Z. Wahrscheinlichkeitstheorie verw. Gebiete
59, 249–276 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Some Singular Diffusion Processes and Their Associated Stochastic Differential Equations

Yōichi Ōshima

Department of Mathematics, Faculty of Engineering, Kumamoto University, Kumamoto, 860, Japan

§1. Introduction

Let $a(x)=\{a_{ij}(x);\, i,j=1,\ldots,d\}$ and $\alpha(x)=\{\alpha_{ij}(x);\, i,j=2,\ldots,d\}$ be two systems of $C_b^2(R^d)$ functions, where $C_b^k(R^d)$ is the class of all functions which has bounded continuous derivatives up to the k-th order. We assume that $a(x)$ and $\alpha(x)$ are non-negative definite matrices for each x and $a_{11}(x)\geqq c$ for some positive constant c. Suppose that a bounded measure μ is given which is singular with respect to Lebesgue measure and set $\eta(dx)=\mu(dx_1)\,dx_2\ldots dx_d$. Consider the following symmetric form $\mathscr{E}$ defined by

$$\mathscr{E}(f,g)=\tfrac{1}{2}\sum_{i,j=1}^{d}\int_{R^d} a_{ij}(x)\,\partial_i f(x)\,\partial_j g(x)\,dx + \tfrac{1}{2}\sum_{i,j=2}^{d}\int_{R^d} \alpha_{ij}(x)\,\partial_i f(x)\,\partial_j g(x)\,\eta(dx) \tag{1.1}$$

for $f,g\in C_0^\infty(R^d)$, where $C_0^\infty(R^d)$ is the class of all $C^\infty(R^d)$ functions with compact support.

By a result of Fukushima [1], if $(\mathscr{E}, C_0^\infty(R^d))$ is closable on $L^2(dx)$, then there exists a dx-symmetric diffusion process $X^0(t)$, outside some set of capacity zero, associated with the smallest closed extension $(\mathscr{E}^0,\mathscr{D}(\mathscr{E}^0))$ of $(\mathscr{E}, C_0^\infty(R^d))$ on $L^2(dx)$.

The purpose of this paper is to characterize $X^0(t)$ as a unique solution of a stochastic differential equation (SDE) (2.1). As a consequence of this result, $X^0(t)$ can be supposed to be a diffusion process without exceptional set. The idea of the proof is as follows. In § 3, we shall prove that the SDE (2.1) has a unique (in the sense of distribution) solution $X(t)$ and that the Dirichlet form of $X(t)$ on $L^2(dx)$ coincides with $\mathscr{E}$ on $C_0^\infty(R^d)$. Hence, for the proofs of existence of $X^0(t)$ and equivalence of $X^0(t)$ and $X(t)$, it is enough to show that $X(t)$ is dx-symmetric and that $C_0^\infty(R^d)$ is a core of the Dirichlet space as-

0044-3719/82/0059/0249/$05.60

sociated with $X(t)$. But the direct proof of these facts seems to be difficult[1]. Hence we shall show that $X^0(t)$ exists and satisfies (2.1) for quasi-everywhere (q.e.) starting points, where quasi-everywhere means except on a set of capacity zero. If these results have been proved, the equivalence of $X^0(t)$ and $X(t)$ follows from the uniqueness of the solution of (2.1).

The SDE (2.1) contains the continuous additive functional (CAF) $\ell_\mu(t)$ which is the CAF associated with the smooth measure $a_{11}(x)\eta(dx)$. Generally, for a given smooth measure, to describe the concrete form of the associated CAF is not easy but it is easy if the smooth measure is absolutely continuous relative to the basic measure. Moreover, it is easy to show that $(\mathscr{E}, C_0^\infty(R^d))$ is closable on $L^2(d\nu)$, where $\nu(dx)=dx+\eta(dx)$. By these reasons, we shall first take $d\nu$ as the basic measure, that is, we shall consider $(\mathscr{E}, C_0^\infty(R^d))$ on $L^2(d\nu)$.

In §4 we shall show the existence of the diffusion process $X^\nu(t)$ associated with the smallest closed extension $(\mathscr{E}^\nu, \mathscr{D}(\mathscr{E}^\nu))$ of $(\mathscr{E}, C_0^\infty(R^d))$ on $L^2(d\nu)$. Also, by using the stochastic calculus due to Fukushima [1], we shall show that the support of the CAF $\phi_A(t)$ of $X^\nu(t)$ associated with the smooth measure dx is equal to R^d q.e. In §5 we shall show that $X^0(t)\equiv X^\nu(\phi_A^{-1}(t))$ is the diffusion process associated with $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$. This follows from the general result concerning to the time change of Markov processes associated with Dirichlet forms. Such a problem is discussed by Silverstein [9]. In §6 we shall also start with $X^\nu(t)$. By an argument similar to Stroock and Varadhan [10; Theorem 4.5.2], we can represent $X^\nu(t)$ by using Brownian motions, $\phi_A(t)$ and the CAF associated with $d\eta$. Changing the time by $\phi_A(t)$, we can show that $X^0(t)$ satisfies (2.1). Similar equations are treated by S. Watanabe [12].

Analytically, Tomisaki [11] constructed the diffusion process without exceptional set in the case $a_{ij}=\alpha_{ij}$ for $i,j\geqq 2$, $a_{ij}\in C^{[(d-1)/2]+1}(R^d)$ and $a(x)$ is strictly positive definite. As for the probabilistic constructions of diffusion processes such as $X^0(t)$, there is a work of Ikeda and Watanabe [2].

§ 2. Main Results

Let $\sigma(x)=\{\sigma_{ij}(x); i,j=1,\dots,d\}$ and $\tau(x)=\{\tau_{ij}(x); i,j=2,\dots,d\}$ be two matrices satisfying $\sigma\cdot{}^t\sigma=a$, $\tau\cdot{}^t\tau=\alpha/a_{11}$, $\|\sigma(x)-\sigma(y)\|\leqq K|x-y|$ and $\|\tau(x)-\tau(y)\|\leqq K|x-y|$ for some constant K, where

$$\|\sigma(x)-\sigma(y)\| = \sum_{i,j=1}^{d} |\sigma_{ij}(x)-\sigma_{ij}(y)|$$

and

$$\|\tau(x)-\tau(y)\| = \sum_{i,j=2}^{d} |\tau_{ij}(x)-\tau_{ij}(y)|.$$

In our case, such matrices exist [10; Theorem 5.2.3]. Set $b_i=\frac{1}{2}\sum_{j=1}^{d}\partial_j a_{ji}$, $\beta_i=\frac{1}{2a_{11}}\sum_{j=2}^{d}\partial_j\alpha_{ji}$ and $\tau_{i1}=\tau_{1j}=\beta_1=0$. Consider the following SDE

---

[1] Concerning to the symmetry of the solution of martingale problems, there is a recent work of Fukushima and Stroock [13]. Some parts of our arguments may be simplified by using their result

$$dX_i(t) = \sum_{j=1}^{d} \sigma_{ij}(X(t))\, dB_j(t) + b_i(X(t))\, dt$$
$$+ \sum_{j=1}^{d} \tau_{ij}(X(t))\, dM_j(t) + \beta_i(X(t))\, \ell_\mu(dt), \tag{2.1}$$

where $B(t) = \{B_i(t); i=1, \ldots, d\}$, $M(t) = \{M_i(t); i=2, \ldots, d\}$ and $\ell_\mu(t)$ are stochastic processes satisfying the following conditions.

(i) $B(t)$ is a d-dimensional Brownian motion.

(ii) $\ell_\mu(t) = \int \ell(t, x_1)\, \mu(dx_1)$, where $L(t) = \{\ell(t, x_1); x_1 \in R^1\}$ is a family of (t, x_1)-continuous non-negative increasing processes satisfying

$$\int_0^t I_{\{X_1(s) = x_1\}}\, \ell(ds, x_1) = \ell(t, x_1)$$

for all $x_1 \in R^1$ and $t \geqq 0$, and

$$\int_0^t f(X_1(s))\, a_{11}(X(s))\, ds = \int_{R^1} \ell(t, x_1) f(x_1)\, dx_1$$

for all $t \geqq 0$ and $f \in C_0(R^1)$.

(iii) $M(t)$ is a family of continuous local martingales satisfying $\langle M_i, M_j \rangle (t) = \delta_{ij} \ell_\mu(t)$ and $\langle B_i, M_j \rangle (t) = 0$.

The *solution of* (2.1) is defined as a system $\{X(t), B(t), M(t), L(t)\}$ satisfying the conditions (i)–(iii) and Eq. (2.1). If the distribution of $X(t)$ is uniquely determined by that of $X(0)$, then we shall say that the solution of (2.1) is *unique* (in the sense of distribution). Then we have the following

Theorem 1. *The* SDE (2.1) *has a unique solution* $X(t)$. *Moreover, the Dirichlet form of* $X(t)$ *on* $L^2(dx)$ *is an extension of* $(\mathscr{E}, C_0^\infty(R^d))$.

Let $d\nu$, $(\mathscr{E}^\nu, \mathscr{D}(\mathscr{E}^\nu))$ and X^ν be those introduced in §1. Let Λ be a Borel set in R^1 with full Lebesgue measure satisfying $\mu(\Lambda) = 0$ and let $\phi_\Lambda(t)$ be the CAF of $X^\nu(t)$ defined by

$$\phi_\Lambda(t) = \int_0^t I_\Lambda(X_1^\nu(s))\, ds, \tag{2.2}$$

where $X_1^\nu(s)$ is the first coordinate of $X^\nu(s)$. Since $I_\Lambda(x_1)$ is a density of dx relative to $d\nu$, $\phi_\Lambda(t)$ is the CAF associated with the smooth measure dx. It is shown that $\phi_\Lambda(t)$ is strictly increasing. Let $X^0(t)$ be the time changed process of $X^\nu(t)$ by $\phi_\Lambda(t)$, that is, $X^0(t) = X^\nu((\phi_\Lambda)^{-1}(t))$. Then we have the following.

Theorem 2. $X^0(t)$ *is the diffusion process associated with the Dirichlet form* $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$.

Let (Ω, P_x) be the probability space on which $X^0(t)$ is defined. A probability space $(\tilde{\Omega}, \tilde{P}_x)$ is called an *enlargement* of (Ω, P_x) if there exists a mapping i of $\tilde{\Omega}$ onto Ω such that $P_x = \tilde{P}_x \circ i^{-1}$. In this case we shall write $X^0(t, \tilde{\omega})$ in place of $X^0(t, i \circ \tilde{\omega})$.

Theorem 3. *There exists an enlargement* $(\tilde{\Omega}, \tilde{P}_x)$ *of* (Ω, P_x) *such that, for q.e.x,* $(X^0(t), \tilde{\Omega}, \tilde{P}_x)$ *satisfies the* SDE (2.1) *with initial condition* $X(0)=x$.

§ 3. The SDE (2.1)

In this section, we shall prove Theorem 1. But we shall discuss under slightly more general setting. Let $\sigma(x)=\{\sigma_{ij}(x); 1\leqq i,j\leqq d\}$ and $\tau(x)=\{\tau_{ij}(x); 2\leqq i,j\leqq d\}$ be the matrices in §2 and let $b(x)=\{b_i(x); 1\leqq i\leqq d\}$ and $\beta(x)=\{\beta_i(x); 2\leqq i\leqq d\}$ be arbitrary systems of functions on R^d satisfying

$$\sum_{i=1}^{d} |b_i(x)-b_i(y)|\leqq K|x-y| \quad \text{and} \quad \sum_{i=2}^{d} |\beta_i(x)-\beta_i(y)|\leqq K|x-y|$$

for some constant K. Note that the functions $b_i(x)$ and $\beta_i(x)$ in §2 satisfy these conditions. Transforming by an orthogonal matrix, we may assume that $\sigma_{1j}(x) =\delta_{1j}\sqrt{a_{11}(x)}$ for $1\leqq j\leqq d$. In this section, unless otherwise stated, we shall consider the SDE (2.1) having these coefficients. For the definition of the solution and its uniqueness, see §2. Then we have the following theorem.

Theorem 3.1. *For any probability measure* ξ *on* R^d, *there exists a solution of the Eq.* (2.1) *which has* ξ *as the initial distribution. Moreover, the uniqueness of the solution of* (2.1) *holds.*

Proof. The proof is similar to the proof of [3; Theorem IV-7.2], so that we shall only sketch it. If (X,B,M,L) is a solution of (2.1) corresponding to the coefficients (σ,b,τ,β) then the time changed process (X^a,B^a,M^a,L^a) of (X,B,M,L) by $\phi_a(t)=\int_0^t a_{11}(X(s))ds$, that is, $X^a(t)=X(\phi_a^{-1}(t))$,

$$B^a(t)=\int_0^t \sqrt{a_{11}(X^a(s))}\, dB(\phi_a^{-1}(s)), \qquad M^a(t)=M(\phi_a^{-1}(t))$$

and $L^a(t)=\{\ell^a(t,x_1); x_1\in R^1\}$, where $\ell^a(t,x_1)=\ell(\phi_a^{-1}(t),x_1)$, is a solution corresponding to the coefficients $(\tilde{\sigma},\tilde{b},\tau,\beta)$, where $\tilde{\sigma}=\sigma/\sqrt{a_{11}}$ and $\tilde{b}=b/a_{11}$. Conversely, (X,B,M,L) is obtained from (X^a,B^a,M^a,L^a) by a time change by $\psi_a(t) =\int_0^t (1/a_{11})(X^a(s))\,ds$. Hence it is enough to, and will, assume that $a_{11}=1$.

