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On the Continuity of the *L*-Distribution Functions

By

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1. Introduction and Summary

The distribution function (d.f.) $F(x)$ is called infinitely divisible (i.d.) if its characteristic function (ch.f.) $\varphi(t)$ satisfies for every positive integer n the relation $\varphi(t) = [\varphi_n(t)]^n$ with $\varphi_n(t)$ a ch.f. KHINTCHIN [4] has shown that the ch.f. $\varphi(t)$ of an i.d.d.f. is representable in the form

(1) $$\log \varphi(t) = i\gamma t + \int_{-\infty}^{\infty} A(u, t)\,[(1 + u^2)/u^2]\, dG(u)$$

where
$$A(u, t) = \exp(iut) - 1 - itu/(1 + u^2)$$

and where γ is a constant, $G(u)$ is a non-decreasing function of bounded variation and the integrand at $u = 0$ equals $-t^2/2$. The representation (1) is unique.

An alternative formula for $\log \varphi(t)$ has been given by LÉVY [6]

(1′) $$\log \varphi(t) = i\gamma t - \frac{1}{2}\delta^2 t^2 + \left[\int_{-\infty}^{0-} + \int_{0+}^{\infty} A(u, t)\, dH(u)\right]$$

where γ and $\delta \geqq 0$ are constants, $H(u)$ is defined and non-decreasing for $u < 0$ and $u > 0$, $H(-\infty) = H(+\infty) = 0$ and, for any finite $\varepsilon > 0$,

$$\left[\int_{-\varepsilon}^{0-} + \int_{0+}^{\varepsilon} u^2 dH(u)\right] < \infty$$

As has been shown by KHINTCHIN [5], the class of i.d.d.f. is equivalent to the class of all limits in the sense of weak convergences (iwc) of sequences $F_n(x)$ of the form

(2) $$F_n(x) = P\left(\sum_{k=1}^{k_n} y_{nk} - A_n < x\right),$$

where y_{nk} is a double sequence of independent and infinitesimal random variables (r.v.) and A_n is some sequence of constants.

The i.d.d.f. $F(x)$ is said to belong to the class L ($F \in L$) if it is the limit iwc of $F_n(x)$ given by (2) with $k_n = n$ and $y_{nk} = y_k/B_n$ $(k = 1, \ldots, n)$ where B_n is some sequence of constants.

If $F \in L$, the function $H(u)$ assigned to F by formula (1′), has at any point $u < 0$ and $u > 0$ right and left derivatives, and $uH'(u)$ is nonincreasing for

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$u < 0$ and $u > 0$, where $H'(u)$ denotes either the right or the left derivative. The function $H(u)$ satisfies for arbitrary $u_1 < u_2 < 0$ and for arbitrary $0 < u_1 < u_2$ the inequality

$$H(u_2) - H(u_1) \geqq H\left(\frac{u_2}{\alpha}\right) - H\left(\frac{u_1}{\alpha}\right) \tag{3}$$

for any $0 < \alpha < 1$. (See GNEDENKO and KOLMOGOROV [2], § 30.)

It has been stated that all d.f. $F \in L$ are unimodal. A counter example, due to IBRAGIMOV [3], invalidated this statement. It is the purpose of this note to prove that all non-degenerate L-distribution functions satisfy a weaker property, namely that they are continuous.

2. The Theorem and its Proof

Theorem. *Any non-degenerate* d.f. $F \in L$ *is continuous.*

Proof. If δ in formula (1') is positive, F is continuous because it is a convolution of two d.f. one of which is Gaussian. Before considering the case $\delta = 0$, we shall prove the following

Lemma. *Let the distribution function $F \in L$ and let $H(u)$ correspond to F by formula* (1'). *Then for $u < 0$ $(u > 0)$ the relation*

$$\lim_{u \uparrow 0-} H(u) = \infty \qquad (\lim_{u \downarrow 0+} H(u) = -\infty) \tag{4}$$

holds, unless $H(u) \equiv 0$ for $u < 0$ $(u > 0)$.

Proof of Lemma. Suppose that $H(u) \not\equiv 0$ for $u < 0$ and that relation (4) does not hold. Since $H(u)$ is nondecreasing, it would be

$$\lim_{u \uparrow 0-} H(u) = a < \infty, \tag{5}$$

and, by the continuity of $H(u)$, it would be possible to find for arbitrary $\varepsilon > 0$ and $\eta > 0$ two numbers $u_1 < u_2 < 0$ such that $|u_1| < \eta$ and $H(u_2) - H(u_1) <$ $< \varepsilon$. Since η is arbitrary, it would then follow from formula (3) that the increment of H on an arbitrary large interval $\left[\frac{u_1}{\alpha}, \frac{u_2}{\alpha}\right]$ is less than ε. Taking into account that $\varepsilon > 0$ may be arbitrarily small, we would get $H(u) \equiv 0$ for $u < 0$, contrary to the assumption; relation (4), therefore, holds.

The case of $u > 0$ may be proved in the same way. The Lemma has thus been proved.

Let now in formula (1') be $\delta = 0$. By assumption, F is nondegenerate and, therefore, $H(u) \not\equiv 0$ either for $u < 0$ or for $u > 0$. Suppose that $H(u) \not\equiv 0$ for $u < 0$. By the Lemma proved, we have

$$\int_{-\infty}^{0-} dH(u) = \infty. \tag{6}$$

Since, for $u < 0$,

$$\int_{-\infty}^{0-} dH(u) = \int_{-\infty}^{-0} \frac{1+u^2}{u^2}\, dG(u),$$

and taking into account that $G(u)$ has bounded variation, relation (6) implies

$$\int_{-\infty}^{0-} \frac{1}{u^2}\, dG(u) = \infty. \tag{7}$$

By a theorem of BLUM and ROSENBLATT [1], relation (7) implies continuity of F.

References

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