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On Stochastic Horizontal Lifts

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1. Introduction

The notion of stochastic parallel displacement was introduced by K. Itô [4, 6]. He treated a stochastic parallel displacement of a tensor along a random curve. A stochastic parallel displacement of a frame was also defined [3, 11]. Let M be a Riemannian manifold and $O(M)$ be an orthonormal frame bundle. The stochastic parallel displacement of a frame along a random curve on M is defined by using the covariant derivative (determined by the Riemannian metric) on M. Then it defines a curve on $O(M)$. This is a natural extension of a horizontal lift for a smooth curve in differential geometry. But $O(M)$ has a structure of a principal fibre bundle. So we extend above result to principal fibre bundles.

Let $\{P, M, G, \pi\}$ be a principal fibre bundle where P is a bundle space, M is a base space, G is a structure group and π is a projection. On the principal fibre bundle a notion equivalent to the covariant derivative exists, i.e., a notion of a connection form. Let a connection form ω be given. Let $c=(x_t;\ 0\leqq t\leqq 1)$ be a smooth curve in M. Then a horizontal lift of c is a smooth curve $c^*=(u_t;\ 0\leqq t\leqq 1)$ in P such that $\pi(u_t)=x_t$ and $\omega(\dot{u}_t)=0$ for $0\leqq t\leqq 1$ where $\dot{u}_t$ is a vector tangent to the curve c^* at u_t. If (u_t) is a random curve, e.g., a path of a diffusion process on P, $\dot{u}_t$ may have no meaning. Hence we rewrite it in the integral form, i.e., if we define $c_s^*=(u_{st};\ 0\leqq t\leqq 1)$ for $s\in[0,1]$, then $\int_{c_s^*}\omega=0$ for all $s\in[0,1]$. Now we can generalize it to random curves because the integral of 1-form along the path of the diffusion process was defined by N. Ikeda and S. Manabe [2]. So if we replace the line integral by the integral of 1-form along the path of the diffusion process, we can define the stochastic horizontal lift. Precisely speaking, let $(X_t)_{t\geqq 0}$ be an M-valued continuous semimartingale (definition is given in Sect. 2). Then the stochastic horizontal lift of (X_t) is an $O(M)$-valued continuous semimartingale $(\tilde{X}_t)_{t\geqq 0}$ such that

(i) $\pi(\tilde{X}_t)=X_t$ for all $t\geqq 0$ a.e.,

(ii) $\int_{\tilde{X}[0,t]}\omega=0$ for all $t\geqq 0$ a.e.,

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where $\int_{\tilde{X}[0,t]} \omega$ is a stochastic line integral along a path $(\tilde{X}_t)$ (see [2, 3]). We will give a proof of the existence and the uniqueness of the stochastic horizontal lift under given initial condition. A stochastic parallel displacement on an associated vector bundle is easily obtained from the stochastic horizontal lift on the principal fibre bundle in the same way as in differential geometry. Itô's formula for the stochastic parallel displacement is also discussed in Sect. 5.

2. Stochastic Horizontal Lifts

Let $\{P, M, G, \pi\}$ be a principal fibre bundle. We assume that all the manifolds discussed in this paper are smooth, connected and paracompact. We denote the Lie algebra of G by $\mathfrak{g}$. G acts freely on P on the right. Hence for any $A \in \mathfrak{g}$ one parameter subgroup $\{\exp tA;\ t \in \mathbf{R}\}$ defines a one parameter transformation group on P and induces a vector field on P. We denote it by A^*. Then the mapping $A \mapsto A^*$ is a Lie algebra homomorphism from $\mathfrak{g}$ into $\Gamma^\infty(T(P))$ where $\Gamma^\infty(T(P))$ is a set of all C^∞ vector fields on P. Since G acts freely on P, this mapping is injective. A *connection form* is a $\mathfrak{g}$-valued 1-form on P satisfying the following conditions;

(i) $\omega(A^*) = A \quad$ for $A \in \mathfrak{g}$,

(ii) $R_g^* \omega = \mathrm{Ad}(g^{-1})\,\omega \quad$ for $g \in G$,

where R_g is defined by $R_g(u) = ug$. Suppose that a connection form ω is given. The *canonical 1-form* on G is the $\mathfrak{g}$-valued 1-form θ defined by $\theta_g(A_g) = A$ for $g \in G$ and $A \in \mathfrak{g}$. If we define the mapping $\phi_u\colon G \to P$ by $\phi_u(g) = ug$ for fixed $u \in P$, then $\phi_u^* \omega = \theta$. We use this fact later.

