Z. Wahrscheinlichkeitstheorie verw. Gebiete
59, 169–177 (1982)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Asymptotic Behaviour of Gaussian Random Fields

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Summary. Let $X=\{X(t), t\in\mathbb{R}^N\}$ be a centred Gaussian random field with covariance $\mathscr{E}X(t)X(s)=r(t-s)$ continuous on $\mathbb{R}^N\times\mathbb{R}^N$ and $r(0)=1$. Let $\sigma(t,s)=(\mathscr{E}(X(t)-X(s))^2)^{1/2}$; $\sigma(t,s)$ is a pseudometric on $\mathbb{R}^N$. Assume X is σ-separable. Let D_1 be the unit cube in $\mathbb{R}^N$ and for $0<k\in\mathbb{R}$, $D_k=\{x\in\mathbb{R}^N: k^{-1}x\in D_1\}$, $Z(k)=\sup\{X(t), t\in D_k\}$. If X is sample continuous and $|r(t)| =o(1/\log|t|)$ as $|t|\to\infty$ then

$$Z(k)-(2N\log k)^{1/2}\to 0 \quad \text{as } k\to\infty \text{ a.s.}$$

§1. Introduction

Let $X=\{X(t), t\in\mathbb{R}^N\}$ be a centred Gaussian random field with covariance $\mathscr{E}X(t)X(s)=r(t-s)$ continuous on $\mathbb{R}^N\times\mathbb{R}^N$ and $r(0)=1$. Let $\sigma(t, s)$ be the increments variance

$$\sigma(t, s)=(\mathscr{E}(X(t)-X(s))^2)^{1/2}=\sqrt{2}(1-r(t-s))^{1/2} \tag{1.1}$$

then $\sigma(t,s)$ is a pseudometric on $\mathbb{R}^N$ which we shall use frequently, in particular X will be taken to be σ-separable. For $x=(x_1,\dots,x_n)$ let $|x|=(\sum_i x_i^2)^{1/2}$ and for $D\subset\mathbb{R}^N$, $\varepsilon>0$, denote by $N(D,\varepsilon)$ the minimal number of σ-balls with centres in D and radii $\leqq\varepsilon$ needed to cover D. The function $H(D,\varepsilon)=\log N(D,\varepsilon)$ is known as the metric entropy of D.

Let D_1 be the unit cube in $\mathbb{R}^N$ centred at 0. Define D_k by $D_k =\{x\in\mathbb{R}^N: k^{-1}x\in D_1\}$, $k>0$. Then, if λ is Lebesgue measure on $\mathbb{R}^N$, $\lambda(D_k)=k^N$; Let $Z_k=\sup\{X(t), t\in D_k\}$. Our main result, obtained as a consequence of Theorems 2 and 3, is the following:

Theorem 1. *Let X be as above. Assume that X is sample continuous and*

$$|r(t)|=o(1/\log|t|) \quad \text{as } |t|\to\infty$$

then

$$Z(k)-(2N\log k)^{1/2}\to 0 \quad \text{as } k\to\infty$$

with probability one.

0044-3719/82/0059/0169/$01.80

This extends results known for Gaussian processes [6, 8, 13].

The asymptotic behaviour of fields has been studied previously by several authors under certain local conditions on the covariance and stronger mixing conditions [5, 6, 11]. While their results were more precise our interest was to obtain information for as wide a class of fields as possible.

In Sect. 2 we get an upper bound for the tail of the distribution of the supremum of a process having an arbitrary parameter space that satisfies certain conditions on its metric entropy. This inequality is used in Sect. 3 to obtain upper bounds for the supremum, and may be of independent interest.

Some of the results in this work were obtained at Imperial College, London and were included in a Ph.D. thesis submitted at the University of London. The help and encouragement of Prof. G.E.H. Reuter is gratefully acknowledged. The author wishes to thank the referees for many helpful comments.

We end this section with some notation. $C>0$ is a constant that may change from line to line, $\phi(x)=(2\pi)^{-1/2}\exp(-x^2/2)$, $\psi(x)=x^{-1}\phi(x)$.

