and from (15) and (16)

$$\tilde{\sigma}^2(t) = o(\tilde{\psi}_t^{1/2}). \tag{17}$$

3. The Behaviour of the Transformed Process

The aim of this section is the proof of the following result.

Theorem 1. *Assume* (A1)–(A4). *Then* $t^{-1}Y_t \to 1$ a.s. *on* $\{X_t \to \infty\}$, *as* $t \to \infty$. *Moreover*

i) *If* $\psi(\infty) < \infty$, *then* $Y_t - \alpha_t$ *converges* a.s. *on* $\{X_t \to \infty\}$. *The distribution of the limit, conditioned on* $\{X_t \to \infty\}$, *has a unimodal density, which is strictly positive everywhere.*

ii) *If* $\psi(\infty) = \infty$, *then* a.s. *on* $\{X_t \to \infty\}$

$$Y_t = \alpha_t + Z_t + o(\tilde{\psi}_t^{1/2}).$$

Furthermore there is a standard Brownian motion $B(t)$, *such that* a.s. *on* $\{X_t \to \infty\}$

$$Z_t = B(\tilde{\psi}_t + o(\tilde{\psi}_t)) \tag{18}$$

and for any real b, as $t \to \infty$,

$$P(Z_t \leqq b\,\tilde{\psi}_t^{1/2} \mid X_t \to \infty) \to P(B(1) \leqq b). \tag{19}$$

Before we come to the proof, we give some consequences, being of some independent interest. We use the theorem to analyse the behaviour of the first hitting time of $x > 0$, namely

$$t_x^* = \inf\{t \geqq 0 \mid X_t = x\}. \tag{20}$$

Without loss of generality let us assume $X_t \to \infty$ a.s. We concentrate our attention on the case $\psi(\infty) = \infty$. From $Y_t \sim t$ we get $t_x^* \sim G(x)$ a.s., as $x \to \infty$. From (14) $\tilde{\psi}(t_x^*) \sim \tilde{\psi}(G(x)) = \psi(x)$. If we denote $Z(t_x^*)$ by Z_x^*, from (18)

$$Z_x^* = B(\psi_x + o(\psi_x)). \tag{21}$$

Thus if we replace t by t_x^*, we obtain a.s.

$$G(x) - \alpha(t_x^*) = Z_x^* + o(\psi_x^{1/2}).$$

Let

$$\eta_x = \int_{\mu(\alpha_0)}^{x} \frac{dt}{g(t)\,(1-h(t))}, \quad \text{or } \tilde{\eta}_x = \int_{\alpha_0}^{x} \frac{dt}{1-\tilde{h}(t)}. \tag{22}$$

$\tilde{\eta}_x$ is the inverse function of α_t, thus by the mean-value theorem $G(x) - \alpha(t_x^*) = (\tilde{\eta}(G(x)) - t_x^*)\,\alpha'(\bar{x})$ for a suitable $\bar{x}$ between $G(x)$ and t_x^*. Since $\alpha'(t) \to 1$, as

$t\to\infty$, we get a.s.

$$t_x^* = \eta_x - Z_x^* + o(Z_x^*) + o(\psi_x^{1/2}). \tag{23}$$

Thus, from (21) and (23), $\psi_x^{-1/2}(t_x^* - \eta_x)$ is asymptotically standard normal, as $x\to\infty$. There are other results in the literature on the asymptotic normality of the hitting times of diffusions (compare [8]). The interesting feature of (23) is that the normalization of t_x^* explicitly is given, whereas in the known results t_x^* is normalized by its mean and variance. There seems to be no easy way to deduce our result from these. – In the same way one proves that $t_x^* - \eta_x$ converges a.s. on $\{X_t\to\infty\}$, if $\psi(\infty)<\infty$, the limit being the same as that of $\alpha_t - Y_t$.

From the theorem we may also deduce a criterion on the triviality of the tail σ-field $\mathscr{F}_\infty$ of X_t reply. Y_t, which does not follow in a simple manner from known results. If $\psi(\infty)<\infty$, $\mathscr{F}_\infty$ is non-trivial because of part i) of Theorem 1. The same is true, if $0<P(X_t\to\infty)<1$, $\{X_t\to\infty\}$ being a tail event. Thus let us assume $\psi(\infty)=\infty$ and $P(X_t\to\infty)=1$. We apply a criterion, which is developed in [4] and [9]. Instead of Y_t we have to consider the diffusion $\bar{Y}_t$ on the restricted state-space $[G(\delta), G(\infty))$, $0<\delta<1$, having the same local characteristics and driving Wiener process as Y_t and a reflecting left boundary $G(\delta)$. If now $\mathscr{F}_\infty$ is non-trivial, it follows from [4] or [9] that $\limsup_{y\to\infty} \operatorname{Var}(\bar{t}_y^*)<\infty$, where $\bar{t}_y^*$ is the first hitting time of y for the $\bar{Y}_t$ process. Thus $\bar{t}_y^* - E\bar{t}_y^*$ is stochastically bounded. Now $Y_t = \bar{Y}_t$ as long as Y_t does not pass through $G(\delta)$. Since $Y_t\to\infty$ a.s., $Y_t=\bar{Y}_t$ for all $t\geqq 0$ holds with positive probability. On this event $t_y^* - E\bar{t}_y^*$ thus is also stochastically bounded, which is not compatible with (21) and (23). Therefore the tail σ-field of X_t is trivial, iff $P(X_t\to\infty)=1$ and $\psi(\infty)=\infty$. $\psi(\infty)<\infty$ may be viewed as the situation, where asymptotically the random effects vanish on $\{X_t\to\infty\}$.

The rest of the section is concerned with the

Proof of Theorem 1. i) We start with showing $Y_t\sim t$ on $\{X_t\to\infty\}$ under the additional assumptions that Y_t has finite second moments and $\tilde{h}$ is bounded. In view of (12) Z_t has finite second moment, too. Being a stochastic integral, Z_t is a zero mean martingale. Since $\tilde{h}$ is bounded, there is a $c_1>0$ such that for large t

$$EY_t \leqq c_1 t.$$

Suppose now that $\tilde{\sigma}^2$ is ultimately concave and increasing. Then there is a $c_2>0$ such that $EY_t\geqq c_2$ implies because of Jensen's inequality

$$E\tilde{\sigma}^2(Y_t) \leqq \tilde{\sigma}^2(EY_t) \leqq \tilde{\sigma}^2(c_1 t).$$

From (13) we see that $EY_t\leqq c_2$ implies $E\tilde{\sigma}^2(Y_t)\leqq c_3$ for some suitable $c_3>0$. From these estimates and the properties of stochastic integration

$$E((Z_n - Z_{n-1})^2) = \int_{n-1}^{n} E\tilde{\sigma}^2(Y_t)\,dt \leqq n^2 \int_{n-1}^{n} t^{-2}(\tilde{\sigma}^2(c_1 t) + c_3)\,dt.$$

If $\tilde{\sigma}^2$ is ultimately decreasing and thus bounded, the same estimate remains trivially true. Thus in any case, using (A3)

$$\sum_1^\infty n^{-2} E((Z_n - Z_{n-1})^2) < \infty. \tag{24}$$

From the martingale convergence theorem the a.s. convergence of $\sum_1^\infty n^{-1}(Z_n - Z_{n-1})$ follows, and by means of Kronecker's lemma we get a.s.

$$Z_n = o(n). \tag{25}$$

Furthermore from Doob's inequality for any $\varepsilon > 0$

$$P(\sup_{n-1 \leqq t \leqq n} |Z_t - Z_{n-1}| \geqq \varepsilon n) \leqq (\varepsilon n)^{-2} E((Z_n - Z_{n-1})^2).$$

This estimate, (24), (25) and the Borel-Cantelli lemma imply a.s.

$$Z_t = o(t).$$

Because of (A2) $\int_0^t \tilde{h}(s)\,ds = o(t)$ a.s. on $\{X_t \to \infty\}$. Thus the desired result follows from (12).