Let $(\Omega, \mathscr{F}, P)$ be a standard probability space and $(\mathscr{F}_t)_{t \geq 0}$ be a family of nondecreasing sub σ-fields. We assume that $(\mathscr{F}_t)_{t \geq 0}$ is right continuous and $\mathscr{F}_t$ contains all P-null sets for $t \geq 0$. Let $X = (X_t;\ 0 \leq t < \infty)$ be an M-valued semimartingale[1] (we assume that semimartingales are always continuous). Assume that $X_0 = x$ a.e. for some $x \in M$. Take any point $u \in P$ such that $\pi(u) = x$ and fix it. Now we define the stochastic horizontal lift as the following.

Definition 2.1. The *stochastic horizontal lift* of X starting at u is the P-valued semimartingale $\tilde{X} = (\tilde{X}_t;\ 0 \leq t < \infty)$ such that

(i) $\tilde{X}_0 = u$ a.e.,

(ii) $\pi(\tilde{X}_t) = X_t$ for all $t \geq 0$ a.e.,

(iii) $\int_{\tilde{X}[0,t]} \omega = 0$ for all $t \geq 0$ a.e.,

where $\int_{\tilde{X}[0,t]} \omega$ is the integral of 1-form ω along the path $\tilde{X}_t$ (c.f. N. Ikeda, S. Manabe [2]).

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[1] For any C^∞ function f on M, $f(X_t)$ is a continuous semimartingale i.e., a sum of a continuous local martingale and a continuous bounded variation process. This definition is applicable to general manifolds

In this paper we shall prove the existence and the uniqueness of the stochastic horizontal lift. So we state a result as a theorem.

Theorem 2.1. *For any M-valued semimartingale $X=(X_t)$ such that $X_0=x$ a.e. for some $x\in M$, there exists a unique stochastic horizontal lift $\tilde{X}=(\tilde{X}_t)$ of X starting at u such that $\pi(u)=x$.*

We prepare some results in Sect. 3 and give a proof of above theorem in Sect. 4.

3. The Integrals of 1-forms

The integrals of 1-forms along the path of diffusion process were defined by N. Ikeda and S. Manabe. Here we extend them slightly. Let M be a d-dimensional C^∞ manifold and $X=(X_t;\ 0\leqq t<\infty)$ be an M-valued semimartingale such that $X_0=x$ a.e. for some $x\in M$. We denote the cotangent bundle of M by $T^*(M)$.

Definition 3.1. Let $\Lambda(X)$ be a set of all α: $[0,\infty)\times M\times\Omega\to T^*(M)$ such that

(i) α is measurable,

(ii) for any fixed $(t,w)\in[0,\infty)\times\Omega$, $\alpha(t,\cdot,w)$ is a cross section of $T^*(M)$, i.e., a 1-form (we do not assume the smoothness),

(iii) $t\mapsto\alpha(t,X_t,w)$ is a $T^*(M)$-valued semimartingale.

In the sequel we sometimes omit the variables w and x and $\alpha(t)$ is denoted by α_t. Now we shall define the integral of $\alpha\in\Lambda(X)$ along the path X_t. We only define it locally: the global definition is easily obtained by using the partition of unity. Let $(x^1,\dots,x^d)$ be a local coordinate and set $X_t^i=x^i(X_t)$. From Definition 3.1, α can be represented as $\alpha(t,X_t,w)=\sum_{i=1}^{d}\alpha_i(t)(dx^i)_{X_t}$ where $\alpha_i(t)$ $(i=1,\dots,d)$ is a semimartingale. Then the integral of α along the path X_t is defined by $\sum_{i=1}^{d}\int_0^t\alpha_i(s)\circ dX_s^i$ which we denote by $\int_0^t\alpha_s\circ dX_s$, where the symbol $\circ$ means Fisk-Stratnovich's symmetric integral. It is easy to check that this definition does not depend on a particular choice of the local coordinate. We summarize some properties of the integrals of 1-forms. To do so, we need to introduce another class $S(X)$.

Definition 3.2. Let $S(X)$ be a set of all f: $[0,\infty)\times M\times\Omega\to\mathbf{R}$ such that

(i) f is measurable,

(ii) $t\mapsto f(t,X_t,w)$ is a semimartingale.

Then we have the following lemma.

