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On the Symmetric Wiener-Hopf Factorization for Markov Additive Processes*

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1. Introduction

The classical symmetric Wiener-Hopf factorization of a probability measure was given by Spitzer [20] and Feller [6], and has a strong connection to random walks. Namely, given the characteristic function $\psi(\theta)$ of a distribution on $(-\infty, \infty)$, there is a unique factorization

$$1-\psi(\theta)=[1-\psi_+(\theta)]\,[1-\psi_-(\theta)],$$

where ψ_+ is the characteristic function of a distribution concentrated on $[0, \infty]$ and ψ_- is the characteristic function of a distribution concentrated on $(-\infty, 0]$. This result was generalized later by Fristedt [7], Greenwood and Pitman [8], Silverstein [19], and Prabhu [16] to generators of Lévy processes. Namely, if $\phi(\theta)$ is the Lévy exponent of a Lévy process X_t and if $\lambda>0$, then

$$\lambda+\phi(\theta)=\phi_+(\lambda, \theta)\,\phi_-(\lambda, \theta)$$

where $\theta\to\phi_+(\lambda, \theta)$ is the Lévy exponent of an increasing Lévy process and $\theta\to\phi_-(\lambda, \theta)$ is that of a decreasing Lévy process. It was further shown in [7, 19], that $\phi_+(\lambda, \theta)$ and $\phi_-(\lambda, \theta)$ are the Lévy exponents of the increasing and decreasing ladder processes, respectively, associated with (X_t). Equivalently, if A is the generator of a Lévy process, then for any $f\in C^2$

$$(\theta I-A)f=(\theta_1 I-A_+)(\theta_2 I-A_-)f$$

where $\theta=\theta_1\theta_2$, A_+ is the generator of an increasing Lévy process, and A_- is that of a decreasing Lévy process (Prabhu [16]). Later, Presman [18] and Arjas and Speed [1] generalized this property to Markov additive processes in discrete time.

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It is the purpose of this paper to carry the result one step further, namely to the case of continuous time parameter Markov additive processes, where the Markovian component has a finite state space and is ergodic.

Two main tools are used for proving the factorization. One is the existence of a Lévy system for the excursion process, from a general Borel set, of a Markov process[1]. The other is the use of a dual Markov additive process, and duality relation between the potentials of the process and its dual. The use of the dual process and the Wiener-Hopf factorization enables us to obtain a duality relation between the minimum of the additive part, Y, of a Markov additive process (X, Y), and the "content" (namely, $\hat{Y}_t - \inf_{0\leqq s\leqq t} \hat{Y}_s$) of the dual process $(\hat{X}, \hat{Y})$.

In some dam and queueing models, $Y_t - \inf_{0\leqq s\leqq t} Y_s$ is the content of the dam and the virtual waiting time respectively. Thus, if $\inf_{0\leqq s\leqq t} Y_s$ decreases to a finite limit, we can use the duality relation mentioned above to obtain a limiting distribution for the content functional.

This paper is organized as follows. Section 2 is devoted to the ladder set, the set where the additive part achieves its maximum. In Sect. 3 we construct the ladder process - a Markov additive process, whose range is the ladder set, and prove the Wiener Hopf factorization. We close in Sect. 4, with some applications. We treat the maximum and content functionals and find conditions under which they have a limiting distribution.

2. The Ladder Set

We use notations of Çinlar [3-5], and Blumenthal and Getoor [2].

Let $(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, Y_t, \theta_t, P^x)$ be a perfect MAP with $(F, \mathcal{F}) = (R, \mathcal{R})$. Define

(2.1) $$M_t = \sup_{0\leqq s\leqq t} Y_s,$$

(2.2) $$M = \{t: Y_t - M_t = 0\}.$$

Note that M is closed from the right. M is called the ladder set of the additive part of the MAP. Let $\overline{M}$ be the closure of M, and define

(2.3) $$D_t = \inf\{s > t: s \in M\},$$

(2.4) $$R_t = D_t - t,$$

(2.5) $$G_t = \sup\{s \leqq t: s \in \overline{M}\},$$

(2.6) $$R = \inf\{t > 0: t \in \overline{M}\}.$$

(2.7) **Lemma.** *The random variables D_t are $\mathcal{L}_t$ stopping times.*

[1] The theory of excursions from a point, of a Markov process, was used in [19] to prove the Wiener Hopf factorization for a Lévy process

Proof. Y_t is right continuous a.s. and hence separable. Hence $M_t \in \mathscr{L}_t$, and $\sup_{t \leq u \leq s} Y_u \in \mathscr{L}_s$. Therefore

$$\{\omega: D_t(\omega) > s\} = \{\omega: M_t(\omega) > \sup_{t < u \leq s} Y_u\} \in \mathscr{L}_s.$$

Let

$$\mathscr{L}_{D_s} = \{A \in \mathscr{L}: A \cap \{D_s \leq t\} \in \mathscr{L}_t\}. \tag{2.8}$$

We shall now prove the regeneration property at the ladder set.

(2.9) **Theorem.** *Let S be an $\mathscr{L}_{D_t}$ stopping time, then for all $A \in \mathscr{L}_{D_S}$, $B \in \mathscr{L}$, μ finite measure on $\mathscr{E}$.*

$$P^\mu(A \cap \theta_{D_S}^{-1} B) = \int_A P^{X_{D_S}}(B)\, dP^\mu. \tag{2.10}$$

Proof. Using the right continuity of Y_t it can be easily verified that D_t is right continuous. $\mathscr{L}_t$ is right continuous by the definition of MAP. Therefore, it follows from Maisonneuve [12] that D_S is an $\mathscr{L}_t$ stopping time and both sides of (2.10) are defined. Since D_S is an $\mathscr{L}_t$ stopping time, (2.10) follows from the strong Markov property proved in [4].

In Maisonneuve's [12] treatment of regenerative systems, the systems involved were time homogeneous. In this context, time homogeneity means that, for a.e. ω, $M(\theta_t \omega) = (M(\omega) - t) \cap R_+$ for all $t \geq 0$, or equivalently, that R is a perfect terminal time [2]. In our case time homogeneity will hold for all $t \in M$. Suppose $t \in M(\omega)$, then

$$\begin{aligned} M(\theta_t \omega) &= \{s: Y_s(\theta_t \omega) \geq Y_u(\theta_t \omega) \text{ for all } 0 \leq u \leq s\} \\ &= \{s: Y_{t+s}(\omega) - Y_t(\omega) \geq Y_{t+u}(\omega) - Y_t(\omega) \text{ for all } 0 \leq u \leq s\} \\ &= \{s: Y_{t+s}(\omega) > Y_{t+u}(\omega) \text{ for all } 0 \leq u \leq s\}. \end{aligned}$$

But since $t \in M(\omega)$, $Y_t(\omega) \geq Y_s(\omega)$ for all $0 \leq s \leq t$, and hence, the last set is equal to

$$\{s > t: Y_s(\omega) \geq Y_u(\omega) \text{ for all } 0 \leq u \leq s\} = (M(\omega) - t) \cap R_+.$$

If, on the other hand, t is not in $M(\omega)$, the last equality does not hold, because then the fact that $Y_{t+s}(\omega) \geq Y_{t+u}(\omega)$ for all $0 \leq u \leq s$ does not imply that $Y_{t+s}(\omega) \geq Y_u(\omega)$ for all $0 \leq u \leq t+s$.

