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On the Concentration Function of a Sum of Independent Random Variables

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1. Introduction

The concentration function $Q(X;\lambda)$ of a random variable X is a function of the positive variable λ defined by

$$Q(X;\lambda) = \sup_{-\infty < X < \infty} P(x \leqq X \leqq x+\lambda), \qquad \lambda > 0.$$

The following fundamental properties of the concentration function are easy to prove. It is a bounded non-decreasing function of λ. If α is a real positive number, then

$$Q(X;\alpha\lambda) \leqq ([\alpha]+1)\, Q(X;\lambda)$$

where $[\alpha]$ is the integer part of α. Further, if X and Y are independent random variables

$$Q(X+Y;\lambda) \leqq \mathrm{Min}(Q(X;\lambda);\, Q(Y;\lambda)).$$

In the following — with the exception of the last section where random vectors are considered — let $X_1, X_2, \ldots$ be a sequence of independent random variables and $S_n = \sum_{k=1}^{n} X_k$. By $C_1, C_2, \ldots$ we denote positive absolute constants but for the sake of brevity we also let the same C without index denote generally different positive absolute constants.

The uniform distance between the distribution function $\bar{F}_n$ of S_n and the family of infinitely divisible distribution functions was studied by Kolmogorov [7], [9] and between $\bar{F}_n$ and suitably chosen Poisson exponentials by Le Cam [10], [11]. In Kolmogorov's as well as in Le Cam's investigations a certain inequality for $Q(S_n;\lambda)$ plays an important part. The original Kolmogorov version of this inequality, stated in [7] and proved in [8], was later improved and generalized by Rogozin [12], [13] who, combining Kolmogorov's methods with a combinatorial lemma of Sperner, was able to prove the following inequality.

(Kolmogorov-Rogozin). For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n \leqq \lambda$, one has

$$\text{(A)} \qquad Q(S_n;\lambda) \leqq C\lambda \left(\sum_{k=1}^{n} \lambda_k^2 (1 - Q(X_k;\lambda_k))\right)^{-1/2}.$$

A somewhat more general Kolmogorov type inequality is the inequality (B) below, from which (A) easily follows.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive numbers less than or equal to λ. Assume that there are numbers b_k and α_k such that

$$P\left(X_k \leqq b_k - \frac{\lambda_k}{2}\right) \geqq \alpha_k, \quad P\left(X_k \geqq b_k + \frac{\lambda_k}{2}\right) \geqq \alpha_k, \quad k = 1, 2, \ldots, n.$$

Then

$$Q(S_n; \lambda) \leqq C\lambda \left(\sum_{k=1}^{n} \lambda_k^2 \alpha_k\right)^{-1/2}. \tag{B}$$

A proof of this inequality in the case $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ can be found in Le Cam [10] where Kolmogorov's method of proof is used.

In an earlier paper [5] the present author, using characteristic functions, was able to give a new proof of the inequality (A); at the same time some multi-dimensional generalizations were obtained. With respect to one of the results of the present paper it should be noticed that the inequality (C) stated below was implicitly proved in [5]. Throughout this paper the following notations will be used. If X is a random variable with the distribution function $F(x)$ we denote by X' a random variable independent of X and with the same distribution and by X^s the symmetrically distributed random variable $X^s = X - X'$. The distribution function of X^s is denoted by $F^s(x)$.

For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n \leqq \lambda$, one has

$$Q(S_n; \lambda) \leqq C\lambda \left(\sum_{k=1}^{n} \lambda_k^2 P\left(|X_k^s| \geqq \frac{\lambda_k}{2}\right)\right)^{-1/2}. \tag{C}$$

Incidentally, let us show that the inequalities (A) and (B) are consequences of (C). From

$$P\left(|X_k^s| \geqq \frac{\lambda_k}{2}\right) \geqq 1 - Q(X_k - X_k'; \lambda_k) \geqq 1 - Q(X_k; \lambda_k)$$

it is seen that (C) implies (A). To show that (B) follows from (C) we apply the following simple lemma:

Lemma 1.1. *If X is a random variable such that*

$$P\left(X \leqq b - \frac{\lambda}{2}\right) \geqq \alpha, \quad P\left(X \geqq b + \frac{\lambda}{2}\right) \geqq \alpha, \quad \lambda \geqq 0,$$

then

$$P\left(|X^s| \geqq \frac{\lambda}{2}\right) \geqq \frac{\alpha}{2}.$$

Proof. Evidently we can assumme that $b = 0$. Let μ be the median of X and suppose first that $\mu \leqq 0$. Then

$$P\left(|X - X'| \geqq \frac{\lambda}{2}\right) \geqq P\left(X \geqq \frac{\lambda}{2}\,;\ X' \leqq 0\right) = P\left(X \geqq \frac{\lambda}{2}\right) P(X' \leqq 0) \geqq \frac{\alpha}{2}.$$

The case $\mu \geqq 0$ is treated similarly.

In this paper we shall obtain some further estimations of $Q(S_n; \lambda)$. Our starting point is the Main Lemma stated and proved in the next section by means of which we get lower and upper bounds of the concentration function $Q(X; \lambda)$ of a random variable X; these bounds depend explicitly on the characteristic function of X. To prove the lemma we use a certain convolution method which was applied in [5] and earlier by ROSÉN [14] to obtain the upper bound. In the same section we also prove a rather general inequality by means of which the upper bound in the Main Lemma can be estimated. In this way we get an inequality for $Q(S_n;\lambda)$ which contains (C) and new proofs of two inequalities for the concentration function of an infinitely divisible distribution earlier obtained by LECAM [10]. These results are contained in section 3. In the following two sections we restrict ourselves to identically distributed non-degenerated random variables. It is shown that $Q(S_n;\lambda)$ is exactly of order $n^{-1/2}$ if and only if the second order moments of the summands are finite. Further, estimations of $Q(S_n;\lambda)$ are obtained in the case where the common distribution function of the summands belongs to the domain of attraction of a stable law. Finally, in section 6, we consider some multi-dimensional generalizations.

2. Two Basic Lemmas

We shall prove two lemmas the first of which, the Main Lemma, gives us a lower and an upper bound for the concentration function; the second lemma will be applied to the upper bound in the Main Lemma.

Main Lemma. *Let X be a random variable with the concentration function $Q(X;\lambda)$ and the characteristic function $f(t)$. There are two absolute constants C_1 and C_2 such that*

$$C_1 \frac{\lambda}{1+b\lambda} \int_{-b/2}^{b/2} |f(t)|^2\, dt \leqq Q(X;\lambda) \leqq C_2 a^{-1} \int_{-a}^{a} |f(t)|\, dt \tag{2.1}$$

where b is an arbitrary positive parameter and a is a parameter satisfying $0 < a\lambda \leqq \pi$ but otherwise arbitrary.

Proof. We introduce the auxiliary functions

$$H(x) = \left(\frac{\sin x/2}{x/2}\right)^2, \quad h(t) = (1-|t|)^+.$$

Then

$$H(x) = \int_{-\infty}^{\infty} e^{itx} h(t)\, dt.$$

Let $F(x)$ be the distribution function of X. It is easily seen that

$$\int_{-\infty}^{\infty} H(a(x-\xi))\, dF(x) = a^{-1} \int_{-a}^{a} f(t)\, h(t/a)\, e^{-it\xi}\, dt \tag{2.2}$$

where a and ξ are real parameters and $a > 0$.

