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Brownian Approximations to First Passage Probabilities*

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Summary. By direct probabilistic argument one term of an Edgeworth type asymptotic expansion is obtained for certain first passage distributions for random walks. These results provide partial justification for and extensions of approximations suggested earlier as a heuristic consequence of Laplace transform calculations.

1. Introduction and Summary

Let $x_1, x_2, \ldots$ be independent and identically distributed with mean $E(x_1)=\mu$. Let $s_n = x_1 + \ldots + x_n$, and for $a<0\leqq b$ define the stopping times

$$\tau=\tau(b)=\inf\{n\colon s_n>b\} \qquad (\tau_+=\tau(0))$$

and

$$T=T(a,b)=\inf\{n\colon s_n\notin[a,b]\}.$$

The probabilities

$$P\{\tau<m\} \tag{1}$$

and

$$P\{T<m,\ s_T>b\} \tag{2}$$

with $m\leqq\infty$ are of interest in a variety of probability models. For example, with the proper identifications (1) is the probability that the waiting time for the $(m-1)$-th customer in a single server queue exceeds b; with $m=\infty$ it is the stationary waiting time probability or alternatively the ruin probability of an insurance risk process. The two-sided probabilities (2) arise in sequential statistical analysis and in dam theory. (See Feller, 1966, for a more complete discussion.) A commonly used approximation to (1) and (2) is the Brownian

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motion approximation (heavy traffic approximation in queueing theory). Siegmund (1979) gave a heuristic argument based on Laplace transforms and numerical examples to show that this approximation can be considerably improved by obtaining the next term of an Edgeworth type expansion of (1) and (2), with the Brownian motion approximation as the leading term. This method has been extended by Yuh (1981) for studying joint probabilities of the form

$$P\{\tau<m,\ s_m<b-x\} \tag{3}$$

and

$$P\{T<m,\ s_T>b,\ s_m<b-x\}. \tag{4}$$

In addition to being closely related to (1) and (2) the probabilities (3) and (4) are the essential ingredients in the conditional probabilities $P\{\tau<m|s_m=b-x\}$ and $P\{T<m,\ s_T>b|s_m=b-x\}$, which arise in the study of the Kolmogorov-Smirnov statistics.

The purpose of this paper is to give a direct probabilistic calculation of a one-term Edgeworth expansion to probabilities like (3). The method is in principle applicable to (4) although the computations are much more involved, and no details are given in this case. An example of our results is as follows.

Theorem 1. *Suppose* $\mu=0$, $Ex_1^2=1$, *and* $\gamma=Ex_1^3$ *is finite. Let* $b=\zeta m^{1/2}$. *If the distribution of* x_1 *is strongly non-lattice in the sense that* $\limsup_{|t|\to\infty}|E\exp(itx_1)|<1$, *then for each* $x>0$ *as* $m\to\infty$

$$\begin{aligned}&P\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}\\&\quad=1-\Phi(\zeta+x)-m^{-1/2}\phi(\zeta+x)[2\beta+(\gamma/6)(x^2-\zeta^2-1)]+o(m^{-1/2}).\end{aligned} \tag{5}$$

Here $\beta=E(s_{\tau_+}^2/2Es_{\tau_+})$ *if* $\zeta>0$ *and* $\beta=Es_{\tau_+}$ *if* $\zeta=0$; ϕ *and* Φ *denote the standard normal density and distribution functions.*

Remarks. (a) Since $P\{\tau\leqq m\}=P\{s_m>b\}+P\{\tau<m,\ s_m\leqq b\}$, if $P\{\tau<m,\ s_m<b-x\}$ were known exactly for all $x>0$, then (at least for continuous distributions) one would obtain $P\{\tau\leqq m\}$ by letting $x\to 0$. Although it seems plausible that (5) should hold uniformly in x for x near 0, and in fact the right hand side of (5) with $x=0$ agrees with the result obtained heuristically by Siegmund (1979), we have been unable to prove this uniformity.