§2. The Tail of the Distribution of the Supremum

Lemma 3 below gives an upper bound for the tail when the field has an arbitrary parameter space whose metric entropy satisfies Dudley's condition. A similar inequality was obtained independently by M. Weber [13].

The method used in the proof is a combination of the well-known procedure of Sirao [12], used by many authors, with the more recent methods of metric entropy introduced by Dudley [2] and also used in other works, [3, 4]. In this section S is a metric space. We start by giving two preliminary lemmas.

Lemma 1 ([12] Lemma 2). *Let X and Y be jointly Gaussian r.v.'s with means* 0, *variance* 1 *and correlation r. Then $P\{X>a+h, Y\leqq a\}$ is a nonincreasing function of r for a, h fixed, $a>0$, $h>0$.*

Lemma 2. *Under the assumptions of Lemma* 1, *if*

$$r h-a(1-r)>0 \tag{2.1}$$

$$P\{X>a+h, Y\leqq a\}\leqq\psi(a)\frac{(1-r^2)^{1/2}}{r h-a(1-r)}\exp(-h^2/2(1-r^2)).$$

Proof. Let ξ and η be i.i.d. Gaussian r.v.'s, means 0, variances 1. Then $(\xi, r\xi+(1-r^2)^{1/2}\eta)$ have the same distribution as (X, Y). Therefore

$$\begin{aligned}
P\{X>a+h, Y\leqq a\}&=P\{\xi>a+h, r\xi+(1-r^2)^{1/2}\eta\leqq a\}\\
&\leqq P\{\xi>a+h, \eta\leqq(a-r(a+h))(1-r^2)^{-1/2}\}\\
&=P\{\xi>a+h\}\,P\{\eta>(r h-a(1-r))(1-r^2)^{-1/2}\}\\
&\leqq\frac{(1-r^2)^{1/2}}{2\pi(a+h)(h r-a(1-r))}\exp\left\{-\frac{(a+h)^2}{2}-\frac{(r h-a(1-r))^2}{2(1-r^2)}\right\}\\
&\leqq\frac{\psi(a)(1-r^2)^{1/2}}{hr-a(1-r)}\exp\left\{-\frac{h^2}{2(1-r^2)}-\frac{a^2(1-r)}{2(1+r)}-\frac{a h}{1+r}\right\}
\end{aligned}$$

Since $r \leqq 1$ the result follows. □

Define $\varepsilon_0(x) = \inf\left\{\varepsilon: \frac{H^{1/2}(\varepsilon)}{\varepsilon} \leqq x\right\}$. Note that $\varepsilon_0(x) \to 0$ as $x \to \infty$

Lemma 3. *Let $X = \{X(t), t \in S\}$ be a σ-separable centred Gaussian random field with $\mathscr{E} X(t) X(s) = r(t, s)$ continuous on $S \times S$ and $r(t, t) = 1$ for all t. Assume that for $T \subset S$ and some $\tau > 0$*

$$I(T, \tau) = \int_0^\tau H^{1/2}(T, u)\, du < \infty.$$

Then

$$P\{\sup_{t \in T} X(t) > x + A_1 I(T, \varepsilon_0(x)) \leqq C\psi(x)$$

where A_1 is a constant, $A_1 > 4$.

Proof. The result is immediate if $H(\varepsilon)$ is bounded as $\varepsilon \to 0$, therefore we assume that

$$H(\varepsilon) \uparrow \infty \quad \text{as } \varepsilon \to 0 \tag{2.2}$$

We start by defining a monotone decreasing sequence $\varepsilon_n, n = 1, 2, \ldots$ in terms of x and $H(\varepsilon)$. Let $\alpha > 0$ and

$$\delta_n = \sqrt{2} \inf\{\varepsilon: H(\varepsilon) \leqq (1+\alpha) H(\varepsilon_n/2)\} \tag{2.3}$$

$$\varepsilon_1 = \varepsilon_0/2$$

$$\varepsilon_{n+1} = \min(\varepsilon_n/2, \delta_n), \qquad n \geqq 1. \tag{2.4}$$

then $\varepsilon_n \to 0$ as $n \to \infty$. Let $T_n = \{t_i^n;\ 1 \leqq i \leqq N(\varepsilon_n)\}$ be a minimal ε_n - net of T and