First, we shall consider the case $b_1=0$. On a suitable probability space (Ω,P), take mutually independent random variables $X(0)$, $B(t)$ and $\hat{B}(t)$ satisfying the following conditions; $X(0)=(X_1(0),\dots,X_d(0))$ is a d-dimensional random variable with ξ as the distribution, $B(t)=(B_1(t),\dots,B_d(t))$ is a d-dimensional Brownian motion such that $B(0)=0$ and $\hat{B}(t)=(\hat{B}_2(t),\dots,\hat{B}_d(t))$ is a $(d-1)$-dimensional Brownian motion such that $\hat{B}(0)=0$. Set $X_1(t)=X_1(0)+B_1(t)$ and let $\mathfrak{t}(t,x_1)$ be the local time of $B_1(t)$ at x_1. Set $\ell(t,x_1)=2\,\mathfrak{t}(t,x_1-X_1(0))$. Using these processes, define $\ell_\mu(t)$ and $M(t)$ by $\ell_\mu(t)=\int \ell(t,x_1)\,\mu(dx_1)$ and $M_i(t) =\hat{B}_i(\ell_\mu(t))$. Then they satisfy (i)–(iii) of the definition of the solution. Hence it is enough to construct $(X_2(t),\dots,X_d(t))$ satisfying (2.1). It is constructed by the usual successive approximation method. For $2\leqq i\leqq d$ and $n\geqq 0$, define $X^{(n)}(t)=(X_2^{(n)}(t),\dots,X_d^{(n)}(t))$ inductively by

$$X_i^{(0)}(t)=X_i(0),$$

$$\begin{aligned}X_i^{(n)}(t)=X_i(0)&+\sum_{j=1}^{d}\int_0^t\sigma_{ij}(X_1(s),X^{(n-1)}(s))\,dB_j(s)\\&+\int_0^t b_i(X_1(s),X^{(n-1)}(s))\,ds+\sum_{j=2}^{d}\int_0^t\tau_{ij}(X_1(s),X^{(n-1)}(s))\,dM_j(s)\\&+\int_0^t\beta_i(X_1(s),X^{(n-1)}(s))\,\ell_\mu(ds).\end{aligned}$$

Then, by an obvious modification of the proof of [3; Theorem IV-7.2], for almost all ω, $X_i^{(n)}(t)$ converges uniformly on every finite t-interval to $X_i(t)$ ($i=2,\dots,d$) satisfying (2.1).

For given $X(0)$, it is easy to see that the distribution of $X(t)$ is uniquely determined.

If $b_1\neq 0$, then the theorem follows from the transformation of the drift. That is, for a solution $(X,\hat{B},M,L)$ on $(\Omega,\hat{P})$ of (2.1) corresponding to the coefficients $(\sigma,b-\sigma_{\cdot 1}b_1,\tau,\beta)$, define the measure P and a system of processes $B(t)=(B_1(t),\dots,B_d(t))$ by

$$P(D)=\hat{E}\left[\exp\left\{\int_0^t b_1(X(s))\,d\hat{B}_1(s)-\tfrac{1}{2}\int_0^t b_1^2(X(s))\,ds\right\}:D\right]\tag{3.1}$$

$$D\in\sigma(X(s);\,s\leqq t),\quad B_1(t)=\hat{B}_1(t)-\int_0^t b_1^2(X(s))\,ds$$

and $B_i(t)=\hat{B}_i(t)$ for $i\geqq 2$. Then the process (X,B,M,L) on (Ω,P) is a solution of (2.1) corresponding to the coefficients (σ,b,τ,β). The proof of the uniqueness is similar to [3; Theorem IV-7.2].

In this section (X,B,M,L) on (Ω,P_x) denotes the solution of (2.1) corresponding to the coefficients (σ,b,τ,β) and initial condition $X(0)=x$. Also denote by $E_x[\cdot]$ the expectation relative to P_x. As in the proof of Theorem 3.1, denote by (X^a,B^a,M^a,L^a) the solution corresponding to the coefficients $(\tilde{\sigma},\tilde{b},\tau,\beta)$ obtained by a time change of (X,B,M,L) by $\phi_a(t)$. Then $X_1^a(t)=X_1^a(0)+B_1^a(t)$ is a Brownian motion and $\frac{1}{2}\ell^a(t,x_1)$ is its local time at x_1. Thus, for all $x=(x_1,\dots,x_d)$, since

$$P_x[\ell^a(t,x_1)\in ds]=\frac{2}{\sqrt{2\pi t}}e^{-s^2/2t}\,ds\tag{3.2}$$

(see Ito and McKean [4; p. 45]), we have

$$E_x[\ell^a(t,x_1)]=\sqrt{\frac{2t}{\pi}}\quad\text{and}\quad E_x[\ell^a(t,x_1)^2]=t.\tag{3.3}$$

Moreover, since

$$E_x[\ell^a(t, y_1)] = \int_0^t \sqrt{\frac{2(t-s)}{\pi}} \frac{|y_1 - x_1|}{2\sqrt{\pi s^3}} \exp\left(-\frac{|y_1 - x_1|^2}{2s}\right) ds$$

$$= \int_0^1 \frac{\sqrt{1-s}}{\sqrt{2\pi}} \frac{|x_1 - y_1|}{\sqrt{s^3}} \exp\left(-\frac{|x_1 - y_1|^2}{2ts}\right) ds \tag{3.4}$$

(see [4; p. 25]), it follows that

$$\int_{R^1} E_x[\ell^a(t, y_1)]\, dx_1 = \frac{t}{2} \quad \text{and} \quad \int_{R^1} E_x[\ell^a(t, y_1)^2]\, dx_1 = C_1 t \sqrt{t}, \tag{3.5}$$

where $C_1 = \frac{4}{3}\sqrt{\frac{2}{\pi}}$.

Lemma 3.1. *For any positive constant* C *and* $t \geqq 0$,

$$E_x[\exp\{C\ell_\mu^a(t)\}] \leqq 2\exp(C_2 t), \tag{3.6}$$

where $C_2 = \mu(R^1)^2 C^2/2$ *and* $\ell_\mu^a(t) = \int \ell^a(t, x_1)\, \mu(dx_1)$.

Proof. Since $E_x[\exp\{C\ell_\mu^a(t)\}] \leqq E_x[\exp\{C\mu(R^1)\ell^a(t, x_1)\}]$, (3.6) follows easily from (3.2).

To calculate the Dirichlet form of $X(t)$, we shall provide a lemma similar to Stroock and Varadhan [10; Theorem 4.2.1].

Lemma 3.2. *For all* $T > 0$, *there exist positive constants* C_3 *and* C_4 *such that*

$$P[\max_{0 \leqq s \leqq t} |X(s) - X(0)| \geqq \lambda] \leqq C_3 \exp(-C_4 \lambda^{4/3} t^{-1/3}), \tag{3.7}$$

for all $\lambda > 0$ and $t \leqq T$.

Proof. Suppose that $a_{11} = 1$ and $b_1 = 0$. For $\rho > 0$ and $\theta = (\theta_1, \dots, \theta_d)$ such that $|\theta| = 1$, set

$$Y_{\rho\theta}(t) = \sum_{i,j=1}^{d} \rho \int_0^t \theta_i \sigma_{ij}(X(s))\, dB_j(s)$$

$$+ \sum_{i,j=2}^{d} \rho \int_0^t \theta_i \tau_{ij}(X(s))\, dM_j(s).$$

Then

$$\langle Y_{\rho\theta}\rangle(t) = \rho^2 \int_0^t \langle\theta, a(X(s))\,\theta\rangle\, ds + \rho^2 \int_0^t \langle\theta, \alpha(X(s))\,\theta\rangle\, \ell_\mu(ds).$$

Hence, for an upper bound C_5 of a, b, α and β, we have

$$\begin{aligned}
&P[\max_{0\leqq s\leqq t}\langle\rho\,\theta, X(s)-X(0)\rangle\geqq\rho\,\lambda]\\
&\quad=P\Big[\max_{0\leqq s\leqq t}\exp\{Y_{\rho\theta}(s)-\tfrac{1}{2}\langle Y_{\rho\theta}\rangle(s)\\
&\qquad+\int_0^s(\rho\langle\theta, b(X(u))\rangle+\tfrac{1}{2}\rho^2\langle\theta, a(X(u))\,\theta\rangle)\,du\\
&\qquad+\int_0^s(\rho\langle\theta,\beta(X(u))\rangle+\tfrac{1}{2}\rho^2\langle\theta,\alpha(X(u))\,\theta\rangle)\,\ell_\mu(du)\}\geqq e^{\rho\lambda}\Big]\\
&\quad\leqq e^{-\rho\lambda}E\Big[\max_{0\leqq s\leqq t}\exp\Big\{Y_{\rho\theta}(s)-\tfrac{1}{2}\langle Y_{\rho\theta}\rangle(s)\\
&\qquad+\int_0^s(\rho\langle\theta, b(X(u))\rangle+\tfrac{1}{2}\rho^2\langle\theta, a(X(u))\,\theta\rangle)\,du\\
&\qquad+\int_0^s(\rho\langle\theta,\beta(X(u))\rangle+\tfrac{1}{2}\rho^2\langle\theta,\alpha(X(u))\,\theta\rangle)\,\ell_\mu(du)\Big\}\Big]\\
&\quad\leqq e^{-\rho\lambda}E\Big[\max_{0\leqq s\leqq t}\exp\{Y_{\rho\theta}(s)-\tfrac{1}{2}\langle Y_{\rho\theta}\rangle(s)\}\exp\Big\{C_5\Big(\rho+\frac{\rho^2}{2}\Big)(t+\ell_\mu(t))\Big\}\Big]\\
&\quad\leqq e^{-\rho\lambda}E[\max_{0\leqq s\leqq t}\exp\{2\,Y_{\rho\theta}(s)-\langle Y_{\rho\theta}\rangle(s)\}]^{1/2}\\
&\qquad\times E[\exp\{C_5(2\rho+\rho^2)(t+\ell_\mu(t))\}]^{1/2}.
\end{aligned}$$

By Lemma 3.1 and Novikov's result [3], for any constant C, the process $\exp\left\{C\,Y_{\rho\theta}(t)-\frac{C^2}{2}\langle Y_{\rho\theta}\rangle(t)\right\}$ $(t\geqq 0)$ is a martingale and

$$E\left[\exp\left\{C\,Y_{\rho\theta}(t)-\frac{C^2}{2}\langle Y_{\rho\theta}\rangle(t)\right\}\right]=1.$$

Hence, by the martingale inequality

$$\begin{aligned}
&E[\max_{0\leqq s\leqq t}\exp\{2\,Y_{\rho\theta}(s)-\langle Y_{\rho\theta}\rangle(s)\}]\\
&\quad\leqq 4E[\exp\{2\,Y_{\rho\theta}(t)-\langle Y_{\rho\theta}\rangle(t)\}]\\
&\quad=4E[\exp\{2\,Y_{\rho\theta}(t)-4\langle Y_{\rho\theta}\rangle(t)\}\exp\{3\langle Y_{\rho\theta}\rangle(t)\}]\\
&\quad\leqq 4E[\exp\{4\,Y_{\rho\theta}(t)-8\langle Y_{\rho\theta}\rangle(t)\}]^{1/2}E[\exp\{6\langle Y_{\rho\theta}\rangle(t)\}]^{1/2}\\
&\quad=4E[\exp\{6\langle Y_{\rho\theta}\rangle(t)\}]^{1/2}\leqq 4E[\exp\{6\,C_5\,\rho^2(t+\ell_\mu(t))\}]^{1/2}.
\end{aligned}$$

Therefore, by Lemma 3.1,

$$\begin{aligned}
&P[\max_{0\leqq s\leqq t}\langle\theta, X(s)-X(0)\rangle\geqq\lambda]\\
&\quad\leqq 2e^{-\rho\lambda}E[\exp\{6\,C_5\,\rho^2(t+\ell_\mu(t))\}]^{1/4}\\
&\qquad\times E[\exp\{C_5(2\rho+\rho^2)(t+\ell_\mu(t))\}]^{1/2}\\
&\quad\leqq 4\exp\Big(-\rho\,\lambda+C_6\,\rho\,t+C_6\,\rho^2\,t+C_6\,\rho^3\,t+\frac{C_6}{2}\rho^4\,t\Big)
\end{aligned}$$

for some constant C_6. Set $\rho=(\lambda/C_6 t)^{1/3}$. Then we have

$$P[\max_{0\leqq s\leqq t}\langle\theta, X(s)-X(0)\rangle\geqq\lambda]$$
$$\leqq 4\exp\{-\tfrac{1}{2}\lambda^{4/3}(C_6 t)^{-1/3}+(C_6 t)^{2/3}\lambda^{1/3}+(C_6 t)^{1/3}\lambda^{2/3}+\lambda\}$$
$$\leqq C_7\exp(-C_8\lambda^{4/3}t^{-1/3})$$

for suitable constants C_7 and C_8 depending on T. Substituting $-\theta$ for θ we have

$$P[\max_{0\leqq s\leqq t}|\langle\theta, X(s)-X(0)\rangle|\geqq\lambda]\leqq 2C_7\exp(-C_8\lambda^{4/3}t^{-1/3}).$$

Therefore

$$P[\max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda]\leqq 2dC_7\exp(-C_8\lambda^{4/3}t^{-1/3}).$$

Secondly we shall suppose that $a_{11}(x)=1$ and $b_1(x)\neq 0$. Let $(X,\hat{B},M,L)$ on $(\Omega,\hat{P})$ be the solution of (2.1) corresponding to the coefficients $(\sigma, b-\sigma._1 b_1, \tau, \beta)$ and P be the measure defined by (3.1). Then

$$P[\max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda]$$
$$=\hat{E}\left[\exp\left\{\int_0^t b_1(X(s))\,d\hat{B}_1(s)-\tfrac{1}{2}\int_0^t b_1^2(X(s))\,ds\right\};\ \max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda\right]$$
$$\leqq\hat{E}\left[\exp\left\{2\int_0^t b_1(X(s))\,d\hat{B}_1(s)-\int_0^t b_1^2(X(s))\,ds\right\}\right]^{1/2}$$
$$\times\hat{P}[\max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda]^{1/2}.$$

Since

$$\hat{P}[\max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda]\leqq 2dC_7\exp(-C_8\lambda^{4/3}t^{-1/3}),$$

by the previous result and

$$\hat{E}\left[\exp\left\{2\int_0^t b_1(X(s))\,d\hat{B}_1(s)-\int_0^t b_1^2(X(s))\,ds\right\}\right]$$
$$\leqq\hat{E}\left[\exp\left\{2\int_0^t b_1(X(s))\,d\hat{B}_1(s)-2\int_0^t b_1^2(X(s))\,ds\right\}\right]\exp(\|b_1\|^2 t)$$
$$\leqq\exp(\|b_1\|^2 t)\leqq\exp(\|b_1\|^2 T),$$

we have the result.