From now on we shall restrict our attention to MAP's in which the X component is a regular step process. For each $x \in E$.

$$P^x\{X_t = x \text{ for all } 0 \leq t \leq s\} = e^{-c(x)s} \tag{2.11}$$

where $c(x)$ is some finite positive number. Let $\hat{L}_t^x$ be the amount of time X has spent at x during $[0, t]$, and $\hat{\tau}_t^x$ be its inverse. We define now a new process as follows:

$$V_t = Y_{\hat{\tau}_t^x} - \sum_{s \leq t} 1_{\{\hat{\tau}_t^x \neq \hat{\tau}_{t-}^x\}} (Y_{\hat{\tau}_t^x} - Y_{\hat{\tau}_{t-}^x}) = \int_0^{\hat{\tau}_t^x} 1_x(X_s)\, dY_s. \tag{2.12}$$

Since $X_{\hat{\tau}_t^x}=x$ a.s., this process is a Lévy process. We call it the Lévy process associated with x.

(2.13) *Definition.* A point $x \in E$ is regular for M if

$$P^x\{R=0\}=1.$$

Note. We assume here $Y_0=0$ a.s.

(2.14) **Lemma.** *Let X_t be a regular step process. A point x is regular for M if and only if*

$$R^x=\inf\{t>0\colon V_t \geqq V_0\}=0 \qquad P^x-\text{a.s.}$$

Proof.

$$\begin{aligned} P^x\{R=0\} &= \lim_{\varepsilon\downarrow 0} P^x\{Y_t \geqq Y_0 \text{ for some } 0<t<\varepsilon\} \\ &= \lim_{\varepsilon\downarrow 0} P^x\{Y_t \geqq Y_0 \text{ for some } 0<t<\varepsilon, \hat{\tau}_\varepsilon^x=\varepsilon\} \\ &\quad + \lim_{\varepsilon\downarrow 0} P^x\{Y_t \geqq Y_0 \text{ for some } 0<t<\varepsilon, \hat{\tau}_\varepsilon^x>\varepsilon\}. \end{aligned}$$

The second summand is smaller than

$$\lim_{\varepsilon\downarrow 0} P^x\{\hat{\tau}_\varepsilon^x>\varepsilon\}=0 \qquad \text{because } c(x)<\infty.$$

Thus

$$\begin{aligned} P^x\{R>0\} &= \lim_{\varepsilon\downarrow 0} P^x\{V_t \geqq V_0 \text{ for some } 0<t<\varepsilon, \hat{\tau}_\varepsilon^x=\varepsilon\} \\ &\geqq \lim_{\varepsilon\downarrow 0} [P^x\{V_t \geqq V_0 \text{ for some } 0<t<\varepsilon\} - P^x\{\hat{\tau}_\varepsilon^x>\varepsilon\}] \\ &= \lim_{\varepsilon\downarrow 0} P^x\{V_t \geqq V_0 \text{ for some } 0<t<\varepsilon\} = P^x\{R^x=0\} \end{aligned}$$

and

$$P^x\{R=0\} \leqq \lim_{\varepsilon\downarrow 0} P^x\{V_t \geqq V_0 \text{ for some } 0<t<\varepsilon\} = P^x\{R^x=0\}.$$

Hence

$$P^x\{R=0\}=P^x(R^x=0)=0 \qquad \text{or } 1$$

or equivalently, the property of a point being regular for the ladder set is a property of the Lévy process associated with that point.

In order to construct the ladder process, one needs to construct an additive functional that increases on M. This will be done in the next section using excursion theory. We shall work with the canonical realization of the process $\{X_t, Y_t, U_t\colon t \geqq 0\}$ where $U_t=Y_t-M_t$, and its excursions from the set $E \times R \times \{0\}$.

3. Ladder Process and the Wiener-Hopf Factorization

Let (X, Y) be a MAP for which the Markov part X, is ergodic, has finite state space, and Y takes values in $\mathbb{R}$. Define

(3.1) $$U_t = Y_t - M_t.$$

The process $\{X_t, Y_t, U_t: t \geqq 0\}$ is a Hunt process on the space $(E \times R \times R, \mathscr{E} \times \mathscr{R} \times \mathscr{R})$, whose transition semi-group is given by

(3.2) $$P_t f(x, y, u) = P^x[f(X_t, Y_t + y, Y_t - (M_t - u)^+ - u)].$$

We shall work with its canonical realization. Measurability, predictability, etc. will be with respect to the canonical σ-algebras. The regeneration set defined above is

(3.3) $$M = \{t: U_t = 0\}$$

Denote by

(3.4) $$\tilde{F} = \{(x, y, u): P^{x,y,u}\{R = 0\} = 1\}.$$

We note that $(x, y, u) \in \tilde{F}$ if, and only if, $u = 0$ and x is regular for $\bar{M}$. Therefore

(3.5) $$\tilde{F} = F \times R \times \{0\}$$

where

(3.6) $F = \{x \in E: x$ regular for $M\}$ (regularity defined in (2.13)).

Define

(3.7) $$G = \{t: R_{t-} = 0, R_t > 0\} \qquad \text{where } R_{0-} = 0.$$

G is the set of left endpoints of intervals that are contiguous to M. Let

(3.8) $$G_1 = \{t \in G: x_{t-} \in F\},$$

(3.9) $$G_2 = \{t \in G: t \in M, X_t \notin F\}.$$

The following theorem follows from Maissoneuve [13] (Theorems 4.1 and 9.2)

(3.10) **Theorem.** *There exists a continuous additive functional K, with 1-potential smaller than 1, carried by $F \times R \times \{0\}$, and a kernel $\tilde{P}$ from $(E \times R \times R, \mathscr{E} \times \mathscr{R} \times \mathscr{R})$ into $(\Omega, \mathscr{M}^*)$ satisfying $\tilde{P}^{x,y,u}\{R = 0\} = 0$ and $\tilde{P}^{x,y,u}(1 - e^{-R}) \leqq 1$ for all (x, y, u) such that*

(3.11) $$E^{\cdot}\Big[\sum_{s \in G_1} e^{-s} Z_s f(\theta_s)\Big] = E^{\cdot}\left[\int_0^{\infty} e^{-s} Z_s \tilde{P}^{X_s, Y_s, U_s}(f)\, dK_s\right]$$

for all positive predictable processes Z_s and all positive measurable functions f.