The right hand side of (2.1) was already proved in [5] where our starting point was the relation (2.2). In that paper we chose $a\lambda = \pi$ but the inequality is still true with the same constant C_2 if $0 < a\lambda \leqq \pi$.

The left hand side of the inequality (2.1) may also be proved by means of the relation (2.2) if we replace $F(x)$ by $F^s(x)$, thus $f(t)$ by $|f(t)|^2$, and set $\xi = 0$. The following method, however, is simpler.

Let the random variable U have the characteristic function $h(t/b)$ and be independent of X^s. Consider the random variable $V = X^s + U$ with the characteristic function $|f(t)|^2 h(t/b)$. Then

$$Q(X;\lambda) \geqq Q(V;\lambda). \tag{2.3}$$

From the inversion formula for characteristic functions we get

$$P(|V| \leqq b^{-1}) = (\pi b)^{-1} \int_{-b}^{b} |f(t)|^2 (1 - |t|/b) \frac{\sin t/b}{t/b} dt.$$

Applying the inequality

$$\sin x/x \geqq 2\pi^{-1} \quad \text{for} \quad |x| \leqq \pi/2$$

we have

$$Q(V; 2b^{-1}) \geqq P(|V| \leqq b^{-1}) \geqq b^{-1}\pi^{-2} \int_{-b/2}^{b/2} |f(t)|^2 dt. \tag{2.4}$$

But

$$Q(V; 2b^{-1}) \leqq (2(b\lambda)^{-1} + 1) Q(V;\lambda) \leqq 2(1 + b\lambda)(b\lambda)^{-1} Q(V;\lambda),$$

and hence from (2.3) and (2.4)

$$Q(X;\lambda) \geqq \tfrac{1}{2}\pi^{-2} \frac{\lambda}{1+b\lambda} \int_{-b/2}^{b/2} |f(t)|^2 dt$$

which is the desired inequality.

Remark. The factor $\lambda(1 + b\lambda)^{-1}$ of the left hand side of (2.1) tends to zero as const. λ as $\lambda \to 0$. This must necessarily be the case since, for instance, for the normal distribution $Q(X;\lambda) \sim$ const. $\cdot\, \lambda$ as $\lambda \to 0$. On the other hand $\lambda(1 + b\lambda)^{-1}$ is bounded as $\lambda \to \infty$. This is also necessary since $Q(X;\lambda) \leqq 1$.

The following lemma is stated in such a way that it can be immediately applied to an infinitely divisible distribution. The measures M_k of the lemma have the same properties as the measure occurring in Lévy's canonical representation of the characteristic function of an infinitely divisible distribution.

Lemma 2.1. *Let $M_1, M_2, \ldots, M_n$ be non-negative measures on the Borel sets of the real line deprived of the origin with the properties*

$$M_k((-\infty, -\varepsilon)) + M_k((\varepsilon, +\infty)) < \infty \tag{i}$$

for every $\varepsilon > 0$,

$$\int_{-1}^{1} x^2 M_k(dx) < \infty. \tag{ii}$$

Let $\varkappa$ be a positive constant and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ non-negative constants.

For arbitrary positive $\lambda_1, \lambda_2, \ldots, \lambda_n \leqq \lambda$ the following inequality is valid:

$$\int_{-1/\lambda}^{1/\lambda} \exp\left\{-\varkappa \sum_{k=1}^{n} \left(\tfrac{1}{2}\sigma_k^2 t^2 + \int_{-\infty}^{\infty} (1 - \cos tx) M_k(dx)\right)\right\} dt \tag{2.5}$$
$$\leqq C\varkappa^{-1/2} \left(\sum_{k=1}^{n} \left(\sigma_k^2 + \int_{|x| < \lambda_k} x^2 M_k(dx) + \lambda_k^2 \int_{|x| \geqq \lambda_k} M_k(dx)\right)\right)^{-1/2}.$$

Proof. We shall prove the inequality (2.5) by a method which is very similar to that used to prove an inequality occurring in [5]. This latter inequality is an immediate consequence of (2.5). However, since the paper [5] was written we have realized that a certain simplification of the proof is possible. We shall use this simplified method to prove (2.5).

Applying the inequality

$$1-\cos x \geqq \frac{11}{24}x^2 \quad \text{for} \quad |x| \leqq 1$$

we have for $|t| \leqq \lambda^{-1}$

$$(2.6)\qquad \int_{-\infty}^{\infty}(1-\cos t x)\, M_k(dx) = \int_{|x|<\lambda_k}(1-\cos tx)\, M_k(dx) + \int_{|x|\geqq\lambda_k}(1-\cos t x)\, M_k(dx) \geqq \frac{11}{24}\, t^2 \int_{|x|<\lambda_k} x^2\, M_k(dx) + \int_{|x|\geqq\lambda_k}(1-\cos t x)\, M_k(dx).$$

Let us denote by I the left hand side of (2.5) and let us introduce the quantities

$$(2.7)\qquad \gamma_k = \int_{|x|<\lambda_k} x^2\, M_k(dx), \qquad \sigma^2 = \sum_{k=1}^{n}(\sigma_k^2+\gamma_k), \qquad p_k = \int_{|x|\geqq\lambda_k} M_k(dx).$$

From (2.6) we obtain

$$(2.8)\qquad I \leqq \int_{-1/\lambda}^{1/\lambda} \exp\left\{-\frac{11}{24}\varkappa\,\sigma^2 t^2\right\} \prod_{k=1}^{n} \exp\left\{-\varkappa \int_{|x|\geqq\lambda_k}(1-\cos t x)\, M_k(dx)\right\} dt.$$

Let

$$\alpha_0 = \varkappa\,\sigma^2, \qquad \alpha_k = \varkappa\,\lambda_k^2 p_k \quad \text{for} \quad k = 1, 2, \ldots, n,$$

$$(2.9)\qquad A = \sum_{k=0}^{n}\alpha_k = \varkappa \sum_{k=1}^{n}\left(\sigma_k^2 + \int_{|x|<\lambda_k} x^2\, M_k(dx) + \lambda_k^2 \int_{|x|\geqq\lambda_k} M_k(dx)\right),$$

$$\beta_k = \alpha_k / A \quad \text{for} \quad k = 0, 1, \ldots, n.$$

As is easily seen we may suppose without loss of generality that all β_k are positive. From Hölder's inequality applied to (2.8) we get

$$(2.10)\qquad I \leqq \left(\int_{-1/\lambda}^{1/\lambda} \exp\left\{-\frac{11}{24} A t^2\right\} dt\right)^{\beta_0} \cdot \prod_{k=1}^{n}\left(\int_{-1/\lambda}^{1/\lambda} \exp\left\{-A\,\lambda_k^{-2}\int_{-\infty}^{\infty}(1-\cos t x)\, N_k(dx)\right\} dt\right)^{\beta_k},$$

where

$$N_k(dx) = \begin{cases} M_k(dx)/p_k & \text{for} \quad |x| \geqq \lambda_k \\ 0 & \text{for} \quad |x| < \lambda_k \end{cases}$$

is a probability measure.