In the general case, since

$$P[\max_{0\leqq s\leqq t}|X(s)-X(0)|\geqq\lambda]$$
$$\leqq P[\max_{0\leqq s\leqq\|a_{11}\|t}|X^a(s)-X^a(0)|\geqq\lambda],$$

the result follows from the above case.

Denote by P_t the transition function of $(X(t), P_x)$. Then the following corollary holds.

Corollary. *If f is a bounded measurable function which vanishes outside a compact set, then $P_t f$ is $d\nu$-integrable.*

Proof. Denote by B_n the sphere with center 0 and radius n. Then it is enough to show that $\int P_t(x, B_n)\, \nu(dx) < \infty$ for all $n \geqq 1$. Fix $n \geqq 1$. Then, by Lemma 3.2,

$$P_t(x, B_n) \leqq P_x[|X(t) - X(0)| \geqq k - n] \\ \leqq C_3 \exp\{-C_4 (k-n)^{4/3} t^{-1/3}\}$$

for all $x \in B_{k+1} - B_k$ $(k \geqq n)$. Since $\nu(B_{k+1}) - \nu(B_k) \leqq \text{constant} \times (k+1)^d$,

$$\int P_t(x, B_n)\, \nu(dx) \leqq \nu(B_n) + C_3 \sum_{k=n}^{\infty} \exp\{-C_4 (k-n)^{4/3} t^{-1/3}\} \\ \times (\nu(B_{k+1}) - \nu(B_k)) < \infty.$$

Now we shall show the fundamental lemma in this section. Roughly speaking, the lemma shows that the measure $a_{11}(x)\eta(dx)$ is the smooth measure associated with the CAF $\ell_\mu(t)$ of $X(t)$.

Lemma 3.3. *For all $f, g \in C_0^\infty(R^d)$,*

$$\lim_{t \to 0} \frac{1}{t} \int_{R^d} f(x) E_x\left[\int_0^t g(X(s))\, \ell_\mu(ds)\right] dx \\ = \int_{R^d} a_{11}(x) f(x) g(x)\, \eta(dx). \tag{3.8}$$

Proof. Firstly, we shall suppose that $b_1 = 0$. By the definition of ϕ_a, X^a and ℓ^a,

$$\begin{aligned} &\int f(x) E_x\left[\int_0^t g(X(s))\, \ell_\mu(ds)\right] dx \\ &= \int f(x) E_x\left[\int_0^{\phi_a(t)} g(X^a(s))\, \ell_\mu^a(ds)\right] dx \\ &= \int f(x) \int \mu(dy_1) E_x\left[\int_0^{\phi_a(t)} g(y_1, X_2^a(s), \ldots, X_d^a(s))\, \ell^a(ds, y_1)\right] dx \\ &= \int f(x) \int \mu(dy_1) E_x\left[\int_0^{\phi_a(t)} g(y_1, X_2^a(0), \ldots, X_d^a(0))\, \ell^a(ds, y_1)\right] dx \\ &\quad + \int f(x) \int \mu(dy_1) E_x\left[\int_0^{\phi_a(t)} \{g(y_1, X_2^a(s), \ldots, X_d^a(s)) \right. \\ &\quad \left. - g(y_1, X_2^a(0), \ldots, X_d^a(0))\}\, \ell^a(ds, y_1)\right] dx \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Note that $\phi_a(t)$ is the inverse function of the CAF $\psi_a(t) = \int_0^t (1/a_{11})\,(X^a(s))\, ds$, and hence a stopping time, of X^a. Set $Y(t) = (X_2^a(t), \ldots, X_d^a(t))$. Then by Ito's formula,

$$\int_0^{\phi_a(t)} \{g(y_1, Y(s)) - g(y_1, Y(0))\} \ell^a(ds, y_1)$$

$$= \int_0^{\phi_a(t)} \left\{ \sum_{i=2}^{d} \int_0^s \partial_i g(y_1, Y(u))\, dX_i^a(u) + \frac{1}{2} \sum_{i,j=2}^{d} \int_0^s \partial_i \partial_j g(y_1, Y(u))\, d\langle X_i^a, X_j^a\rangle(u) \right\} \ell^a(ds, y_1)$$

$$= \ell^a(\phi_a(t), y_1) \left\{ \sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(u))\, dX_i^a(u) + \frac{1}{2} \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_i \partial_j g(y_1, Y(u))\, d\langle X_i^a, X_j^a\rangle(u) \right\}$$

$$- \sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, \ell^a(s, y_1)\, dX_i^a(s)$$

$$- \frac{1}{2} \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_i \partial_j g(y_1, Y(s))\, \ell^a(s, y_1)\, d\langle X_i^a, X_j^a\rangle(s).$$

Since $\phi_a(t) \leqq \|a_{11}\|\, t$, we have

$$\left| E_x\left[\int_0^{\phi_a(t)} \{g(y_1, Y(s)) - g(y_1, Y(0))\} \ell^a(ds, y_1) \right] \right|$$

$$\leqq E_x\left[\ell^a(\phi_a(t), y_1) \left| \sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, dX_i^a + \frac{1}{2} \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_1 \partial_j g(y_1, Y(s))\, d\langle X_i^a, X_j^a\rangle(s) \right| \right]$$

$$+ \left| E_x\left[\sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, \ell^a(s, y_1) \{(b_i/a_{11})(X^a(s))\, ds + \beta_i(X^a(s))\, \ell_\mu^a(ds)\} \right] \right|$$

$$+ \left| E_x\left[\frac{1}{2} \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_i \partial_j g(y_1, Y(s))\, \ell^a(s, y_1) \times \{(a_{ij}/a_{11})(X^a(s))\, ds + \alpha_{ij}(X^a(s))\, \ell_\mu^a(ds)\} \right] \right|$$

$$\leqq \sqrt{6}\, E_x[\ell^a(\phi_a(t), y_1)^2]^{1/2}$$

$$\times E_x\left[\left\{ \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, \tilde{\sigma}_{ij}(X^a(s))\, dB_j^a(s) \right\}^2 + \left\{ \sum_{i,j=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, \tau_{ij}(X^a(s))\, dM_j^a(s) \right\}^2 \right.$$

$$+ \left\{ \sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, (b_i/a_{11})(X^a(s))\, ds \right\}^2$$

$$+ \left\{ \sum_{i=2}^{d} \int_0^{\phi_a(t)} \partial_i g(y_1, Y(s))\, \beta_i(X^a(s))\, \ell_\mu^a(ds) \right\}^2$$

$$+\frac{1}{4}\left\{\sum_{i,j=2}^{d}\int_{0}^{\phi_a(t)}\partial_i\partial_j g(y_1, Y(s))(a_{ij}/a_{11})(X^a(s))\,ds\right\}^2$$

$$+\frac{1}{4}\left\{\sum_{i,j=2}^{d}\int_{0}^{\phi_a(t)}\partial_i\partial_j g(y_1, Y(s))\,\alpha_{ij}(X^a(s))\,\ell_\mu^a(ds)\right\}^2\Bigg]^{1/2}$$

$$+\sum_{i=2}^{d}E_x\left[\ell^a(\phi_a(t), y_1)\left\{\int_{0}^{\phi_a(t)}|\partial_i g(y_1, Y(s))|\,(|(b_i/a_{11})(x^a(s))|\,ds\right.\right.$$

$$\left.\left.+|\beta_i(X^a(s))|\,\ell_\mu^a(ds))\right\}\right]$$

$$+\frac{1}{2}\sum_{i,j=2}^{d}E_x\left[\ell^a(\phi_a(t), y_1)\left\{\int_{0}^{\phi_a(t)}|\partial_i\partial_j g(y_1, Y(s))|\right.\right.$$

$$\left.\left.\times(|(a_{ij}/a_{11})(X^a(s))|\,ds+|\alpha_{ij}(X^a(s))|\,\ell_\mu^a(ds))\right\}\right]$$

$$\leqq C_9\{E_x[\ell^a(\|a_{11}\|\,t, y_1)^2]^{1/2}E_x[t+t^2+\ell_\mu^a(\|a_{11}\|\,t)$$

$$+\ell_\mu^a(\|a_{11}\|\,t)^2]^{1/2}+E_x[\ell^a(\|a_{11}\|\,t, y_1)(t+\ell_\mu^a(\|a_{11}\|\,t)]\},$$

for some constant C_9 depending on g. Since $f\in C_0^\infty(R^d)$, there exist two functions $f_1(x_1)\in C_0(R^1)$ and $f_0(x_2,\ldots,x_d)\in C_0(R^{d-1})$ such that

$$|f(x)|\leqq f_1(x_1)\,f_0(x_2,\ldots,x_d).$$

Hence, by (3.5),

$$\lim_{t\to 0}\frac{1}{t}\left|\int f(x)\,\mathrm{II}\,dx\right|$$

$$\leqq\lim_{t\to 0}C_9\int\mu(dy_1)\int f_0(x_2,\ldots,x_d)\,dx_2\ldots dx_d$$

$$\times\left\{\frac{1}{t}\int f_1(x_1)\,E_x[\ell_\mu^a(\|a_{11}\|\,t, y_1)^2]\,dx_1\right\}^{1/2}$$

$$\times\left\{\frac{1}{t}\int f_1(x_1)\,E_x[t+t^2+\ell_\mu^a(\|a_{11}\|\,t)+\ell_\mu^a(\|a_{11}\|\,t)^2]\,dx_1\right\}^{1/2}$$

$$+\lim_{t\to 0}C_9\int\mu(dy_1)\int f_0(x_2,\ldots,x_d)\,dx_2\ldots dx_d$$

$$\times\left\{\frac{1}{t}\int f_1(x_1)\,E_x[\ell^a(\|a_{11}\|\,t, y_1)(t+\ell_\mu^a(\|a_{11}\|\,t))]\,dx_1\right\}=0.$$

Let $\phi_{a,1}(t)$ be the inverse function of the CAF $\psi_{a,1}(t)$ of $X_1^a(t)$ defined by

$$\psi_{a,1}(t)=\int_0^t(1/a_{11})(X_1^a(s), Y(0))\,ds.$$

Write the term I as

$$\mathrm{I}=\int f(x)\,E_x\left[\int_0^{\phi_{a,1}(t)}g(X_1^a(s), Y(0))\,\ell_\mu^a(ds)\right]dx+\int f(x)\,E_x\left[\int_{\phi_{a,1}(t)}^{\phi_a(t)}g(X_1^a(s), Y(0))\,\ell_\mu^a(ds)\right]dx$$

$$=\mathrm{III}+\mathrm{IV}.$$

The term III is written as

$$\mathrm{III}=\int dx_2\dots dx_d\int g(y_1,x_2,\dots,x_d)\,\mu(dy_1)\\ \times\int dx_1 f(x_1,\dots,x_d)\,E_{(x_1,\dots,x_d)}[\ell^a(\phi_{a,1}(t),y_1)].$$

For fixed $(x_2,\dots,x_d)$, since $\frac{1}{2}\ell^a(\phi_{a,1}(t),x_1)$ is the local time at x_1 of the 1-dimensional diffusion process $X_1^a(\phi_{a,1}(t))$ with speed measure $(1/a_{11})(x_1,\dots,x_d)\,dx_1$ (see [4; §5.4]), we have

$$\begin{aligned}\lim_{t\to0}\frac{1}{t}\,\mathrm{III}&=\lim_{t\to0}\frac{1}{t}\int dx_2\dots dx_d\int g(y_1,x_2,\dots,x_d)\,\mu(dy_1)\\&\quad\times\int f(x_1,\dots,x_d)\,a_{11}(x_1,\dots,x_d)\\&\quad\times E_{(y_1,x_2,\dots,x_d)}[\ell^a(\phi_{a,1}(t),x_1)]\,(1/a_{11})(x_1,\dots,x_d))\,dx_1\\&=\lim_{t\to\infty}\frac{1}{t}\int dx_2\dots dx_d\int g(y_1,x_2,\dots,x_d)\,\mu(dy_1)\\&\quad\times E_{(y_1,x_2,\dots,x_d)}\left[\int_0^t(f\cdot a_{11})(X_1^a(\phi_{a,1}(s)),x_2,\dots,x_d)\,ds\right]\\&=\int g(y_1,x_2,\dots,x_d)(f\cdot a_{11})(y_1,x_2,\dots,x_d)\,\mu(dy_1)\,dx_2\dots dx_d\\&=\int g(x)f(x)\,a_{11}(x)\,\eta(dx).\end{aligned}$$