Next we remove the extra assumption on $\tilde{h}$ and the moments of Y_t. Since the left boundary is absorbing, $\{X_t \to \infty\}$ is the union of the events $A_n = \{X_t \to \infty, \inf_t X_t \geqq n^{-1}\}$. The paths belonging to A_n remain unaffected, if we modify g and σ in the interval $(0, n^{-1})$. We do this in such a way that $G(0) > -\infty$ and $G(0)$ is unattainable for Y_t. Because of (A2) we may also achieve that $\tilde{h}$ is bounded. Our assumptions on $\tilde{h}$ and $\tilde{\sigma}^2$ entail that all moments of the modified process Y_t will be finite (compare [5], p. 48). From the result above $Y_t \sim t$ on A_n for the modified and thus also for the original process Y_t. The proof of the first statement thus is finished.

ii) We come to the proof of assertion i) of the theorem. Without loss let $X_t \to \infty$ a.s. From (8) and (12)

$$Y_t - \alpha_t = -\int_0^t (\tilde{h}(Y_s) - \tilde{h}(\alpha_s))\,ds + Z_t - \alpha_0. \tag{26}$$

We shall estimate the integral. Let

$$f(t) = |Y_t - \alpha_t|, \qquad \lambda(t) = \left|\frac{d\tilde{h}}{dt}\left(\frac{t}{2}\right)\right|.$$

Because of (A2) and (A4) $|\tilde{h}|$ has to be ultimately convex and decreasing. Thus λ_t is ultimately decreasing. Further $\dfrac{d\tilde{h}}{dt}$ is ultimately positive or negative, which implies

$$\int_0^\infty \lambda(t)\,dt < \infty. \tag{27}$$

Now, since $Y_t \sim t$ and $\alpha_t \sim t$, from the mean-value theorem a.s.

$$|\tilde{h}(Y_s) - \tilde{h}(\alpha_s)| \leqq \lambda_s f_s, \tag{28}$$

if s is large enough. From (26) a.s.

$$f(t) \leqq \int_0^t \lambda_s f_s \, ds + |Z_t| + c$$

for all $t \geqq 0$ with some random c. The Bellman-Gronwall lemma ([3], p. 198) implies

$$f(t) \leqq \int_0^t \lambda_s \exp\left(\int_s^t \lambda_u \, du\right)(c + |Z_s|) \, ds + |Z_t| + c,$$

and because of (27)

$$f(t) \leqq C \int_0^t \lambda_s |Z_s| \, ds + |Z_t| + c \tag{29}$$

with suitable C and c. Let us now assume that Z_t is a.s. convergent. From (29) and (27) we see that $f(t)$ is uniformly bounded with probability one. From (28) and (27) we deduce a.s.

$$\int_0^\infty |\tilde{h}(Y_s) - \tilde{h}(\alpha_s)| \, ds < \infty.$$

Therefore we get the a.s. convergence of $Y_t - \alpha_t$ from (26) and Lebesgues convergence theorem.

We proceed with showing that Z_t is a.s. convergent. We use the fact that (after enlarging the probability space, if necessary) a standard Wiener process $B(t)$ exists such that ([5], p. 31)

$$Z_t = \int_0^t \tilde{\sigma}(Y_s) \, dW_s = B\left(\int_0^t \tilde{\sigma}^2(Y_s) \, ds\right). \tag{30}$$

Now $Y_s \sim s$ a.s. Thus, if $\tilde{\psi}(\infty) < \infty$, $\int_0^\infty \tilde{\sigma}^2(Y_s) \, ds < \infty$ with probability one. From (30) the a.s. convergence of Z_t follows.

We show now the properties of the a.s. limit $\bar{Y}$ of $\alpha_t - Y_t$. As shown above $\bar{Y}$ is also the limit of $t_x^* - \eta_x$, with t_x^*, η_x as in (20), (22). Now the distribution function of t_x^* is unimodal, i.e. there is a t_0 such that $P(t_x^* \leqq t)$ is concave for $t < t_0$ and convex $t > t_0$ (compare [10]). Further $P(t_x^* = t_0) \leqq P(X_{t_0} = x) = 0$. Therefore t_x^* has an unimodal density. For $1 \leqq x \leqq y$ $t_y^* - \eta_y - t_x^*$ is independent of t_x^*. Letting $y \to \infty$ we see that $\bar{Y} - t_x^*$ and t_x^* are independent. Therefore $\bar{Y}$ possesses also a density. It is unimodal, since the weak limit of unimodal distribution functions is again unimodal ([6]). It remains to show that the density is everywhere positive. Since $t_x^* - \eta_x$ is a.s. convergent, $P(|t_y^* - \eta_y - t_x^* + \eta_x| < 1) > 1/2$ for $1 \leqq x \leqq y$ and x large enough.

Furthermore $P(t_x^* < 1) > 0$. From the independence of t_x^* and $t_y^* - t_x^*$

$$\begin{aligned} P(t_y^* - \eta_y < 2 - \eta_x) &\geqq P(|t_y^* - \eta_y - t_x^* + \eta_x| < 1) \, P(t_x^* < 1) \\ &\geqq P(t_x^* < 1)/2. \end{aligned}$$

Letting $y \to \infty$

$$P(\bar{Y} \leqq 2 - \eta_x) > 0.$$

Now from (22) $\eta_x \to \infty$, as $x \to \infty$. Thus $\bar{Y}$ is not bounded from below. A similar argument shows that $\bar{Y}$ is not bounded from above. This finishes the proof of assertion i).

iii) Let now $\tilde{\psi}(\infty)=\infty$. Since $Y_t \sim t$ a.s., from the convexity assumptions on $\tilde{\sigma}^2(t)$ a.s.

$$\int_0^t \tilde{\sigma}^2(Y_s)\, ds \sim \int_0^t \tilde{\sigma}^2(s)\, ds = \tilde{\psi}(t).$$

Now (18) follows, if we use the Wiener process, given in (30). We like to show that a.s.

$$\int_0^t \lambda_s |Z_s|\, ds = o(\tilde{\psi}_t^{1/2}). \tag{31}$$

It is sufficient to prove

$$\lambda_s Z_s = o(\tilde{\psi}_s^{-1/2}\, \tilde{\sigma}_s^2). \tag{32}$$

Let A be the inverse function of $\tilde{\psi}$. From (14) $A(t/4) \leqq A(t)/2 \leqq A(t)$, thus because of our assumptions on λ and $\tilde{\sigma}^2$

$$\lambda(A_t)\, \tilde{\sigma}^{-2}(A_t) \leqq \left| \frac{d}{dt} (\tilde{h} \circ A) \left(\frac{t}{4}\right) \right| \cdot 5 \tilde{\sigma}^{-2} \left(A \left(\frac{t}{4}\right)\right)$$

for large t. The right hand expression is the derivative of $-5|\tilde{h} \circ A|$, evaluated at $t/4$. Because of (A2) and (A4) it is finitely integrable and ultimately decreasing, just as λ. Because of (18), inserting $A(t)$ for s in (32), it is enough to prove

$$\lambda_t B(t+o(t)) = o(t^{-1/2}).$$

This will follow, if we show that for any $d>0$ with probability 1 only finitely many of the events

$$B_r = \{\lambda(2^r)\, 2^{r/2} |B(t+o(t))| \geqq d \text{ for some } 2^r \leqq t \leqq 2^{r+1}\}$$

will occur. In fact it is sufficient that the same is true for the events

$$C_r = \{\sup_{t \leqq 2^{r+2}} |B(t)| \geqq d \lambda(2^r)^{-1}\, 2^{-r/2}\}.$$

Since

$$\begin{aligned} P(C_r) &\leqq \frac{1}{d} \lambda(2^r)\, 2^{r/2}\, E|B(2^{r+2})| \\ &\leqq \frac{1}{d} 2^{r+1} \lambda(2^r) \leqq \frac{4}{d} \int_{2^{r-1}}^{2^r} \lambda(s)\, ds \end{aligned}$$

for large r, the desired result follows from (27) and the Borel-Cantelli lemma and (31) is established.

From (31) and (29)

$$f(t) \leqq |Z_t| + o(\tilde{\psi}_t^{1/2}). \tag{33}$$

For any $a>0$ $\int_0^t \lambda_s \tilde{\psi}_s^{1/2}\, ds \leqq c+\tilde{\psi}_t^{1/2} \int_a^\infty \lambda_s\, ds$, if c is large enough. Therefore

$$\int_0^t \lambda_s \tilde{\psi}_s^{1/2}\, ds = o(\tilde{\psi}_t^{1/2})$$

and from (28), (29) and (31)

$$\int_0^t |\tilde{h}(Y_s)-\tilde{h}(\alpha_s)|\, ds \leqq \int_0^t \lambda_s |Z_s|\, ds + O\left(\int_0^t \lambda_s \tilde{\psi}_s^{1/2}\, ds\right)$$
$$= o(\tilde{\psi}_t^{1/2}).$$