We shall estimate each of the integrals of the right hand side of (2.10). We easily find that

$$(2.11)\qquad I_0 = \int_{-1/\lambda}^{1/\lambda} \exp\left\{-\frac{11}{24} A t^2\right\} dt \leqq C A^{-1/2}.$$

To estimate

$$I_k = \int_{-1/\lambda}^{1/\lambda} \exp\left\{-A\,\lambda_k^{-2}\int_{-\infty}^{\infty}(1-\cos t x)\, N_k(dx)\right\} dt$$

we apply Jensen's inequality for continuous convex functions. Let g be a continuous convex function and φ a real function of a random variable X. Then

$$g(E\varphi(X)) \leqq Eg(\varphi(X)),$$

provided the two sides of the inequality have a meaning. Since e^{-x} is convex we obtain from Jensen's inequality with $\varphi(x) = A\lambda_k^{-2}[(1 - \cos tx)$ that

$$\exp\left\{-A\lambda_k^{-2}\int_{-\infty}^{\infty}(1-\cos tx)N_k(dx)\right\} \leqq \int_{-\infty}^{\infty}\exp\{-A\lambda_k^{-2}(1-\cos tx)\}N_k(dx).$$

(The simplification of the proof mentioned above consists of the use of Jensen's inequality. In [5] a Riemann-Stieltjes integral was approximated by a Riemann-Stieltjes sum.) Thus, changing the order of integration we obtain

$$I_k \leqq \int_{|x|\geqq\lambda_k}\left(\int_{-1/\lambda}^{1/\lambda}\exp\{-A\lambda_k^{-2}(1-\cos tx)\}\,dt\right)N_k(dx).$$

For $|x| \geqq \lambda_k$ it is easily shown as in [5] that

$$\int_{-1/\lambda}^{1/\lambda}\exp\{-A\lambda_k^{-2}(1-\cos tx)\}\,dt \leqq CA^{-1/2}$$

and hence

$$I_k \leqq CA^{-1/2}. \tag{2.12}$$

From (2.10), (2.11) and (2.12) we finally get

$$I \leqq CA^{-1/2},$$

A being defined by (2.9), and the lemma is proved.

3. Bounds for the Concentration Function

In this section we shall combine the Main Lemma and Lemma 2.1 in order to obtain further estimations of the concentration function. We begin by proving an inequality for $Q(S_n;\lambda)$; it will turn out that the inequality (C) and hence the inequalities (A) and (B) in the introduction are consequences of this inequality.

By the censored variance (at λ) of a random variable X with the distribution function $F(x)$ we understand the quantity $D^2(X;\lambda)$ defined by

$$D^2(X;\lambda) = \lambda^{-2}\int_{|x|<\lambda}x^2\,dF(x) + \int_{|x|\geqq\lambda}dF(x), \qquad \lambda > 0.$$

(For $\lambda = 0$ we set $D^2(X;0) = P(X \neq 0)$.) It is not difficult to show that

(i) $D^2(X;\lambda)$ is a non-increasing function of λ, $\lambda \geqq 0$,

(ii) $D^2(X;\lambda) = 0$ for some $\lambda \geqq 0$ if and only if the distribution of X is degenerated at zero,

(iii) $D^2(X;\lambda) \geqq u^{-2}\int_{|x|\geqq u}x^2\,dF(x)$ for $u \geqq \lambda$.

Theorem 3.1. *For any positive $\lambda_1, \lambda_2, \ldots, \lambda_n \leqq \lambda$, one has*

$$Q(S_n;\lambda) \leqq C\lambda\left(\sum_{k=1}^{n}\lambda_k^2 D^2(X_k^s;\lambda_k)\right)^{-1/2}. \tag{3.1}$$

We observe the following special cases of the inequality (3.1). For $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$ we get

$$Q(S_n; \lambda) \leqq C\left(\sum_{k=1}^{n} D^2(X_k^s; \lambda)\right)^{-1/2}. \tag{3.2}$$

If, furthermore, the summands are identically distributed, then

$$Q(S_n; \lambda) \leqq C(D(X_1^s; \lambda))^{-1} n^{-1/2}. \tag{3.3}$$

((From the property (ii) of $D^2(X; \lambda)$ we know that $D(X_1^s; \lambda) = 0$ iff X_1 has a degenerated distribution.)

Proof. Let X_k have the distribution function $F_k(x)$ and the characteristic function $f_k(t)$. Then S_n has the characteristic function $\prod_{k=1}^{n} f_k(t)$. From the right hand side of the Main Lemma applied to S_n we get with $a = \lambda^{-1}$

$$Q(S_n; \lambda) \leqq C_2 \lambda \int_{-1/\lambda}^{1/\lambda} \prod_{k=1}^{n} |f_k(t)|\, dt. \tag{3.4}$$

Using the inequality

$$|f_k(t)| \leqq \exp\{-\tfrac{1}{2}(1 - |f_k(t)|^2)\}$$

and observing that

$$1 - |f_k(t)|^2 = \int_{-\infty}^{\infty} (1 - \cos t x)\, dF_k^s(x)$$

we obtain from (3.4)

$$Q(S_n; \lambda) \leqq C_2 \lambda \int_{-1/\lambda}^{1/\lambda} \exp\left\{-\tfrac{1}{2} \sum_{k=1}^{n} \int_{-\infty}^{\infty} (1 - \cos t x)\, dF_k^s(x)\right\} dt. \tag{3.5}$$

The integral above is of the type encountered in Lemma 2.1 with the exception that $F_k^s(x)$ may have a jump at the origin. It is, however, easily seen that the lemma is still applicable. Using (2.5) we get

$$Q(S_n; \lambda) \leqq C\lambda\left(\sum_{k=1}^{n}\left(\int_{|x|<\lambda_k} x^2\, dF_k^s(x) + \lambda_k^2 \int_{|x|\geqq\lambda_k} dF_k^s(x)\right)\right)^{-1/2}$$

and the theorem is proved.

With regard to later applications we shall state another inequality for $Q(S_n; \lambda)$. In the same way as (3.5) was obtained we get from the Main Lemma

$$Q(S_n; \lambda) \leqq C_2 a^{-1} \int_{-a}^{a} \exp\left\{-\tfrac{1}{2} \sum_{k=1}^{n} \int_{-\infty}^{\infty} (1 - \cos t x)\, dF_k^s(x)\right\} dt$$

where $0 < a\lambda \leqq 1$. To the right hand side of the above inequality we apply the inequalities

$$\int_{-\infty}^{\infty} (1 - \cos t x)\, dF_k^s(x) \geqq \int_{-1/|t|}^{1/|t|} (1 - \cos t x)\, dF_k^s(x) \geqq \frac{11}{24} t^2 \psi_k(t)$$

where

$$\psi_k(t) = \int_{-1/|t|}^{1/|t|} x^2\, dF_k^s(x). \tag{3.6}$$

Thus we have proved the following lemma.

Lemma 3.1.

$$Q(S_n;\lambda) \leqq C a^{-1} \int_{-a}^{a} \exp\left\{-\frac{11}{48} t^2 \sum_{k=1}^{n} \psi_k(t)\right\} dt, \tag{3.7}$$

where $0 < a\lambda \leqq 1$ *and* $\psi_k(t)$ *is defined by* (3.6).

Since $\psi_k(t)$ is an even function and non-increasing for $t > 0$ we get from (3.7)

$$Q(S_n;\lambda) \leqq C a^{-1} \int_{-a}^{a} \exp\left\{-\frac{11}{48} t^2 \sum_{k=1}^{n} \psi_k(a)\right\} dt \leqq C a^{-1} \left(\sum_{k=1}^{n} \psi_k(a)\right)^{-1/2},$$

where $0 < a\lambda \leqq 1$. Putting $a = u^{-1}$ we obtain from the last inequality and the definition of $\psi_k(t)$ the following theorem.