We note that since $\{X_t, U_t: t \geqq 0\}$ is also a Hunt process, one can show, repeating the proof of [13], that K_t is an additive functional of (X, U), and if f is measurable with respect to $\mathscr{U}$, the usual completion of $\sigma\{X_s, U_s: s \geqq 0\}$, then

(3.12) $$\tilde{P}^{x,y,0} f = \tilde{P}^{x,0} K a \cdot e$$

where $\tilde{P}^{x,u}$ are transition kernels defined for (X, U). Let

(3.13) $$M_1 = \{t: t \in M, X_t \in F\},$$

(3.14) $$\bar{R}=\inf\{t>0, t\in M_1\}.$$

Using the quasi left continuity of the MAP (X, Y), one can show that all stopping times T such that $[T]\in \overline{M}_1 - M_1$ are totally inaccessible. Further, M_1 defined above has no isolated points. Applying (3.11) to $f(\omega)=1-e^{-\bar{R}(\omega)}$ we get

(3.15) $$\begin{aligned} E^{\cdot}(e^{-\bar{R}}) &= E^{\cdot}\sum_{s\in G_1} e^{-s}(1-e^{-\bar{R}_s}) + E^{\cdot}\int_0^{\infty} e^{-s} 1_{M_1}(s)\,ds \\ &= E^{\cdot}\int_{s=0}^{\infty} e^{-s}\tilde{P}^{X_s, 0}(1-e^{-\bar{R}})\; dK_s + E^{\cdot}\int_0^{\infty} e^{-s} 1_{M_1}(s)\,ds. \end{aligned}$$

It was shown in [12] that $\tilde{P}^{(x,0)}(1-e^{-\bar{R}})=1$ except on a set of K potential 0.

Define

(3.16) $$L_t^c=\int_0^t 1_{M_1}(s)\,ds + K_t$$

L_t^c is the local time of equilibrium of order 1 at M_1 and increases on M_1. Let

(3.17) $$M_2=\{t: t\in M, X_t\notin F\}.$$

It was shown in [13] that the set

$$\{(t, \omega): t\in G(\omega), X_t(\omega)\notin F\}$$

is well measurable. Therefore there exist stopping times T_n such that this set is equal to $\bigcup [T_n]$ (where $[T_n]$ are the graphs of T_n). Define now

(3.18) $$S_n=\begin{cases} T_n & \text{on } \{U_{T_n}=0\} \\ \infty & \text{otherwise} \end{cases}$$

then $\{(t, \omega): t\in M(\omega), X_t(\omega)\notin F\} = \bigcup_{n=1}^{\infty} [S_n]$.

Therefore one can define

(3.19) $$L_t^d = \sum_{n: S_n<t} \lambda(X_{S_n}) J_n$$

where $J_1, J_2 \dots$ are i.i.d. exponential (1) random variables and

(3.20) $$\lambda(x)=E^x(1-e^{-R}).$$

To make this functional additive and measurable, one has to enlarge the probability space to include the exponential random variables. This was done in [11] in some detail, and for the sake of conciseness will be omitted in this paper. Let

(3.21) $$L_t=L_t^c+L_t^d$$

L_t is an additive functional that increases on M. Let τ_s be its inverse,

$$\text{(3.22)}\qquad \begin{aligned} \mathcal{M}_t^0 &= \sigma\{X_{\tau_s}, Y_{\tau_s}, \tau_s : s \leqq t\} \\ \mathcal{M}^0 &= \sigma\{X_{\tau_s}, Y_{\tau_s}, \tau_s : s \geqq 0\} \end{aligned}$$

and $\mathcal{M}'$, $\mathcal{M}_t'$ the usual completions of those σ-algebras. One can easily verify that $(\Omega, \mathcal{M}', \mathcal{M}_t', X_{\tau_t}, Y_{\tau_t}, \tau_t, \theta_{\tau_t}, P^x)$ is a MAP. We call it the ladder process associated with the MAP (X, Y).

Since X is ergodic, there exists a unique invariant measure π for the process X. Consider the MAP $(\hat{X}, \hat{Y})$ whose transition function is given by

$$\text{(3.23)}\qquad \hat{P}^i(\hat{X}_t = j, \hat{Y}_t \in B) = \frac{\pi(j)}{\pi(i)} P^j\{X_t = i, Y_t \in -B\}.$$

Following Arjas and Speed [1], we call this process the dual MAP. In the same manner we have constructed the ladder process for (X, Y) we construct a ladder process $\hat{X}_{\hat{\tau}_s}, \hat{Y}_{\hat{\tau}_s}, \hat{\tau}_s$ for $(\hat{X}, \hat{Y})$. We call it the ladder process of the dual process.

Having constructed the ladder process we define the following operators

$$\text{(3.24)}\qquad A_\theta f(j) = \lim_{t \downarrow 0} \frac{f(j) - E^j(e^{i\theta Y_t} f(X_t))}{t},$$

$$\text{(3.25)}\qquad U_{\lambda,\theta} f(j) = E^j \int_0^\infty e^{-\lambda t} e^{i\theta Y_t} f(X_t)\, dt,$$

$$\text{(3.26)}\qquad H_{\lambda,\theta} f(j) = \lim_{s \downarrow 0} \frac{f(j) - E^j(e^{-\lambda \tau_s} e^{i\theta Y_{\tau_s}} f(X_{\tau_s}))}{s},$$

$$\text{(3.27)}\qquad \hat{H}_{\lambda,\theta} f(j) = \lim_{s \downarrow 0} \frac{f(j) - \hat{E}^j(e^{-\lambda \hat{\tau}_s} e^{i\theta \hat{Y}_{\hat{\tau}_s}} f(\hat{X}_{\hat{\tau}_s}))}{s}.$$

We assume here that $f(\Delta) = 0$, and hence we do not need the condition $1_{\{\tau_s \neq \infty\}}$ in (3.26) and (3.27). Note that since $e^{i\theta y} f(j)$ is in the domain of the infinitesimal generator of the original process (3.24) is well defined. Similarly $f(j)\, e^{i\theta y - \lambda t}$ is in the domain of the infinitesimal generators of both $(X_{\tau_s}, Y_{\tau_s}, \tau_s)$ and $(\hat{X}_{\hat{\tau}_s}, \hat{Y}_{\hat{\tau}_s}, \hat{\tau}_s)$, and therefore (3.26), (3.27) are well defined. We now state the main result of this paper.