Lemma 3.1. *Let f be in $S(X)$ and α be in $\Lambda(X)$. Then $(f\cdot\alpha)(t,x,w)=f(t,x,w)\,\alpha(t,x,w)$ is in $\Lambda(X)$. Moreover if we define $M_t=\int_0^t\alpha_s\circ dX_s$, then*

$$\int_0^t f(s,X_s)\circ dM_s=\int_0^t(f\cdot\alpha)_s\circ dX_s. \tag{3.1}$$

Proof. Let $\alpha(t, X_t, w)=\sum_{i=1}^{d}\alpha_i(t)\,(dx^i)_{X_t}$ in a local coordinate $(x^1, \dots, x^d)$. It is easy to see that $(f\cdot\alpha)_t$ belongs to $\Lambda(X)$. Furthermore,

$$\begin{aligned}\int_0^t f(s, X_s)\circ dM_s &= \sum_{i=1}^{d}\int_0^t f(s, X_s)\circ(\alpha_i(s)\circ dX_s^i)\\ &= \sum_{i=1}^{d}\int_0^t (f(s, X_s)\,\alpha_i(s))\circ dX_s^i\\ &= \int_0^t (f\cdot\alpha)_s\circ dX_s.\end{aligned}$$

This completes the proof.

Next we assume that X is a solution of the following stochastic differential equation

$$\begin{aligned}dX_t &= \sum_{i=1}^{r} A_i(X_t)\circ dM_t^i\\ X_0 &= x\end{aligned} \tag{3.2}$$

where M^i is a semimartingale and A_i is a C^∞ vector field on M.

Lemma 3.2. *Let α be in $\Lambda(X)$. Then*

$$\int_0^t \alpha_s\circ dX_s=\sum_{i=1}^{r}\int_0^t \alpha(s, X_s)\,(A_i(X_s))\circ dM_s^i. \tag{3.3}$$

Proof. Let $(x^1, \dots, x^d)$ be a local coordinate, $\alpha(t, X_t, w)=\sum_{j=1}^{d}\alpha_j(t)\,(dx^j)_{X_t}$ and $A_i(x)=\sum_{j=1}^{d}A_i^j(x)\,\partial_j$ where $\partial_j=\partial/\partial x^j$. Since X is a solution of (3.2), we have

$$X_t^j=\sum_{i=1}^{r}\int_0^t A_i^j(X_s)\circ dM_s^i.$$

Hence

$$\begin{aligned}\int_0^t \alpha_s\circ dX_s &= \sum_{j=1}^{d}\sum_{i=1}^{r}\int_0^t \alpha_j(s)\circ(A_i^j(X_s)\circ dM_s^i)\\ &= \sum_{i=1}^{r}\int_0^t \left(\sum_{j=1}^{d}\alpha_j(s)\,A_i^j(X_s)\right)\circ dM_s^i\\ &= \sum_{i=1}^{r}\int_0^t \alpha(s, X_s)\,(A_i(X_s))\circ dM_s^i.\end{aligned}$$

Thus we have (3.3).

Let G be a p-dimensional Lie group and $\mathfrak{g}$ be its Lie algebra. We consider the semimartingales on G. Let θ be the canonical 1-form on G and $X=(X_t)$ be a solution of the following stochastic differential equation

$$\begin{aligned}dX_t &= \sum_{i=1}^{r} B_i(X_t)\circ dM_t^i\\ X_0 &= g\end{aligned} \tag{3.4}$$

where M^i is a semimartingale such that $M_0^i=0$ and B_i is in $\mathfrak{g}$. Then Lemma 3.2 yields $\int_0^t \theta\circ dX_s=\sum_{i=1}^r M_t^i B_i$. We shall prove the converse. Let (X_t) be a G-valued semimartingale such that $X_0=g$ a.e. and $\{A_1,\ldots,A_p\}$ be a basis of $\mathfrak{g}$. Since $\int_0^t \theta\circ dX_s$ is a $\mathfrak{g}$-valued semimartingale, there exist semimartingales $M^1,\ldots,M^p$ such that $M_0^i=0$ a.e. $(i=1,\ldots,p)$ and $\int_0^t \theta\circ dX_s=\sum_{i=1}^p M_t^i A_i$. Then we have the following lemma.

Lemma 3.3. *(X_t) is a solution of the following stochastic differential equation*

$$\begin{aligned} dX_t &= \sum_{i=1}^p A_i(X_t)\circ dM_t^i \\ X_0 &= g. \end{aligned} \tag{3.5}$$

In particular, if $\int_0^t \theta\circ dX_s=0$ for all $t\geqq 0$ a.e. then $X_t=g$ for all $t\geqq 0$ a.e.

Proof. Take any C^∞ function f on G. For each $g\in G$ we define $F(g)\in\mathfrak{g}^*$ by $F(g)X=(df)_g(X_g)$ where $\mathfrak{g}^*$ is a dual space of $\mathfrak{g}$. Then F is a C^∞ mapping from G into $\mathfrak{g}^*$. Hence the mapping $g\mapsto F(g)\circ\theta_g$ is a 1-form on G. From Lemma 3.1, we have

$$\begin{aligned} \int_0^t (F\circ\theta)\circ dX_s &= \sum_{i=1}^p \int_0^t F(X_s)A_i\circ dM_s^i \\ &= \sum_{i=1}^p \int_0^t (df)_{X_s}(A_i(X_s))\circ dM_s^i \\ &= \sum_{i=1}^p \int_0^t (A_i f)(X_s)\circ dM_s^i. \end{aligned}$$

On the other hand

$$f(X_t)-f(X_0)=\int_0^t df\circ dX_s=\int_0^t (F\circ\theta)\circ dX_s.$$

Hence we have

$$f(X_t)-f(X_0)=\sum_{i=1}^p \int_0^t (A_i f)(X_s)\circ dM_s^i.$$

This implies that X is a solution of (3.5).