Theorem 3.2.

$$Q(S_n;\lambda) \leqq C\left(\sup_{u \geqq \lambda} u^{-2} \sum_{k=1}^{n} \int_{|x| \leqq u} x^2\, dF_k^s(x)\right)^{-1/2}. \tag{3.8}$$

Remark. Theorem 3.2 is a consequence of Theorem 3.1 and relation (3.2) in particular, if we observe the property (iii) of the censored variance stated in the beginning of the section. We have, however, preferred to prove the inequality (3.8) directly since the multi-dimensional generalization of the proof is almost immediate.

By means of the Main Lemma and Lemma 2.1 we shall give a new proof of two inequalities for the concentration function of an infinitely divisible distribution obtained by Le Cam [10].

Theorem 3.3. (Le Cam). *Let the random variable Y have the infinitely divisible characteristic function $g(t)$ with the Lévy canonical representation*

$$g(t) = \exp\left\{i a t - \tfrac{1}{2}\sigma^2 t^2 + \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) M(dx)\right\}, \tag{3.9}$$

where a and σ are real constants and the non-negative measure M satisfies the same conditions as the measures M_k in Lemma 2.1.

There are two absolute constants C_3 and C_4 such that

$$\begin{aligned} C_3 \operatorname{Min}\left(1;\lambda\left(\sigma^2 + \int_{|x|<\lambda} x^2 M(dx)\right)^{-1/2}\right) \exp\left\{-\int_{|x|\geqq\lambda} M(dx)\right\} \\ \leqq Q(Y;\lambda) \leqq C_4 \lambda \left(\sigma^2 + \int_{|x|<\lambda} x^2 M(dx) + \lambda^2 \int_{|x|\geqq\lambda} M(dx)\right)^{-1/2}. \end{aligned} \tag{3.10}$$

Proof. We begin by proving the right hand side inequality. Since

$$|g(t)| = \exp\left\{-\tfrac{1}{2}\sigma^2 t^2 - \int_{-\infty}^{\infty}(1 - \cos tx)\, M(dx)\right\},$$

it follows from the Main Lemma with $a = \lambda^{-1}$ that

$$Q(Y;\lambda) \leqq C_2 \lambda \int_{-1/\lambda}^{1/\lambda} \exp\left\{-\tfrac{1}{2}\sigma^2 t^2 - \int_{-\infty}^{\infty}(1 - \cos tx)\, M(dx)\right\} dt$$

and the proposed inequality immediately results from Lemma 2.1 with $n = 1$.

To prove the left hand side inequality we write $g(t)$ as

$$g(t) = \exp\left\{i a' t - \tfrac{1}{2}\sigma^2 t^2 + \int\limits_{|x|<\lambda}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) M(dx)\right\}$$

$$\exp\left\{\int\limits_{|x|\geqq\lambda}(e^{itx} - 1)\, M(dx)\right\} = g_1(t)\, g_2(t),$$

where a' is a new constant. Thus $Y = U + V$ where U and V are independent random variables with the characteristic function $g_1(t)$ and $g_2(t)$ respectively. From the expression of $g_2(t)$ it is easily seen that

$$P(V = 0) = \exp\left\{-\int\limits_{|x|\geqq\lambda} M(dx)\right\}. \tag{3.11}$$

But

$$Q(Y;\lambda) = Q(U+V;\lambda) \geqq Q(U;\lambda)\, P(V=0). \tag{3.12}$$

A lower bound for $Q(U;\lambda)$ is now obtained from the Main Lemma. We find with $b = \lambda^{-1}$ that

$$\begin{aligned} Q(U;\lambda) &\geqq \tfrac{1}{2} C_1 \lambda \int\limits_{-1/2\lambda}^{1/2\lambda} \exp\left\{-\sigma^2 t^2 - 2\int\limits_{|x|<\lambda}(1 - \cos tx)\, M(dx)\right\} dt \\ &\geqq \tfrac{1}{2} C_1 \lambda \int\limits_{-1/2\lambda}^{1/2\lambda} \exp\left\{-\sigma^2 t^2 - t^2 \int\limits_{|x|<\lambda} x^2 M(dx)\right\} dt. \end{aligned} \tag{3.13}$$

From (3.11), (3.12) and (3.13) the left hand side of the inequality is easily obtained.

Remark 1. If we apply the Main Lemma directly to $g(t)$ in order to find a lower bound we get the factor $\exp\left\{-4\int\limits_{|x|\geqq\lambda} M(dx)\right\}$ instead of $\exp\left\{-\int\limits_{|x|\geqq\lambda} M(dx)\right\}$.

Remark 2. By means of the inequalities (3.10) the following theorem of Doeblin [3] is easily proved: In order that the infinitely divisible distribution function with the characteristic function (3.9) have at least one point of discontinuity it is necessary and sufficient that $\sigma = 0$ and $\int\limits_{-\infty}^{\infty} M(dx) < \infty$.

4. The Concentration Function of a Sum of Identically Distributed Independent Random Variables

In this and the following section we assume that the random variables $X_1, X_2, \ldots$ *have the same non-degenerated distribution function* $F(x)$. The corresponding characteristic function is denoted by $f(t)$. The function $\psi(t)$ is defined by

$$\psi(t) = \int\limits_{-1/|t|}^{1/|t|} x^2\, dF^s(x); \tag{4.1}$$

evidently $\psi(t)$ is an even for $t > 0$ non-increasing function. It is easily seen that

$$\lim_{t\to 0} \psi(t) < \infty$$

if and only if $E(X_1^2) < \infty$.

From Lemma 3.1 we find that

$$Q(S_n; \lambda) \leqq C a^{-1} \int_{-a}^{a} \exp\left\{-\frac{11}{48} n t^2 \psi(t)\right\} dt, \tag{4.2}$$

where a is fixed but so small that $0 < a\lambda \leqq 1$ and $\psi(a) > 0$. This is always possible since the distribution function $F(x)$ is non-degenerated.

Let us now suppose that $E(X_1^2) = +\infty$, i.e. $\lim_{t\to 0} \psi(t) = +\infty$. Then for $0 < \varepsilon < a$ we get from (4.2)

$$\begin{aligned} Q(S_n; \lambda) &\leqq C a^{-1} \int_{-\varepsilon}^{\varepsilon} \exp\left\{-\frac{11}{48} n t^2 \psi(\varepsilon)\right\} dt \\ &\quad + C a^{-1} \int_{\varepsilon}^{a} \exp\left\{-\frac{11}{48} n t^2 \psi(a)\right\} dt \\ &\leqq \theta_1 (n \psi(\varepsilon))^{-1/2} + \theta_2 n^{-1/2} \int_{\varepsilon(n\psi(a))^{1/2}}^{\infty} \exp\left\{-\frac{11}{48} u^2\right\} du, \end{aligned}$$

where θ_1 and θ_2 are constants not depending on n or ε. Choosing $\varepsilon = n^{-1/4}$ we have $\lim_{n\to\infty} \psi(n^{-1/4}) = +\infty$ and thus

$$Q(S_n; \lambda) = o(n^{-1/2}), \qquad n \to \infty.$$

Theorem 4.1. *Let $X_1, X_2, \ldots$ be independent, non-degenerated, identically distributed random variables and $E(X_1^2) = +\infty$. Then for every fixed λ*

$$Q(S_n; \lambda) = o(n^{-1/2}) \quad \text{as} \quad n \to \infty.$$

It is seen from the proof of Theorem 4.1 that the faster the integral $\int x^2 dF^s(x)$ diverges the faster will $Q(S_n; \lambda)$ tend to zero as $n \to \infty$. This observation is confirmed by the next theorem.