From (26) the desired representation of assertion (ii) of the theorem follows. It remains to prove (19), which follows at once from (18), if $X_t \to \infty$ a.s. If $0<P(X_t\to\infty)<1$, we have to investigate some extra work. Let us denote by P_x the probability measure, belonging to the diffusion X_t, starting at time 0 in $x>0$. For any $\varepsilon>0$, because of (18)

$$\begin{aligned}
&\liminf_{t\to\infty} P_x(Z_t \leqq b\, \tilde{\psi}_t^{1/2})\\
&\qquad \geqq \liminf_{t\to\infty} P_x(X_t\to\infty,\ B(\tilde{\psi}_t + o(\tilde{\psi}_t)) \leqq b\, \tilde{\psi}_t^{1/2})\\
&\qquad \geqq P(B(1)\leqq b) - P_x(X_t \nrightarrow \infty)\\
&\qquad \geqq P(B(1)\leqq b) - \varepsilon
\end{aligned}$$

if x is large enough, since $P_x(X_t \nrightarrow \infty)\to 0$ as $x\to\infty$. We also need the following fact: Since ∞ is attracting, with probability 1 either $X_t\to\infty$ or $X_t\to 0$ ([5], p. 119). Thus it is possible to find a function δ_t going to ∞, as $t\to\infty$, such that the probability of the symmetric difference of $\{X_{t_0}\geqq \delta_{t_0}\}$ and $\{X_t\to\infty\}$ goes to zero, if $t_0\to\infty$. From the Markov property, with $t>t_0$ and suitable $C>0$

$$\begin{aligned}
&P(X_{t_0}\geqq \delta_{t_0},\ Z_t \leqq b\, \tilde{\psi}_t^{1/2})\\
&\qquad \geqq P(X_{t_0}\geqq \delta_{t_0},\ Z_t - Z_{t_0} \leqq b\, \tilde{\psi}_t^{1/2} - C) - \varepsilon\\
&\qquad = \int_{\{X_{t_0}\geqq \delta_{t_0}\}} P_{X_{t_0}}(Z_{t-t_0} \leqq b\, \tilde{\psi}_t^{1/2} - C)\, dP - \varepsilon.
\end{aligned}$$

Using Fatou's lemma and the estimate from above we get, if t_0 is large enough

$$\begin{aligned}
&\liminf_{t\to\infty} P(X_{t_0}\geqq \delta_{t_0},\ Z_t \leqq b\, \tilde{\psi}_t^{1/2})\\
&\qquad \geqq P(X_{t_0}\geqq \delta_{t_0})\, P(B(1)\leqq b) - 2\varepsilon.
\end{aligned}$$

Letting $t_0\to\infty$ and then $\varepsilon\to 0$

$$\liminf_{t\to\infty} P(X_t\to\infty,\ Z_t \leqq b\, \tilde{\psi}_t^{1/2}) \geqq P(X_t\to\infty)\, P(B(1)\leqq b).$$

The lim sup is estimated similarly. Thus (19) is proved. q.e.d.

4. The Asymptotic Behaviour of X_t

In this section we treat the question, under which conditions representations, similar to those given in the last section for Y_t, are valid for X_t. Smooth results only hold, if one contents oneself with approximations in probability. We start with characterizing the case, where a certain weak law of large numbers holds.

Theorem 2. *Assume* (A1)–(A4). *Then the following statements are equivalent*

i) $t^{-1} g(t) = o(\psi_t^{-1/2})$.

ii) $X_t/\mu_t \to 1$ *in probability on* $\{X_t \to \infty\}$, *as* $t \to \infty$.

iii) *There are positive numbers* β_t, $t \geqq 0$, *such that* $X_t/\beta_t \to 1$ *in probability on* $\{X_t \to \infty\}$.

We shall see in Sect. 6 that the a.s. versions of the latter statements are not equivalent to the given ones. From the definition of $\psi(t)$ we see that condition i) is a restriction on the rate of growth of $\sigma^2(t)$ depending on $g(t)$ in a slightly vague fashion. We shall discuss this further in Sect. 5. Note that i) implies $g(t) = o(t)$, thus $\log \mu_t = o(t)$. Therefore we may expect $X_t \sim \mu_t$ only in the case of subexponential growth. We continue with characterizing the cases, where X_t, properly normalized, converges in distribution.

Theorem 3. *Assume* (A1)–(A4). *Then the following is equivalent*

i) *There is a* $0 \leqq c < \infty$ *such that* $\psi^{1/2}(t)\, t^{-1} g(t) \to c$, *as* $t \to \infty$.

ii) *There are numbers* γ_t, δ_t, $t \geqq 0$, *such that the distribution of* $\gamma_t X_t + \delta_t$, *conditioned on* $\{X_t \to \infty\}$, *converges weakly to a nondegenerate probability measure.*

Furthermore, if these conditions are satisfied, then

a) *If* $\psi(\infty) < \infty$, $(X_t - \mu_t)/\tilde{g}(t)$ *converges* a.s. *on* $\{X_t \to \infty\}$ *to a nondegenerate limit.*

b) *If* $\psi(\infty) = \infty$ *and* $c = 0$, *then on* $\{X_t \to \infty\}$

$$(X_t - \mu_t)/\tilde{g}(t) = B(\tilde{\psi}_t) + o_p(\tilde{\psi}_t^{1/2})$$

in probability, where $B(t)$ *is the standard Brownian motion, given in* (30).

c) *If* $\psi(\infty) = \infty$ *and* $c > 0$, *then on* $\{X_t \to \infty\}$

$$\log X_t/\mu_t = c\, \tilde{\psi}_t^{-1/2} B(\tilde{\psi}_t) - c^2 + o_p(1)$$

in probability.

The statements of this theorem again only are valid in the domain of subexponential growth with one exception. If $t^{-1} g(t) \to c' > 0$ and $\psi(\infty) < \infty$, we get from a) the a.s. convergence of X_t/μ_t to a non-degenerate limit on $\{X_t \to \infty\}$. This behaviour is well-known from many stochastic growth models as the Galton-Watson process, but rather atypical in our approach.

Let us compare Theorems 1 and 3. If $\psi(\infty) < \infty$ (small random effects), the corresponding statements coincide up to the scaling. If $\psi(\infty) = \infty$, Y_t is asymptotically normal. The same is true for X_t only, if the magnitude of $\sigma^2(t)$ is limited by the condition $t^{-1}g(t)=o(\psi_t^{-1/2})$, which already occurred in Theorem 2. Additionally X_t may be log-normal, but this possibility is an exceptional one, since here the asymptotic magnitude of $\sigma^2(t)$ is practically determined by $g(t)$. Notice the occurrence of the term $-c^2$ in the representation in this case. This implies that for large t the event $\{X_t \geqq \mu_t\}$ takes places with probability smaller than 1/2. Thus, if the order of $\sigma^2(t)$ exceeds a certain boundary, X_t is becoming a tendency to be smaller than μ_t. This, in a more drastical form, follows from the next theorem.

Theorem 4. *Assume* (A1)–(A4). *Furthermore let* $\psi_t^{1/2} t^{-1} g(t) \to \infty$, *as* $t \to \infty$, *and ultimately increase. If* $t=o(g(t))$, *assume additionally* $\varphi(\infty)=\infty$. *Then in probability* $X_t = o_p(\mu_t)$.

An inspection of the proof shows that the conditions of this theorem cannot be weakened substantially. The rest of this section contains the proof of these results. We start with three analytical statements:

Lemma 1. *If* (A1) *and* (A4) *hold and* $g(t)$ *is ultimately decreasing, then* $g'(t) = O(G(t)^{-1})$ *and* $t^{-1}g(t)=O(G(t)^{-1})$.

Proof. In this case g is ultimately convex, thus $g'(t) \leqq 0$, $g''(t) \geqq 0$ for large t. Because of (A4) $-g'(t)$ is ultimately convex, too, thus for large t

$$\int_t^{\bar{t}} l(s)\,ds \leqq -\int_t^{\infty} g'(s)\,ds,$$

where $l(s)$ is the tangent of $g'(s)$ at point t, and $\bar{t}>t$ is determined by $l(\bar{t})=0$. Calculating both integrals and taking reciprocal values we get

$$2\frac{g''(t)}{g'(t)^2} \geqq \frac{1}{g(t)-g(\infty)} \geqq \frac{1}{g(t)}.$$

A further integration yields, with a suitable C,

$$-\frac{2}{g'(t)} \geqq G(t)+C,$$

which is the first statement. Further for decreasing g, from the definition of $G(t)$,

$$G(t) \leqq t/g(t),$$

which entails the second statement. q.e.d.

Lemma 2. *If* $\varphi(\infty)=\int_1^{\infty}\frac{h(s)}{g(s)}ds$ *is finite, then* $t-\alpha_t$ *converges to a finite limit. Otherwise* $t-\alpha_t \sim \tilde{\varphi}_t$, *as* $t\to\infty$.

Proof. $\tilde{\varphi}(\infty)$ is finite, if and only if $\int_0^{\infty} |\tilde{h}(s)|\,ds < \infty$, since $h(t)$ ultimately keeps its sign. Since $\alpha_t \sim t$, the first statement follows from (8). If $\tilde{\varphi}(\infty)=\pm\infty$,

$$\int_0^t \tilde{h}(\alpha_s)\,ds \sim \int_0^t \tilde{h}(s)\,ds = \tilde{\varphi}_t$$

from the convexity assumption on $\tilde{h}$. Again from (8) the second assertion follows. q.e.d.

Lemma 3. *If for some* $c \geqq 0$ $\psi_t^{1/2} t^{-1} g(t) \to c$, *as* $t \to \infty$, *then* $\psi_t^{1/2} g'(t) \to c$. *If* $\psi(\infty) = \infty$, *also* $\psi_t^{-1/2} \varphi_t \to c$.