Let M, N, L be C^∞ manifolds and $X=(X_t)$ and $Y=(Y_t)$ be semimartingales on M and N respectively such that $X_0=x_0$ and $Y_0=y_0$ a.e. Let ϕ: $M\times N\to L$ be a C^∞ mapping and α be a C^∞ 1-form on L. For each $x\in M$, we define ϕ_x: $N\to L$ by $\phi_x(y)=\phi(x,y)$. Similarly ${}_y\phi$: $M\to L$ by ${}_y\phi(x)=\phi(x,y)$ for each $y\in N$. Put $Z_t=\phi(X_t,Y_t)$. Then $Z=(Z_t)$ is an L-valued semimartingale. Let $\phi_x^*\alpha$ be a pull back of α by ϕ_x. Since a mapping $(x,y)\mapsto(\phi_x^*\alpha)_y$ is a C^∞ mapping from $M\times N$ into $T^*(N)$, $\phi_{X_t}^*\alpha$ is in $\Lambda(Y)$. Similarly ${}_{Y_t}\phi^*\alpha$ is in $\Lambda(X)$.

Lemma 3.4. *The following equality holds.*

$$\int_0^t \alpha \circ dZ_s = \int_0^t \phi^*_{X_s} \alpha \circ dY_s + \int_0^t {}_{Y_s}\phi^* \alpha \circ dX_s. \tag{3.6}$$

Proof. Let $(x^1, \ldots, x^m)$, $(y^1, \ldots, y^n)$ and $(z^1, \ldots, z^l)$ be local coordinates of M, N and L respectively. Let $X_t^i = x^i(X_t)$, $Y_t^j = y^j(Y_t)$, $Z_t^k = z^k(Z_t)$ and $\alpha = \sum_{k=1}^{l} \alpha_k dz^k$. Then

$$(\phi^*_x \alpha)_y = \sum_{k=1}^{l} \sum_{j=1}^{n} \alpha_k(\phi(x, y)) \frac{\partial \phi^k}{\partial y^j} dy^j,$$

$$({}_y\phi^* \alpha)_x = \sum_{k=1}^{l} \sum_{i=1}^{m} \alpha_k(\phi(x, y)) \frac{\partial \phi^k}{\partial x^i} dx^i.$$

By Itô's formula we have

$$\begin{aligned} Z_t^k - Z_0^k &= \phi^k(X_t^1, \ldots, X_t^m, Y_t^1, \ldots, Y_t^n) - \phi^k(X_0^1, \ldots, X_0^m, Y_0^1, \ldots, Y_0^n) \\ &= \sum_{i=1}^{m} \int_0^t \frac{\partial \phi^k}{\partial x^i}(X_s^1, \ldots, X_s^m, Y_s^1, \ldots, Y_s^n) \circ dX_s^i \\ &\quad + \sum_{j=1}^{n} \int_0^t \frac{\partial \phi^k}{\partial y^j}(X_s^1, \ldots, X_s^m, Y_s^1, \ldots, Y_s^n) \circ dY_s^j. \end{aligned}$$

Hence we have

$$\begin{aligned} \alpha \circ dZ_t &= \sum_{k=1}^{l} \alpha_k(\phi(X_t, Y_t)) \circ dZ_t^k \\ &= \sum_{k=1}^{l} \sum_{i=1}^{m} \alpha_k(\phi(X_t, Y_t)) \circ \left(\frac{\partial \phi^k}{\partial x^i}(X_t, Y_t) \circ dX_t^i \right) \\ &\quad + \sum_{k=1}^{l} \sum_{j=1}^{n} \alpha_k(\phi(X_t, Y_t)) \circ \left(\frac{\partial \phi^k}{\partial y^j}(X_t, Y_t) \circ dY_t^j \right) \\ &= ({}_{Y_t}\phi^* \alpha) \circ dX_t + (\phi^*_{X_t} \alpha) \circ dY_t. \end{aligned}$$

This completes the proof.