Theorem 4.2. *Let X_1, $X_2, \ldots$ be independent, identically distributed random variables such that*

$$\beta_r = E(|X_1|^r) < \infty,$$

where r is a constant and $0 < r \leqq 2$. Then

$$Q(S_n; \lambda) \geqq K(r)\, \lambda (\lambda + (n \beta_r(a))^{1/r})^{-1}, \tag{4.3}$$

where $\beta_r(a) = E(|X_1 - a|^r)$ and a is arbitrary.

The constant $K(r)$ depends only on r and may be given the value

$$K(r) = \begin{cases} 1/4\,(r+1)^{-(1+1/r)} & \text{if} \quad 0 < r < 2 \\ 1/3\sqrt{3} & \text{if} \quad r = 2. \end{cases}$$

Remark. For moderately large λ it may be preferrable to write (4.3) in the form

$$Q(S_n; \lambda) \geqq K(r)\, \lambda (\lambda + (\beta_r(a)^{1/r})^{-1} n^{-1/r}, \tag{4.4}$$

valid for all λ but less suitable if λ is large.

If, in particular, $\operatorname{var} X_1 = \sigma^2 < \infty$ we have

$$Q(S_n; \lambda) \geqq 1/3\sqrt{3}\, \lambda (\lambda + \sigma\sqrt{n})^{-1} \geqq 1/3\sqrt{3}\, \lambda (\lambda + \sigma)^{-1} n^{-1/2}. \tag{4.5}$$

Proof. It is possible to prove Theorem 4.2 by means of the Main Lemma and the inequality $|f(t)|^2 \geqq 1 - K_1(r)\beta_r(a)|t|^r$, $0 < r \leqq 2$, where $K_1(r)$ is a positive constant only depending on r. The following direct method is, however, simpler. We restrict ourselves to the case $0 < r < 2$.

Let us write

$$S_n^s = S_n - S_n' = \sum_{k=1}^{n} X_k^s$$

and $\beta_r^s = E(|X_1^s|^r)$. From a moment inequality proved in [1] and valid for symmetrically distributed independent random variables we get

$$E(|S_n^s|^r) \leqq n\,\beta_r^s$$

and hence from the Markov inequality

$$P(|S_n^s| \leqq (k\,n\,\beta_r^s)^{1/r}) \geqq 1 - k^{-1},$$

where $k > 1$. Thus

$$Q(S_n^s; 2(k\,n\,\beta_r^s)^{1/r}) \geqq 1 - k^{-1}.$$

Since

$$Q(S_n^s; 2(k\,n\,\beta_r^s)^{1/r}) \leqq (2\lambda^{-1}(k\,n\,\beta_r^s)^{1/r} + 1)\,Q(S_n^s; \lambda)$$

we get

$$Q(S_n; \lambda) \geqq Q(S_n^s; \lambda) \geqq \lambda(\lambda + 2(k\,n\,\beta_r^s)^{1/r})^{-1}(1 - k^{-1}).$$

But

$$\beta_r^s = E(|X_1 - X_1'|^r) \leqq 2^r E(|X_1 - a|^r)$$

whence

$$Q(S_n; \lambda) \geqq \tfrac{1}{4}\lambda(\lambda + (n\,\beta_r(a))^{1/r})^{-1} k^{-1/r}(1 - k^{-1}).$$

For $k = r + 1$ the function $k^{-1/r}(1 - k^{-1})$ is as large as possible whence the constant $K(r)$ of the theorem.

If $r = 2$ we proceed similarly but apply Chebyshev's inequality.

The next theorem is an immediate consequence of Theorems 4.1, 4.2 and 3.1.

Theorem 4.3. *Let $X_1, X_2, \ldots$ be a sequence of independent, non-degenerated random variables with the same distribution function $F(x)$. If and only if $E(X_1^2) < \infty$ there exist positive constants $K_1(\lambda, F)$ and $K_2(\lambda, F)$ only depending on λ and the distribution function F such that*

$$K_1(\lambda, F)\,n^{-1/2} \leqq Q(S_n; \lambda) \leqq K_2(\lambda, F)\,n^{-1/2}, \quad n \geqq 1.$$

5. Stable Limit Laws

In this section we shall suppose that the common distribution function $F(x)$ of the independent, non-degenerated random variables $X_1, X_2, \ldots$ belongs to the domain of attraction of a stable law with exponent α, $0 < \alpha \leqq 2$. A positive function $L(x)$ defined on $(0, +\infty)$ is called slowly varying at infinity if for every $a > 0$

$$\lim_{x \to +\infty} \frac{L(a\,x)}{L(x)} = 1.$$

For a thorough discussion of the properties of slowly varying functions, see Feller [6, Ch. 8].

To begin with, let us consider the case $0 < \alpha < 2$. As is well known (see for instance FELLER [6, p. 544]), in order that $F(x)$ belong to the domain of attraction of a stable law with exponent α, $0 < \alpha < 2$, it is necessary and sufficient that

$$1 - F(x) + F(-x) \sim x^{-\alpha} L(x), \quad x \to +\infty, \tag{5.1}$$

where $L(x)$ is slowly varying at infinity, and

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p, \quad \frac{F(-x)}{1 - F(x) + F(-x)} \to q \quad \text{as} \quad x \to +\infty, \tag{5.2}$$

where $p \geqq 0$, $q \geqq 0$, $p + q = 1$.

From known properties of such a distribution function $F(x)$ and its corresponding characteristic function $f(t)$ it is not difficult to prove the following lemma.

Lemma 5.1. *Let the distribution function $F(x)$ satisfy the conditions* (5.1) *and* (5.2). *Then*

$$|f(t)|^2 \sim 1 - K_2(\alpha)|t|^\alpha L(|t|^{-1}) \quad as \quad t \to 0,$$

where

$$K_2(\alpha) = \begin{cases} 2\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2} & if \quad 0 < \alpha < 2, \ \alpha \neq 1 \\ \pi & if \quad \alpha = 1. \end{cases}$$

For a corresponding result, see FELLER [6, p. 562, problem 12].

From Lemma 5.1 we easily get:

Lemma 5.2. *Let the distribution function $F(x)$ satisfy the conditions* (5.1) *and* (5.2). *There are positive constants a and b (depending on F) such that*

$$|f(t)|^2 \leqq \exp\{-\tfrac{1}{2}K_2(\alpha)|t|^\alpha L(|t|^{-1})\} \quad for \quad |t| \leqq a$$

$$|f(t)|^2 \geqq \exp\{-2K_2(\alpha)|t|^\alpha L(|t|^{-1})\} \quad for \quad |t| \leqq b.$$

We now choose the parameters a and b in the Main Lemma so small that the inequalities of Lemma 5.2 are valid and obtain

$$\begin{aligned} C_1 \frac{\lambda}{1 + b\lambda} \int_{-b/2}^{b/2} \exp\{-2nK_2(\alpha)|t|^\alpha L(|t|^{-1})\}\,dt &\leqq Q(S_n; \lambda) \\ &\leqq C_2 a^{-1} \int_{-a}^{a} \exp\left\{-\frac{n}{4}K_2(\alpha)|t|^\alpha L(|t|^{-1})\right\} dt, \end{aligned} \tag{5.3}$$

where, furthermore, a is so small that $0 < a\lambda \leqq 1$.