Proof. If g is decreasing, the assumption of the lemma holds with $c=0$ because of Lemma 1 and (16). Similarly from Lemma 1 $\psi_t^{1/2} g'(t) \to 0$. Thus let g be ultimately increasing. If the assumption of the lemma holds with $c=0$, $g(t) = o(t)$. In this case our regularity assumptions imply $0 \leqq g'(t) \leqq g(t)\, t^{-1}$ for large t, and the assertion on $g'(t)$ follows again. Thus let us suppose that $c>0$. We start with showing that then $G(t)$ is a slowly varying function. Rewrite the assumption as

$$s^{-1} \sim c\, g(s)^{-1} \tilde{\psi}(G(s))^{-1/2}.$$

Integrating both sides from t to bt for some $b>1$, we get

$$c^{-1} \log b \sim A(G(b\,t)) - A(G(t))$$

with $A(t) = \int_0^t \tilde{\psi}(s)^{-1/2}\, ds$. The mean-value theorem, applied on the function $A(e^t)$, yields

$$A(G(b\,t)) - A(G(t)) = \frac{\bar{t}}{\tilde{\psi}(\bar{t})^{1/2}} (\log G(b\,t) - \log G(t))$$

with $G(t) \leqq \bar{t} \leqq G(b\,t)$. Taking (16) into account, as $t \to \infty$,

$$\log \frac{G(b\,t)}{G(t)} \to 0.$$

Thus $G(t)$ is slowly varying. From (14)

$$\psi(b\,t)/\psi(t) = \tilde{\psi}(G(b\,t))/\tilde{\psi}(G(t)) \to 1,$$

thus ψ_t is slowly varying, too. From the assumption of the lemma we see that $g(t)$ itself varies regularly with exponent 1. Since $g'(t)$ is ultimately monotone, $g'(t) \sim t^{-1} g(t)$ ([2], p. 446, lemma), and the assertion of $g'(t)$ is fully proved. It is now easy to prove the second statement. Let $\psi(\infty) = \infty$ and $c>0$. Then

$$\varphi(t) = \tfrac{1}{2} \int_1^t g'(s)\, \psi'(s)\, ds \sim \tfrac{1}{2} c \int_1^t \psi(s)^{-1/2} \psi'(s)\, ds = c\, \psi_t^{1/2}.$$

The case $c=0$ is treated similarly. q.e.d.

Proof of Theorem 3. Without loss of generality let us assume $X_t \to \infty$ a.s.

i) Let us first suppose that $t^{-1} g(t) = o(\psi_t^{-1/2})$. Therefore $g(t) = o(t)$ and in view of (A4) $g'(t) = o(1)$. We prove that

$$g'(\mu(t/2))\, (Y_t - t) = o_p(1) \tag{34}$$

in probability. This is easy, if $\psi(\infty)<\infty$. Since $g'(t)$ is bounded, because of (7) $\varphi(\infty)$ is finite, and Y_t-t is a.s. convergent because of Theorem 1 and Lemma 2. In this case (34) holds a.s. Now let $\psi(\infty)=\infty$. From Lemma 2 and 3 and from (14)

$$g'(\mu(t/2))=o(\tilde{\psi}(t/2)^{-1/2})=o(\tilde{\psi}_t^{-1/2})$$

and

$$t-\alpha_t=o(\tilde{\psi}_t^{1/2}).$$

These two statements and Theorem 1 again imply (34). Now from the mean value theorem

$$X_t-\mu_t=g(U_t)(Y_t-t) \tag{35}$$

with U_t between X_t and μ_t, and similarily

$$\begin{aligned} g(U_t)-g(\mu_t)&=\tilde{g}(G(U_t))-\tilde{g}(t)\\ &=g'(V_t)\,g(V_t)\,(G(U_t)-t)\end{aligned}$$

with V_t between U_t and μ_t. Since g and g' ultimately are monotone, we have a.s. for large t

$$\begin{aligned}\frac{|g(U_t)-g(\mu_t)|}{\max(g(U_t),g(\mu_t))}&\leqq|g'(V_t)\,(G(U_t)-t)|\\ &\leqq|g'(\mu(t/2))\,(Y_t-t)|.\end{aligned}$$

The last inequality follows, since a.s. $G(V_t)\geqq\min(t,Y_t)\sim t$.

From (34) we get

$$g(U_t)\sim g(\mu_t)=\tilde{g}(t)$$

in probability, thus from (35)

$$(X_t-\mu_t)/\tilde{g}(t)\sim Y_t-t. \tag{36}$$

Let now $\psi(\infty)<\infty$. Then, as noted above, (34) and therefore (36) hold a.s. and Y_t-t converges a.s. Thus statement a) of the theorem is true. Next consider the case $\psi(\infty)=\infty$. Here we know from above that $t-\alpha_t=o(\tilde{\psi}_t^{1/2})$. Thus assertion b) of the theorem follows from (36) and Theorem 1.

ii) Let now $t^{-1}g(t)\sim c\,\psi_t^{-1/2}$ with $c>0$. From the mean value theorem, applied on $\log\mu_t$,

$$\log X_t-\log\mu_t=\frac{g(V_t)}{V_t}(Y_t-t) \tag{37}$$

with V_t between X_t and μ_t. In view of Theorem 1 $G(V_t)\sim t$ a.s., and by means of (14)

$$g(V_t)/V_t\sim c\,\tilde{\psi}(G(V_t))^{-1/2}\sim c\,\tilde{\psi}_t^{-1/2}. \tag{38}$$

Let $\psi(\infty)=\infty$. From Lemma 3 $\varphi(\infty)=\pm\infty$ and taking into account Lemma 2, $t-\alpha_t\sim c\,\tilde{\psi}_t^{1/2}$. From (37) and (38)

$$\log(X_t/\mu_t)\sim c\,\tilde{\psi}_t^{-1/2}(Y_t-\alpha_t-c\,\tilde{\psi}_t^{1/2}).$$

From Theorem 1 assertion c) of the theorem follows.

Let now $\psi(\infty)<\infty$. In this case $g(t)\sim t\,c\,\psi(\infty)^{-1/2}=c' t$ and thus $g'(t)\to c'>0$. From (7) we see that $\varphi(\infty)$ is finite, too. From Theorem 1 and Lemma 2 we get the a.s. convergence of Y_t-t. The same is true for X_t/μ_t because of (37). Since $g(t)\sim c' t$, that is equivalent to the a.s. convergence of $(X_t-\mu_t)/\tilde{g}(t)$. Thus again statement a) of the theorem holds.

It remains to prove that condition i) of the theorem follows from condition ii). This is a consequence of the next result.

Proposition 1. *Assume* (A1), (A4) *and that g is ultimately increasing. Let X_n be a sequence of non-negative random variables and ρ_n, τ_n be non-negative numbers such that $(X_n-\rho_n)/\tau_n$ converges in distribution to a non-degenerate distribution function F. Suppose further that there are non-negative numbers r_n, s_n such that $s_n/s_{n+1}\to 1$, s_n is increasing and $s_n=o(G(r_n))$ and $(G(X_n)-G(r_n))/s_n$ converges to a distribution function H. Suppose finally that H is continuous and strictly increasing on the whole real line. Then there is a $c\geqq 0$ such that*

$$\frac{g(r_n)\,s_n}{r_n}\to c$$

as $n\to\infty$.

To apply this result on the proof of Theorem 3, let $t_n\to\infty$ and choose $G(r_n)=\alpha(t_n)$ and $s_n=\tilde{\psi}^{1/2}(t_n)\sim\tilde{\psi}^{1/2}(\alpha(t_n))$. The diverse assumptions of the proposition follow from Theorem 1, (14) and (16). If g is decreasing, nothing has to be shown in view of Lemma 1 and (16).

Proof. Let us assume that $d=\lim_{n\to\infty}\rho_n/\tau_n$ exists. For any $\varepsilon>0$

$$F(z)\leqq P\left(\frac{X_n-\rho_n}{\tau_n}\leqq z+\varepsilon\right)+\varepsilon\leqq P(X_n/\tau_n\leqq z+d+2\varepsilon)+\varepsilon$$

with some large n. Since X_n is non-negative, $F(z)=0$ for $z<-d$. Therefore, letting $z_0=\inf\{z\,|\,F(z)>0\}$

$$z_0\geqq -d.$$

This implies $d>-\infty$. Without loss of generality we may assume $d=0$ or $d=\infty$.