Finally we shall show that $\lim_{t\to0}\frac{1}{t}\,\mathrm{IV}=0$. Obviously $\phi_a(t)\leqq\|a_{11}\|\,t$ and $\phi_{a,1}(t)\leqq\|a_{11}\|\,t$. Hence, for all $\varepsilon>0$,

$$\begin{aligned}|\mathrm{IV}|&\leqq\|g\|\int|f(x)|\,dx\int\mu(dy_1)\,E_x[|\ell^a(\phi_a(t),y_1)-\ell^a(\phi_{a,1}(t),y_1)|]\\&\leqq2\|g\|\int|f(x)|\,dx\int\mu(dy_1)\,E_x[\ell^a(\|a_{11}\|\,t,y_1);\\&\qquad\max_{s\leqq\|a_{11}\|t}|X^a(s)-X^a(0)|\geqq\varepsilon]\\&\quad+\|g\|\int|f(x)|\,dx\int\mu(dy_1)\,E_x[|\ell^a(\phi_a(t),y_1)-\ell^a(\phi_{a,1}(t),y_1)|;\\&\qquad\max_{s\leqq\|a_{11}\|t}|X^a(s)-X^a(0)|<\varepsilon]\\&=\mathrm{V}+\mathrm{VI}.\end{aligned}$$

Since

$$\begin{aligned}&E_x[\ell^a(\|a_{11}\|\,t,y_1);\ \max_{s\leqq\|a_{11}\|t}|X^a(s)-X^a(0)|\geqq\varepsilon]\\&\quad\leqq E_x[\ell^a(\|a_{11}\|\,t,y_1)^2]^{1/2}P_x[\max_{s\leqq\|a_{11}\|t}|X^a(s)-X^a(0)|\geqq\varepsilon]^{1/2},\end{aligned}$$

by (3.5) and (3.6) we have

$$\begin{aligned}\lim_{t\to0}\frac{1}{t}\,\mathrm{V}&=\lim_{t\to0}\frac{2}{t}\|g\|\int f_0(x_2,\dots,x_d)\,dx_2\dots dx_d\\&\quad\times\int\mu(dy_1)\{\int f_1(x_1)\,E_x[\ell^a(\|a_{11}\|\,t,y_1)^2]\,dx_1\}^{1/2}\\&\quad\times\{\int f_1(x_1)\,E_x[\max_{s\leqq\|a_{11}\|t}|x^a(s)-X^a(0)|\geqq\varepsilon]\,dx_1\}^{1/2}\\&\leqq C_{10}\lim_{t\to0}t^{-1/4}\exp\{-C_4\,\varepsilon^{4/3}(\|a_{11}\|\,t)^{-1/3}\}=0\end{aligned}$$

for all $\varepsilon>0$.

Set $D=\{\omega;\ |X^a(s)-X^a(0)|<\varepsilon$ for all $s\leqq \|a_{11}\|t\}$. Then, since $|a_{11}(x)-a_{11}(y)|\leqq C_{11}|x-y|$ for some constant C_{11},

$$\begin{aligned}|\phi_{a,1}(t)-a_{11}(X^a(0))\,t| &\leqq \int_0^t |a_{11}(X_1^a(\phi_{a,1}(s), Y(0))-a_{11}(X^a(0))|\,ds\\ &\leqq C_{11}\,t\{\max_{s\leqq \|a_{11}\|t}|X^a(s)-X^a(0)|\}\leqq C_{11}\,t\varepsilon\end{aligned}$$

on D. Hence, if $|y_1-X_1^a(0)|<\varepsilon$,

$$\begin{aligned}|\phi_{a,1}(t)-a_{11}(y_1, Y(0))\,t| &\leqq |\phi_{a,1}(t)-a_{11}(X^a(0))\,t|\\ &\quad +t\,|a_{11}(X^a(0))-a_{11}(y_1, Y(0))|\leqq 2C_{11}\,t\varepsilon\end{aligned}$$

on D. Since $\ell^a(\phi_a(t), y_1)=\ell^a(\phi_{a,1}(t), y_1)=0$ on D if $|y_1-X_1^a(0)|\geqq\varepsilon$, we have

$$\begin{aligned}\mathrm{VI}&\leqq 2\|g\|\,\|f_1\|\int f_0(x_2,\dots,x_d)\,dx_2\dots dx_d\int\mu(dy_1)\\ &\quad\times\int E_x[\ell^a(a_{11}(y_1,x_2,\dots,x_d)+2C_{11}\,t\varepsilon, y_1)\\ &\quad-\ell^a(a_{11}(y_1,x_2,\dots,x_d)-2C_{11}\,t\varepsilon, y_1)]\,dx_1\\ &=4C_{11}\,\mu(R^1)\,t\varepsilon\|g\|\,\|f_1\|\int f_0(x_2,\dots,x_d)\,dx_2\dots dx_d=C_{12}\,t\varepsilon,\end{aligned}$$

by (3.5). Hence $\frac{1}{t}\mathrm{VI}\leqq C_{12}\,\varepsilon$. Thus we have the result.

Let A and L be the differential operators defined by

$$Ag(x)=\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\,\partial_i\,\partial_j\,g(x)+\sum_{i=1}^{d} b_i(x)\,\partial_i\,g(x)$$

and

$$Lg(x)=\frac{1}{2}\sum_{i,j=2}^{d}\alpha_{ij}(x)\,\partial_i\,\partial_j\,g(x)+a_{11}(x)\sum_{i=2}^{d}\beta_i\,\partial_i\,g(x),$$

respectively. Then we have

Theorem 3.1. *For all $f, g\in C_0^\infty(R^d)$,*

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t}\int_{R^d} f(x)\,(I-P_t)\,g(x)\,dx\\ &\qquad=-\int_{R^d} f(x)\,Ag(x)\,dx-\int_{R^d} f(x)\,Lg(x)\,\eta(dx).\end{aligned}\tag{3.9}$$

Proof. By Ito's formula,

$$\begin{aligned}&g(X(t))-g(X(0))\\ &=\sum_{i=1}^{d}\int_0^t\partial_i\,g(X(s))\,dX_i(s)+\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\partial_i\,\partial_j\,g(X(s))\,d\langle X_i, X_j\rangle(s)\\ &=\sum_{i,j=1}^{d}\int_0^t\partial_i\,g(X(s))\,\sigma_{ij}(X(s))\,dB_j(s)+\int_0^t Ag(X(s))\,ds\\ &\quad+\sum_{i,j=2}^{d}\int_0^t\partial_i\,g(X(s))\,\tau_{ij}(X(s))\,dM_j(s)+\int_0^t Lg(X(s))/a_{11}(X(s))\,\ell_\mu(ds).\end{aligned}$$

Hence by Lemmy 3.2,

$$\begin{aligned}\lim_{t\to 0}\frac{1}{t}\int_{R^d} f(x)(I-P_t)\,g(x)\,dx &= -\lim_{t\to 0}\frac{1}{t}\int_{R^d} f(x)\,E_x\left[\int_0^t Ag(x(s))\,ds\right]dx \\ &\quad -\lim_{t\to 0}\frac{1}{t}\int_{R^d} f(x)\,E_x\left[\int_0^t Lg(X(s))/a_{11}(X(s))\,\ell_\mu(ds)\right]dx \\ &= -\int_{R^d} f(x)\,Ag(x)\,dx - \int_{R^d} f(x)\,Lg(x)\,\eta(dx).\end{aligned}$$

Corollary. *Let b_i and β_i be those defined in §2. Then the Dirichlet form on $L^2(dx)$ of the solution $X(t)$ of (2.1) is an extension of $(\mathscr{E}, C_0^\infty(R^d))$.*

§4. The Process X^ν

Let Λ and Γ be two Borel sets in R^1 such that $\Gamma=R^1-\Lambda$, $\mu(\Lambda)=0$ and $\int_\Gamma dx_1=0$. Since $\nu(dx)=dx+\eta(dx)$, we have $dx=I_\Lambda(x_1)\,\nu(dx)$ and $\eta(dx)=I_\Gamma(x_1)\,\nu(dx)$. Hence, for all $f, g\in C_0^\infty(R^d)$, the form $\mathscr{E}$ can be written as

$$\mathscr{E}(f,g)=-\tfrac{1}{2}\int_{R^d} f(x)\left\{I_\Lambda(x_1)\sum_{i,j=1}^{d}\partial_i(a_{ij}\partial_j g)(x) + I_\Gamma(x_1)\sum_{i,j=2}^{d}\partial_i(\alpha_{ij}\partial_j g)(x)\right\}\nu(dx). \tag{4.1}$$

If $f_n\in C_0^\infty(R^d)$ converges to 0 in $L^2(d\nu)$ then (4.1) implies that $\mathscr{E}(f_n, g)$ converges to 0 for all $g\in C_0^\infty(R^d)$. This implies that the symmetric form $(\mathscr{E}, C_0^\infty(R^d))$ is closable on $L^2(d\nu)$ (see [1; Problem 1.1.2]). Moreover, by the results of Fukushima [1; Theorems 2.1.1 and 2.1.2], its smallest closed extension $(\mathscr{E}^\nu, \mathscr{D}(\mathscr{E}^\nu))$ is a regular Dirichlet form on $L^2(d\nu)$ with local property. By another result of Fukushima [1; Chap. 6], there exists a $d\nu$-symmetric diffusion process $X^\nu(t)$ on a probability space (Ω, P_x) associated with $(\mathscr{E}^\nu, \mathscr{D}(\mathscr{E}^\nu))$. As noted in §2, the increasing process $\phi_\Lambda(t)$ defined by (2.2) is the CAF of X^ν associated with the smooth measure dx. Set

$$\phi_\Gamma(t)=\int_0^t I_\Gamma(X_1^\nu(s))\,ds. \tag{4.2}$$

Then ϕ_Γ is the CAF associated with $d\eta$. In the rest of this section, we shall show that the increasing process $\phi_\Lambda(t)$ is strictly increasing.

For the purpose we shall use the stochastic calculus related to X. Necessary facts are presented in [1; §5.4]. By noting (4.1), for $f, g\in C_0^\infty(R^d)$, we have

$$\mathscr{E}(f,g)+p(f,g)_\nu=\int f(x)\,(p-\mathrm{g}^\nu)\,g(x)\,\nu(dx), \tag{4.3}$$

where

$$\mathfrak{g}^{\nu} g(x)=\frac{1}{2} \sum_{i,j=1}^{d} I_{A}(x_1)\, \partial_i(a_{ij}\partial_j g)(x) + \frac{1}{2} \sum_{i,j=2}^{d} I_{\Gamma}(x_1)\, \partial_i(\alpha_{ij}\partial_j g)(x).$$

Denote by V^p the resolvent of X^{ν}. Then (4.3) implies $g=V^p(p-\mathfrak{g}^{\nu})g$ q.e. Hence, by [1; Theorem 5.2.2], the process $M^{\nu,[g]}(t)$ defined by

$$M^{\nu,[g]}(t)=g(X^{\nu}(t))-g(X^{\nu}(0))-\int_0^t \mathfrak{g}^{\nu} g(X^{\nu}(s))\,ds \tag{4.4}$$

is a martingale CAF such that

$$\langle M^{\nu,[g]}\rangle(t)=\int_0^t \sum_{i,j=1}^{d} a_{ij}\partial_i g\partial_j g(X^{\nu}(s))\,\phi_A(ds) + \int_0^t \sum_{i,j=2}^{d} \alpha_{ij}\partial_i g\partial_j g(X^{\nu}(s))\,\phi_{\Gamma}(ds).$$

In particular, by taking $g\in C_0^{\infty}(R^d)$ such that $g(x)=x_i$ locally, there exists a system $\{M_i^{\nu}(t);\ i=1,\dots,d\}$ of local martingale CAFs satisfying

$$X_i^{\nu}(t)-X_i^{\nu}(0)=M_i^{\nu}(t)+\int_0^t b_i(X^{\nu}(s))\,\phi_A(ds) + \int_0^t \tilde{\beta}_i(X^{\nu}(s))\,\phi_{\Gamma}(ds) \tag{4.5}$$

and

$$\langle M_i^{\nu}, M_j^{\nu}\rangle(t)=\tfrac{1}{2}\{\langle M_i^{\nu}+M_j^{\nu}\rangle(t)-\langle M_i^{\nu}\rangle(t)-\langle M_j^{\nu}\rangle(t)\} = \int_0^t a_{ij}(X^{\nu}(s))\,\phi_A(ds)+\int_0^t \alpha_{ij}(X^{\nu}(s))\,\phi_{\Gamma}(ds), \tag{4.6}$$

where $b_i(x)=\frac{1}{2}\sum_{j=1}^{d}\partial_j a_{ji}(x)$, $\tilde{\beta}_i(x)=\frac{1}{2}\sum_{j=2}^{d}\partial_j \alpha_{ji}(x)$ and $\alpha_{1j}=\alpha_{i1}=\tilde{\beta}_1=0$.