4. Proof of Theorem 2.1

We will prove the existence first. Let the notations be as in Sect. 2. Take an open neighborhood U of x such that there exists a trivializing diffeomorphism $\Phi: \pi^{-1}(U) \to U \times G$. For simplicity we assume that $X_t \in U$ for all $t \geqq 0$ a.e. Let u be in P as in Theorem 2.1. Put $\Phi(u) = (x, g) \in U \times G$. Define a P-valued semimartingale $\tilde{Y} = (\tilde{Y}_t)$ by $\tilde{Y}_t = \Phi^{-1}(X_t, g)$. Let $\{A_1, \ldots, A_p\}$ be a basis of $\mathfrak{g}$. Since $\int_0^t \omega \circ d\tilde{Y}_s$ is a $\mathfrak{g}$-valued semimartingale there exist semimartingales $M^1, \ldots, M^p$ such that $\int_0^t \omega \circ d\tilde{Y}_s = \sum_{i=1}^{p} M_t^i A_i$. Let $\tilde{g}_t$ be a solution of the following stochastic differential equation

$$\begin{aligned} d\tilde{g}_t &= \sum_{i=1}^{p} A_i(\tilde{g}_t) \circ dM_t^i \\ \tilde{g}_0 &= e. \end{aligned} \tag{4.1}$$

If $g_t = \tilde{g}_t^{-1}$ then (g_t) satisfies the following stochastic differential equation (see, e.g., [9])

$$\begin{aligned} dg_t &= -\sum_{i=1}^{p} (\mathrm{Ad}(g_t^{-1}) A_i)(g_t) \circ dM_t^i \\ g_0 &= e. \end{aligned} \tag{4.2}$$

Define a P-valued semimartingale $(\tilde{X}_t)$ by $\tilde{X}_t = \tilde{Y}_t g_t = \Phi^{-1}(X_t, g g_t)$. We will show that $(\tilde{X}_t)$ is a stochastic horizontal lift. It is easy to show that $\pi(\tilde{X}_t) = X_t$ for all $t \geqq 0$ a.e. We need to show that $\int_0^t \omega \circ d\tilde{X}_s = 0$. Define ϕ: $P \times G \to P$ by $\phi(v, h) = v h$. For v in P and h in G, we can define ϕ_v and ${}_h\phi$ respectively in the same way as in Lemma 3.4. Note that $\phi_v^* \omega = \theta$ and ${}_h\phi = R_h$. From Lemma 3.4 we have

$$\int_0^t \omega \circ d\tilde{X}_s = \int_0^t \theta \circ dg_s + \int_0^t R_{g_s}^* \omega \circ d\tilde{Y}_s.$$

From a property of connection form ω, $R_{g_s}^* \omega = \mathrm{Ad}(g_s^{-1}) \omega$. On the other hand, from Lemma 3.2 and (4.2) we have

$$\int_0^t \theta \circ dg_s = -\sum_{i=1}^{p} \int_0^t \mathrm{Ad}(g_s^{-1}) A_i \circ dM_s^i.$$

Combining these facts and Lemma 3.1 we obtain

$$\begin{aligned} \int_0^t \omega \circ d\tilde{X}_s &= -\sum_{i=1}^{p} \int_0^t \mathrm{Ad}(g_s^{-1}) A_i \circ dM_s^i + \int_0^t \mathrm{Ad}(g_s^{-1}) \omega \circ d\tilde{Y}_s \\ &= -\sum_{i=1}^{p} \int_0^t \mathrm{Ad}(g_s^{-1}) A_i \circ dM_s^i + \sum_{i=1}^{p} \int_0^t \mathrm{Ad}(g_s^{-1}) A_i \circ dM_s^i \\ &= 0. \end{aligned}$$

This implies that $(\tilde{X}_t)$ is a stochastic horizontal lift of X.

Next we will show the uniqueness. Suppose that $(\tilde{X}_t)$ and $(\tilde{Y}_t)$ are horizontal lifts of (X_t) such that $\tilde{X}_0 = \tilde{Y}_0 = u$ a.e. Since $\pi(\tilde{X}_t) = \pi(\tilde{Y}_t)$ for all $t \geqq 0$ a.e. there exists $g_t \in G$ such that $\tilde{X}_t = \tilde{Y}_t g_t$ for all $t \geqq 0$ a.e. We can easily check that (g_t) is a G-valued semimartingale such that $g_0 = e$ a.e. By a similar argument we have

$$0 = \int_0^t \omega \circ d\tilde{X}_s = \int_0^t \theta \circ dg_s + \int_0^t \mathrm{Ad}(g_s^{-1}) \omega \circ d\tilde{Y}_s.$$

Since $\int_0^t \omega \circ d\tilde{Y}_s = 0$, Lemma 3.1 implies that $\int_0^t \mathrm{Ad}(g_s^{-1}) \omega \circ d\tilde{Y}_s = 0$. Hence we have

$\int_0^t \theta \circ dg_s = 0$. From Lemma 3.3 we have that $g_t = e$ for all $t \geqq 0$ a.e. Thus we obtain $\tilde{X}_t = \tilde{Y}_t$ for all $t \geqq 0$ a.e. This completes the proof.