(3.28) **Theorem.** (*The symmetric Wiener-Hopf factorization*). *Let* $(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, Y_t, \theta_t, P^x)$ *be a perfect MAP, where* X_t *is defined on a finite state space and is ergodic. Let* $(\Omega, \hat{\mathcal{M}}, \hat{\mathcal{M}}_t, \hat{X}_t, \hat{Y}_t, \hat{\theta}_t, P^x)$ *be its dual process. Then for all* $\lambda > 0$, $\theta \in R$

$$\lambda I - A(\theta) = \hat{H}^*_{\lambda,\theta} \Lambda(\lambda)^{-1} H_{\lambda,\theta}$$

where Λ *is an invertible operator that does not depend on* θ, *and* H^* *is the adjoint of the operator* H *with respect to the invariant measure* π.

The proof of this Theorem consists of two main steps. The first one uses Theorem (3.10) to express the operator $U_{\lambda,\theta}$ as a product of two operators. As functions of θ, the first one will be the Fourier transform of a finite measure concentrated on $(-\infty, 0]$ and the other the Fourier transform of a finite measure concentrated on $[0, \infty)$. Silverstein [19] has used Theorem (3.10) in a similar fashion to prove the Wiener-Hopf factorization for Lévy processes. The second step identifies those operators as the resolvent of the ladder processes and the adjoint of the resolvent of the dual process.

Define

$$dJ_t = \sum_{s \in G_2} \lambda(X_s)\, 1_{\{s\}}(dt) \tag{3.29}$$
$$dB_t^0 = dJ_t + dL_t^c$$

where $\lambda(x)$ is defined in (3.20).

Note that

$$\begin{aligned} G_1 &= \{t\colon t \in G, X_{t-} \in F\} \\ &= \{t\colon t \in G, X_{t-} \in F, X_t \in F\} \cup \{t\colon t \in G, X_{t-} \in F, X_t \notin F\}. \end{aligned} \tag{3.30}$$

The first set in the union was denoted in [13] by G^r. We call the second G_1^i. Note that $G_1^i \cap G_2$ is not necessarily empty. To avoid counting points that are in it twice, we shall have to elaborate more on the structure of $(K, \tilde{P})$ of Theorem (3.10).

The following was shown in [13].

(i) There exists a continuous additive functional (CAF) A and a transition kernel $\tilde{P}_1$ satisfying the same conditions as $\tilde{P}$ of Theorem (3.10), such that

$$E^{\cdot} \sum_{s \in G^r} e^{-s}(1 - e^{-R(\theta_s)}) f(\theta_s) = E^{\cdot} \int_0^\infty e^{-s} \tilde{P}_1^{X_s, 0}((1 - e^{-R}) f)\, dA_s. \tag{3.31}$$

(ii) There exists a CAF B, and a transition kernel $q((x,u), \cdot)$ from $(E \times R, \mathscr{E} \times \mathscr{R})$ into itself, such that

$$\begin{aligned} &E^{\cdot} \sum_{s \in G_1^i} e^{-s}(1 - e^{-R(\theta_s)}) f(\theta_s) \\ &= E^{\cdot} \int_0^\infty e^{-s} \int_{E \times R \setminus F \times \{0\}} q[(X_s, 0), dx', dy']\, E^{x', y'}((1 - e^{-R}) f)\, dB_s. \end{aligned} \tag{3.32}$$

We now let h, k, l be the derivatives of

(i) $\int_0^t \tilde{P}_1^{X_s, 0}(1 - e^{-R})\, dA_s$,

(ii) $\int_0^t \int_{E \times R \setminus F \times \{0\}} q[(X_s, 0), dx', dy')\, E^{x', y'}(1 - e^{-R})\, dB_s$,

(iii) $C_t = m[(0, t) \cap M]$ where m is the Lebesgue measure on $\mathbb{R}$,

with respect to L_t^c, respectively. Let $P^{\cdot}$ be the transition kernel of (X, U), then $\tilde{P}$ defined in (3.12), (3.13) is equal to

$$h(x, 0)\, \tilde{P}_1((x, 0), \cdot) + k(x, 0)\, q((x, 0), P^{\cdot}(\cdot)). \tag{3.33}$$

To avoid counting points in $G_1^i \cap G_2$ twice we redefine $q((x,0),(E\setminus F)\times\{0\})$ to be 0 for all x. We assume from now on that $\tilde{P}$ is defined by (3.33) with q adjusted as above. Let

$$Q_t(x,g)=\begin{cases}\dfrac{E^{x,0}[g(X_t,U_t)\times 1_{\{R>t\}}]}{E^x(1-e^{-R})} & x\notin F\\ \tilde{P}^{(x,0)}[g(X_t,U_t)\,1_{\{R>t\}}] & x\in F.\end{cases} \tag{3.34}$$

Note that

$$M_t=\begin{cases}Y_{G_t-} & \text{if } G_t\notin G_2\\ Y_{G_t} & \text{if } G_t\in G_2.\end{cases} \tag{3.35}$$

We now use the last exit decomposition which follows from (3.10) to obtain

$$\begin{aligned}E^x[e^{i\theta Y_t}f(X_t)]&=E^x[e^{i\theta Y_t}f(X_t)\,1_M(t)]\\&\quad+E^x\{e^{i\theta Y_t}f(X_t)\,1_{\{G_t<t\}}\}\\&=E^x[e^{i\theta Y_t}f(X_t)\,1_M(t)]\\&\quad+E^x(e^{i\theta Y_{G_t-}}f(X_t)\,e^{i\theta U_t}\,1_{\{G_t<t,\,G_t\notin G_2\}})\\&\quad+E^x[e^{i\theta Y_{G_t}}f(X_t)\,e^{i\theta U_t}\,1_{\{G_t<t,\,G_t\in G_2\}}]\\&=E^x\int_0^t e^{i\theta Y_s}Q_{t-s}(X_s,h)\,dB_s^0+E^x(e^{i\theta Y_t}f(X_t)\,1_M(t))\end{aligned}$$

where $h(x,u)=f(x)\,e^{i\theta u}$.