$$Z_n(\omega) = \sup(X(t, \omega), t \in T_n); \qquad Z(\omega) = \sup(X(t, \omega), t \in T)$$

$$A = \left\{\omega: Z(\omega) > x + \sum_{i=0}^{\infty} x_i\right\}; \qquad A_n = \{\omega: Z_n(\omega) > y_n\}$$

with $y_n = x + \sum_0^n x_i$, $x_0 = H(\varepsilon_0)/x$, $x_i = (1+\alpha)\varepsilon_{i-1} H^{1/2}(\varepsilon_i)$ for $i \geqq 1$. Since we have assumed (2.2)

$$x_i > 0 \quad \text{all } i. \tag{2.5}$$

$T^1 = \bigcup_{i=0}^{\infty} T_i$ is a countable σ-dense subset of T and we have assumed X to be σ-separable and continuous in probability. Hence T^1 is a separating set for X. From $A \subseteqq A_0 \cup (A_1 - A_0) \cup (A_2 - A_1) \cup \ldots$ we get

$$P(A) \leqq P(A_0) + \sum_{n=1}^{\infty} P(A_n - A_{n-1})$$

Clearly, for X a standard normal r.v.

$$P(A_0)=P\{Z_0>x+H(\varepsilon_0)/x\}$$
$$\leqq N(\varepsilon_0)P\{X>x+H(\varepsilon_0)/x\}\leqq\psi(x)$$

On the other hand

$$P(A_n-A_{n-1})=P\{Z_n>y_n;\ Z_{n-1}\leqq y_{n-1}\}$$
$$\leqq\sum_{i=1}^{N(\varepsilon_n)}P\{X(t_i^n)>y_n;\ X(s)\leqq y_{n-1}\ \text{all}\ s\in T_{n-1}\}$$
$$\leqq\sum_{i=1}^{N(\varepsilon_n)}P\{X(t_i^n)>y_n;\ X(t_j^{n-1})\leqq y_{n-1}\}$$

where $t_j^{n-1}\in T_{n-1}$ and $\sigma(t_i^n,t_j^{n-1})\leqq\varepsilon_{n-1}$. From (1.1) $r(t_i^n,t_j^{n-1})\geqq 1-\varepsilon_{n-1}^2/2$. Define r_n by $r_n=1-\varepsilon_{n-1}^2/2$ then

$$r_n\leqq r(t_i^n,t_j^{n-1}) \tag{2.6}$$

for $1\leqq i\leqq N(\varepsilon_n)$. For $0<\gamma<1$, $n\geqq 1$ and x large $r_n>(1-\gamma)$.

Let ξ and η be centred Gaussian r.v.'s with variances 1 and $\mathscr{E}\xi\eta=r_n$. Then (2.5), (2.6) and Lemma 1 imply

$$P\{X(t_i^n)>y_n;\ X(t_j^{n-1})\leqq y_{n-1}\}\leqq P\{\xi>y_n;\ \eta\leqq y_{n-1}\}$$

and

$$P(A_n-A_{n-1})\leqq N(\varepsilon_n)P\{\xi>y_n;\ \eta\leqq y_{n-1}\}$$

suppose we can show that

$$r_n x_n-(1-r_n)y_{n-1}>C\varepsilon_{n-1}\varepsilon_0 x \tag{2.7}$$

then (2.1) is satisfied and Lemma 2 gives

$$P(A_n-A_{n-1})\leqq N(\varepsilon_n)\frac{\psi(y_{n-1})(1-r_n^2)^{1/2}}{r_n x_n-(1-r_n)y_{n-1}}\exp\left\{-\frac{x_n^2}{2(1-r_n)}\right\}$$

but

$$\frac{(1-r_n^2)^{1/2}}{r_n x_n-(1-r_n)y_{n-1}}\leqq\frac{C\varepsilon_{n-1}}{\varepsilon_{n-1}\varepsilon_0 x}\leqq CH^{-1/2}(\varepsilon_0)\to 0\qquad\text{as } x\to\infty$$