We shall next apply the above discussion to a function which is not smooth. Set $m(dx_1)=dx_1+\mu(dx_1)$ and let $k(x_1)$ be a bounded continuous function on R^1 satisfying the following conditions (a), (b) and (c); (a) $k(x_1)\in L^2(dm)$, (b) $k(x_1)$ is absolutely continuous and $\frac{d^+k}{dx_1}\in L^2(dx_1)$, (c) $d\left(\frac{d^+k}{dx_1}\right)$ is absolutely continuous relative to dm and there exists a version of $\frac{d}{dm}\frac{d+k}{dx_1}$ which belongs to $L^2(dm)\cap C_b(R^1)$.

Let $k_0(x_2,\dots,x_d)$ be a $C_0^{\infty}(R^{d-1})$ function and set $h(x)=k(x_1)\,k_0(x_2,\dots,x_d)$. Denote by ρ_n the 1-dimensional mollifier supported by $\left\{x_1;\ |x_1|\le\frac{1}{n}\right\}$. Set $k_n(x_1)=\rho_n * k(x_1)$ and $h_n(x)=k_n(x_1)\,k_0(x_2,\dots,x_d)$ $(n=1,\dots,d)$.

Lemma 4.1. *The function h belongs to $\mathcal{D}(\mathcal{E}^{\nu})$.*

Proof. Obviously h_n is approximated by $C_0^\infty(R^d)$ functions in the norm $\mathscr{E}^\nu(\cdot,\cdot)+(\cdot,\cdot)_\nu$. Hence it is enough to remark that $\mathscr{E}^\nu(h_n-h, h_n-h)+(h_n-h, h_n-h)_\nu$ tends to 0 as n tends to ∞.

Lemma 4.2. *If we understand* $\partial_1 h(x)$ *as* $\dfrac{d^+k}{dx_1}(x_1)k_0(x_2,\dots,x_d)$ *then, for all* $f\in C_0^\infty(R^d)$,

$$\mathscr{E}^\nu(f,h)=\frac{1}{2}\sum_{i,j=1}^{d}\int_{R^d}a_{ij}(x)\,\partial_i f(x)\,\partial_j h(x)\,dx+\frac{1}{2}\sum_{i,j=2}^{d}\int_{R^d}\alpha_{ij}(x)\,\partial_i f(x)\,\partial_j h(x)\,\eta(dx). \quad (4.7)$$

Proof. It is easy to see that $\mathscr{E}^\nu(f,h_n)=\mathscr{E}(f,h_n)$. In this equality, since $\partial_i h_n$ tends to $\partial_i h$ in $L^2(d\nu)$ as n tends to infinity, the lemma follows.

For any compact set K of R^d, choose $k_0\in C_0^\infty(R^{d-1})$ so that $k_0(x_2,\dots,x_d)=1$ for every $x=(x_1,\dots,x_d)\in K$. Since $h(x)=k(x_1)k_0(x_2,\dots,x_d)=k(x_1)$ for $x\in K$ we can see that $k(x_1)$ belongs to $\mathscr{D}_{\text{loc}}(\mathscr{E}^\nu)$, that is, k equals locally to a function which belongs to $\mathscr{D}(\mathscr{E}^\nu)$. Thus we can decompose $k(X_1^\nu(t))-k(X_1^\nu(0))$ as

$$k(X_1^\nu(t))-k(X_1^\nu(0))=M^{\nu,[k]}(t)+N^{\nu,[k]}(t), \quad (4.8)$$

where $M^{\nu,[k]}(t)$ is a CAF which equals locally to a martingale additive functional and $N^{\nu,[k]}(t)$ is a CAF which is locally of zero energy ([1; (5.4.41)]). As in (4.5), we have an explicit representation of $N^{\nu,[k]}(t)$.

Lemma 4.3. $N^{\nu,[k]}(t)$ *is given by*

$$\begin{aligned}N^{\nu,[k]}(t)=&\int_0^t a_{11}(X^\nu(s))\frac{d}{2dm}\frac{d^+k}{dx_1}(X_1^\nu(s))\,ds\\&+\int_0^t b_1(X^\nu(s))\frac{d^+k}{dx_1}(X_1^\nu(s))\,\phi_A(ds).\end{aligned} \quad (4.9)$$

Proof. Let $p>0$. By (4.7) we have

$$\begin{aligned}&\mathscr{E}^\nu(f,h)+p(f,h)_\nu\\&=\tfrac{1}{2}\int_{R^d}a_{11}(x)\,\partial_1 f(x)\,\partial_1 h(x)\,dx\\&\quad-\tfrac{1}{2}\sum_{i\text{ or }j\neq 1}\int_{R^d}f(x)\,\partial_i(a_{ij}\partial_j h)(x)\,dx\\&\quad-\tfrac{1}{2}\sum_{i,j=2}^{d}\int_{R^d}f(x)\,\partial_i(\alpha_{ij}\partial_j h)(x)\,\eta(dx)+p\int_{R^d}f(x)h(x)\,\nu(dx)\\&=-\int_{R^d}f(x)\,a_{11}(x)\frac{d}{2dm}\frac{d^+k}{dx_1}(x_1)\,k_0(x_2,\dots,x_d)\,\nu(dx)\\&\quad-\tfrac{1}{2}\int_{R^d}f(x)\,\partial_1 a_{11}(x)\,\partial_1 h(x)\,dx\\&\quad-\tfrac{1}{2}\sum_{i\text{ or }j\neq 1}\int_{R^d}f(x)\{a_{ij}(x)\,\partial_i\partial_j h(x)+\partial_i a_{ij}(x)\,\partial_j h(x)\}\,dx\\&\quad-\tfrac{1}{2}\sum_{i,j=2}^{d}\int_{R^d}f(x)\{\alpha_{ij}(x)\,\partial_i\partial_j h(x)+\partial_i\alpha_{ij}(x)\,\partial_j h(x)\}\,\eta(dx)\\&\quad+p\int_{R^d}f(x)h(x)\,\nu(dx)\\&=\int_{R^d}f(x)(p-\mathrm{g}^\nu)h(x)\,\nu(dx),\end{aligned}$$

where

$$\mathfrak{g}^{\nu} h(x)=a_{11}(x)\frac{d}{2dm}\frac{d^{+}k}{dx_1}(x_1)\,k_0(x_2,\dots,x_d)$$
$$+\tfrac{1}{2}I_{\Lambda}(x_1)\sum_{i\text{ or }j\neq 1} a_{ij}(x)\,\partial_i\,\partial_j\,h(x)$$
$$+\tfrac{1}{2}I_{\Gamma}(x_1)\sum_{i,j=2}^{d}\alpha_{ij}(x)\,\partial_i\,\partial_j\,h(x)+I_{\Lambda}(x_1)\sum_{i=1}^{d}b_i(x)\,\partial_i\,h(x)$$
$$+I_{\Gamma}(x_1)\sum_{i=2}^{d}\tilde{\beta}_i(x)\,\partial_i\,h(x).$$

This implies that $h(x)=V^{p}(p-\mathfrak{g}^{\nu})\,h(x)$ q.e. and hence $N^{\nu,[h]}(t)=\int_0^t \mathfrak{g}^{\nu} h(X^{\nu}(s))\,ds$, where $N^{\nu,[h]}(t)$ is a CAF of zero energy appearing in the decomposition of $h(X^{\nu}(t))-h(X^{\nu}(0))$. Setting $k_0=1$ we have the result.

As for the martingale part $M^{\nu,[k]}(t)$, by (4.7), it satisfies

$$\langle M^{\nu,[k]}\rangle(t)=\int_0^t a_{11}(X^{\nu}(s))\left\{\frac{d^{+}k}{dx_1}(X_1^{\nu}(s))\right\}^2 \phi_{\Lambda}(ds). \tag{4.10}$$

Lemma 4.4. *The local martingale* CAFs $M^{\nu,[k]}(t)$ and $M_1^{\nu}(t)$ *are related by*

$$M^{\nu,[k]}(t)=\int_0^t \frac{d^{+}k}{dx_1}(X_1^{\nu}(s))\,dM_1^{\nu}(s). \tag{4.11}$$

Proof. If $k(x_1)$ belongs to $C^1(R^1)$, this result is contained in the result of Fukushima [1; Theorem 5.4.3]. In the present case, although k does not belong to $C^1(R^1)$, this can be proved similarly. In fact, it is enough to show that, for any compact set K, if f, g and v are $C_0^{\infty}(R^d)$ functions supported by K and, for all $i(i=1,\dots,d)$, u_i is a $C_0^{\infty}(R^d)$ function which is equal to the coordinate function x_i on K then

$$\int f(x)\,g(x)\,\mu_{\langle h,v\rangle}(dx)=\sum_{i=1}^{d}\int f(x)\,g(x)\,\partial_i\,h(x)\,\mu_{\langle u_i,v\rangle}(dx),$$

where $\mu_{\langle h,v\rangle}$ and $\mu_{\langle u_i,v\rangle}$ are the signed measures associated with $\langle M^{\nu,[h]}, M^{\nu,[v]}\rangle$ and $\langle M^{\nu,[u_i]}, M^{\nu,[v]}\rangle$, respectively. This equality follows easily from

$$\mu_{\langle h,v\rangle}(dx)=\sum_{i,j=1}^{d} a_{ij}(x)\,\partial_i\,h(x)\,\partial_j\,v(x)\,dx$$
$$+\sum_{i,j=2}^{d}\alpha_{ij}(x)\,\partial_i\,h(x)\,\partial_j\,v(x)\,\eta(dx)$$

and

$$\mu_{\langle u_i,v\rangle}(dx)=\sum_{j=1}^{d} a_{ij}(x)\,\partial_j\,v(x)\,dx+\sum_{j=2}^{d}\alpha_{ij}\,\partial_j\,v(x)\,\eta(dx)$$

on K.

Let $\phi_a^\nu(t)$ be the CAF of $X^\nu(t)$ defined by

$$\phi_a^\nu(t)=\int_0^t a_{11}(X^\nu(s))\,ds \tag{4.12}$$

and let $X^a(t)$, $M_i^a(t)$, $M^{a,[k]}(t)$ and $N^{a,[k]}(t)$ be the time changed processes of $X^\nu(t)$, $M_i^\nu(t)$, $M^{\nu,[k]}(t)$ and $N^{\nu,[k]}(t)$ by $\phi_a^\nu(t)$, that is, $X^a(t)=X^\nu((\phi_a^\nu)^{-1}(t))$, etc. Then, by (4.6) and (4.11), $M_1^a(t)$ and $M^{a,[k]}(t)$ are martingale CAFs on (Ω, P_x) such that

$$\langle M_1^a\rangle(t)=\int_0^t a_{11}(X^a(s))\,\phi_A^a(ds) \tag{4.13}$$

and

$$M^{a,[k]}(t)=\int_0^t \frac{d^+k}{dx_1}(X_1^a(s))\,dM_1^a(s). \tag{4.14}$$

Also, by (4.9), we have

$$\begin{aligned} N^{a,[k]}(t)=&\int_0^t \frac{d}{2dm}\frac{d^+k}{dx_1}(X_1^a(s))\,ds \\ &+\int_0^t b_1(X^a(s))\frac{d^+k}{dx_1}(X_1^a(s))\,\phi_A^a(ds), \end{aligned} \tag{4.15}$$

where $\phi_A^a(t)=\phi_A((\phi_a^\nu)^{-1}(t))$.

Set $\tilde{b}_1(x)=b_1(x)/a_{11}(x)$. Let P_x^a be the measure defined by

$$P_x^a(D)=E_x\left[\exp\left\{-\int_0^t \tilde{b}_1(X^a(s))\,dM_1^a(s)-\tfrac{1}{2}\int_0^t \tilde{b}_1(X^a(s))^2\,d\langle M_1^a\rangle(s)\right\};D\right]$$

for $D\in\sigma(X^a(s);\ s\leq t)$. Then, by Girsanov's theorem, for any local martingale CAF $M(t)$ on (Ω, P_x), the stochastic process $M(t)+\int_0^t \tilde{b}_1(X^a(s))\,d\langle M, M_1^a\rangle(s)$ is a local martingale CAF on (Ω, P_x^a).

Lemma 4.5. *The process*

$$k(X_1^a(t))-k(X_1^a(0))-\int_0^t \frac{d}{2dm}\frac{d^+k}{dx_1}(X_1^a(s))\,ds \tag{4.16}$$

is a martingale CAF *on* (Ω, P_x^a).

Proof. Applying the above remark to $M(t)=M^{a,[k]}(t)$ and using (4.13) and (4.14) we can show that

$$\begin{aligned} &M^{a,[k]}(t)+\int_0^t \tilde{b}_1(X^a(s))\,d\langle M^{a,[k]}, M_1^a\rangle(s) \\ &=M^{a,[k]}(t)+\int_0^t b_1(X^a(s))\frac{d^+k}{dx_1}(X_1^a(s))\,\phi_A^a(ds) \end{aligned}$$

is a martingale CAF on (Ω, P_x^a). Hence, by (4.8) and (4.15), the result follows.