The distribution function $F(x)$ belongs to the so called normal domain of attraction of the stable law if the conditions (5.1) and (5.2) are satisfied and, furthermore, $\lim_{x \to +\infty} L(x) = c$ where $c > 0$ is a constant. The next theorem follows from (5.3) after some easy calculations.

Theorem 5.1. *Let the distribution function $F(x)$ belong to the normal domain of attraction of a stable law with exponent α, $0 < \alpha < 2$. There exist positive constants $K_3(\lambda, F)$ and $K_4(\lambda, F)$ only depending on λ and the distribution function F such that*

$$K_3(\lambda, F)\, n^{-1/\alpha} \leqq Q(S_n; \lambda) \leqq K_4(\lambda, F)\, n^{-1/\alpha}, \quad n \geqq 1. \tag{5.4}$$

Remark 1. Theorem 5.1 is also true if $\alpha = 2$ and $F(x)$ belongs to the normal domain of attraction of the normal law. Then $E(X_1^2) < \infty$ and by Theorem 4.3 the inequalities (5.4) are still valid with $\alpha = 2$.

Remark 2. From (5.4) it follows that the jumps, if any, of the distribution function of S_n are $0(n^{-1/\alpha})$.

Remark 3. Let $F(x)$ satisfy the conditions of Theorem 5.1. Then there are norming constants a_n such that the distribution function $\bar{F}_n(x)$ of

$$\frac{X_1 + X_2 + \cdots + X_n}{n^{1/\alpha}} - a_n$$

tends to an infinitely divisible distribution function $D_\alpha(x)$ with exponent α as $n \to \infty$. If one could prove that

$$|\bar{F}_n(x) - D_\alpha(x)| \leqq K(F)\, n^{-1/\alpha}, \quad -\infty < x < \infty, \tag{5.5}$$

where $K(F)$ depends only on F, the inequalities (5.4) would be immediate consequences. It is possible to show that (5.5) is true if one assumes that $F(x)$ satisfies certain further conditions. For the best known results in this direction, see Cramér [2].

Let us now consider the general case where it is only known that $F(x)$ belongs to the domain of attraction of a stable law with $0 < \alpha < 2$. Applying the inequality (5.3) and the inequality

$$x^{-\varepsilon} < L(x) < x^{\varepsilon}$$

valid for every fixed $\varepsilon > 0$ if x is sufficiently large one obtains the following result.

Theorem 5.2. *Let the distribution function $F(x)$ belong to the domain of attraction of a stable law with exponent α, $0 < \alpha < 2$. Then to every ε such that $0 < \varepsilon < \alpha$ there correspond positive constants $K_5(\lambda, \varepsilon, F)$ and $K_6(\lambda, \varepsilon, F)$ only depending on λ, ε and the distribution function F such that*

$$K_5(\lambda, \varepsilon, F)\, n^{-1/(\alpha-\varepsilon)} \leqq Q(S_n; \lambda) \leqq K_6(\lambda, \varepsilon, F)\, n^{-1/(\alpha+\varepsilon)}, \quad n \geqq 1.$$

Let now $\alpha = 2$ and suppose that $F(x)$ belongs to the domain of attraction of the normal law. This is the case if and only if

$$\int_{-x}^{x} y^2\, dF(y) = L(x), \tag{5.6}$$

where $L(x)$ is slowly varying at infinity. The case $E(X_1^2) < \infty$ has already been treated by Theorem 4.3 and Remark 1 of Theorem 5.1. In the sequel we thus suppose that $E(X_1^2) = +\infty$. Then the function $L(x)$ defined by (5.6) is non-decreasing for $x > 0$ and $\lim\limits_{x \to +\infty} L(x) = +\infty$. From (5.6) it is not difficult to show that

$$\int_{-x}^{x} y^2\, dF^s(y) \sim 2L(x) \quad \text{as} \quad x \to +\infty. \tag{5.7}$$

The next lemma is easily proved by means of (5.7) and the relation

$$1 - |f(t)|^2 = \int_{-\infty}^{\infty} (1 - \cos tx)\, dF^s(x).$$

Lemma 5.3. *If $F(x)$ satisfies the condition* (5.6) *there are constants a and b such hat*

$$|f(t)|^2 \leqq \exp\{-\tfrac{1}{2}t^2 L(|t|^{-1})\} \quad \text{for} \quad |t| \leqq a$$
$$|f(t)|^2 \geqq \exp\{-4t^2 L(|t|^{-1})\} \quad \text{for} \quad |t| \leqq b\,.$$

In the same way as (5.3) and Theorems 5.1 and 5.2 were obtained we use the Main Lemma and Lemma 5.3 to prove the following theorem, observing that $L(x)$ is non-decreasing.

Theorem 5.3. *Let the distribution function $F(x)$ belong to the domain of attraction of a normal law and suppose that the function $L(x)$ defined by* (5.6) *tends to infinity as $x \to +\infty$. Then to every ε such that $0 < \varepsilon < 1/2$ there correspond positive constants $K_7(\lambda, \varepsilon, F)$ and $K_8(\lambda, \varepsilon, F)$ only depending on λ, ε and the distribution function F such that*

$$K_7(\lambda, \varepsilon, F)\,(n\,L(n^{1/2+\varepsilon}))^{-1/2} \leqq Q(S_n;\lambda) \leqq K_8(\lambda, \varepsilon, F)\,(n\,L(n^{1/2-\varepsilon}))^{-1/2}, \quad n \geqq 1.$$

6. Some Multi-Dimensional Results

Most of the theorems obtained in the previous sections have more or less straight-forward multi-dimensional generalizations. We will confine ourselves to multi-dimensional versions of Theorems 3.1 and 3.2.

A point or vector $(t_1, t_2, \ldots, t_r)$ in R^r will be denoted by t and we write $dt = dt_1 dt_2 \ldots dt_r$. The norm of t is defined as $|t| = \left(\sum_{k=1}^{r} t_k^2\right)^{1/2}$ and the inner product of two vectors t and x in R^r as $(t, x) = \sum_{k=1}^{r} t_k x_k$. By the same $C(r)$ we shall understand generally different, positive constants only depending on r.

A multi-dimensional generalization of the one-dimensional concept of concentration function can be defined in several ways. We shall restrict ourselves to the simple case where the concentration function is defined with regard to spheres. Let X be a random vector with values in R^r and $\Sigma_\varrho(\xi)$ a sphere in R^r with radius ϱ and center ξ. Then the concentration function of X with regard to spheres in R^r of radius ϱ is defined by

$$Q(X;\Sigma_\varrho) = \sup_{\xi \in R^r} P(X \in \Sigma_\varrho(\xi))\,.$$

Let us first state a generalization of the right hand side inequality of the Main Lemma.