If z is a point of continuity of F,

$$F(z)=\lim_{n\to\infty}P\left(\frac{X_n-\rho_n}{\tau_n}\leqq z\right)=\lim_{n\to\infty}P\left(\frac{Y_n-q_n}{s_n}\leqq\frac{G(\rho_n+z\,\tau_n)-q_n}{s_n}\right)$$

with $q_n=G(r_n)$, $Y_n=G(X_n)$. Because of our assumptions on the distribution function H

$$\lim_{n\to\infty}\frac{G(\rho_n+z\,\tau_n)-q_n}{s_n}=(H^{-1}\circ F)(z). \tag{39}$$

Let $z_1<z_2$ be both points of continuity of F with $F(z_1)<1$ and $F(z_2)>0$ (the set of pairs (z_1,z_2) with these properties will be denoted by M). Then

$$\lim_{n\to\infty} \frac{G(\rho_n+z_2\tau_n)-G(\rho_n+z_1\tau_n)}{s_n}=H^{-1}(F(z_2))-H^{-1}(F(z_1)), \tag{40}$$

the right hand expression being possibly infinite. From the mean-value theorem, with a suitable t_n between z_1 and z_2

$$\lim_{n\to\infty} \frac{\tau_n}{s_n\, g(\rho_n+t_n\tau_n)}=\frac{(H^{-1}\circ F)(z_2)-(H^{-1}\circ F)(z_1)}{z_2-z_1}=Q(z_1,z_2). \tag{41}$$

Notice that $Q(z_1,z_2)=\infty$, if either $F(z_1)=0$ or $F(z_2)=1$. Let now

$$f_n(z)=\frac{s_n\, g(\rho_n+z\tau_n)}{\tau_n}, \qquad z>-\frac{\rho_n}{\tau_n}$$

and

$$f_-(z)=\liminf_{n\to\infty} f_n(z), \qquad f_+(z)=\limsup_{n\to\infty} f_n(z), \qquad z>-d.$$

f_- and f_+ are increasing functions, since the same is true for g ultimately. From (41), with $(z_1,z_2)\in M$,

$$-d<z\leqq z_1<z_2 \quad \text{implies} \quad f_+(z)\leqq Q(z_1,z_2)^{-1}, \tag{42}$$

$$-d<z_1<z_2\leqq z \quad \text{implies} \quad f_-(z)\geqq Q(z_1,z_2)^{-1}. \tag{43}$$

Let us assume that $z_\infty=\sup\{z \mid f_+(z)<\infty\}<\infty$. If $F(z_\infty)<1$, we may find $(z_1,z_2)\in M$ with $z_\infty\leqq z_1<z_2$ and arbitrary large z_2. From (42) $Q(z_1,z_2)=0$. Since H^{-1} is strictly increasing, $F(z_1)=F(z_2)$. Thus, letting $z_2\to\infty$, $F(z)=1$ for $z>z_\infty$. Since F is non-degenerate, we may find $-d<z_1<z_2<z_3$ with $0<F(z_1)\leqq F(z_2)<1$, $F(z_3)=1$. From (42) and (43) we deduce the contradiction

$$0<Q(z_1,z_2)^{-1}\leqq f_-(z_2)\leqq f_+(z_2)\leqq Q(z_2,z_3)^{-1}=0.$$

Thus

$$f_+(z)<\infty \qquad \text{for all } z>-d. \tag{44}$$

Now $\rho_n+\tau_n\to\infty$, since otherwise X_n is stochastically bounded, at least along a subsequence. But from the assumptions $G(r_n)\to\infty$ and $G(X_n)/G(r_n)$ converges to 1 in probability. If g is ultimately convex, from (44) $0<\lim_{t\to\infty} t^{-1}g(t)<\infty$ and also $\lim_{n\to\infty} s_n<\infty$. From this the assertion of the proposition follows immediately.

We treat now the case that g is ultimately concave. Then the same is true for f_n, and f_- will be concave everywhere. If $f_-(z)=0$ for some $z>-d$, f_- has to vanish everywhere. From (43) $Q(z_1,z_2)=\infty$ for all $(z_1,z_2)\in M$. This is only possible, if F is degenerated. Thus

$$f_-(z)>0 \qquad \text{for all } z>-d.$$

Suppose $z_0>-d$. Then one may choose $(z_1,z_2)\in M$ with $-d<z_1<z_0<z_2$. Thus $Q(z_1,z_2)=\infty$ and from (42) we reach the contradiction $f_-(z_1)=0$. Thus

$$z_0=-d.$$

If we do not assume that $\lim_{n\to\infty} \rho_n/\tau_n$ exists, one may show that the lim sup as well as the lim inf of the sequence ρ_n/τ_n is equal to $-z_0$ by means of the same argument, applied on suitable sub-sequences. Thus our assumption on the existence of $\lim \rho_n/\tau_n$ at the beginning of the proof is no restriction.

Next we draw a conclusion from the fact that f_- is continuous as a concave function. For $z>z_0$ and $\varepsilon>0$ choose $\delta>0$ such that $|f_-(z)-f_-(z+\delta)|<\varepsilon$. From (42) and (43) $f_+(z)\leqq Q(z_1,z_2)^{-1}\leqq f_-(z+\delta)\leqq f_-(z)+\varepsilon$ with $z\leqq z_1<z_2\leqq z+\delta$. Letting $\varepsilon\to 0$, $f_+(z)=f_-(z)$. In other words, $\lim_{n\to\infty} f_n(z)=f(z)$ exists for all $z>z_0$. Next, for suitable $z-\delta\leqq z_1<z_2\leqq z+\delta$, $f(z-\delta)\leqq Q(z_1,z_2)^{-1}\leqq f(z+\delta)$. Letting $\delta\to 0$, from the definition of $Q(z_1,z_2)$,

$$(H^{-1}\circ F)'(z)=\frac{1}{f(z)} \qquad z>z_0.$$

Let now $d=\infty$. Then $f(z)$ is defined on the whole real line, and, being strictly positive and concave, is constant. Thus $(H^{-1}\circ F)(z)=Az+B$ with $A>0$. Choose z_1 such that $H^{-1}(F(z_1))<0$. From (39) $\rho_n+z_1\tau_n\leqq r_n$ ultimately. Since $g(t)\,t^{-1}$ is ultimately decreasing

$$0\leqq \lim_{n\to\infty}\frac{g(r_n)\,s_n}{r_n}\leqq \lim_{n\to\infty}\frac{g(\rho_n+z_1\tau_n)\,s_n}{\tau_n}\lim_{n\to\infty}\frac{\tau_n}{\rho_n+z_1\tau_n}$$
$$=f(z_1)\cdot\frac{1}{d+z_1}=0,$$

and the desired result follows with $c=0$. Let d be finite. Without loss $d=z_0=0$ and $\rho_n=0$. In this case $\tau_n\to\infty$. Since additionally $g(t)\,t^{-1}$ is ultimately monotone, $s_{n+1}/s_n\to 1$ and

$$\frac{s_n\,g(z\,\tau_n)}{z\,\tau_n}=\frac{f_n(z)}{z}\to\frac{f(z)}{z}, \qquad z>0,$$

from [2], p. 277, Lemma 3 g is regularly varying with exponent r and $f(z)=c\,z^r$ with $c>0$. Since f is concave, $0\leqq r\leqq 1$.

Let $r<1$. Then G is regularly varying with exponent $1-r>0$. Since from our assumptions $G(X_n)/G(r_n)\to 1$ in probability, also $X_n/r_n\to 1$ in probability. This is not compatible with the fact that X_n/τ_n converges to a non-degenerate limit.

Thus $r=1$. In this case $(H^{-1}\circ F)(z)=A\log z+B$ with $A>0$. Thus we may choose $z_1>0$ such that $H^{-1}(F(z_1))=0$. From (39)

$$\lim_{n\to\infty}\frac{\rho_n+z_1\tau_n}{r_n}=1$$

such that

$$\lim_{n\to\infty}\frac{g(r_n)\,s_n}{r_n}=\lim_{n\to\infty}\frac{g(\rho_n+z_1\tau_n)}{z_1\tau_n}=\frac{1}{z_1}\lim_{n\to\infty} f_n(z_1)$$
$$=\frac{f(z_1)}{z_1}.$$

The proof of the proposition is finished. q.e.d.

Proof of Theorem 2. Without loss of generality $X_t \to \infty$ a.s. Let us assume statement i) of Theorem 2. If $\psi(\infty) < \infty$, $g(t) = o(t)$. Thus from statement a) of Theorem 3 $(X_t - \mu_t)/\mu_t = o_p(1)$. If $\psi(\infty) = \infty$, from statement b) of Theorem 3

$$\frac{X_t - \mu_t}{\mu_t} = \frac{\tilde{g}(t)}{\mu_t}(B(\tilde{\psi}_t) + o_p(\tilde{\psi}_t^{1/2})) = o_p(1).$$

Thus we have proved the implication i)⇒ii). It remains to prove iii)⇒i) of the theorem. Let $X_t \sim \beta_t$ in probability. We show that we may assume $\beta_t = v_t$ with v_t as in (9). Let $\varepsilon < 1$. Then

$$P(Y_t \leqq G(\varepsilon \beta_t)) = P(X_t \leqq \varepsilon \beta_t) \to 0.$$

In view of Theorem 1 $G(\varepsilon \beta_t) \leqq \alpha_t$ or $\varepsilon \beta_t \leqq v_t$ ultimately. Thus $\beta_t \sim v_t$. – Now, if g is decreasing, nothing has to be shown in view of Lemma 1 and (16). Thus we assume that g increases ultimately. Let $g(t)t^{-1}$ decrease for large t. From the mean-value theorem, since ultimately $v_t = \mu(\alpha_t) \leqq \mu(2t)$,

$$G(v_t/2) = G(v_t - (v_t/2)) \geqq \alpha_t - \frac{v_t/2}{g(v_t/2)} \geqq \alpha_t - \frac{\mu_{2t}}{g(\mu_{2t})}.$$

Therefore

$$P(Y_t - \alpha_t \leqq -\mu_{2t}/g(\mu_{2t})) \leqq P(X_t \leqq v_t/2) \to 0.$$

From Theorem 1 and (14)

$$g(\mu_{2t})/\mu_{2t} = o(\tilde{\psi}_t^{-1/2}) = o(\tilde{\psi}_{2t}^{-1/2}),$$

or $g(t)t^{-1} = o(\psi_t^{-1/2})$. If $g(t)t^{-1}$ is ultimately increasing, use $\mu(t/2)$ instead of $\mu(2t)$. The proof is finished. q.e.d.