Lemma 4.6. *Denote by $\{W^p; p>0\}$ and $\{V_a^p; p>0\}$ the resolvents of 1-dimensional diffusion process with speed measure $2dm$ and $X^a(t)$, respectively. Then there exists a properly exceptional set N of $X^\nu(t)$ such that $V_a^p(x, D\times R^{d-1}) = W^p(x_1, D)$ for all Borel set $D\subset R^1$ and $x\notin N$, where x_1 is the first coordinate of x.*

Proof. Since $C_0(R^1)$ is separable, it is enough to show that, for all $f_1\in C_0(R^1)$, there exists a properly exceptional set N such that $V_a^p f_1(x)=W^p f_1(x_1)$ for $x\notin N$, where $V_a^p f_1(x)=V_a^p(f_1\times I_{R^{d-1}})(x)$. Set $k(x_1)=W^p f_1(x_1)$. Then it satisfies the conditions (*a*), (*b*) and (*c*) preceding Lemma 4.1. Hence, by (4.16), we can show that there exists a properly exceptional set N such that

$$E_x^a[W^p f_1(X_1^a(t))]-W^p f_1(x_1)=\int_0^t E_x^a\left[\frac{d}{2dm}\frac{d^+}{dx_1}W^p f_1(X_1^a(s))\right]ds$$

for $x\notin N$. Multiplying e^{-pt} and integrating by t we have

$$V_a^p\left(p-\frac{d}{2dm}\frac{d^+}{dx_1}\right)W^p f_1(x)=W^p f_1(x_1).$$

Since $\left(p-\frac{d}{2dm}\frac{d^+}{dx_1}\right)W^p f_1=f_1$, the result follows.

Theorem 4.1. *For all $x_1\in R^1$ there exists at least one point $y\notin N$ such that $y_1=x_1$. Moreover, for $x\notin N$, the distribution of $X_1^a(t)$ under P_x^a is independent of $(x_2,\dots,x_d)$. If we denote by $P_{x_1}^a$ instead of P_x^a for $x\notin N$ whenever we consider $X_1^a(t)$, then $(X_1^a(t), P_{x_1}^a)$ is a 1-dimensional diffussion process with speed measure $2dm$.*

Proof. Let z be an arbitrary point of R^d-N and let (c_1, c_2) be an arbitrary open interval of R^1. Let f_1 be a non-negative function supported by $[c_1, c_2]$ such that $\langle m, f_1\rangle>0$. Then $W^p f_1(z_1)>0$. Hence, by Lemma 4.6, $V_a^p f_1(z)>0$. This implies that the process $X^a(t)$ started from z hits the set $\{y\in R^d;\ c_1\leqq y_1\leqq c_2\}$ and hence it hits the hyperplane $\{y\in R^d;\ y_1=x_1\}$ if $z_1\leqq x_1\leqq c_1$ or $c_2\leqq x_1\leqq z_1$. Therefore the hyperplane is not contained in N, that is, there exists at least one point $y\notin N$ such that $y_1=x_1$. Since (c_1, c_2) is arbitrary, the first part of the theorem holds. The other parts are obvious by Lemma 4.6.

By the theorem, the support of the CAF $\phi_A^a(t)$ coincides with R^d-N, that is, $\inf\{t; \phi_A^a(t)>0\}=0$ a.s. P_x^a for all $x\in R^d-N$. Turning back to the process $X^\nu(t)$, we have the following theorem.

Theorem 4.2. *The support of the* CAF *$\phi_A(t)$ defined by* (2.2) *coincides with R^d-N for a suitable properly exceptional set N.*

§5. Proof of Theorem 2

As was proved in Theorem 4.2, the CAF $\phi_A(t)$ is strictly increasing a.s. P_x for $x\in R^d-N$. Hence the time changed process $X^0(t)=X^\nu(\phi_A^{-1}(t))$ is a diffusion process on the probability space $(\Omega, P_x;\ x\in R^d-N)$. To prove that $X^0(t)$ is the dx-symmetric diffusion process associated with $(\mathscr{E}^0, \mathscr{D}(\mathscr{E}^0))$, it is enough to prove the following result: If $\phi(t)$ is a CAF associated with a measure $\xi(dx) = a(x)\,\nu(dx)$ for some bounded measurable function $a(x)$ and if $P_x[\phi(t)>0$ for all $t>0]=1$ q.e., then the time changed process $X^{\xi}(t)=X^{\nu}(\phi^{-1}(t))$ is a $d\xi$-symmetric diffusion process such that the Dirichlet form of X^{ξ} on $L^2(d\xi)$ is the smallest closed extension of $(\mathscr{E}, C_0^{\infty}(R^d))$ on $L^2(d\xi)$. Except for Theorem 5.1, we shall suppose that ξ is a general Radon measure associated with a strictly increasing CAF $\phi(t)$.

Some parts of the following results will follow from the results of Silverstein [9; I.8], but we shall present it since, in our case, it follows from elementary calculations.

Denote by $V_{t\phi}^{pq}$ and $V_{\phi t}^{qp}$ the kernels defined by

$$V_{t\phi}^{pq} f(x) = E_x\left[\int_0^{\infty} \exp(-p\,t - q\,\phi(t))\,f(X^{\nu}(t))\,d\phi(t)\right],$$

$$V_{\phi t}^{qp} f(x) = E_x\left[\int_0^{\infty} \exp(-p\,t - q\,\phi(t))\,f(X^{\nu}(t))\,dt\right].$$

If $V_{t\phi}^{r0}|f|$ is bounded for some $r\geqq 0$ and a bounded measurable function f, then

$$V_{t\phi}^{pq} f - V_{t\phi}^{rs} f + (p-r)\,V_{\phi t}^{qp}\,V_{t\phi}^{rs} f + (q-s)\,V_{t\phi}^{pq}\,V_{t\phi}^{rs} f = 0 \tag{5.1}$$

for all $p, q, r, s\geqq 0$ such that $p+q>0$ and $r+s>0$ (see [5, 6]). Similarly, if $V_{\phi t}^{s0}|f|$ is bounded for some $s\geqq 0$ and a bounded measurable function f, then

$$V_{\phi t}^{qp} f - V_{\phi t}^{sr} f + (p-r)\,V_{\phi t}^{qp}\,V_{\phi t}^{sr} f + (q-s)\,V_{t\phi}^{pq}\,V_{\phi t}^{sr} f = 0 \tag{5.2}$$

for all $p, q, r, s\geqq 0$ such that $p+q>0$ and $r+s>0$. Note that (5.1) [resp. (5.2)] holds for all bounded measurable function f if $q, s>0$ [resp. $p, r>0$]. If $p>0$ and $f>0$, then by (5.2),

$$p\,V_{t\phi}^{p0}\,V_{\phi t}^{pp} f = V^p f - V_{\phi t}^{pp} f \leqq V^p f \leqq \|f\|/p.$$

Moreover, since $V_{\phi t}^{pp} f$ is finely continuous, the set $F_n^{(1)} = \left\{x;\, V_{\phi t}^{pp} f(x) \geqq \frac{1}{n}\right\}$ is a finely closed set satisfying $F_n^{(1)} \to R^d - N$ and $V_{t\phi}^{p0}(x, F_n^{(1)}) \leqq n\|f\|/p^2$ for $x\in R^d - N$. By a similar argument, there exists a sequence $\{F_n^{(2)}\}_{n\geqq 1}$ of finely closed sets such that $F_n^{(2)} \to R^d - N$ and $V_{\phi t}^{p0}(x, F_n^{(2)})$ is bounded on $R^d - N$. Furthermore, by [1; Theorem 3.2.3], there exists a sequence $\{F_n^{(3)}\}$ of closed sets such that the measure $\xi(dx\cap F_n^{(3)})$ is a measure with finite energy integral. Set $F_n = F_n^{(1)}\cap F_n^{(2)}\cap F_n^{(3)}\cap(R^d - N)$. Then $\{F_n\}_{n\geqq 1}$ is an increasing sequence of finely closed subsets of $R^d - N$ satisfying $F_n \to R^d - N$, $V_{t\phi}^{p0}(x, F_n)$ and $V_{\phi t}^{p0}(x, F_n)$ are bounded on $R^d - N$ for all $p>0$ and $\xi_n = I_{F_n}\xi$ is a measure with finite energy integral. Set $\phi_n(t) = \int_0^t I_{F_n}(X^{\nu}(s))\,\phi(ds)$. Then it is the CAF associated with the measure ξ_n. Hence, by [1; Lemma 5.1.4(ii)], $(f, V_{tn}^{p0} g)_{\nu} = (V^p f, g)_{\xi_n}$, where

$$V_{tn}^{pq} f(x) = E_x\left[\int_0^{\infty} \exp(-p\,t - q\,\phi_n(t))\,f(X^{\nu}(t))\,d\phi_n(t)\right].$$

Letting $n \to \infty$ we have

$$(f, V_{t\phi}^{p0} g)_\nu = (V^p f, g)_\xi \tag{5.3}$$

for all $p > 0$ and bounded non-negative measurable functions f and g. Denote by $U^p(f\xi_n)$ the potential of the measure $f\xi_n$, that is, $U^p(f\xi_n) \in \mathscr{D}(\mathscr{E}^\nu)$ such that

$$\mathscr{E}_p^\nu(U^p(f\xi_n), g) = \int g(x) f(x)\, \xi_n(dx)$$

for all $g \in C_0^\infty(R^d)$, where $\mathscr{E}_p^\nu(\cdot\,,\cdot) = \mathscr{E}^\nu(\cdot\,,\cdot) + p(\cdot\,,\cdot)_\nu$.

Since $V_{tn}^{p0} f$ is a quasi continuous modification of $U^p(f\xi_n)$ (see [1; Lemma 5.1.3]), for all $f, g \in C_0^\infty(R^d)$

$$(f, V_{tn}^{p0} g)_{\xi_n} = \mathscr{E}_p^\nu(V_{tn}^{p0} f, V_{tn}^{p0} g) = (V_{tn}^{p0} f, g)_{\xi_n},$$

by [1; Theorem 3.2.2]. Letting $n \to \infty$, we have

$$(f, V_{t\phi}^{p0} g)_\xi = (V_{t\phi}^{p0} f, g)_\xi \tag{5.4}$$

for all $p > 0$ and non-negative measurable functions f and g.

Lemma 5.1. *For all $p, q > 0$ and non-negative measurable functions f and g,*

$$(f, V_{t\phi}^{pq} g)_\xi = (V_{t\phi}^{pq} f, g)_\xi. \tag{5.5}$$

Proof. Similarly to the above discussion, without loss of generality, we can assume that $V_{t\phi}^{p0} 1$ and $V_{\phi t}^{q0} 1$ are bounded. By (5.1),

$$V_{t\phi}^{p0} g = (I + q V_{t\phi}^{p0}) V_{t\phi}^{pq} g.$$

Hence, for $p, q > 0$ such that $q \| V_{t\phi}^{p0} 1 \| < 1$,

$$V_{t\phi}^{pq} g = \sum_{n=0}^{\infty} (-q V_{t\phi}^{p0})^n V_{t\phi}^{p0} g. \tag{5.6}$$

Thus (5.5) follows from (5.4) in this case. For fixed $p > 0$, since $\{V_{t\phi}^{pq}; q > 0\}$ satisfies the resolvent equation, (5.5) holds for all $p, q > 0$.

Denote by $\{V_\phi^p; p > 0\}$ the resolvent of $X^\xi(t)$. Then $V_\phi^p = V_{t\phi}^{0p}$. Hence we have the following

Corollary. *$X^\xi(t)$ is a $d\xi$-symmetric diffusion process.*

Lemma 5.2. *Let p, q, f and g be as in Lemma 5.1. Then*

$$(f, V_{t\phi}^{pq} g)_\nu = (V_{\phi t}^{qp} f, g)_\xi. \tag{5.7}$$

Proof. As in the proof of Lemma 5.1, we shall suppose that $V_{t\phi}^{p0} 1$ and $V_{\phi t}^{q0} 1$ are bounded. By (5.3), (5.4) and (5.6)

$$(f, V_{t\phi}^{pq} g)_\nu = \sum_{n=0}^{\infty} (f, (-q V_{t\phi}^{p0})^n V_{t\phi}^{p0} g)_\nu = \sum_{n=0}^{\infty} ((-q V_{t\phi}^{p0})^n V^p f, g)_\xi$$

for $p, q > 0$ such that $q \| V_{t\phi}^{p0} 1 \| < 1$. On the other hand, since $V^p f = (I + q V_{t\phi}^{p0}) V_{\phi t}^{qp} f$ by (5.1), we have

$$V_{\phi t}^{qp} f = \sum_{n=0}^{\infty} (-q V_{t\phi}^{p0})^n V^p f$$

for such $p, q>0$. Hence (5.7) holds in this case. If (5.7) holds for some p, $q>0$, then by Lemma 5.1,

$$V_{t\phi}^{pq+r} g = \sum_{n=0}^{\infty} (-r V_{t\phi}^{pq})^n V_{t\phi}^{pq} g \quad \text{and} \quad V_{\phi t}^{q+rp} f = \sum_{n=0}^{\infty} (-r V_{t\phi}^{pq})^n V_{\phi t}^{pq} f$$

for $r<1/\| V_{t\phi}^{p0} 1 \| \leqq 1/\| V_{t\phi}^{pq} 1 \|$. Hence (5.7) holds for $q+r$ instead of q. Repeating this argument, we have the result.

Let $(\mathscr{E}^{\xi}, \mathscr{D}(\mathscr{E}^{\xi}))$ be the Dirichlet form on $L^2(d\xi)$ associated with the diffusion process $X^{\xi}(t)$. Then we have the following lemma (cf. [9; Lemma 8.4]).