(b) In the case $\zeta=0$, Theorem 1 is equivalent to a result of Iglehart (1974). In this case the asymptotic behavior of $P\{\tau_+>m\}$ is known from fluctuation theory, e.g., Feller (1966), p. 399, which shows that (5) is true with $x=0$, although a completely different proof is involved.

(c) Siegmund (1979) has given a method for calculating β numerically.

(d) Under the stronger assumptions that the x's have a finite moment generating function, Borovkov (1962) gave a complete asymptotic expansion of (3). His methods use complex analysis, and the results are not given in a form which permits simple comparisons with (5). Also Borovkov's methods appear to handle the case $x=0$ without difficulty, although they do not seem to adapt readily to two-sided stopping rules.

(e) Analogous results may be obtained for arithmetic distributions.

(f) If for some n the characteristic function of s_n is integrable, one can obtain a similar expansion for the density $-d/dx\,P\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}$, which can be formally calculated by differentiating (5). As an application, one can improve the limiting distribution of the one sample Kolmogorov-Smirnov statistic (see Yuh, 1982, for details).

The remainder of this paper is arranged as follows. Theorem 1 is proved in Sect. 2. Section 3 considers the case $Ex_1 \neq 0$. In Sect. 4 we discuss our original method for studying these problems, in which direct approximation of conditional probabilities such as

$$P\{T<m,\ s_T>b \mid s_m=b-x\}$$

plays a primary role. Although this method does not seem as general as the method of Sects. 2 and 3 for obtaining asymptotic expansions, it has the conceptual advantage of yielding the Brownian motion approximation and the next Edgeworth term simultaneously. The method is not developed here completely, but is used to give a simple derivation of Anderson's (1960) results for Brownian motion. More recently Siegmund (1982) has adapted this method to deal with large deviations for boundary crossing probabilities, including those for non-linear boundaries.

2. Proof of Theorem 1

Let F_n denote the distribution function of s_n, $n=0,1,\ldots$. Here $s_0=0$ and F_0 denotes the point mass at 0. It is easy to see that

$$\begin{aligned} P\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\} &= \int_{\{\tau<m\}} P\{s_m<(\zeta-x)m^{1/2} \mid \tau, s_\tau\}\,dP \\ &= \int_{\{\tau<m\}} F_{m-\tau}((\zeta-x)m^{1/2}-s_\tau)\,dP \\ &= \int_{\{\tau\leqq m\}} F_{m-\tau}(-xm^{1/2}-R_m)\,dP, \end{aligned} \tag{6}$$

where $R_m=s_\tau-\zeta m^{1/2}$. Similarly,

$$\begin{aligned} P\{s_m\geqq(\zeta+x)m^{1/2}\} &= P\{\tau\leqq m,\ s_m\geqq(\zeta+x)m^{1/2}\} \\ &= \int_{\{\tau\leqq m\}} [1-F_{m-\tau}(xm^{1/2}-R_m)]\,dP, \end{aligned} \tag{7}$$

so

$$\begin{aligned} &P\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\} \\ &= P\{s_m\geqq(\zeta+x)m^{1/2}\} - \int_{\{\tau\leqq m\}} [1-F_{m-\tau}(xm^{1/2}-R_m)-F_{m-\tau}(-xm^{1/2}-R_m)]\,dP. \end{aligned} \tag{8}$$

The customary Edgeworth expansion applies to the first term on the right hand side of (8); hence the remainder of the proof is a detailed expansion of the integrand in (8) and an asymptotic evaluation of the resulting integral.