Proof of Theorem 4. Since $P(X_t \to \infty) > 0$, on the complement of $\{X_t \to \infty\}$ $X_t \to 0$ a.s. ([5], p. 119), thus without loss $X_t \to \infty$ a.s. From (16) and the assumption of the theorem it is clear that $g(t)$ has to be ultimately increasing. Let us first look at the case $g(t) = o(t)$, thus $\psi(\infty) = \infty$. Let us write

$$g(t) = t\,\psi_t^{-1/2}\,q(t) \tag{45}$$

with a suitable $q(t)$, ultimately increasing and tending to infinity. For large t

$$g'(t) \geqq q(t)\left(\psi_t^{-1/2} - \frac{t}{2}\psi_t^{-3/2}\frac{\sigma^2(t)}{g(t)}\right).$$

Because of (17) and our assumption

$$t\,\psi_t^{-3/2}\frac{\sigma^2(t)}{g(t)} = o(\psi_t^{-1}\sigma^2(t)) = o(\psi_t^{-1/2}),$$

thus ultimately

$$g'(t) \geqq q(t)\,\psi_t^{-1/2}/2.$$

For large t

$$\varphi(t)=\frac{1}{2}\int_1^t \psi'(s)\,g'(s)\,ds \geqq \frac{1}{4}\int_1^t \psi'(s)\,\psi(s)^{-1/2}\,q(s)\,ds.$$

Since $\psi(\infty)=\infty$ and $q(s)\to\infty$, also $\varphi(\infty)=\infty$ and $\psi_t^{1/2}=o(\varphi(t))$. From Lemma 2 as $t\to\infty$,

$$\tilde{\psi}_t^{-1/2}(t-\alpha_t)\to\infty.$$

By means of Theorem 1 and (14), in probability

$$\tilde{\psi}_{2t}^{-1/2}(Y_t-t)\to-\infty,$$

thus because of (45), in probability

$$\frac{g(\mu_{2t})}{\mu_{2t}}(Y_t-t)\to-\infty.$$

In particular $P(t/2\leqq Y_t<t)\to 1$. If t is large enough, on this event

$$\log X_t-\log\mu_t\leqq\frac{g(\mu_{2t})}{\mu_{2t}}(Y_t-t).$$

This follows from the mean-value theorem, applied on $\log\mu_t$, and since $g(t)\,t^{-1}$ is ultimately decreasing. Thus in probability

$$\log(X_t/\mu_t)\to-\infty,$$

which is the desired result.

Next we consider the case that $g(t)\,t^{-1}$ converges to a positive number or infinity. The same is true for $g'(t)$, which thus is ultimately positive and bounded away from zero. Therefore $\psi(t)=O(\varphi(t))$. If $\varphi(\infty)=\infty$, we get $\psi_t^{1/2}=o(\varphi(t))$, and we finish the proof as above by means of Theorem 1 and Lemma 2. q.e.d.

5. Examples

In this section we discuss the conditions, occurring in the theorems of the last section.

A. Let us first consider the question, under which conditions on μ_t one has $g(t)\,t^{-1}=o(\psi_t^{-1/2})$, irrespectively of the choice of $\sigma^2(t)$. Because of (16) this reduces to the question, under which circumstances $g(\mu_t)/\mu_t=O(t^{-1})$. Because of (A4) this will be true only if $g(t)\,t^{-1}$ ultimately decreases. Now the answer comes from the estimate

$$t\frac{g(\mu_{2t})}{\mu_{2t}}\leqq\int_t^{2t}\frac{g(\mu_s)}{\mu_s}\,ds=\log\frac{\mu_{2t}}{\mu_t}\leqq t\frac{g(\mu_t)}{\mu_t},$$

holding for large t. Thus necessary and sufficient for $g(\mu_t)/\mu_t = O(t^{-1})$ is the existence of a $C>0$ such that for t large enough

$$\mu_{2t} \leqq C\mu_t. \tag{46}$$

It is easy to see that this regularity property implies

$$\mu_t = O(t^r)$$

for a suitable $r>0$, on the other hand (46) is fulfilled by any power of t. In this situation, where the rate of growth is polynomial, always $X_t \sim \mu_t$ on $\{X_t \to \infty\}$ in probability and log-normality of X_t will not occur. This becomes clear also from the following consideration: If $g(t)\,t^{-1} \sim c\,\psi_t^{-1/2}$ for some $c>0$, we know from the proof of Lemma 3 that $g(t)$ varies regularly with exponent 1. In this case μ_t has to grow quicker than any power of t. On the other hand X_t will only be asymptotically log-normal, if $\psi(\infty)=\infty$. Thus $g(t)=o(t)$ or equivalently $\log \mu_t = o(t)$ is necessary for the occurence of log-normality.

B. Let us study the example

$$dX_t = g(X_t)\,dt + dW_t,$$

i.e. $\sigma^2(t) = g(t)^{-2}$. First let $g(t)$ be ultimately decreasing. Then $\tilde{\psi}_t = O(t\,\tilde{\sigma}^2(t))$, and from (15) we see that ψ_t is of the same order as $\sigma^2(t)\,G(t) = g(t)^{-2}\,G(t)$. Thus the condition $g(t)\,t^{-1} = o(\psi_t^{-1/2})$ is equivalent to $G(t) = o(t^2)$ or $t^{1/2} = o(\mu_t)$. Since

$$\mu_t - \mu_0 = \int_0^t g(\mu_s)\,ds \geqq t\,g(\mu_t)$$

for large t, these conditions are also equivalent to $t^{-1/2} = o(\tilde{g}(t)) = o(\tilde{\sigma}^{-1}(t))$, thus in essential to (A3). Furthermore from Lemma 1

$$\tilde{g}'(t)\,\tilde{\sigma}^2(t) = O(t^{-1}\,\tilde{\sigma}^2(t)) = o(1),$$

such that (A2) is also fulfilled.

Finally $\psi(\infty) = \int_1^\infty g(s)^{-3}\,ds = \infty$. Thus our theory is applicable, iff $\mu_t t^{-1/2} \to \infty$ as $t \to \infty$. From Theorem 2 and 3 $X_t \sim \mu_t$ on $\{X_t \to \infty\}$ in this situation, and X_t is asymptotically normal.

Let now $g(t)$ be ultimately increasing. Then $\sigma^2(t) = O(1)$, thus (A3) holds. (A2) reduces to $g'(t)\,g(t)^{-2} \to 0$. If $g'(t) = O(1)$, this is trivially true. Let $g'(t) \to \infty$. Then it has to be ultimately concave because of (A4), otherwise $g(t)\,t^{-2}$ will be bounded away from zero and $G(\infty) < \infty$. Therefore $t\,g'(t)/2 \leqq \int_1^t g'(s)\,ds \leqq g(t)$ and

$$g'(t)\,g(t)^{-2} = O(g(t)^{-1}\,t^{-1}) = o(1).$$

Thus (A3) is fulfilled and Theorem 1 may be applied, if $G(\infty) = \infty$ and g is smooth enough. We show that $g(t)\,t^{-1} = o(\psi_t^{-1/2})$, iff $g(t) = o(t)$. In fact, since $g(t)\,t^{-1}$ will be decreasing for large t, $g(t) = o(t)$ implies

$$\frac{g(t)^2}{t^2}\psi_t = \frac{g(t)^2}{t^2}\int_1^t \frac{ds}{g(s)^3} \leq \int_a^\infty \frac{ds}{s^2 g(s)} + o(1)$$

for any $a>0$, thus $g(t)t^{-1}=o(\psi_t^{-1/2})$. $\psi(\infty)<\infty$ may occur now.

Altogether, $X_t \sim \mu_t$ on $\{X_t \to \infty\}$, if μ_t grows quicker than $t^{1/2}$, but not at an exponential rate (compare [5], p. 132, Theorem 5).