We first note that since the exponential random variables were independent of the MAP

$$E^{\cdot}\int_0^t e^{i\theta Y_s}Q_{t-s}(X_s,h)dJ_s=E^{\cdot}\int_0^t e^{i\theta Y_s}Q_{t-s}(X_s,h)dL_t^d$$

hence

$$E^{\cdot}(e^{i\theta Y_t}f(X_t)\,1_{G_t<t})=E^{\cdot}\int_0^t e^{i\theta Y_s}Q_{t-s}(X_s,h)\,dL_s. \tag{3.36}$$

We now go back to the resolvent $U_{\lambda,\theta}f$

$$\begin{aligned}U_{\lambda,\theta}f(i)&=E^i\int_0^\infty e^{-\lambda t}e^{i\theta Y_t}f(X_t)\,dt\\&=E^i\int_0^\infty e^{-\lambda t}e^{i\theta Y_t}1_M(t)f(X_t)\,dt+E^i\int_0^\infty e^{-\lambda t}\left[\int_0^t e^{i\theta Y_s}Q_{t-s}(X_s,h)\,dL_s\right]dt\\&=E^i\int_0^\infty e^{-\lambda t}e^{i\theta Y_t}\,l(X_t,0)f(X_t)\,dL_t\\&\quad+E^i\int_0^\infty e^{-\lambda t}\left[\int_{s=0}^t e^{i\theta Y_s}Q_{t-s}(X_s,h)\,dL_s\right]dt.\end{aligned} \tag{3.37}$$

We treat the two summands in the last expression separately. Using the time change $u=L_t$ the first one is equal to

$$E^i \int_0^\infty e^{-\lambda\tau_s} e^{i\theta Y_{\tau_s}} l(X_{\tau_s}, 0) f(X_{\tau_s})\, ds.$$

As for the second summand, we use the fact that $(t,x)\to Q_t(x,\cdot)$ is jointly universally measurable, the fact that $\int_0^\infty e^{-\lambda u} Q_u(x,1)\, du \leq \frac{1}{\lambda}$ and Fubini's theorem to obtain:

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \int_{s=0}^t e^{i\theta Y_s} Q_{t-s}(X_s, h)\, dL_s\, dt \\
&= \int_{s=0}^\infty e^{-\lambda s} e^{i\theta Y_s} \left(\int_{t=s}^\infty e^{-\lambda(t-s)} Q_{t-s}(X_s, h)\, dt \right) dL_s \\
&= \int_{s=0}^\infty e^{-\lambda s} e^{i\theta Y_s} \int_{u=0}^\infty e^{-\lambda u} Q_u(X_s, h)\, du\, dL_s
\end{aligned}$$

and using the time change $u=L_t$ this last expression is equal to

$$\int_0^\infty e^{-\lambda\tau_s} e^{i\theta Y_{\tau_s}} \left[\int_{u=0}^\infty e^{-\lambda u} Q_u(X_{\tau_s}, h)\, du \right] ds.$$

To summarize, we have shown that

$$U_{\lambda,\theta} f(i) = E^i \int_0^\infty e^{-\lambda\tau_s} e^{i\theta Y_{\tau_s}} \Big[l(X_{\tau_s}, 0) f(X_{\tau_s}) + \int_0^\infty e^{-\lambda u} Q_u(X_{\tau_s}, h)\, du \Big]\, ds. \tag{3.38}$$

To avoid long formulae we define

$$G_{\lambda,\theta} f(i) = l(i,0) f(i) + \int_0^\infty e^{-\lambda u} Q_u(i,h)\, du. \tag{3.39}$$

We recall now the definition of $H_{\lambda,\theta} f(i)$, the infinitesimal generator of $(X_{\tau_s}, Y_{\tau_s}, \tau_s)$ applied to $e^{i\theta y} e^{-\lambda t} f(j)$. This operator on $\mathbb{R}^m$ is invertible and its inverse is given by the resolvent

$$E^i \int_0^\infty e^{-\lambda\tau_s} e^{i\theta Y_{\tau_s}} f(X_{\tau_s})\, ds. \tag{3.40}$$

This resolvent is finite for $\lambda > 1$, by the choice of the local time, and using the fact that E is finite we can prove that it is finite for all $\lambda > 0$.

Combining (3.38), (3.39) and (3.40) we obtain the following

(3.14) **Lemma.**

$$U_{\lambda,\theta} f(x) = H_{\lambda,\theta}^{-1} G_{\lambda,\theta} f(x).$$

Note that $\theta \to H_{\lambda,\theta}^{-1} f$ is the Fourier transform of a finite measure concentrated on $[0, \infty)$, and $\theta \to G_{\lambda,\theta} f$ is the Fourier transform of a finite measure concentrated on $(-\infty, 0]$. We also note that

$$\theta \to \int_0^\infty e^{-\lambda u} Q_u(X_s, h)\, du$$

is a Fourier transform of a measure that has no mass at 0.

Remark. Using the identity

$$\frac{1}{s}\left[U_{\lambda,\theta} f(x) - E^x\left(e^{\{-\lambda \tau_s + i\theta Y_{\tau_s}} U_{\lambda,\theta} f(X_{\tau_s})\right)\right]$$

$$= \frac{1}{s} E^x \int_0^{\tau_s} e^{\{-\lambda t + i\theta Y_t\}} f(X_t)\, dt$$

for all $s \geqq 0$, and the fact that the limit as $s \to 0$ on the left side exists, one can show that $G_{\lambda,\theta} f(x)$, defined above, is equal to

$$\lim_{s \downarrow 0} \frac{1}{s} E^x \int_0^{\tau_s} e^{\{-\lambda t + i\theta Y_t\}} f(X_t)\, dt. \tag{3.42}$$

We now turn to the dual process. Let

$$\hat{U}_{\lambda,\theta} f(x) = \hat{E}^x \int_0^\infty e^{-\lambda t} e^{i\theta \hat{Y}_t} f(\hat{X}_t)\, dt. \tag{3.43}$$

Then

$$\hat{U}_{\lambda,\theta} = U_{\lambda,\theta}^* \tag{3.44}$$

where

$$U_{\lambda,\theta}^*(i,j) = U_{\lambda,\theta}^* 1_{\{i\}}(j) = \frac{\pi(j)}{\pi(i)} \overline{U}_{\lambda,\theta} 1_{\{j\}}(i).$$

($\bar{z}$ being to complex conjugate of z). The analogue of (3.14) in this case is

$$\hat{U}_{\lambda,\theta} = \hat{H}_{\lambda,\theta}^{-1} \hat{G}_{\lambda,\theta}, \tag{3.45}$$

where $\hat{H}_{\lambda,\theta}, \hat{G}_{\lambda,\theta}$ are defined for the dual process as $H_{\lambda,\theta}$ and $G_{\lambda,\theta}$ were defined for the original process. Since all operators here are invertible we get from (3.41), (3.44) and (3.45)

$$G_{\lambda,\theta} \hat{H}_{\lambda,\theta}^* = H_{\lambda,\theta} \hat{G}_{\lambda,\theta}^*. \tag{3.46}$$

We note that an equation similar to this appears in Arjas and Speed's [1] proof. In the discrete time parameter case the corresponding operators were Fourier transforms of finite measures in θ, i.e. all entries in the corresponding matrices were Fourier transforms. As we shall see below, this is not so in our case, and this causes part of the difficulty in passing from the discrete time parameter to the continuous time parameter case.