therefore for x large,

$$\sum_{n=1}^{\infty}P(A_n-A_{n-1})\leqq\psi(x)\sum_{n=1}^{\infty}N(\varepsilon_n)\exp\{-x_n^2/2(1-r_n)\}$$
$$=\psi(x)\sum_{n=1}^{\infty}\exp\{H(\varepsilon_n)-x_n^2/\varepsilon_{n-1}^2\}$$
$$\leqq\psi(x)\sum_{n=1}^{\infty}\exp\{-2\alpha H(\varepsilon_n)\}$$
$$\leqq 2\psi(x)\sum_{n=1}^{\infty}\exp\{-2\alpha H(\varepsilon_{2n-1})\}$$

but $\varepsilon_{n+2}<\delta_n/\sqrt{2}$ and (2.3) implies $H(\varepsilon_{n+2})>(1+\alpha)H(\varepsilon_n)$. Hence

$$\sum_{n=1}^{\infty} P(A_n-A_{n-1}) \leqq 2\psi(x) \sum_{n=1}^{\infty} \exp\{-2\alpha(1+\alpha)^{n-1} H(\varepsilon_1)\}.$$

Using (2.2), since $\varepsilon_1 \to 0$ as $x\to\infty$ the series is convergent and uniformly bounded as $x\to\infty$, therefore

$$\sum_{n=1}^{\infty} P(A_n-A_{n-1}) \leqq C\psi(x) \quad \text{and} \quad P(A) \leqq C'\psi(x).$$

It only remains to show that (2.7) holds. We begin by looking at the sum $\sum_{i=0}^{\infty} x_i$.

Lemma 4.

$$\sum_{i=0}^{\infty} x_i \leqq 4(1+\alpha)^2 I(T,\varepsilon_0).$$

Proof.

$$\sum_{i=0}^{\infty} x_i = H(\varepsilon_0)/x+(1+\alpha)\sum_{i=1}^{\infty} \varepsilon_{i-1} H^{1/2}(\varepsilon_i)$$

$$\leqq \varepsilon_0 H^{1/2}(\varepsilon_0)+(1+\alpha)\sum_{i=1}^{\infty} \varepsilon_{i-1} H^{1/2}(\varepsilon_i)$$

but $2\varepsilon_i \leqq \varepsilon_{i-1} \Rightarrow \varepsilon_{i-1} \leqq 2(\varepsilon_{i-1}-\varepsilon_i)$. Therefore

$$\sum_{i=0}^{\infty} x_i \leqq 2(1+\alpha)\left[(\varepsilon_0-\varepsilon_1)H^{1/2}(\varepsilon_0)+\sum_{i=1}^{\infty}(\varepsilon_{i-1}-\varepsilon_i)H^{1/2}(\varepsilon_i)\right].$$

If $\varepsilon_i=\varepsilon_{i-1}/2$,

$$(\varepsilon_{i-1}-\varepsilon_i)H^{1/2}(\varepsilon_i) \leqq 2\int_{\varepsilon_i/2}^{\varepsilon_{i-1}/2} H^{1/2}(u)\,du.$$

If $\varepsilon_i=\delta_{i-1}$

$$(\varepsilon_{i-1}-\varepsilon_i)H^{1/2}(\varepsilon_i) \leqq (1+\alpha)(\varepsilon_{i-1}-\varepsilon_i)H^{1/2}(\varepsilon_{i-1}/2)$$

$$\leqq 2(1+\alpha)\int_{\varepsilon_i/2}^{\varepsilon_{i-1}/2} H^{1/2}(u)\,du$$

therefore

$$\sum_{i=0}^{\infty} x_i \leqq 2(1+\alpha)\left[\int_{\varepsilon_0/2}^{\varepsilon_0} H^{1/2}(u)\,du+2(1+\alpha)\int_0^{\varepsilon_0/2} H^{1/2}(u)\,du\right]$$

$$\leqq 4(1+\alpha)^2 I(T,\varepsilon_0). \quad \square$$

Since $\varepsilon_0 \to 0$ as $x\to\infty$ this lemma implies that $\sum_0^{\infty} x_i \to 0$ as $x\to\infty$. Therefore for x large, $y_n < 3x/2$ for all n and

$$\begin{aligned} r_n x_n - (1-r_n) y_{n-1} &> r_n x_n - 3(1-r_n) x/2 \\ &> (1-\gamma) x_n - 3x \varepsilon_{n-1}^2/4 \\ &= (1-\gamma)(1+\alpha) \varepsilon_{n-1} H^{1/2}(\varepsilon_n) - 3x \varepsilon_{n-1}^2/4 \\ &\geqq \varepsilon_{n-1}((1-\gamma)(1+\alpha) H^{1/2}(\varepsilon_1) - 3x\varepsilon_0/4). \end{aligned}$$