To carry out the following analysis it is technically useful to modify (8) to insure that in the integrand $m-\tau$ is not too small and R_m is not too large. Let $m_1=m(1-(\log m)^{-2})$. A consequence of Lemmas 2 and 3 below is that for some $\varepsilon_m \to 0$

$$\begin{aligned} &P\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}\\ &\quad =P\{\tau<m_1,\ R_m<m^{1/2}\varepsilon_m,\ s_m<(\zeta-x)m^{1/2}\}+o(m^{-1/2}), \end{aligned}$$

and hence by an argument similar to that leading to (8)

$$\begin{aligned} P\{\tau&<m,\ s_m<(\zeta-x)m^{1/2}\}\\ &=P\{s_m\geqq(\zeta+x)m^{1/2}\}+o(m^{-1/2})\\ &\quad -\int_{\{\tau<m_1,\, R_m<m^{1/2}\varepsilon_m\}}[1-F_{m-\tau}(xm^{1/2}-R_m)-F_{m-\tau}(-xm^{1/2}-R_m)]\,dP. \end{aligned} \tag{9}$$

According to Petrov (1972) VI.3 Theorem 3

$$F_n(xn^{1/2})=\Phi(x)-(\gamma/6n^{1/2})(x^2-1)\,\phi(x)+(1+|x|^3)^{-1}\,o(n^{-1/2}),$$

where $o(\cdot)$ is uniform in x. This may be used to expand the integrand in (9), and a subsequent expansion of the normal distribution function Φ by Taylor's theorem shows that uniformly on $\{\tau<m_1,\ R_m<m^{1/2}\varepsilon_m\}$ the integrand in (9) equals

$$\begin{aligned} &m^{-1/2}(1-\tau/m)^{-1/2}\,\phi(x/(1-\tau/m)^{1/2})\{2R_m+(\gamma/3)[x^2/(1-\tau/m)-1]\}\\ &\quad +O(m^{-1/2}\varepsilon_m R_m)+o(m^{-1/2}). \end{aligned}$$

To complete the proof we shall find the limiting joint distribution of τ/m and R_m and show that this integrand is uniformly integrable, so that the integral in (9) may be evaluated asymptotically by integrating with respect to the limiting distribution. Suppose initially that $\zeta>0$. The uniform integrability follows from Lemma 1 below and the fact that $(1-\tau/m)^{-3/2}\,\phi[x/(1-\tau/m)^{1/2}]$ is a bounded continuous function of τ/m for $\tau<m$. (Here it is important that $x>0$.) By Lemma 3 of Siegmund (1979), τ/m and R_m are asymptotically independent; so it suffices to consider the two variables separately. By the Erdös-Kac theorem and simple calculus

$$\begin{aligned} \lim_{m\to\infty} P\{\tau/m\leqq t\} &= \lim_{m\to\infty} P\{\max_{1\leqq n\leqq[mt]} s_n>\zeta m^{1/2}\}=2[1-\Phi(\zeta/t^{1/2})]\\ &=\int_0^t \zeta u^{-3/2}\,\phi(\zeta/u^{1/2})\,du. \end{aligned}$$

Since $R_m=s_\tau-\zeta m^{1/2}$ is the residual waiting time for the renewal process determined by the process of ladder heights of the random walks s_n, $n=1,2,\ldots$, the renewal theorem yields

$$\lim_{m\to\infty} P\{R_m<w\}=\int_0^w P\{s_{\tau_+}\geqq y\}\,dy/E s_{\tau_+}$$

(cf. Feller, 1966, XI and XII). Using these results shows that the integral in (9) is asymptotically equal to

$$m^{-1/2}\int_0^1 [2\beta+(\gamma/3)\{x^2/(1-t)-1\}](1-t)^{-1/2}$$
$$\cdot\,\phi[x/(1-t)^{1/2}]\,\zeta t^{-3/2}\,\phi[\zeta/t^{1/2}]\,dt,$$

which after some calculus yields (5).

In the case $\zeta=0$ the argument is similar but much simpler. It is obvious that $\tau/m\to 0$ in probability and $R_m=s_{\tau_+}$.