We return now to the operators $H_{\lambda,\theta}$. $H_{\lambda,\theta}f(j)$ is the value of the infinitesimal generator of $(X_{\tau_s}, Y_{\tau_s}, \tau_s)$ applied to $f(j)\,\mathrm{e}^{\{i\theta y-\lambda t\}}$. We now use a result of Çinlar [5] for this generator to get:

$$(3.47)\quad \begin{aligned} H_{\lambda,\theta}(j,j) &= H_{\lambda,\theta}1_{\{j\}}(j) = k_j + \lambda\bar{a}(j) - i\theta\bar{b}(j) - v^j(\infty) \\ &\quad + \int_{t=0}^{\infty}\int_{y=0}^{\infty}(1-\mathrm{e}^{\{-\lambda t+i\theta y\}})\,v^j(dt\times dy) \\ &= k_j + \lambda\bar{a}(j) - i\theta\bar{b}(j) - v^j(\infty) + \int_{y=0}^{\infty}(1-\mathrm{e}^{i\theta y})\,v_\lambda^j(dy) \\ &\quad + \int_{y=0}^{\infty}\int_{t=0}^{\infty}(1-\mathrm{e}^{-\lambda t})\,v^j(dt\times dy) \end{aligned}$$

where $v^j(dt\times dy)$ are Lévy measures on R_+^2 and $v_\lambda^j(dx)$ is the Lévy measure given by

$$(3.48)\qquad v_\lambda^j(dx) = \int_0^{\infty}\mathrm{e}^{-\lambda t}\,v^j(dt\times dx).$$

This yields

$$(3.49)\qquad \begin{aligned} H_{\lambda,\theta}(j,j) &= A_\lambda(j) - \int_{y=a}^{\infty}\mathrm{e}^{i\theta y}\,v_\lambda^j(dy) - i\theta[\bar{b}(j) \\ &\quad + \int_{y=0}^{a}\mathrm{e}^{i\theta y}\,v_\lambda^j(y,a]\,dy \end{aligned}$$

where

$$\bar{A}_\lambda(j) = k_j + \lambda\bar{a}(j) + \int_{t=0}^{\infty}\int_{y=0}^{\infty}(1-\mathrm{e}^{-\lambda t})\,v^j(dt\times dy) - v^j(\infty).$$

For $j \neq k$

$$(3.50)\qquad H_{\lambda,\theta}(j,k) = H_{\lambda,\theta}1_{\{k\}}(j) = -k_{jk}\int_{y=0}^{\infty}\int_{t=0}^{\infty}\mathrm{e}^{\{-\lambda t+i\theta y\}}F_{jk}(dt\times dy)$$

where

$$k_{jk} = \lim_{t\downarrow 0}\frac{P^j(X_{\tau_t}=k)}{t},\qquad j\neq k,$$

$$k_j = -\sum_{k\neq j}k_{jk},$$

and $F_{jk}(t,y)$ is the distribution of the jump size of (τ_s, Y_{τ_s}) due to a jump of X_{τ_s} from j to k.

For the dual process

(3.51)
$$(\hat{H}_{\lambda,\theta})_{jj} = \bar{\bar{A}}_\lambda(j) - \int_{y=a}^{\infty} e^{i\theta y}\, \mu_\lambda^j(dy) - i\theta\left[\bar{\bar{b}}(j) + \int_{y=0}^{a} e^{i\theta y}\, \mu_\lambda^j(y, a]\, dy\right]$$

$$\hat{H}_{\lambda,\theta})_{jk} = -\hat{k}_{jk} \int_{y=0}^{\infty}\int_{t=0}^{\infty} e^{\{-\lambda t + i\theta y\}}\, \hat{F}_{jk}(dt \times dy).$$

(3.52) **Lemma.** *There exists an invertible operator Λ_λ on R^m, which does not depend on θ, such that*

$$G_{\lambda,\theta} f = H^{*-1}_{\lambda,\theta} \Lambda_\lambda f. \tag{3.53}$$

Proof. Let $\bar{\mu}^j(A) = \mu^j(-A)$, and $\bar{F}_{jk}(dt \times A) = \hat{F}_{jk}(dt \times -A)$. Substituting (3.49), (3.50), (3.51) into (3.46), we get for every k and j

$$\bar{F}(\theta) - i\theta \bar{K}(\theta) = \bar{\bar{F}}(\theta) + i\theta \bar{\bar{K}}(\theta) \tag{3.54}$$

where

(3.55)
$$\bar{F}(\theta) = \sum_{m \neq j} (H_{\lambda,\theta})_{jm} (\hat{G}^*_{\lambda,\theta})_m + (\hat{G}^*_{\lambda,\theta})_{jk} A_\lambda(j) - (\hat{G}^*_{\lambda,\theta})_{jk} \int_{y=a}^{\infty} e^{i\theta y}\, v_\lambda^j(dy).$$

$$\bar{K}(\theta) = (\hat{G}^*_{\lambda,\theta})_{jk}\left[\bar{b}(j) + \int_{y=0}^{a} e^{i\theta y}\, v_\lambda^j(y, a]\, dy\right]$$

$$\bar{\bar{F}}(\theta) = \sum_{m \neq k} (G_{\lambda,\theta})_{jm} (\hat{H}^*_{\lambda,\theta})_m + (G_{\lambda,\theta})_{jk} \bar{A}_\lambda(k) - (G_{\lambda,\theta})_{jk} \int_{y=-\infty}^{-a} e^{i\theta y}\, \bar{\mu}_\lambda^k(dy),$$

$$\bar{\bar{K}}(\theta) = (\hat{G}_{\lambda,\theta})_{jk}\left[\bar{\bar{b}}(k) + \int_{y=-a}^{0} e^{i\theta y}\, \bar{\mu}_\lambda^k([-a, y))\, dy\right].$$

Note that $\theta \to \bar{F}(\theta)$, $\theta \to \bar{K}(\theta)$ are the Fourier transforms of finite measures concentrated on $[0, \infty)$, while $\theta \to \bar{\bar{F}}(\theta)$, $\theta \to \bar{\bar{K}}(\theta)$ are the Fourier transforms of finite measures concentrated on $(-\infty, 0]$. (measure in this context may be negative.)

Remark. $\theta \to (G_{\lambda,\theta})_{jk}$ is the Fourier transform of a measure that puts mass at 0 if and only if $j = k$ and $\{t: t \in M, X_t = k\}$ has positive Lebesgue measure. In this case, k is regular for the maximum of the dual process if and only if V_t^k (the Lévy process associated with k) is compound Poisson (see [16]). Thus $\theta \to (G_{\lambda,\theta})_{kk}$ is the Fourier transform of a measure that puts mass at 0 if and only if $\theta \to (\hat{H}^*_{\lambda,\theta})_{kk}$ is the Fourier transform of a measure concentrated on $(-\infty, 0]$.