5. Examples

Here we give some examples.

Example 5.1. Let $\{P, M, G, \pi\}$ be a G-principal fibre bundle and ω be a connection form on P. Let $A_1, \ldots, A_r$ be C^∞ vector fields on M and $M^1, \ldots, M^r$ be semimartingales. We consider the solution of the following stochastic differential equation

$$\begin{aligned} dX_t &= \sum_{i=1}^r A_i(X_t) \circ dM_t^i \\ X_0 &= x \in M. \end{aligned} \tag{5.1}$$

Then the stochastic horizontal lift $(\tilde{X}_t)$ of (X_t) is given by the solution of the following stochastic differential equation

$$\begin{aligned} d\tilde{X}_t &= \sum_{i=1}^r \tilde{A}_i(\tilde{X}_t) \circ dM_t^i \\ \tilde{X}_0 &= u \in P \end{aligned} \tag{5.2}$$

where $\pi(u) = x$ and $\tilde{A}_1, \ldots, \tilde{A}_r$ are horizontal lifts of $A_1, \ldots, A_r$ respectively i.e., $\tilde{A}_i$ is a vector fields on P such that $\pi_* \tilde{A}_i = A_i$ and $\omega(\tilde{A}_i) = 0$. This fact is an easy consequence of the definition of the stochastic horizontal lift and Lemma 3.2.

Example 5.2. Let M be a d-dimensional Riemannian manifold, $O(M)$ be the orthonormal frame bundle of M, π: $O(M) \to M$ be the natural projection and ω be the Riemannian connection form. We regard an element u of $O(M)$ as a linear isomorphism from $\mathbf{R}^d$ onto $T_{\pi(u)}(M)$ which preserves the inner product. Since ω is an $\mathfrak{o}(d)$-valued 1-form ($\mathfrak{o}(d)$ is a set of all skew symmetric real matrices of degree d), we can write $\omega = (\omega_j^i)$ by its components. Let U be a coordinate neighborhood and $(x^1, \ldots, x^d)$ be a local coordinate of U. Let $(x^i, e_k^j;$ $i, j, k = 1, \ldots, d)$ be a local coordinate of $\pi^{-1}(U)$ and Γ_{jk}^i, $i, j, k = 1, \ldots, d$, be components of the Riemannian connection. Then the connection form $\omega = (\omega_j^i)$ is given by

$$\omega_j^i = \sum_{k=1}^d f_k^i \left(de_j^k + \sum_{l,m=1}^d \Gamma_{ml}^k e_j^l dx^m \right), \tag{5.3}$$

where (f_j^i) is the inverse matrix of (e_j^i). Let $X = (X_t)$ be the Brownian motion on M. Malliavin [11] defined the horizontal Brownian motion $\tilde{X}_t = [X_t, E_t]$ by the solution of

$$dE_j^i(t) = -\sum_{k,l=1}^d \Gamma_{lk}^i(X_t) E_j^k(t) \circ dX_t^l. \tag{5.4}$$

By noting that

$$\omega_j^i \circ d\tilde{X}_t = \sum_{k=1}^{d} f_k^i(X_t) \circ \left\{ dE_j^k(t) + \sum_{l,m=1}^{d} \Gamma_{ml}^k(X_t) E_j^l(t) \circ dX_t^m \right\}$$

we can easily see that $\omega_j^i \circ d\tilde{X}_t = 0$ for any $i, j = 1, \dots, d$, if and only if (5.4) holds. Thus these definitions are equivalent.

Example 5.3. Let the notations be as above. Let $X = (X_t)$ be an M-valued semimartingale such that $X_0 = x \in M$ and $\tilde{X} = (\tilde{X}_t)$ be a stochastic horizontal lift of X such that $\tilde{X}_0 = u \in O(M)$ with $\pi(u) = x$. Let η be the canonical 1-form on $O(M)$, i.e., $\mathbf{R}^d$-valued 1-form defined by $\eta_v(\xi) = v^{-1}(\pi_*)_v(\xi)$ for $v \in O(M)$ and $\xi \in T_v(O(M))$. Define semimartingales $M^1, \dots, M^d$ by $(M_t^1, \dots, M_t^d) = \int_0^t \eta \circ d\tilde{X}_s$. Let $L_1, \dots, L_d$ be standard horizontal vector fields on $O(M)$, i.e., $\omega(L_i) = 0$ and $\eta(L_i) = (0, \dots, \overset{i}{1}, \dots, 0)$ for $i = 1, \dots, d$. Then $\tilde{X}$ satisfies the following stochastic differential equation

$$\begin{aligned} d\tilde{X}_t &= \sum_{i=1}^{d} L_i(\tilde{X}_t) \circ dM_t^i \\ \tilde{X}_0 &= u. \end{aligned} \tag{5.5}$$