Lemma 1. *$Es^2_{\tau_+}<\infty$ and $\{s_{\tau(b)}-b,\ b>0\}$ is uniformly integrable.*

Proof. That $Es^2_{\tau_+}<\infty$ is known from random walk theory, e.g., problem 6, p. 232 of Spitzer (1976). As above, the renewal theorem implies that $P\{s_\tau-b\leqq x\}\to(Es_{\tau+})^{-1}\int_0^x P\{s_{\tau+}>y\}\,dy$; and since $Es^2_{\tau+}<\infty$, *renewal theory also yields* $E(s_\tau-b)\to Es^2_{\tau+}/2Es_{\tau+}$ (Feller, 1966, p. 353 ff.), which proves uniform integrability. (The uniform integrability may alternatively be proved from first principles by an elaboration of the indicated idea for proving $Es^2_{\tau_+}<\infty$. This may be the best approach to use in Lemma 4 below.)

Lemma 2. *For each $\varepsilon>0$, $P\{R_m>\varepsilon m^{1/2}\}=o(m^{-1/2})$.*

Proof. By the Markov inequality

$$P\{R_m>\varepsilon m^{1/2}\}=\varepsilon^{-1}m^{-1/2}\int_{\{R_m>\varepsilon m^{1/2}\}} R_m\,dP,$$

which is $o(m^{-1/2})$ by Lemma 1.

Lemma 3. *Let $m_1=m(1-(\log m)^{-2})$ as in the proof of Theorem 1. Then as $m\to\infty$,*

$$P\{m_1<\tau<m,\ s_m<(\zeta-x)m^{1/2}\}=o(m^{-1/2})$$

and

$$P\{m_1<\tau\leqq m,\ s_m\geqq(\zeta+x)m^{1/2}\}=o(m^{-1/2}).$$

Proof. By Lemma 2

$$\begin{aligned}&P\{m_1<\tau\leqq m,\ s_m\geqq(\zeta+x)m^{1/2}\}\\&\quad=P\{m_1<\tau\leqq m,\ R_m<\tfrac{1}{2}xm^{1/2},\ s_m\geqq(\zeta+x)m^{1/2}\}\\&\qquad+o(m^{-1/2})\leqq\sup_{m_1<n\leqq m}P\{s_{m-n}\geqq\tfrac{1}{2}xm^{1/2}\}+o(m^{-1/2}),\end{aligned}$$

which is easily seen to be $o(m^{-1/2})$ by Nagaev's (1965) improvement of the Berry-Esseen theorem or by the related result of Petrov quoted earlier. A similar but easier argument shows that the first probability in Lemma 3 is also $o(m^{-1/2})$.

3. The Case $Ex_1 \neq 0$

When $Ex_1 \neq 0$, results analogous to Theorem 1 are more complicated technically. Although it is probably possible to formulate a comprehensive theorem, it would be extremely cumbersome. Therefore, in this section we briefly discuss two important special cases: (i) when the distribution of x_1 can be imbedded in an exponential family, and (ii) when Ex_1 is a location parameter. Although the treatment of these two cases is slightly different, it should be apparent that modulo certain technicalities the methods can be applied to other similar problems.

To consider briefly the simpler case of an exponential family, suppose that the distribution of x_1 is given by $F_\theta(dx) = \exp[\theta x - \psi(\theta)] F_0(dx)$, where F_0 is a strongly non-lattice distribution having mean 0 and variance 1. It is easily seen that $\psi(0)=0$, $\psi'(\theta)=E_\theta x_1$, and $\psi''(\theta)=\operatorname{var}_\theta(x_1)$. By taking F_0 to have mean 0, ψ has been standardized so that $\psi'(0)=0$, and thus $\operatorname{sgn} E_\theta x_1 = \operatorname{sgn}\theta$. For $\theta_0<0\,(\theta_1>0)$ it will be assumed that there exists a $\theta_1>0\,(\theta_0<0)$, necessarily unique for which $\psi(\theta_0)=\psi(\theta_1)$.