Lemma 5.3. *For all p, $q>0$, $V_{t\phi}^{pq}(C_0(R^d))$ and $V_{\phi t}^{qp}(C_0(R^d))$ are contained in $\mathscr{D}(\mathscr{E}^{\nu}) \cap \mathscr{D}(\mathscr{E}^{\xi})$ and*

$$\mathscr{E}^{\nu}(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g) = \mathscr{E}^{\xi}(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g), \tag{5.8}$$

$$\mathscr{E}^{\nu}(V_{\phi t}^{qp} f, V_{\phi t}^{qp} g) = \mathscr{E}^{\xi}(V_{\phi t}^{qp} f, V_{\phi t}^{qp} g) \tag{5.9}$$

for all f, $g \in C_0(R^d)$.

Proof. If $f \in C_0(R^d)$ then $V_{t\phi}^{pq}|f| \leqq V_{\phi}^{q}|f|$ implies $V_{t\phi}^{pq} f \in L^2(d\xi)$. Also since

$$(V_{t\phi}^{pq}|f|)^2 \leqq (\|f\|/q)\, V_{t\phi}^{pq}|f|,$$

(5.7) implies $V_{t\phi}^{pq} f \in L^2(d\nu)$. Similarly $V_{\phi t}^{qp} f \in L^2(d\nu) \cap L^2(d\xi)$. Hence, for the proof of (5.8), it is enough to show that

$$\lim_{r\to\infty} (V_{t\phi}^{pq} f, r(I - r V^r) V_{t\phi}^{pq} g)_{\nu} = \lim_{r\to\infty} (V_{t\phi}^{pq} f, r(I - r V_{\phi}^r) V_{t\phi}^{pq} g)_{\xi}.$$

By (5.1),

$$(I - r V^r) V_{t\phi}^{pq} g = V_{t\phi}^{r0} g - p V^r V_{t\phi}^{pq} g - q V_{t\phi}^{r0} V_{t\phi}^{pq} g.$$

Therefore, by noting (5.7) we have

$$\begin{aligned}
&\lim_{r\to\infty} (V_{t\phi}^{pq} f, r(I - r V^r) V_{t\phi}^{pq} g)_{\nu} \\
&\quad = \lim_{r\to\infty} (V_{t\phi}^{pq} f, r V_{t\phi}^{r0} g - r p V^r V_{t\phi}^{pq} g - r q V_{t\phi}^{r0} V_{t\phi}^{pq} g)_{\nu} \\
&\quad = \lim_{r\to\infty} \{(r V^r V_{t\phi}^{pq} f, g)_{\xi} - p(V_{t\phi}^{pq} f, r V^r V_{t\phi}^{pq} g)_{\nu} - q(r V^r V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\xi}\} \\
&\quad = (V_{t\phi}^{pq} f, g)_{\xi} - p(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\nu} - q(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\xi},
\end{aligned}$$

where, in the last equality, we used the fine continuity of $V_{t\phi}^{pq} f$ and $V_{t\phi}^{pq} g$. Similarly, by (5.2) and (5.7),

$$\begin{aligned}
&\lim_{r\to\infty} (V_{t\phi}^{pq} f, r(I - r V_{\phi}^r) V_{t\phi}^{pq} g)_{\xi} \\
&\quad = \lim_{r\to\infty} (V_{t\phi}^{pq} f, r V_{\phi}^r g - p r V_{\phi t}^{r0} V_{t\phi}^{pq} g - q r V_{\phi}^r V_{t\phi}^{pq} g)_{\xi} \\
&\quad = \lim_{r\to\infty} \{(r V_{\phi}^r V_{t\phi}^{pq} f, g)_{\xi} - p(r V_{\phi}^r V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\nu} - q(r V_{\phi}^r V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\xi}\} \\
&\quad = (V_{t\phi}^{pq} f, g)_{\xi} - p(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\nu} - q(V_{t\phi}^{pq} f, V_{t\phi}^{pq} g)_{\xi},
\end{aligned}$$

since the support of ϕ coincides with $R^d - N$. Thus (5.8) has been proved. The proof of (5.9) is similar, in fact it becomes

$$\begin{aligned}\mathscr{E}^{\nu}(V^{qp}_{\phi t} f, V^{qp}_{\phi t} g) &= \mathscr{E}^{\xi}(V^{qp}_{\phi t} f, V^{qp}_{\phi t} g)\\ &=(V^{qp}_{\phi t} f, g)_{\nu} - q(V^{qp}_{\phi t} f, V^{qp}_{\phi t} g)_{\xi} - p(V^{qp}_{\phi t} f, V^{qp}_{\phi t} g)_{\nu}.\end{aligned}$$

Lemma 5.4. *For all $q>0$ and $f \in C_0(R^d)$,*

$$\lim_{p\to 0} \mathscr{E}^{\xi}_1(V^{pq}_{t\phi} f - V^q_{\phi} f, V^{pq}_{t\phi} f - V^q_{\phi} f) = 0, \tag{5.10}$$

where $\mathscr{E}^{\xi}_1(\cdot,\cdot) = \mathscr{E}^{\xi}(\cdot,\cdot) + (\cdot,\cdot)_{\xi}$.

Proof. Obviously $\lim_{p\to 0}(V^{pq}_{t\phi} f - V^q_{\phi} f, V^{pq}_{t\phi} f - V^q_{\phi} f)_{\xi} = 0$. By (5.1) and (5.7),

$$\begin{aligned}&\mathscr{E}^{\xi}(V^{pq}_{t\phi} f - V^q_{\phi} f, V^{pq}_{t\phi} f - V^q_{\phi} f)\\ &= \lim_{r\to\infty}(V^{pq}_{t\phi} f - V^q_{\phi} f, r(I - rV^r_{\phi})(V^{pq}_{t\phi} f - V^q_{\phi} f))_{\xi}\\ &= \lim_{r\to\infty}(V^{pq}_{t\phi} f - V^q_{\phi} f, r\,q V^r_{\phi} V^q_{\phi} f - r\,p V^{r0}_{\phi t} V^{pq}_{t\phi} f - r\,q V^r_{\phi} V^{pq}_{t\phi} f)_{\xi}\\ &= (V^{pq}_{t\phi} f - V^q_{\phi} f, q V^q_{\phi} f)_{\xi} - \lim_{r\to\infty}(r V^r_{\phi}(V^{pq}_{t\phi} f - V^q_{\phi} f), p V^{pq}_{t\phi} f)_{\nu}\\ &\quad - (V^{pq}_{t\phi} f - V^q_{\phi} f, q V^{pq}_{t\phi} f)_{\xi}\\ &= (V^{pq}_{t\phi} f - V^q_{\phi} f, q V^q_{\phi} f - q V^{pq}_{t\phi} f)_{\xi} - (V^{pq}_{t\phi} f - V^q_{\phi} f, p V^{pq}_{t\phi} f)_{\nu}.\end{aligned}$$

Since $\lim_{p\to 0} V^{pq}_{t\phi} f = V^q_{\phi} f$ boundedly and q.e., $\|p V^{pq}_{t\phi} f\|_{L^1(\nu)} \leqq \|f\|_{L^1(\xi)}$ and

$$\|q V^{pq}_{t\phi} f\|_{L^1(\xi)} \leqq \|q V^q_{\phi} f\|_{L^1(\xi)} \leqq \|f\|_{L^1(\xi)},$$

(5.10) follows.

By a similar argument, we have

Lemma 5.5. *For all $p>0$ and $f \in C_0(R^d)$,*

$$\lim_{q\to 0} \mathscr{E}^{\nu}_1(V^{qp}_{\phi t} f - V^p f, V^{qp}_{\phi t} f - V^p f) = 0. \tag{5.11}$$

According to Lemmas 5.3, 5.4 and 5.5, the set $\mathscr{D} = \{V^{pq}_{t\phi} f;\ p, q>0, f\in C_0(R^d)\} \cup \{V^{qp}_{\phi t} f;\ p,q>0,\ f \in C_0(R^d)\}$ is contained in $\mathscr{D}(\mathscr{E}^{\nu}) \cap \mathscr{D}(\mathscr{E}^{\xi})$ and the forms $\mathscr{E}^{\nu}$ and $\mathscr{E}^{\xi}$ coincide on $\mathscr{D}$. Moreover $\mathscr{D}$ is dense in $\mathscr{D}(\mathscr{E}^{\nu})$ [resp. $\mathscr{D}(\mathscr{E}^{\xi})$] relative to the norm $\mathscr{E}^{\nu}_1$ [resp. $\mathscr{E}^{\xi}_1$].

Theorem 5.1. *Suppose that $\xi(dx) = a(x)\,\nu(dx)$ for some bounded measurable function $a(x)$ and that the associated* CAF *$\phi(t) = \int_0^t a(X^{\nu}(s))\,ds$ is strictly increasing for q.e. starting points. Then the Dirichlet form $(\mathscr{E}^{\xi}, \mathscr{D}(\mathscr{E}^{\xi}))$ is the smallest closed extension of $(\mathscr{E}, C_0^{\infty}(R^d))$ on $L^2(d\xi)$.*

Proof. Since $C_0^{\infty}(R^d) \subset L^2(d\xi)$, for the proof of $C_0^{\infty}(R^d) \subset \mathscr{D}(\mathscr{E}^{\xi})$, it is enough to show that $\lim_{p\to\infty} p(f, (I - pV^p_{\phi})f)_{\xi} < \infty$ for all $f \in C_0^{\infty}(R^d)$. Let $f \in C_0^{\infty}(R^d)$. Then, since $f \in \mathscr{D}(\mathscr{E}^{\nu})$, there exists a sequence $\{f_n\}_{n\geqq 1} \subset \mathscr{D}$ such that $\lim_{n\to\infty} \mathscr{E}^{\nu}_1(f_n - f, f_n$

$-f)=0$. Hence, by the triangle inequality (see the proof of [9; Lemma 1.7]),

$$\lim_{p\to\infty} |\{p(f,(I-pV_\phi^p)f)_\xi\}^{1/2}-\{p(f_n,(I-pV_\phi^p)f_n)_\xi\}^{1/2}|$$
$$\leqq \limsup_{m\to\infty} \{\mathscr{E}^\xi(f_m-f_n,f_m-f_n)\}^{1/2}$$
$$=\limsup_{m\to\infty} \{\mathscr{E}^\nu(f_m-f_n,f_m-f_n)\}^{1/2}.$$

Therefore

$$\lim_{p\to\infty} p(f,(I-pV_\phi^p)f)_\xi=\lim_{n\to\infty}\lim_{p\to\infty} p(f_n,(I-pV_\phi^p)f_n)_\xi$$
$$=\lim_{n\to\infty}\mathscr{E}^\xi(f_n,f_n)=\lim_{n\to\infty}\mathscr{E}^\nu(f_n,f_n)=\mathscr{E}^\nu(f,f)<\infty.$$

To prove the denseness of $C_0^\infty(R^d)$ in $\mathscr{D}(\mathscr{E}^\xi)$, suppose $f\in\mathscr{D}(\mathscr{E}^\xi)$. Since f is approximated by the functions in $\mathscr{D}$ relative to $\mathscr{E}_1^\xi$-metric by Lemma 5.4, it is enough to suppose that $f\in\mathscr{D}$. Since $f\in\mathscr{D}(\mathscr{E}^\nu)$ and $C_0^\infty(R^d)$ is dense in $\mathscr{D}(\mathscr{E}^\nu)$, there exists a sequence $\{f_n\}_{n\geqq 1}\subset C_0^\infty(R^d)$ such that $\mathscr{E}_1^\nu(f_n-f,f_n-f)\to 0$ as $n\to\infty$. Since $\mathscr{E}^\xi=\mathscr{E}^\nu$ on $\mathscr{D}\cup C_0^\infty(R^d)$ and $\xi(dx)\leqq\|a\|\,\nu(dx)$, we can see that $\mathscr{E}_1^\xi(f_n-f,f_n-f)\to 0$ as $n\to\infty$. This completes the proof of the theorem.

The proof of Theorem 2 is obvious by the corollary of Lemma 5.1 and Theorem 5.1, in fact, it is enough to set $\xi(dx)=I_A(x_1)\,dx$ and $\phi(t)=\phi_A(t)$.

§ 6. Proof of Theorem 3

In this section, $X^0(t)$, $X^\nu(t)$, $M_i^\nu(t),\dots$ denote those given in §4 and §5. By (4.6), $M^\nu(t)=(M_1^\nu(t),\dots,M_d^\nu(t))$ is a system of martingale CAFs on (Ω,P_x) satisfying

$$\langle M_i^\nu, M_j^\nu\rangle\langle t\rangle=\int_0^t a_{ij}(X^\nu(s))\,\phi_A(ds)+\int_0^t \alpha_{ij}(X^\nu(s))\,\phi_\Gamma(ds).$$

Firstly, we shall give a representation of $M^\nu(t)$ by Brownian motions.