Conversely, if $\tilde{X}$ is a solution of (5.5) where $M^1, \dots, M^d$ are semimartingales such that $M_0^i = 0$ for $i = 1, \dots, d$, then $(\tilde{X}_t)$ is a stochastic horizontal lift of $(X_t = \pi(\tilde{X}_t))$ and $\int_0^t \eta \circ d\tilde{X}_s = (M_t^1, \dots, M_t^d)$. This fact is essentially due to Y. Yamato[2] who discussed the case X is the Brownian motion on M.

Proof. We shall prove that

$$f(\tilde{X}_t) - f(\tilde{X}_0) = \sum_{i=1}^{d} \int_0^t (L_i f)(\tilde{X}_s) \circ dM_s^i \tag{5.6}$$

for any f in $C^\infty(O(M))$. Since the components of ω_v, η_v form a basis of $T_v^*(O(M))$ for any point $v \in O(M)$, there exist C^∞ functions $E\colon O(M) \to \mathfrak{o}(d)^*$ and $F\colon O(M) \to \mathbf{R}^*$ such that $(df)_v = E(v)\,\omega_v + F(v)\,\eta_v$. Then from Lemma 3.1 we have

$$\begin{aligned} f(\tilde{X}_t) - f(\tilde{X}_0) &= \int_0^t df \circ d\tilde{X}_s \\ &= \int_0^t (E\,\omega + F\,\eta) \circ d\tilde{X}_s \\ &= \sum_{i=1}^{d} \int_0^t F_i(\tilde{X}_s) \circ dM_s^i \\ &= \sum_{i=1}^{d} \int_0^t (L_i f)(\tilde{X}_s) \circ dM_s^i. \end{aligned}$$

Thus we have (5.6). Converse is easily obtained from Lemma 3.2.

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[2] Private communication

Example 5.4. We will discuss here the stochastic parallel displacements of tensor fields. Let M be a d-dimensional C^∞ manifold, $L(M)$ be its linear frame bundle and π: $L(M)\to M$ be the natural projection. Then $\{L(M), M, GL(d,\mathbf{R}), \pi\}$ is a $GL(d,\mathbf{R})$ principal fibre bundle. We regard an element u of $L(M)$ as a linear isomorphism from $\mathbf{R}^d$ onto $T_{\pi(u)}(M)$. Assume that a connection form ω is given. We denote the tensor bundle of type (p,q) by $T_q^p(M)$ and the set of all C^∞ cross sections of $T_q^p(M)$ by $\Gamma^\infty(T_q^p(M))$. We consider the following stochastic differential equation

$$\begin{aligned} dX_t &= \sum_{i=1}^{r} A_i(X_t)\circ dM_t^i \\ X_0 &= x, \end{aligned} \tag{5.7}$$

where $A_1,\dots,A_r$ are C^∞ vector fields on M and $M_t^1,\dots,M_t^r$ are continuous semimartingales. We denote the solution of (5.7) by $X_t(x)$. We assume that (5.7) is conservative for all $x\in M$ and the mapping $x\mapsto X_t(x)$ is a diffeomorphism of M for all $t\geqq 0$. If M is compact, then these conditions are always satisfied. Since u in $L(M)$ is a linear isomorphism from $\mathbf{R}^d$ onto $T(M)_{\pi(u)}$, u can be extended to a linear isomorphism from $\underbrace{\mathbf{R}^d\otimes\dots\otimes\mathbf{R}^d}_{p}\otimes\underbrace{(\mathbf{R}^d)^*\otimes\dots\otimes(\mathbf{R}^d)^*}_{q}$ onto $T_q^p(M)_{\pi(u)}$, i.e., $\underbrace{u\otimes\dots\otimes u}_{p}\otimes\underbrace{(u^{-1})^*\otimes\dots\otimes(u^{-1})^*}_{q}$. Hence we can regard an element u in $L(M)$ as a linear isomorphism as above. From Example 5.1 the stochastic horizontal lift $(\tilde{X}_t)$ of (X_t) is the solution of the following stochastic differential equation

$$\begin{aligned} d\tilde{X}_t &= \sum_{i=1}^{r} \tilde{A}_i(\tilde{X}_t)\circ dM_t^i \\ \tilde{X}_0 &= u \end{aligned} \tag{5.8}$$