The basic identity (6) remains true and in the obvious notation becomes

$$P_\theta\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\} = \int_{\{\tau<m\}} F_{\theta,m-\tau}(-xm^{1/2}-R_m)\,dP_\theta. \tag{10}$$

However, (7) must now be altered. Assume to be specific that $\theta>0$ and write θ_1 for θ. It is easy to see from the exponential family structure that for every function $h \geqq 0$ such that $hI_{\{\tau=n\}}$ is $\mathscr{B}(x_1,\dots,x_n)$ measurable for all n,

$$\int_{\{\tau<\infty\}} h\,dP_{\theta_0} = \int_{\{\tau<\infty\}} h \exp\{-(\theta_1-\theta_0)s_\tau\}\,dP_{\theta_1},$$

and hence

$$\begin{aligned} &e^{(\theta_1-\theta_0)b} P_{\theta_0}\{s_m \geqq (\zeta+x)m^{1/2}\} \\ &\quad= e^{(\theta_1-\theta_0)b} \int_{\{\tau\leqq m\}} [1-F_{\theta_0,m-\tau}(xm^{1/2}-R_m)]\,dP_{\theta_0} \\ &\quad= \int_{\{\tau\leqq m\}} [1-F_{\theta_0,m-\tau}(xm^{1/2}-R_m)] \exp[-(\theta_1-\theta_0)R_m]\,dP_{\theta_1}. \end{aligned} \tag{11}$$

From (10) and (11) one obtains the following analogue of (8):

$$\begin{aligned} &P_{\theta_1}\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\} \\ &\quad= e^{(\theta_1-\theta_0)b} P_{\theta_0}\{s_m \geqq (\zeta+x)m^{1/2}\} \\ &\qquad - \int_{\{\tau\leqq m\}} \{e^{-(\theta_1-\theta_0)R_m}[1-F_{\theta_0,m-\tau}(xm^{1/2}-R_m)] \\ &\qquad - F_{\theta_1,m-\tau}(-xm^{1/2}-R_m)\}\,dP_{\theta_1}. \end{aligned}$$

A similar identity holds for $P_{\theta_0}\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}$.

The technical modification of (8) leading to (9) can be justified under the present assumptions. Subsequent expansion as in the proof of Theorem 1 shows that if $(\theta_1-\theta_0)=2\xi m^{-1/2}$, then

$$\begin{aligned}
&P_{\theta_1}\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}\\
&\quad=\exp[2\xi(\zeta+\beta m^{-1/2})][1-\Phi(\zeta+x+\xi)]\\
&\quad-e^{2\xi\zeta}m^{-1/2}\phi(\zeta+x+\xi)\{2\beta+(\gamma/6)[\xi(\zeta-x)+x^2-\zeta^2-1]\}\\
&\quad+o(m^{-1/2}). \qquad (12)
\end{aligned}$$

For θ_0 the result is formally identical provided ξ is defined as $-\frac{1}{2}m^{1/2}(\theta_1-\theta_0)$. The details of these calculations have been omitted. (For the ideas justifying a version of Lemma 1 in this context, see Siegmund, 1979.)

Since the right hand side of (12) makes sense provided only that the x's have a finite third moment, the exponential family assumption appears to be much too strong. However, the likelihood ratio of the exponential family plays an important role in the derivation of (12), and avoiding its use raises some additional technical problems.

To minimize the number of unpleasant technicalities it will be assumed that θ is a location parameter (which is further restricted below to be non-negative). Hence let F_0 denote a continuous strongly non-lattice distribution function having mean 0, variance 1, and finite third moment γ. Let $F_\theta(x)=F_0(x-\theta)$, and let $F_{\theta,n}$ be the n-fold convolution of F_θ with itself. Let P_θ denote the probability measure under which $x_1, x_2, \ldots$ are independent with common probability distribution F_θ. Except for the indicated convergence in distribution, the following generalization of Lemma 1 may be proved by the method suggested in the parenthetical remark at the end of the proof of Lemma 1. The convergence in distribution can be obtained by straightforward but tedious review of the proof of the renewal theorem given by Breiman (1968, p. 220ff.) to obtain a version which is uniform in θ for θ near 0. The details are omitted.