Since $\varepsilon_1 = \frac{\varepsilon_0}{2}$, $H^{1/2}(\varepsilon_1) \geqq \varepsilon_0 x$ and

$$\begin{aligned} r_n x_n - (1-r_n) y_{n-1} &> \varepsilon_{n-1} \varepsilon_0 x((1+\alpha)(1-\gamma) - 3/4) \\ &= C \varepsilon_{n-1} \varepsilon_0 x \end{aligned}$$

provided we choose γ so that $(1+\alpha)(1-\gamma) > 3/4$. Hence (2.7) is satisfied. Finally, using Lemma 4.

$$P\{Z(\omega) > x + A_1 I(T, \varepsilon_0)\} \leqq P(A) \leqq C\psi(x). \quad \square$$

§ 3. The Asymptotic Behaviour

The results of Sect. 2 are used now to obtain information about $Z(k)$ as $k \to \infty$.

Theorem 2. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centred, σ-separable, sample continuous Gaussian random field with $\mathscr{E} X(t) X(s) = r(t-s)$ and $r(0) = 1$. Given $\varepsilon > 0$ there is, with probability one, a $\tau(\omega) \in \mathbb{R}$ such that for all $k > \tau$*

$$Z(k) < \theta(k) + A_1 I(D_1, \varepsilon_0)$$

where

$$\theta(k) = (2N \log k)^{1/2} + \frac{(\frac{1}{2} + \varepsilon) \log\log k^N}{(2N \log k)^{1/2}}$$

and $\varepsilon_0 = \varepsilon_0(\theta(k))$.

Proof. Let $n \in \mathbb{N}$. Define $E_n = D_{n+2} - D_n$ and divide this set into p_n unit cubes denoted by S_{nj}, $j = 1, \ldots, p_n$ where $p_n = (n+2)^N - n^N \leqq C n^{N-1}$; $E_n = \bigcup_{i=1}^{p_n} S_{nj}$. Let

$$B_{nj} = \{\sup_{t \in S_{nj}} X(t) \geqq \theta(n) + A_1 I(D_1, \varepsilon_0)\},$$

$$B_n = \bigcup_{j=1}^{p_n} B_{nj} = \{\sup_{t \in E_n} X(t) \geqq \theta(n) + A_1 I(D_1, \varepsilon_0)\}.$$

By a theorem of Dudley-Fernique [3], the fact that the field is stationary and sample continuous is equivalent to $I(D_1, v) < \infty$ for some $v > 0$. Therefore, using Lemma 3

$$\sum_{n=n_0}^{\infty} P(B_n) \leqq \sum_{n=n_0}^{\infty} \sum_{j=1}^{p_n} P(B_{nj}) \leqq C \sum_{n=n_0}^{\infty} n^{N-1} \psi(\theta(n)) < \infty$$

and the Borel-Cantelli lemma implies that there is a $n_0(\omega)$ with probability one such that, for $n > n_0$

$$\sup_{t \in E_n} X(t) < \theta(n) + A_1 I(D_1, \varepsilon_0)$$

which in turn implies that there is a $\tau(\omega)$, with probability one, such that for $n > \tau$

$$Z(n+2) < \theta(n) + A_1 I(D_1, \varepsilon_0)$$

but if $k \in [n, n+2]$, $k > \tau$ then

$$Z(k) \leqq Z(n+2) < \theta(n) + A_1 I(D_1, \varepsilon_0) \leqq \theta(k) + A_1 I(D_1, \varepsilon_0).$$

The following corollary is immediate

Corollary. Let X be as in Theorem 2. For any $\varepsilon > 0$ there is with probability one a $\tau(\omega)$ such that for $k > \tau$

$$Z(k) < (2N \log k)^{1/2} - \varepsilon. \tag{3.1}$$

The next theorem gives the "lower half" of Theorem 1. It does not use the results of the previous section and the proof is based on the methods of Pickands [9, 10].