C. Let us now put $\sigma^2(t)\equiv 1$. We may rewrite (2) in the form

$$\dot{X}_t = g(X_t)(1+\dot{W}_t)$$

where $\dot{W}_t$ denotes white noise. Thus X_t may be viewed as the solution of a differential equation, possessing a random varying multiplicative factor. In this situation (A2) reduces to $g'(t)=o(1)$ and (A3) is always true. Furthermore $\tilde{\psi}_t = t$ and $\psi(\infty)=\infty$. We show that Theorem 2 and 3 are applicable, if μ_t is not growing too fast.

Claim. Let $0 \leq c < \infty$. Then the following statements are equivalent:

i) $g(t)\,t^{-1}\psi_t^{1/2} \to c$, as $t\to\infty$,

ii) $t^{-1/2}\log\mu_t \to 2c$,

iii) $g(t)\,t^{-1}\log t \to 2c^2$.

We only have to consider the case $g(t)=o(t)$. For large t

$$\log\mu_t = \int_0^t \frac{g(\mu_s)}{\mu_s}\,ds \geq t\,\frac{g(\mu_t)}{\mu_t}.$$

Thus ii) is equivalent to $g(\mu_t)/\mu_t \sim c\,t^{-1/2} = c\,\tilde{\psi}_t^{-1/2}$, which is nothing else but i). Further, if i) and ii) hold

$$\frac{g(\mu_t)}{\mu_t}\,t^{1/2}\,t^{-1/2}\log\mu_t \to 2c^2,$$

thus iii) holds. On the other hand, if iii) is true,

$$2c^2\,G(t) \sim \int_1^t \frac{\log s}{s}\,ds = \tfrac{1}{2}(\log t)^2.$$

Replacing t by μ_t and taking the square root, we get ii).

Thus $X_t \sim \mu_t$ on $\{X_t \to \infty\}$ and X_t is asymptotic normal, if and only if $\mu_t = o(\exp(c\,t^{1/2}))$ for any $c>0$. log-normality occurs, if $\log\mu_t$ has the critical rate of growth $t^{1/2}$. If μ_t grows even faster, $X_t = o(\mu_t)$ in probability.

6. Approximations to $\log X_t$

In this section we give a a.s. representation of $\log X_t$. We do not give the most general result possible in this direction, nevertheless we have to restrict the rate of divergence of μ_t much less than in Sect. 4. Essentially we assume that $\mu_t = O(\exp(t^\alpha))$ for some $\alpha>0$. Further we require some regularity assumptions.

(A5) There is a $C>0$ such that $\log \mu_{2t} \leqq C \log \mu_t$ for large t. Furthermore the function $e^{-t} g(e^t)$ together with its first derivative is ultimately concave or convex.

Theorem 5. *Assume* (A1)–(A5). *Then*

i) $\log X_t \sim \log \mu_t$ a.s. *on* $\{X_t \to \infty\}$,

ii) *if* $\psi(\infty) < \infty$, *then* $\frac{\mu_t}{\tilde{g}(t)} \log \frac{X_t}{v_t}$ *is* a.s. *convergent on* $\{X_t \to \infty\}$,

iii) *if* $\psi(\infty) = \infty$, *then* a.s. *on* $\{X_t \to \infty\}$

$$\log \frac{X_t}{v_t} = \frac{\tilde{g}(t)}{\mu_t} (Z_t + o(Z_t) + o(\tilde{\psi}_t^{1/2})).$$

v_t is defined in (9), in general it may not be replaced by μ_t in the expressions of the last theorem. One may show that this is allowed, iff again $g(t) t^{-1} = o(\psi_t^{-1/2})$. It is possible to derive similar approximations to $\log \log X_t$ and other functions of X_t. We do not go into this, but discuss a consequence of the theorem.

Let $\psi(\infty) = \infty$ and $g(t) t^{-1} \sim c \{2 \psi_t \log \log \psi_t\}^{-1/2}$ for some $c \geqq 0$. From Theorem 2 $X_t \sim \mu_t$ on $\{X_t \to \infty\}$ in probability. From (18) and the law of iterated logarithms $\limsup_{t \to \infty} \log \frac{X_t}{v_t} = c$, thus a.s.

$$\limsup_{t \to \infty} \frac{X_t}{v_t} = \exp(c)$$

and similarly

$$\liminf_{t \to \infty} \frac{X_t}{v_t} = \exp(-c).$$

Thus the a.s. version of Theorem 2 does not hold. In fact, in view of Theorem 5, $X_t \sim \mu_t$ a.s. on $\{X_t \to \infty\}$, iff $Z_t = o(\mu_t / \tilde{g}(t))$ a.s. on $\{X_t \to \infty\}$. Because of (18) this is equivalent to $B(\psi_t) = o(t/g(t))$ a.s. It is possible to give criteria by means of the Kolmogorov-Petrovski-test. – It is well-known that the classical strong law of large numbers for i.i.d. random variables is not equivalent to the weak law. This has nothing to do with our situation, but with the non-existence of moments. Our problems instead arise from the non-linearity of the functions involved.

Proof of Theorem 5. Let $X_t \to \infty$ a.s. Define

$$k(t) = e^{-t} g(e^t), \qquad K(t) = \int_0^t \frac{ds}{k(s)} = G(e^t).$$

Because of (A5) $k(t)$ fulfills the conditions, which we require for $g(t)$ in (A4). Since $\log \mu_t$ is the inverse of $K(t)$, $k(t)$ belongs to the class of functions, which we discussed in part A of Sect. 5. With the notation

$$f(t) = \frac{g(\mu_t)}{\mu_t} = k(\log \mu_t)$$

we get from the discussion in Sect. 5

$$\frac{f(t)}{\log \mu_t} = O(t^{-1}) \tag{47}$$

Also

$$\frac{f'(t)}{f(t)} = k'(\log \mu_t) = O(t^{-1}). \tag{48}$$

If $k(t)$ is ultimately decreasing, this follows from Lemma 1. If $k(t)$ is increasing, we use (47) and $k'(t) \leqq k(t)\, t^{-1}$ for large t. Now we proceed as above:

$$\log X_t - \log v_t = f(V_t)(Y_t - \alpha_t) \tag{49}$$

with V_t between α_t and Y_t. In view of Theorem 1 it remains to show that $f(V_t) \sim f(t)$ a.s. Now $f(t)$ is ultimately monotone, since this is true for $g(t)\, t^{-1}$. Thus eventually, by means of (48)

$$\begin{aligned}\frac{|f(V_t) - f(t)|}{\max\{f(V_t), f(t)\}} &\leqq \frac{|f'(Z_t)|}{f(Z_t)} |V_t - t| \\ &= O(Z_t^{-1}(V_t - t)) = O(t^{-1}(V_t - t)),\end{aligned}$$

with Z_t between V_t and t, thus $Z_t \sim t$ a.s. Since $|V_t - t| \leqq |Y_t - t| + |\alpha_t - t| = o(t)$ a.s., we obtain $f(V_t) \sim f(t)$ a.s. Assertion ii) and iii) of Theorem 5 follows from (49) and Theorem 1.

Further (47) and (49) imply a.s.

$$(\log X_t - \log v_t)/\log \mu_t = O(t^{-1}(Y_t - \alpha_t)) = o(1).$$

It remains to show that $\log \mu_t \sim \log v_t$. Again from (47) and the mean-value theorem

$$\begin{aligned}\frac{|\log \mu_t - \log v_t|}{\max\{\log \mu_t, \log v_t\}} &\leqq \frac{g(\mu(\bar{t}))}{\mu(\bar{t}) \log \mu(\bar{t})} |t - \alpha_t| \\ &= O(\bar{t}^{-1}(t - \alpha_t)) = o(1),\end{aligned}$$

with $\bar{t}$ between t and α_t, thus $\bar{t} \sim t$. q.e.d.