Lemma 4. *As $\theta\to 0$, $\int_{\{\tau_+<\infty\}} s_{\tau_+}^{\lambda}\,dP_\theta\to E_0 s_{\tau_+}^{\lambda}$ for all $0<\lambda\leqq 2$. Let $\zeta>0$ and $\theta=\xi m^{-1/2}$ for some fixed $\xi\geqq 0$. Then the P_θ distributions of $(s_\tau-\zeta m^{1/2})$ converge to the distribution with density function $(E_0 s_{\tau_+})^{-1}P_0\{s_{\tau_+}>y\}$ and have uniformly integrable first moments.*

Assume now that $\theta=\xi m^{-1/2}$ for some fixed $\xi>0$. The identity (10) remains true in the present context. However, instead of (11) consider

$$\begin{aligned}
&\int_{(x,\infty)} \exp(-2\xi y)P_\theta\{s_m\in(\zeta+dy)m^{1/2}\}\\
&\qquad=\int_{\{\tau\leqq m\}}\Big[\int_{(x,\infty)}\exp(-2\xi y)F_{\theta,m-\tau}(m^{1/2}dy-R_m)\Big]\,dP_\theta, \qquad (13)
\end{aligned}$$

which in the exponential family model is actually equivalent to (11). From (10) and (13) one obtains the following analogue of (8):

$$\begin{aligned}
&P_\theta\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}\\
&\quad=\int_{(x,\infty)}\exp(-2\xi y)P_\theta\{s_m\in(\zeta+dy)m^{1/2}\}\\
&\quad-\int_{\{\tau\leqq m\}}\Big[\int_{(x,\infty)}\exp(-2\xi y)F_{\theta,m-\tau}(m^{1/2}dy-R_m)\\
&\quad-F_{\theta,m-\tau}(-x m^{1/2}-R_m)\Big]\,dP_\theta. \qquad (14)
\end{aligned}$$

With the aid of Lemma 4, this identity may now be expanded along established lines. (Expansion of the integral on the left hand side of (13) and the inner integral on the right hand side is facilitated by integration by parts, application of Petrov's theorem, and integration back by parts, which has the formal effect of expanding $F_{\theta,n}$ as if it had a density which had the appropriate local expansion.) In this case the resulting asymptotic expansion is as $m\to\infty$

$$\begin{aligned}&P_\theta\{\tau<m,\ s_m<(\zeta-x)m^{1/2}\}\\&\quad=\exp\{2\xi[\zeta+m^{-1/2}(\beta+2\gamma\xi\zeta/3)]\}[1-\Phi(\zeta+x+\xi)]\\&\qquad-m^{-1/2}e^{2\xi\zeta}\phi(\zeta+x+\xi)\{2\beta+(\gamma/6)[(x+\xi)^2-\zeta^2-1+4\xi^2]\}\\&\qquad+o(m^{-1/2}).\end{aligned}$$

4. Brownian Motion

Our first approach to studying the problems of this paper involved a different method, which unfortunately seems to require stronger assumptions than the method of Sects. 2 and 3. On the other hand, it seems conceptually simpler, and yields the Brownian motion approximation together with the Edgeworth correction in one calculation. In contrast the method of Sects. 2 and 3 requires that we already understand the Brownian approximation in order to subtract (7) from the basic identity (6) (or (11) from (10)). Nothing follows directly from (6) – even for Brownian motion itself.

To illustrate the basic identity from which this method proceeds without the details of another expansion, we restrict ourselves here to Brownian motion and give a new derivation of the principal probabilistic result of Anderson (1960). The argument is constructive in the sense that the process of successive reflection, which is usually the difficult part of two-boundary problems, is accomplished in a purely mechanical way by means of a simple recursion.