where $\pi(u)=x$ and $\tilde{A}_1,\dots,\tilde{A}_r$ are horizontal lifts of $A_1,\dots,A_r$ respectively. We denote the solution of (5.8) by $\tilde{X}_t(u)$. Let ρ_t be a mapping from $T_q^p(M)$ into $T_q^p(M)$ defined by $\rho_t(\xi)=\tilde{X}_t(u)\circ u^{-1}(\xi)$ for ξ in $T_q^p(M)_x$ and u in $L(M)$ such that $\pi(u)=x$. ρ_t is well-defined, i.e., it does not depend on a choice of u such that $\pi(u)=x$ because $\tilde{X}_t(u)g=\tilde{X}_t(ug)$ for any $g\in GL(d,\mathbf{R})$. Then ρ_t is a bundle map such that $\pi\circ\rho_t(\xi)=X_t(\pi(\xi))$ for ξ in $T_q^p(M)$. For any $\xi\in T_q^p(M)_x$ we call $\rho_t\xi$ a stochastic parallel displacement of ξ along the path of $X_s(x)$, $0\leqq s\leqq t$. It is easy to see that this definition is equivalent to that of Itô [7].

Next we will show Itô's formulae for tensor fields which were discussed by Kunita [10]. Let A be a C^∞ vector field on M, $\tilde{A}$ be its horizontal lift and ξ be in $\Gamma^\infty(T_q^p(M))$. Define the mapping F_ξ: $L(M)\to\underbrace{\mathbf{R}^d\otimes\dots\otimes\mathbf{R}^d}_{p}\otimes\underbrace{(\mathbf{R}^d)^*\otimes\dots\otimes(\mathbf{R}^d)^*}_{q}$ by $F_\xi(u)=u^{-1}\xi_{\pi(u)}$ for $u\in L(M)$. Then it holds that $\tilde{A}F_\xi=F_{\nabla_A\xi}$, ∇_A being the covariant derivative (see Kobayashi, Nomizu [8] p.p. 115). Using this fact we have

$$\begin{aligned} F_\xi(\tilde{X}_t(u))-F_\xi(u) &= \sum_{i=1}^{r}\int_0^t \tilde{A}_i F_\xi(\tilde{X}_s(u))\circ dM_s^i \\ &= \sum_{i=1}^{r}\int_0^t F_{\nabla_{A_i}\xi}(\tilde{X}_s(u))\circ dM_s^i. \end{aligned}$$

Hence

$$\tilde{X}_t(u)^{-1}\xi_{X_t(x)} - u^{-1}\xi_x = \sum_{i=1}^{r}\int_0^t \tilde{X}_s(u)^{-1}(\nabla_{A_i}\xi)_{X_s(x)} \circ dM_s^i.$$

By operating u both hands we have

$$(\rho_t^{-1}\xi)_x - \xi_x = \sum_{i=1}^{r}\int_0^t (\rho_s^{-1}\nabla_{A_i}\xi)_x \circ dM_s^i.$$

This implies that if $\nabla_{A_i}\xi = 0$ for $i=1, \ldots, r$, then ξ is invariant under ρ_t. For example if the connection is the Riemannian connection then the Riemannian metric is invariant under the stochastic parallel displacement.

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References

1. Dynkin, E.B.: Diffusion of tensors. Soviet Math. Dokl. **9**, 532–535 (1968)
2. Ikeda, N., Manabe, S.: Integral of differential forms along the path of diffusion processes. Publ. R.I.M.S. Kyoto Univ. **15**, 827–852 (1979)
3. Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. Tokyo: Kodansha 1981
4. Itô, K.: The Brownian motion and tensor fields on Riemannian manifold. Proc. Intern. Cong. of Math. Stockholm, 536–539 (1962)
5. Itô, K.: Stochastic differentials. Appl. Math. Optim. **1**, 347–381 (1975)
6. Itô, K.: Stochastic parallel displacement. Lecture Notes in Math., **451**, pp. 1–7. Berlin-Heidelberg-New York: Springer 1957
7. Itô, K.: Brownian motion on Riemannian manifold and harmonic tensor fields (in Japanese). Sûgaku **28**, 137–146 (1976)
8. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. New York: Interscience, vol. **1** (1963), vol. **2** (1968)
9. Kunita, H.: On the decomposition of solutions of stochastic differential equations. Lecture Notes in Math., vol. **851**, pp. 213–255. Berlin-Heidelberg-New York: Springer 1981
10. Kunita, H.: Some extensions of Itô's formula. Seminaire de Probabilites XV. Lecture Notes in Math., vol. **850**. Berlin-Heidelberg-New York: Springer 1980
11. Malliavin, P.: Formules de la moyenne, calcul de perturbation et théorèmes d'annulation pour les formes harmoniques. J. Func. Anal. **17**, 247–291 (1974)

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