Dividing now both sides of (3.54) by $1 + i\theta$ we get

$$\left[\frac{\bar{F}(\theta)}{1 - i\theta} - \frac{i\theta \bar{K}(\theta)}{1 - i\theta}\right] \frac{1 - i\theta}{1 + i\theta} = \frac{\bar{\bar{F}}(\theta)}{1 + i\theta} + \frac{i\theta \bar{\bar{K}}(\theta)}{1 + i\theta}. \tag{3.56}$$

Since $1/(1+i\theta)$ is the characteristic function of the negative exponential distribution, the right side of (3.56) is the Fourier transform of a finite measure on $(-\infty, 0]$. This measure puts mass at 0 only if $\theta \to \bar{\bar{K}}(\theta)$ is the Fourier transform of a measure that puts mass at 0.

Now if $\theta \to \hat{H}^*_{\lambda,\theta}(k,k)$ is the Fourier transform of a finite measure, we can take $\bar{\bar{a}}$ and $\bar{\bar{b}}(k)$ in the definition of $\bar{\bar{K}}(\theta)$ to be 0, and hence $\bar{\bar{K}}(\theta)=0$. If $\theta \to \hat{H}^*_{\lambda,\theta}(k,k)$ is not the Fourier transform of a finite measure then $\theta \to (G_{\lambda,\theta})_{jk}$ is the Fourier transform of a measure that puts no mass at 0, and hence so is $\bar{\bar{K}}(\theta)$.

The left side of (3.56) is the Fourier transform of a measure concentrated on $[0, \infty)$ convoluted with the measure

$$2\int_A e^x \times 1_{\{x \leqq 0\}} - 1_{\{x=0\}}(A).$$

Let

$$\bar{M}(\theta) = \frac{\bar{F}(\theta)}{1-i\theta} - \frac{i\theta \bar{K}(\theta)}{1-i\theta}$$

$\bar{M}(\theta)$ is the Fourier transform of a measure $\bar{M}(\cdot)$ concentrated on $[0, \infty)$. Since the right side of (3.56) is the Fourier transform of a measure concentrated on $(-\infty, 0)$ we must have

$$2\int_{x=0}^{\infty} e^{i\theta x} \int_{y=x}^{\infty} e^{(x-y)}\, d\bar{M}(y)\, dx - \int_{x=0}^{\infty} e^{i\theta x}\, d\bar{M}(y) = 0$$

for all $\theta \in R$. Using Fubini's theorem, we get

$$2\int_{y=0}^{\infty} e^{i\theta y}\, d\bar{M}(y) \int_{x=0}^{y} e^{(i\theta+1)(x-y)}\, dx = \bar{M}(\theta)$$

or equivalently

$$\frac{2}{1+i\theta}\left[\int_{y=0}^{\infty} e^{i\theta y}\, d\bar{M}(y) - \int_{y=0}^{\infty} e^{-y}\, d\bar{M}(y)\right] = \bar{M}(\theta)$$

and hence

$$\bar{M}(\theta) = \frac{2\int_{y=0}^{\infty} e^{-y}\, d\bar{M}(y)}{1-i\theta} = \frac{\text{const}}{1-i\theta}.$$

Hence

$$\bar{F}(\theta) - i\theta \bar{K}(\theta) = c(\lambda) = \bar{\bar{F}}(\theta) + i\theta \bar{\bar{K}}(\theta)$$

where c is a constant that does not depend on θ. Since the result is true for every j and k in E, we get

$$G_{\lambda,\theta} H^*_{\lambda,\theta} = H_{\lambda,\theta} G^*_{\lambda,\theta} = \Lambda_\lambda \tag{3.57}$$

where Λ_λ is an invertible operator that does not depend on θ. From (3.57), the lemma follows.

Proof of Theorem (3.28). By the previous lemma

$$G_{\lambda,\theta} = \Lambda_\lambda \hat{H}^{*\,-1}_{\lambda,\theta} \tag{3.58}$$

and substituting this into (3.14) we get

$$U_{\lambda,\theta}=H_{\lambda,\theta}^{-1}\Lambda_\lambda \hat{H}_{\lambda,\theta}^{*-1} \tag{3.59}$$

or equivalently since $U_{\lambda,\theta}$ is the inverse of $\lambda I-A_\theta$

$$\lambda I-A_\theta=\hat{H}_{\lambda,\theta}^{*}\Lambda_\lambda^{-1}H_{\lambda,\theta}$$

which is the desired factorization.

4. Maximum and Content Functionals

We now treat the maximum and content functionals, defined for MAP's for which X is ergodic and has finite state space. Let

$$M_t=\sup_{0\leqq s\leqq t} Y_s,\qquad m_t=\inf_{0\leqq s\leqq t} Y_s \tag{4.1}$$

$$W_t=Y_t-m_t.$$

In some dam models, W_t is the content of the dam at time t, and in queuing models W_t is the virtual waiting time at t. We shall derive their distributions in terms of the ladder processes. Using the Wiener-Hopf factorization proved in the previous section, we shall obtain a duality relation between the distributions of W_t and $\hat{m}_t$ - the minimum of the dual MAP. Conditions for existence of a limiting distributions for W_t will then follow from those for $\hat{m}_t$.

(4.2) **Theorem.** *For all* $\theta\in R$, $\lambda>0$, $f\in b\mathscr{E}$, *let*

$$\Phi_{\lambda,\theta}f(i)=E^i\int_0^\infty e^{\{-\lambda t+i\theta M_t\}}f(X_t)\,dt. \tag{4.3}$$

Then

$$\Phi_{\lambda,\theta}=H_{\lambda,\theta}^{-1}G_{\lambda,0}.$$

Proof. Using Theorem (3.11), we let $Z_s=e^{i\theta Y_{s-}}$, $f(\omega)=1_{\{R>t-s>0\}}f(X_{t-s}(\omega))$

$$E^\cdot[e^{i\theta M_t}f(X_t)1_{\{G_t<t\}}]=E^\cdot[e^{\{i\theta Y_{G_t-}\}}f(X_t)1_{\{G_t<t,G_t\in G_1\cap G_2^c\}}]$$

$$+E^\cdot[e^{\{i\theta Y_{G_t}\}}f(X_t)1_{\{G_t<t,G_t\in G_2\}}]=E^\cdot\int_0^t e^{i\theta Y_s}\tilde{Q}_{t-s}^\theta f(X_s)\,dB_s^0.$$

where $\tilde{Q}_u^\theta f(x)=Q_u(x,g)$, and $g(j,y)=e^{i\theta y}f(j)$. Hence

$$E^\cdot[e^{i\theta M_t}f(X_t)]=E^\cdot(e^{i\theta M_t}f(X_t)1_M(t))$$
$$+E^\cdot\int_0^t e^{i\theta Y_s}\tilde{Q}_{t-s}^\theta f(X_s)\,dB_s^0.$$