Let $\{X(t),\ 0\leq t\leq 1\}$ denote standard Brownian motion and for $\zeta_1<0<\zeta_2$ and $\zeta_1+\eta_1\leq\zeta_2+\eta_2$ define $T=\inf\{t\colon X(t)=\zeta_i+\eta_i t \text{ for } i=1 \text{ or } 2\}$. We shall calculate

$$P\{T<1,\ X(T)=\zeta_2+\eta_2 T\,|\,X(1)=\mu\} \tag{15}$$

(except for the case $\mu=\zeta_1+\eta_1=\zeta_2+\eta_2$). There is no loss of generality in assuming $\mu<\zeta_2+\eta_2$, because in the contrary case one can use the argument to calculate the complementary probability, namely

$$P\{T<1,\ X(T)=\zeta_1+\eta_1 T\,|\,X(1)=\mu\}.$$

Let P_μ denote the conditional distribution of $X(t)$, $0\leq t\leq 1$ given that $X(1)=\mu$. Then the probability in (15) equals

$$P_\mu\{T<1,\ X(T)=\zeta_2+\eta_2 T\}.$$

Let $\mathscr{F}(t)=\mathscr{B}(X(s),\ s\leq t)$. For any $t<1$ and $\mu_1\neq\mu$ the measures P_μ and P_{μ_1} contracted to $\mathscr{F}(t)$ are mutually absolutely continuous with an easily computed likelihood ratio:

$$\frac{dP_{\mu,t}}{dP_{\mu_1,t}}=\exp\{(\mu-\mu_1)(X(t)-\tfrac{1}{2}t(\mu+\mu_1)]/(1-t)\}$$
$$=L(X(t),t;\mu,\mu_1), \tag{16}$$

say. Hence by standard likelihood ratio (or martingale) arguments

$$P_\mu\{T<1,\ X(T)=\zeta_2+\eta_2T\}=\int_{\{T<1,\,X(T)=\zeta_2+\eta_2T\}}L(\zeta_2+\eta_2T,T;\mu,\mu_1)\,dP_{\mu_1}.$$

From (16) one easily sees that the choice of μ_1 for which

$$\zeta_2+\eta_2T-(1/2)T(\mu+\mu_1)=\zeta_2(1-T),$$

i.e., $\mu_1=2(\zeta_2+\eta_2)-\mu$ leads to

$$P_\mu\{T<1,X(T)=\zeta_2+\eta_2T\}$$
$$=\exp[-2\zeta_2(\zeta_2+\eta_2-\mu)]P_{\mu_1}\{T<1,\ X(T)=\zeta_2+\eta_2T\}. \tag{17}$$

Since this choice of μ_1 exceeds $\zeta_2+\eta_2$ (by virtue of the assumption $\mu<\zeta_2+\eta_2$), $P_{\mu_1}\{T<1\}=1$; and hence (17) may be rewritten

$$P_\mu\{T<1,\ X(T)=\zeta_2+\eta_2T\}$$
$$=\exp[-2\zeta_2(\zeta_2+\eta_2-\mu)][1-P_{\mu_1}\{T<1,\ X(T)=\zeta_1+\eta_1T\}]. \tag{18}$$

The identity (18) may now be used recursively to calculate (15), since by the same argument

$$P_{\mu_1}\{T<1,\ X(T)=\zeta_1+\eta_1T\}$$
$$=\exp[-2\zeta_1(\zeta_1+\eta_1+\mu-2(\zeta_2+\eta_2))][1-P_{\mu_2}\{T<1,\ X(T)=\zeta_2+\eta_2T\}], \tag{19}$$

where $\mu_2=2(\zeta_1+\eta_1-\zeta_2-\eta_2)+\mu<(\zeta_1+\eta_1)$, etc.

In general the result of carrying out this computation is an infinite series of terms which alternate in sign. For the very special case $\zeta_1+\eta_1=\zeta_2+\eta_2+\mu$, it turns out that $\mu_2=\mu$; and it is only necessary to solve the two Eqs. (18) and (19) simultaneously to obtain

$$P_\mu\{T<1,\ X(T)=\zeta_2+\eta_2T\}=\frac{\exp\{-2\zeta_1(\zeta_2+\eta_2-\mu)\}-1}{\exp\{2(\zeta_2-\zeta_1)(\zeta_2+\eta_2-\mu)\}-1}$$

(cf. Eq. (4.24) of Anderson, 1960, for this result as well as the answer in the general case).

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