Lemma 6.1. *Let X be a random vector with values in R^r and the characteristic function $f(t)$. Then*

$$(6.1) \qquad Q(X;\Sigma_\varrho) \leqq C(r)\,a^{-r} \int_{|t| \leqq a} |f(t)|\,dt \quad \text{for} \quad 0 < a\varrho \leqq 1\,.$$

Proof. We introduce the auxiliary functions

$$H(x) = 2^r \pi^{r/2} \Gamma(1 + r/2)\,|x|^{-r} (J_{r/2}(|x|/2))^2\,,$$

where $J_{r/2}$ is the Bessel function of order $r/2$, and

$$h(t) = (2\pi)^{-r} \int_{R^r} e^{-i(t,x)} H(x)\,dx\,.$$

Then $h(t)$ is a function of $|t|$, $h(0) = 1$, $0 \leqq h(t) \leqq 1$ for all t and $h(t) = 0$ for $|t| \geqq 1$. For a proof of these properties of $H(x)$ and $h(t)$, see for instance ESSEEN [4, p. 101]. From

$$H(x) = \int_{R^r} \cos(t, x)\, h(t)\, dt \geqq \int_{R^r} (1 - \tfrac{1}{2}|x|^2 |t|^2)\, h(t)\, dt \geqq H(0)\,(1 - \tfrac{1}{2}|x|^2),$$

we see that

$$H(x) \geqq \tfrac{1}{2} H(0) = \tfrac{1}{2} \frac{\pi^{r/2}}{2^r \Gamma(1 + r/2)} \quad \text{for} \quad |x| \leqq 1. \tag{6.2}$$

Let $P(B) = P(X \in B)$ where B is a Borel set in R^r. Consider the easily proved relation

$$\int_{R^r} H(a(x - \xi))\, P(dx) = a^{-r} \int_{|t| \leqq a} f(t)\, h(t/a)\, e^{-i(t,\xi)} dt, \tag{6.3}$$

where a is an arbitrary positive parameter and $\xi \in R^r$. Using (6.3) and (6.2) and proceeding as in the one-dimensional case we find that

$$Q(X; \Sigma_\varrho) \leqq 2 (H(0))^{-1} a^{-r} \int_{|t| \leqq a} |f(t)|\, dt \quad \text{for} \quad 0 < a\varrho \leqq 1$$

and the lemma is proved.

Let $X_1, X_2, \ldots$ be a sequence of independent random vectors with values in R^r and denote the corresponding probability distributions and characteristic functions by $P_k(B)$ and $f_k(t)$ respectively. Let $S_n = \sum_{k=1}^{n} X_k$. If X is a random vector with probability distribution $P(B)$, let X' be a random vector independent of X and with the same distribution and let $P^s(B)$ denote the probability distribution of $X - X'$. Further, the quantity $\chi_k(u)$ is defined by

$$\chi_k(u) = \inf_{|t|=1} \int_{|x|<u} (t, x)^2\, P_k^s(dx), \tag{6.4}$$

i.e. $\chi_k(u)$ is the least eigen value (possibly zero) of the non-negative quadratic form

$$\int_{|x|<u} (t, x)^2\, P_k^s(dx)$$

of the variables $t_1, t_2, \ldots, t_r$. Obviously,

$$\int_{|x|<u} (t, x)^2\, P_k^s(dx) \geqq \chi_k(u)\, |t|^2, \tag{6.5}$$

and $\chi_k(u)$ is a non-decreasing function for $u > 0$. If the distribution of X_k is non-singular it is easily seen that $\chi_k(u) > 0$ for all sufficiently large u.

With the above notations we state the following generalization of Theorem 3.1.

Theorem 6.1. *For any positive* $\varrho_1, \varrho_2, \ldots, \varrho_n \leqq \varrho$, *one has*

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\, \varrho^r \left(\sum_{k=1}^{n} \varrho_k^{2r} [P(|X_k^s| \geqq \varrho_k) + \varrho_k^{-2} \chi_k(\varrho_k)] \right)^{-1/2}. \tag{6.6}$$

Corollary 1.

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\, \varrho^r \left(\sum_{k=1}^{n} \varrho_k^{2r} [1 - Q(X_k; \Sigma_{\varrho_k})] \right)^{-1/2}. \tag{6.7}$$

Corollary 2. *If the random vectors are identically distributed and* $\varrho_1 = \varrho_2 = \cdots = \varrho_n = \tau \leqq \varrho$, *then*

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\,(\varrho/\tau)^r (1 - Q(X_1; \Sigma_\tau))^{-1/2} n^{-1/2}. \tag{6.8}$$

Remark. In [5, Th. 2] an inequality similar to (6.7) was stated with Σ_ϱ replaced by a rectangle and with the right hand side depending on the concentration functions of the vectors X_k defined with regard to certain unbounded domains. The concentration functions occurring in (6.7) are all defined in the same way with regard to bounded domains, spheres. In this respect the new inequality is more satisfactory than the older one. From (6.7) a Kolmogorov type inequality for concentration functions defined with regard to rectangles can easily be obtained.

Proof of Theorem 6.1. Since the method of proof is in many respects similar to that used in proving Lemma 2.1 and Theorem 3.1 we confine ourselves to the main parts of the proof. From Lemma 6.1 we get

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\,\varrho^r \int_{|t| \leqq \varrho^{-1}} \exp\left\{-\tfrac{1}{2}\sum_{k=1}^{n} \int_{R^r} (1 - \cos(t,x))\, P_k^s(dx)\right\} dt. \tag{6.9}$$

For $|t| \leqq \varrho^{-1}$

$$\int_{R^r} (1 - \cos(t,x))\, P_k^s(dx) \geqq \frac{11}{24}\chi_k(\varrho_k)\,|t|^2 + \int_{|x| \geqq \varrho_k} (1 - \cos(t,x))\, P_k^s(dx),$$

where $\chi_k(\varrho_k)$ is defined by (6.4). Introducing the quantities $p_k = \int_{|x| \geqq \varrho_k} P_k^s dx)$ (where without loss of generality $\chi_k(\varrho_k) > 0$ and $p_k > 0$),

$$A = \sum_{k=1}^{n} (\varrho_k^{2r} p_k + \varrho_k^{2r-2}\chi_k(\varrho_k)), \tag{6.10}$$

$$\beta_k = \varrho_k^{2r-2}\chi_k(\varrho_k)/A, \quad \gamma_k = \varrho_k^{2r} p_k/A,$$

and the probability measures

$$N_k(dx) = \begin{cases} P_k^s(dx)/p_k & \text{for } |x| \geqq \varrho_k \\ 0 & \text{for } |x| < \varrho_k, \end{cases}$$

we get from Hölder's inequality

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\,\varrho^r \prod_{k=1}^{n} \left(\int_{|t| \leqq \varrho^{-1}} \exp\left\{-\frac{11}{48}\varrho_k^{2-2r} A\,|t|^2\right\} dt\right)^{\beta_k} \tag{6.11}$$

$$\cdot \prod_{k=1}^{n} \left(\int_{|t| \leqq \varrho^{-1}} \exp\left\{-\tfrac{1}{2}\varrho_k^{-2r} A \int_{R^r} (1 - \cos(t,x))\, N_k(dx)\right\} dt\right)^{\gamma_k} = C(r)\,\varrho^r \prod_{k=1}^{n} I_k^{\beta_k} \prod_{k=1}^{n} T_k^{\gamma_k}.$$

Now

$$I_k \leqq (2/\varrho)^{r-1} \int_{-1/\varrho}^{1/\varrho} \exp\left\{-\frac{11}{48}\varrho_k^{2-2r} A\, t_1^2\right\} dt_1 \tag{6.12}$$

$$\leqq C(r)\,(\varrho_k/\varrho)^{r-1} A^{-1/2} \leqq C(r)\, A^{-1/2}.$$

We apply Jensen's inequality to T_k and obtain

$$T_k \leqq \int_{|x| \geqq \varrho_k} \left(\int_{|t| \leqq \varrho^{-1}} \exp\{-\tfrac{1}{2}\varrho_k^{-2r} A\,(1 - \cos(t,x))\}\, dt\right) N_k(dx).$$

Denoting the inner integral by U_k we have because of symmetry

$$U_k = \int\limits_{|t| \leqq \varrho^{-1}} \exp\{-\tfrac{1}{2}\varrho_k^{-2r} A(1-\cos(|x|t_1))\}\, dt_1 \dots dt_r$$
$$\leqq (2/\varrho)^{r-1}|x|^{-1} \int\limits_{-|x|/\varrho}^{|x|/\varrho} \exp\{-\tfrac{1}{2}\varrho_k^{-2r} A(1-\cos u)\}\, du\,.$$

Since $|x| \geqq \varrho_k$ and $\varrho_k \leqq \varrho$ it is not difficult to show that

$$U_k \leqq C(r) A^{-1/2}$$

and hence

$$T_k \leqq C(r) A^{-1/2}\,. \tag{6.13}$$

From (6.11), (6.12), (6.13) and the definition (6.10) of A the proposed inequality follows.