Lemma 6.1. *There exists a properly exceptional set N, enlargement $(\tilde\Omega,\tilde P_x)$ of (Ω,P_x) and mutually independent stochastic processes*

$$\tilde B^\nu(t)=(\tilde B_1^\nu(t),\dots,\tilde B_d^\nu(t))\quad and\quad \hat B^\nu(t)=(\hat B_2^\nu(t),\dots,\hat B_d^\nu(t))$$

such that, for $x\notin N$, $\tilde B^\nu(t)$ is a d-dimensional Brownian motion started from 0, $\hat B^\nu(t)$ is a $(d-1)$-dimensional Brownian motion started from 0, and

$$M_i^\nu(t)=\int_0^t\left\{\sum_{j=1}^d I_A(X_1^\nu(s))\,\sigma_{ij}(X^\nu(s))\,d\tilde B_j^\nu(s)+\sum_{j=2}^d I_\Gamma(X_1^\nu(s))\,\tilde\tau_{ij}(X^\nu(s))\,d\hat B_j^\nu(s)\right\}\tag{6.1}$$

$\tilde P_x$ - a.s., where $\tilde\tau=\sqrt{a_{11}}\,\tau$, σ and τ are the matrices in §2 *and $X^\nu(t)$ and $M^\nu(t)$ are considered as the processes on $\tilde\Omega$ by $X^\nu(t,\tilde\omega)=X^\nu(t,i\circ\tilde\omega)$ and $M^\nu(t,\tilde\omega)=M^\nu(t,i\circ\tilde\omega)$* (see §2).

Proof. This can be proved by a repeated argument of Stroock and Varadhan [10; Theorem 4.5.2], so that we shall only present the outline. Set

$$\Pi(t)=\lim_{\varepsilon\to 0}(I_A a)(\varepsilon I+I_A a)^{-1}(X^\nu(t)),$$
$$q(t)=\lim_{\varepsilon\to 0}(\varepsilon I+I_A a)^{-1}(X^\nu(t))\,\Pi(t)$$

and $r(t)={}^t(I_A\sigma)(X^\nu(t))\,q(t)$, where we set $I_A(X^\nu(t))=I_A(X_1^\nu(t))$. Then $\Pi(t)=(I_A\sigma)(X^\nu(t))\,r(t)$ and $\hat{\Pi}(t)=r(t)(I_A\sigma)(X^\nu(t))$ are the orthogonal projections onto the range of $I_A a(X^\nu(t))$ and $I_A({}^t\sigma\cdot\sigma)(X^\nu(t))$, respectively. Let $B^{(1)}(t)=(B_1^{(1)}(t),\ldots,B_d^{(1)}(t))$ be a d-dimensional Brownian motion on a probability space $(\Omega^{(1)}, P^{(1)})$ such that $B^{(1)}(0)=0$ and set $\Omega^{(2)}=\Omega\times\Omega^{(1)}$ and $P_x^{(2)}=P_x\times P^{(1)}$. Then $(\Omega^{(2)}, P_x^{(2)})$ is an enlargement of (Ω, P_x). For $\omega^{(2)}=(\omega,\omega^{(1)})\in\Omega^{(2)}$, set $i\circ\omega^{(2)}=\omega$, $X^\nu(t,\omega^{(2)})=X^\nu(t,i\circ\omega^{(2)})$ and $M^\nu(t,\omega^{(2)})=M^\nu(t,i\circ\omega^{(2)})$. Then the process

$$\tilde{B}^\nu(t)=\int_0^t r(s)\,dM^\nu(s)+\int_0^t (I-\hat{\Pi}(s))\,dB^{(1)}(s)$$

is a d-dimensional Brownian motion on $(\Omega^{(2)}, P_x^{(2)})$ for all x outside a properly exceptional set. Furthermore the process $M^{(1)}(t)$ defined by

$$M^{(1)}(t)=M^\nu(t)-\int_0^t \Pi(s)\,dM^\nu(s)$$

is a system of local martingale CAFs such that

$$\langle M_i^{(1)}, M_j^{(1)}\rangle(t)=\int_0^t \{(I-\Pi)(s)(I_A a+I_\Lambda\alpha)(X^\nu(s))(I-\Pi)(s)\}_{ij}\,ds$$
$$=\int_0^t \alpha_{ij}(X^\nu(s))\,\phi_\Gamma(ds).$$

Hence, in particular, $M_1^{(1)}=0$. By a similar argument for $M^{(1)}$, there exist an enlargement $(\tilde{\Omega},\tilde{P}_x)$ of $(\Omega^{(2)}, P_x^{(2)})$ and a $(d-1)$-dimensional Brownian motion $\hat{B}^\nu(t)=(\hat{B}_2^\nu(t),\ldots,\hat{B}_d^\nu(t))$ on $(\tilde{\Omega},\tilde{P}_x)$ such that $\hat{B}^\nu(0)=0$ and

$$\int_0^t \Pi^{(1)}(s)\,dM^{(1)}(s)=\int_0^t I_\Gamma\,\tilde{\tau}(X^\nu(s))\,d\hat{B}^\nu(s),$$

where $\Pi^{(1)}(t)$ is the orthogonal projection in R^{d-1} onto the range of $I_\Gamma\alpha(X^\nu(t))$. By using these Brownian motions, $M^\nu(t)$ is represented as

$$M^\nu(t)=\int_0^t I_A\sigma(X^\nu(s))\,d\tilde{B}^\nu(s)+\int_0^t I_\Gamma\,\tilde{\tau}(X^\nu(s))\,d\hat{B}^\nu(s).$$

Combining the lemma with (4.5), we have

$$X_i^\nu(t)=X_i^\nu(0)+\int_0^t\left\{\sum_{j=1}^d I_A(X_1^\nu(s))\,\sigma_{ij}(X^\nu(s))\,d\tilde{B}_j^\nu(s)\right.$$
$$\left.+\sum_{j=2}^d I_\Gamma(X_1^\nu(s))\,\tilde{\tau}_{ij}(X^\nu(s))\,d\hat{B}_j^\nu(s)\right\}$$
$$+\int_0^t b_i(X^\nu(s))\,\phi_A(ds)+\int_0^t \tilde{\beta}_i(X^\nu(s))\,\phi_\Gamma(ds) \tag{6.2}$$

$\tilde{P}_x$ - a.s. for q.e. x.

As was proved in Theorem 4.1, the process $(X_1^a(t), P_{x_1}^a)$ is a one dimensional diffusion process with speed measure $2m(dx_1)$. Let $\frac{1}{2}\ell^a(t,x_1)$ be its local time at x_1. Then it is characterized by

$$\int_0^t I_{\{X_1^a(s)=x_1\}}\ell^a(ds,x_1)=\ell^a(t,x_1) \tag{6.3}$$

and

$$\int_0^t f(X_1^a(s))\,ds=\int \ell^a(t,x_1)f(x_1)\,m(dx_1) \tag{6.4}$$

for all $f\in C_0(R^1)$. Let

$$\psi_a(t)=\int_0^t (1/a_{11})(X^a(s))\,ds$$

be the inverse function of ϕ_a^ν defined by (4.12). Then $X^\nu(t)=X^a(\psi_a^{-1}(t))$. Set $\ell^\nu(t,x_1)=\ell^a(\psi_a^{-1}(t),x_1)$. Then, by (6.3) and (6.4), it satisfies

$$\int_0^t I_{\{X_1^\nu(s)=x_1\}}\ell^\nu(ds,x_1)=\ell^\nu(t,x_1) \tag{6.5}$$

and

$$\int_0^t f(X_1^\nu(s))\,a_{11}(X^\nu(s))\,ds=\int \ell^\nu(t,x_1)f(x_1)\,m(dx_1) \tag{6.6}$$

for all $f\in C_0(R^1)$. By Theorem 4.2, the CAF, $\phi_A(t)$ is strictly increasing. Set $\ell^0(t,x_1)=\ell^\nu(\phi_A^{-1}(t),x_1)$. Then it is a CAF of $X^0(t)=X^\nu(\phi_A^{-1}(t))$ and, by (6.5), it satisfies

$$\int_0^t I_{\{X_1^0(s)=x_1\}}\ell^0(ds,x_1)=\ell^0(t,x_1). \tag{6.7}$$

Also, since (6.6) holds for any bounded measurable function vanishing outside a compact set, by taking $I_A(x_1)f(x_1)$ instead of $f(x_1)$, we have

$$\int_0^t f(X_1^0(s))\,a_{11}(X^0(s))\,ds=\int \ell^0(t,x_1)f(x_1)\,dx_1 \tag{6.8}$$

for all $f\in C_0(R^1)$. Similarly, by setting $f(x_1)=I_\Gamma(x_1)$ and changing the time, we have

$$\int_0^t a_{11}(X^0(s))\,\phi_\Gamma(\phi_A^{-1}(ds))=\int \ell^0(t,x_1)\,\mu(dx_1). \tag{6.9}$$

We shall consider the processes $X^0(t)$, $\ell^0(t,x_1)$, etc. as the processes on the enlarged probability space $(\tilde{\Omega},\tilde{P}_x)$, as before. Define the processes $B^0(t)=(B_1^0(t),\dots,B_d^0(t))$ and $M^0(t)=(M_1^0(t),\dots,M_d^0(t))$ by

$$B_i^0(t)=\int_0^{\phi_A^{-1}(t)} I_A(X^\nu(s))\,d\tilde{B}_i^\nu(s)$$

and

$$M_i^0(t)=\int_0^{\phi_A^{-1}(t)} \sqrt{a_{11}(X^\nu(s))}\,I_\Gamma(X_1^\nu(s))\,d\hat{B}_i^\nu(s).$$

Lemma 6.2. *Let N be the exceptional set in Lemma* 6.1. *Then, for all $x \notin N$, $B^0(t)$ and $M^0(t)$ are systems of martingale* CAFs *on $(\tilde{\Omega}, \tilde{P}_x)$ satisfying $\langle B_i^0, B_j^0 \rangle(t) = \delta_{ij} t$, $\langle B_i^0, M_j^0 \rangle(t) = 0$ and $\langle M_i^0, M_j^0 \rangle(t) = \delta_{ij} \ell_\mu^0(t)$, where*

$$\ell_\mu^0(t) = \int \ell^0(t, x_1)\, \mu(dx_1).$$

Proof. By the definition,

$$\langle B_i^0, B_j^0 \rangle(t) = \delta_{ij} \int_0^{\phi_A^{-1}(t)} I_A(X^\nu(s))\, ds = \delta_{ij} t$$

and $\langle B_i^0, M_j^0 \rangle(t) = 0$. By noting (6.9), we have

$$\begin{aligned}\langle M_i^0, M_j^0 \rangle(t) &= \delta_{ij} \int_0^{\phi_A^{-1}(t)} a_{11}(X^\nu(s))\, I_\Gamma(X_1^\nu(s))\, ds \\ &= \delta_{ij} \int_0^t a_{11}(X^0(s))\, \phi_\Gamma(\phi_A^{-1}(ds)) \\ &= \delta_{ij} \int \ell^0(t, x_1)\, \mu(dx_1) = \delta_{ij} \ell_\mu^0(t).\end{aligned}$$

Proof of Theorem 2. Set $\phi_A^{-1}(t)$ instead of t in (6.2). Then it can be written as

$$\begin{aligned}X_i^0(t) = X_i^0(0) &+ \sum_{j=1}^d \int_0^t \sigma_{ij}(X^0(s))\, dB_j^0(s) \\ &+ \sum_{j=2}^d \int_0^t \tau_{ij}(X^0(s))\, dM_j^0(s) + \int_0^t b_i(X^0(s))\, ds \\ &+ \int_0^t \beta_i(X^0(s))\, \ell_\mu^0(ds),\end{aligned}$$

$\tilde{P}_x^0$ - a.s. for q.e. x, where $\tau_{ij} = \tilde{\tau}_{ij}/a_{11}$ and $\beta_i = \tilde{\beta}_i/a_{11}$. Thus the theorem has been proved.

Acknowledgement. The author wishes to thank S. Watanabe, T. Watanabe and the referee for helpful suggestions.

References

1. Fukushima, M.: Dirichlet forms and Markov processes. Tokyo: Kodansha and North-Holland Publ. Co. 1980
2. Ikeda, N., Watanabe, S.: The local structure of a class of diffusions and related problems. Proc. 2nd Japan-USSR symp. on probability theory. Lecture Notes in Math. **330**, Berlin-Heidelberg-New York: Springer 1973
3. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Tokyo: Kodansha and North-Holland Publ. Co. 1981
4. Ito, K., McKean, H.P.: Diffusion processes and their sample paths. Berlin-Heidelberg-New York: Springer 1965
5. Nagasawa, M., Sato, K.: Some theorems on time change and killing of Markov processes. Kodai Math. Sem. Rep. **15**, 195–219 (1963)

6. Oshima, Y.: On the equilibrium measure of recurrent Markov processes. Osaka J. Math. **15**, 283–310 (1978)
7. Oshima, Y.: On stochastic differential equations characterizing some singular diffusion processes. Proc. Japan Acad. **57**, Ser. A, 151–154 (1981)
8. Revuz, D.: Mesures associées aux fonctionelles additives de Markov. Trans. Amer. Math. Soc. **148**, 501–531 (1970)
9. Silverstein, M.: Symmetric Markov processes. Lecture Notes in Math. **426**, Berlin-Heidelberg-New York: Springer 1974
10. Stroock, D.W., Varadhan, S.R.S.: Multidimensional diffusion processes. Berlin-Heidelberg-New York: Springer 1979
11. Tomisaki, M.: A construction of diffusion processes with singular product measures. Z. Wahrscheinlichkeitstheorie verw. Gebiete **53**, 51–70 (1980)
12. Watanabe, S.: On stochastic differential equations for multidimensional diffusion processes with boundary conditions I, II. J. Math. Kyoto Univ. **11**, 169–180, 545–551 (1971)
13. Fukushima, M., Stroock, D.W.: Reversibility of solutions to martingale problems, to appear

Received February 20, 1981