Going now through the same computations we did when computing $U_{\lambda,\theta}f(i)$ we get

$$\Phi_{\lambda,\theta} f(i)=E^i \int_0^\infty \mathrm{e}^{\{-\lambda t+i\theta Y_t\}}$$

$$\cdot\left[l(X_t,0)f(X_t)+\tilde{P}^{X_t,0}\int_0^\infty \mathrm{e}^{-\lambda u}\, 1_{\{R>u\}} f(X_u)\,du\right] dL_t$$

where l, L, $\tilde{P}$ were defined in the previous section. This, after the time change $u=L_t$, is equal to

$$E^i \int_{s=0}^\infty \mathrm{e}^{\{-\lambda\tau_s+i\theta Y_{\tau_s}\}}\left[l(X_{\tau_s},0)f(X_{\tau_s})\right.$$

$$\left.+\tilde{P}^{X_{\tau_s},0}\int_{u=0}^\infty \mathrm{e}^{-\lambda u}\times 1_{\{R>u\}} f(X_u)\right] ds=H_{\lambda,\theta}^{-1} G_{\lambda,0} f(i).$$

(4.4) **Theorem.** *For all* $\lambda>0$, $\theta\in R$, $f\in b\mathscr{E}$ *let*

$$\Psi_{\lambda,\theta} f(i)=E^i \int_0^\infty \mathrm{e}^{\{-\lambda t+i\theta W_t\}} f(X_t)\,dt. \tag{4.5}$$

Then

$$\Psi_{\lambda,\theta}=\tilde{H}_{\lambda,0}^{-1}\tilde{G}_{\lambda,\theta} \tag{4.6}$$

where $\tilde{H},\tilde{G}$ are defined for the descending ladder process, in the same manner that H and G were defined in Sect. 3 for the ascending ladder process.

Proof. Identical to the proof of Theorem (6.2), with the Markov process X_t, Y_t, Y_t-m_t, $Z=1$, and $f(\omega)=1_{\{R>t-s>0\}} f(X_{t-s})\, e^{i\theta Y_{t-s}}$.

(4.7) **Theorem.** *Let* $\hat{m}_t=\inf_{0\leq s\leq t}\hat{Y}_s$ *and let*

$$\hat{m}_{\lambda,\theta} f(i)=\hat{E}^i \int_{t=0}^\infty \mathrm{e}^{\{-\lambda t+i\theta\hat{m}_t\}} f(\hat{X}_t)\,dt. \tag{4.8}$$

Then for all $\theta\in R$, $\lambda>0$, $f\in b\mathscr{E}$

$$\Psi_{\lambda,\theta} f(i)=\hat{m}_{\lambda,\theta}^* f(i).$$

Proof. We use an analogue of the Wiener-Hopf factorization for the descending ladder processes. Let $\hat{\tilde{H}},\hat{\tilde{G}}$ be defined for the dual process as $\tilde{H},\tilde{G}$ were defined for the original process, then for all $\lambda>0$, $\theta\in R$

$$\tilde{G}_{\lambda,\theta}\hat{\tilde{H}}_{\lambda,\theta}^*=\tilde{H}_{\lambda,\theta}\hat{\tilde{G}}_{\lambda,\theta}^*=\tilde{\Lambda}_\lambda \tag{4.9}$$

where $\tilde{\Lambda}$ is an invertible operator that does not depend on θ. Therefore

$$\Psi_{\lambda,\theta}=\tilde{H}_{\lambda,0}^{-1}\tilde{G}_{\lambda,\theta}=\hat{\tilde{G}}_{\lambda,0}^*\tilde{\Lambda}_\lambda^{-1}\tilde{\Lambda}_\lambda\hat{\tilde{H}}_{\lambda,\theta}^{*-1}$$

$$=\hat{\tilde{G}}_{\lambda,0}^*\hat{\tilde{H}}_{\lambda,\theta}^{*-1}=(\hat{\tilde{H}}_{\lambda,\theta}^{-1}\hat{\tilde{G}}_{\lambda,0})^*=\hat{m}_{\lambda,\theta}^*.$$

Theorem (6.7) gives us a duality relation between W_t and $\hat{m}_t$ as follows

$$(4.10)\qquad P^i\{W_t \in A, X_t = j\} = \frac{\pi(j)}{\pi(i)} \hat{P}^j\{\hat{m}_t \in -A, \hat{X}_t = i\}.$$

(4.11) **Corollary.** *If $\hat{m}_t \downarrow \hat{m}$, and $\hat{m} > -\infty$ a.s., then W_t has a limiting distribution as $t \to \infty$.*

Conditions under which $\hat{m}_t$ goes a.s. to a finite limit were discussed for the discrete time parameter case by Newbould [15], and his proofs carry without difficulty to the continuous case [10]. We shall state here the result without proof.

(4.12) **Theorem.** *Let (X, Y) be a MAP, for which X is defined on a finite state space and is ergodic. Then one, and only one, of the following holds*

(a) *Y_t degenerates (i.e. there exist constants $\beta_1, \ldots, \beta_m$ such that Y_t jumps $\beta_j - \beta_i$ when X jumps from i to j and remains constant otherwise).*

(b) $\limsup_{t\to\infty} Y_t = +\infty$, $\liminf_{t\to\infty} Y_t = -\infty$. *This happens if, and only if*

$$(4.13)\qquad \int_1^\infty \frac{1}{t} P^i\{Y_t > 0\}\, dt = \infty, \qquad \int_1^\infty \frac{1}{t} P^i\{Y_t < 0\}\, dt = \infty \qquad \textit{for some } i \in E.$$

(c) $\lim_{t\to\infty} Y_t = +\infty$. *This happens if, and only if,*

$$(4.14)\qquad \int_1^\infty \frac{1}{t} P^i\{Y_t > 0\}\, dt = \infty, \qquad \int_1^\infty \frac{1}{t} P^i\{Y_t < 0\}\, dt < \infty \qquad \textit{for some } i \in E.$$

(d) $\lim_{t\to\infty} Y_t = -\infty$. *This happens if and only if*

$$(4.15)\qquad \int_1^\infty \frac{1}{t} P^i\{Y_t > 0\}\, dt < \infty, \qquad \int_1^\infty \frac{1}{t} P^i\{Y_t < 0\}\, dt = \infty \qquad \textit{for some } i \in E.$$

(4.16) **Corollary.** *W_t has a limiting distribution if and only if*

$$(4.17)\qquad \int_1^\infty \frac{1}{t} P^i\{Y_t > 0\}\, dt < \infty, \qquad \int_1^\infty \frac{1}{t} P^i\{Y_t < 0\}\, dt = \infty \qquad \textit{for some } i \in E.$$

Using a Tauberian theorem, one can show that this limit is

$$\lim_{\lambda \to 0} \lambda H_{\lambda,0}^{-1} G_{\lambda,\theta} 1(i).$$

It will further be independent of i.

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