The proof of Corollary 1 is immediate.

Let us suppose for a moment that the random vectors X_k are identically distributed and not equal to a constant vector a.s. It follow from Theorem 6.1 that $Q(S_n; \Sigma_\varrho) = 0(n^{-1/2})$. It was pointed out in [5] that this order of magnitude cannot be improved, at least not in the general case. If, however, the random vectors are non-singularly distributed, $Q(S_n; \Sigma_\varrho)$ should be of order $n^{-r/2}$. We have not been able to prove a simple Kolmogorov type inequality from which results that $Q(S_n; \Sigma_\varrho) = 0(n^{-r/2})$ in the non-singular case, except if the distributions of the random vectors satisfy a certain symmetry condition [5, Th. 3]. From an inequality which will be stated in the next theorem and which was hinted at in [5] it follows that $Q(S_n; \Sigma_\varrho)$ is in fact of order $n^{-r/2}$. This inequality is the multidimensional generalization of the inequality (3.8) of Theorem 3.2 and will be proved in a similar way. — From now on we no longer assume that the random vectors X_k are identically distributed.

Theorem 6.2. *Let $\chi_k(u)$ be defined by* (6.4) *and $0 < \tau \leqq \varrho$. Then*

$$Q(S_n; \Sigma_\varrho) \leqq C(r)(\varrho/\tau)^r \left(\sup_{u \geqq \tau} u^{-2} \sum_{k=1}^{n} \chi_k(u)\right)^{-r/2}. \tag{6.14}$$

Corollary. *If the random vectors are identically and non-singularly distributed one has*

$$Q(S_n; \Sigma_\varrho) \leqq C(r)(\varrho/\tau)^r \left(\sup_{u \geqq \tau} u^{-2} \chi_1(u)\right)^{-r/2} n^{-r/2} \quad \textit{for} \quad 0 < \tau \leqq \varrho\,.$$

(As was earlier remarked, $\sup\limits_{u \geqq \tau} u^{-2}\chi_1(u) > 0$ if X_1 is non-singularly distributed.)

Proof of Theorem 6.2. It is evidently sufficient to prove (6.14) for $\tau = \varrho$. From Lemma 6.1 we get

$$Q(S_n; \Sigma_\varrho) \leqq C(r) a^{-r} \int\limits_{|t| \leqq a} \exp\left\{-\tfrac{1}{2}\sum_{k=1}^{n} \int\limits_{R^r} (1-\cos(t,x))\, P_k^s(dx)\right\} dt\,,$$

where $0 < a\varrho \leqq 1$. Using

$$\int\limits_{R^r} (1-\cos(t,x))\, P_k^s(dx) \geqq \int\limits_{|x| \leqq |t|^{-1}} (1-\cos(t,x))\, P_k^s(dx)$$
$$\geqq \frac{11}{24} \int\limits_{|x| \leqq |t|^{-1}} (t,x)^2\, P_k^s(dx) \geqq \frac{11}{24}\chi_k(|t|^{-1})|t|^2$$

we have

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\, a^{-r} \int_{|t| \leqq a} \exp\left\{-\frac{11}{48}\left(\sum_{k=1}^{n} \chi_k(|t|^{-1})\right)|t|^2\right\} dt.$$

Since $\chi_k(u)$ is a non-decreasing function of $u > 0$ it follows that

$$Q(S_n; \Sigma_\varrho) \leqq C(r)\, a^{-r}\left(\sum_{k=1}^{n} \chi_k(a^{-1})\right)^{-r/2}.$$

Putting $a = u^{-1}$ and observing that $u \geqq \varrho$ we get the desired inequality.

Recently SAZONOV [15] has obtained interesting results concerning estimations of concentration functions of a sum of independent, identically distributed random vectors, defined with regard to convex sets. The common probability distribution P is supposed to satisfy certain weak additional conditions. SAZONOV's methods are different from ours. From his results it follows that $Q(S_n; \Sigma_\varrho) \leqq C(P, V_\varrho)\, n^{-r/2}$, where $C(P, V_\varrho)$ is a constant only depending on the non-singular distribution P and on the volume V_ϱ of the sphere. However, an explicit expression of this dependence is not given. In Theorems 6.1 and 6.2 such an explicit expression has been obtained, even in the case of non-identically distributed random vectors, but on the other side we have confined ourselves to spherical domains.

References

[1] BAHR, B. VON, and C. G. ESSEEN: Inequalities for the rth absolute moment of a sum of random variables, $1 \leqq r \leqq 2$. Ann. math. Statistics **36**, 299—303 (1965).

[2] CRAMÉR, H.: On asymptotic expansions for sums of independent random variables with a limiting stable distribution. Sankhya, Ser. A, **25**, 13—24 (1963).

[3] DOEBLIN, W.: Sur les sommes d'un grand nombres de variables aléatoires indépendantes. Bull. Soc. math. France **53**, 23—32, 35—64 (1939).

[4] ESSEEN, C. G.: Fourier analysis of distribution functions. Acta math. **77**, 1—125 (1945).

[5] — On the Kolmogorov-Rogozin inequality for the concentration function. Z. Wahrscheinlichkeitstheorie verw. Geb. **5**, 210—216 (1966).

[6] FELLER, W.: An introduction to probability theory and its applications, Vol. II. New York: Wiley 1966.

[7] KOLMOGOROV, A.: Two uniform limit theorems for sums of independent random variables. Theor. Probab. Appl. **1**, 384—394 (1956).

[8] — Sur les propriétés des fonctions de concentration de M. P. LÉVY. Ann. Inst. Henri Poincaré **16**, 27—34 (1958).

[9] — On the approximation of distributions of sums of independent summands by infinitely divisible distributions. Sankhya, Ser. A, **25**, 159—174 (1963).

[10] LECAM, L.: On the distribution of sums of independent random variables. Bernoulli, Bayes, Laplace. Ed. by J. NEYMAN and L. LECAM. Berlin-Heidelberg-New York: Springer 1965.

[11] — A remark on the central limit theorem. Proc. nat. Acad. Sci. U.S.A. **54**, 354—359 (1965).

[12] ROGOZIN, B. A.: An estimate for concentration functions. Theor. Probab. Appl. **6**, 94—97 (1961).

[13] — On the increase of dispersion of sums of independent random variables. Theor. Probab. Appl. **6**, 97—99 (1961).

[14] Rosén, B.: On the asymptotic distribution of sums of independent random variables. Ark. Mat. **4**, 323—332 (1961).

[15] Sazonov, V. V.: On multi-dimensional concentration functions. Teor. Verojatn. Primen **11**, 683—690 (1966).

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