7. On the Stratonovitch-Solution

Let $\bar{X}_t$ denote the Stratonovitch-(Wong-Zakai-)solution of (1). From a well-known formula (compare [7], p. 351) $\bar{X}_t$ is the Ito-solution of the equation

$$d\bar{X}_t = g(\bar{X}_t)(1 + \tfrac{1}{2}\sigma(\bar{X}_t)(g\sigma)'(\bar{X}_t))\, dt + g(\bar{X}_t)\, \sigma(\bar{X}_t)\, dW_t. \tag{50}$$

Letting $\bar{Y}_t = G(\bar{X}_t)$, from Ito's formula

$$d\bar{Y}_t = (1 + \tfrac{1}{4}(\tilde{\sigma}^2)'(\bar{Y}_t))\, dt + \tilde{\sigma}(\bar{Y}_t)\, dW_t. \tag{51}$$

Now similar considerations are possible as those of Sect. 3, in fact, the situation is much easier. From (13) and (A4) $\frac{d}{dt}\tilde{\sigma}^2(t)$ is ultimately monotone and tending to zero. It is easily checked that $\bar{Y}_t \sim t$ on $\{\bar{X}_t \to \infty\}$, the proof being the same as for Y_t. If $\psi(\infty)=\infty$, from (17) on $\{\bar{X}_t \to \infty\}$ a.s.

$$\int_0^t (\tilde{\sigma}^2)'(\bar{Y}_s)\,ds \sim \int_0^t (\tilde{\sigma}^2)'(s) = \tilde{\sigma}^2(t) - \tilde{\sigma}^2(0) = o(\tilde{\psi}_t^{1/2}).$$

If $\psi(\infty)<\infty$, $\tilde{\sigma}^2(t)\to 0$ and from a similar argument $\int_0^\infty (\tilde{\sigma}^2)'(\bar{Y}_s)\,ds$ is a.s. finite on $\{\bar{X}_t \to \infty\}$. We thus get from (51)

Theorem 6. *If* (A1), (A3) *and* (A4) *hold, then* $\bar{Y} \sim t$ a.s. *on* $\{\bar{X}_t \to \infty\}$. *Further*

i) *If* $\psi(\infty)<\infty$, *then* $\bar{Y}_t - t$ *converges* a.s. *on* $\{\bar{X}_t \to \infty\}$.

ii) *If* $\psi(\infty)=\infty$, *then* a.s. *on* $\{\bar{X}_t \to \infty\}$

$$\bar{Y}_t = t + \bar{Z}_t + o(\tilde{\psi}_t^{1/2}).$$

$\bar{Z}_t = \int_0^t \tilde{\sigma}(\bar{Y}_s)\,dW_s$ *has the same properties as* Z_t *from Theorem 1.*

The main difference between the representations of Y_t and $\bar{Y}_t$ is that α_t is replaced by t. It is no problem to prove now the corresponding version of Theorem 2 and 3. The only difference is that the correction term $-c^2$ in the case of log-normality of X_t does no longer occur, if $\bar{X}_t$ is considered. Theorem 4 is no longer true, if $\psi(\infty)=\infty$, we always have $P(\bar{X}_t \leq \mu_t) = P(\bar{Y}_t \leq t) \to 1/2$ as $t\to\infty$.

Let $\psi(\infty)=\infty$. A comparison of Theorem 1 and 6 shows that in our context Y_t and $\bar{Y}_t$ probabilistically are indistinguishable, iff $t-\alpha_t = o(\tilde{\psi}_t^{1/2})$ or in view of Lemma 2 $\varphi(t) = o(\psi_t^{1/2})$. It turns out that this condition is well-known to us:

Claim. Let $\psi(\infty)=\infty$. Then $\varphi(t)=o(\psi_t^{1/2})$ is equivalent to $g(t)\,t^{-1} = o(\psi_t^{-1/2})$.

Proof. On part of the statement follows from Lemma 3. Let $\varphi(t)=o(\psi_t^{1/2})$. Then $g'(t)\to 0$, as may be seen from (7). Since g' is ultimately monotone

$$g'(t)\,\psi_t = O\left(\int_0^t g'(s)\,\psi'(s)\,ds\right) = O(\varphi(t)) = o(\psi_t^{1/2})$$

and thus, by means of (17)

$$g(t)\,\psi_t^{1/2} = \int_1^t g'(s)\,\psi_s^{1/2}\,ds + \tfrac{1}{2}\int_1^t \sigma^2(s)\,\psi_s^{-1/2}\,ds = o(t). \quad \text{q.e.d.}$$

If $\psi(\infty)<\infty$, Y_t and $\bar{Y}_t$ behave different only, if $\varphi(\infty)=\infty$ because of Lemma 2. In view of (7) this may only happen, if $g'(t)\to\infty$, as $t\to\infty$, which is the case of superexponential growth. A direct comparison of X_t and $\bar{X}_t$ is also possible. We show this in the case $t^{-1}g(t) \sim c\,\psi_t^{-1/2}$. Let X_t and $\bar{X}_t \to \infty$ a.s. We claim that with probability 1

$$X_t/\bar{X}_t \sim \exp(-c^2). \tag{52}$$

We sketch the proof. Let $L(t)=\int_1^t \frac{ds}{g(s)\,\sigma(s)}$, L^{-1} its inverse. $L(\infty)=\int_0^\infty \tilde{\sigma}(s)^{-1}\,ds = \infty$ because of (13). Further let $U_t=L(X_t)$, $\bar{U}_t=L(\bar{X}_t)$. From Ito's formula

$$dU_t=(\sigma(X_t)^{-1}-\tfrac{1}{2}(\sigma g)'(X_t))\,dt+dW_t$$
$$d\bar{U}_t=\sigma(\bar{X}_t)^{-1}\,dt+dW_t,$$

thus

$$\bar{U}_t-U_t=\int_0^t (\sigma(\bar{X}_s)^{-1}-\sigma(X_s)^{-1})\,ds+\tfrac{1}{2}\int_0^t (\sigma g)'(X_s)\,ds.$$

Now the derivative of $\sigma(L^{-1}(t))^{-1}$ is equal to $-(\sigma' g/\sigma)(L^{-1}(t))$ and

$$\sigma(\bar{X}_s)^{-1}-\sigma(X_s)^{-1}=-\left(\frac{d}{dt}\log\tilde{\sigma}\right)(\xi_t)\,(\bar{U}_t-U_t),$$

with ξ_t between Y_t and $\bar{Y}_t$. $\bar{U}_t-U_t$ therefore obeys the differential equation

$$f'(t)=C(\xi_t)\,f(t)+D(Y_t) \tag{53}$$

with $C(t)=\left(\frac{d}{dt}\log\tilde{\sigma}\right)(t)$ and $D(t)=\frac{1}{2}(\sigma g)'(\mu_t)=\frac{1}{2}\tilde{\sigma}(t)\frac{d}{dt}(\log\tilde{g}\,\tilde{\sigma})(t)$. Now, if $\tilde{\sigma}$ is sufficiently smooth, with $\eta_t\sim t$,

$$\exp\left(\int_0^t C(\xi_s)\,ds\right)=\exp(\log\tilde{\sigma}(\eta_t))\sim\tilde{\sigma}(t).$$

Thus, if $f(t)$ denotes the general solution of (53),

$$\begin{aligned} f(t)&\sim\tilde{\sigma}(t)\left\{\int_0^t \frac{D(Y_s)}{\tilde{\sigma}(s)}\,ds+C\right\}\\ &\sim\tfrac{1}{2}\tilde{\sigma}(t)\log(\tilde{g}\,\tilde{\sigma})(t)\end{aligned}$$

for some real C. (If $\log\tilde{g}\,\tilde{\sigma}(t)$ converges, it has to be replaced by a random constant.) Finally, applying the mean-value theorem on $\log L^{-1}$, with ζ_t between X_t and $\bar{X}_t$,

$$\begin{aligned}\log\bar{X}_t-\log X_t&=\frac{\sigma(\zeta_t)\,g(\zeta_t)}{\zeta_t}(\bar{U}_t-U_t)\\ &\sim\tfrac{1}{2}\tilde{\sigma}(t)^2\,c\,\psi(\zeta_t)^{-1/2}\log(\tilde{g}\,\tilde{\sigma})(t)\\ &\sim\tfrac{1}{2}\tilde{\sigma}(t)^2\,c\,\tilde{\psi}(t)^{-1/2}\log(\tilde{g}\,\tilde{\sigma})(t).\end{aligned}$$

Now it is possible to show that this quantity tends to c^2 $\left(\text{use }\log\tilde{g}=\int_0^t \tilde{g}'(s)\,ds+\text{const}\sim\int_0^t c\,\tilde{\psi}_s^{-1/2}\,ds\right)$.

This follows also from Theorem 3c) and the corresponding result for $\bar{X}_t$. Thus (52) is valid.

References

1. Arnold, L.: Stochastic differential equations: Theory and applications. New York: Wiley 1974
2. Feller, W.: An introduction to probability theory and its applications. Vol. II. New York: Wiley 1971
3. Fleming, H.W., Rishel, R.W.: Deterministic and stochastic optimal control. Berlin, Heidelberg, New York: Springer 1975
4. Fristedt, B., Orey, S.: The tail σ-field of one-dimensional diffusion. Stochastic analysis, Friedman, A., Pinsky, M. (eds.). New York: Academic Press 1978
5. Gihman, I.I., Skorohod, A.V.: Stochastic differential equations. Berlin, Heidelberg, New York: Springer 1972
6. Ibragimov, I.: On the composition of unimodal distributions. Theor. Probability Appl. **1**, 255–260 (1956)
7. Karlin, S., Taylor, H.M.: A second course in stochastic processes. New York: Academic Press 1981
8. Mandl, P.: Analytical treatment of one-dimensional Markov processes. Berlin, Heidelberg, New York: Springer 1968
9. Rösler, U.: The tail σ-field of time-homogeneous one-dimensional diffusion-processes. Ann. Probability **7**, 847–857 (1979)
10. Rösler, U.: Unimodality of passage times for one-dimensional strong Markov processes. Ann. Probability **8**, 853–859 (1980)

Received February 29, 1984