Theorem 3. *Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centred Gaussian random field with $\mathscr{E} X(t) X(s) = r(t-s)$ and $r(0) = 1$. Assume that*

$$|r(t)| = o(1/\log|t|) \qquad \text{as } |t| \to \infty$$

then, given $\varepsilon > 0$ there is with probability one a $\tau(\omega)$ such that for all $k > \tau$

$$Z(k) > (2N \log k)^{1/2} - \varepsilon.$$

Proof. Define

$$c_k = (2N \log k)^{1/2},$$
$$L(k) = \exp(\varepsilon c_k / 4N),$$
$$\delta_k = \sup_{t \geqq L(k)} |r(t)|.$$

Then $\delta_k c_k \to 0$ as $k \to \infty$. Let $\{t_i^k, i = 1, \ldots, m_k\}$ be a set of points in D_k with $|t_i^k - t_j^k| \geqq L(k)$ for $i \neq j$, $m_k = [(k/L(k))^N]$. Let ξ_i, η be i.i.d. Gaussian r.v.'s with mean 0 and variance 1, $i = 1, \ldots, m_k$. Define $Y_i = (1-\delta_k)^{1/2} \xi_i + \delta_k^{1/2} \eta$. Then, using Slepian's lemma [4]

$$P\{Z(k) \leqq x\} \leqq P\{X(t_i^k) \leqq x,\ 1 \leqq i \leqq m_k\}$$
$$\leqq P\{Y_i \leqq x, 1 \leqq i \leqq m_k\}$$

and using Pickand's method it is enough to show that for some $\beta > 1$

$$\lim_{k \to \infty} (\log k)^\beta P\{Y_i \leqq c_k - \varepsilon, 1 \leqq i \leqq m_k\} = 0. \tag{3.2}$$

We have

$$\begin{aligned}
&P\{(1-\delta_k)^{1/2}\,\xi_i+\delta_k^{1/2}\,\eta\leqq c_k-\varepsilon,\ 1\leqq i\leqq m_k\}\\
&\quad=\int_{-\infty}^{\infty} P\left\{\xi_i\leqq\frac{c_k-\varepsilon-\delta_k^{1/2}\,u}{(1-\delta_k)^{1/2}},\ 1\leqq i\leqq m_k\right\}\phi(u)\,du\\
&\quad\leqq\psi(A)+\int_{-A}^{\infty} P\left\{\xi_i\leqq\frac{c_k-\varepsilon-\delta_k^{1/2}\,u}{(1-\delta_k)^{1/2}},\ 1\leqq i\leqq m_k\right\}\phi(u)\,du\\
&\quad\leqq\psi(A)+P\left\{\xi_i\leqq\frac{c_k-\varepsilon+A\,\delta_k^{1/2}}{(1-\delta_k)^{1/2}},\ 1\leqq i\leqq m_k\right\}\\
&\quad=\psi(A)+\exp\left\{m_k\log P\left\{\xi_1\leqq\frac{c_k-\varepsilon+A\,\delta_k^{1/2}}{(1-\delta_k)^{1/2}}\right\}\right\}\\
&\quad\leqq\psi(A)+\exp\left\{C m_k\,\psi\left(\frac{c_k-\varepsilon+A\,\delta_k^{1/2}}{(1-\delta_k)^{1/2}}\right)\right\}.
\end{aligned}$$

Let $A=2\log\log k$. Then for $1<\beta\leqq 2$

$$(\log k)^{\beta}\,\psi(A)\to 0 \qquad \text{as } k\to\infty. \tag{3.3}$$

Consider the second term

$$\begin{aligned}
&\exp\left\{C m_k\,c_k^{-1}\exp\left\{-\frac{(c_k-\varepsilon)^2+2A(c_k-\varepsilon)\,\delta_k^{1/2}+A^2\,\delta_k}{2(1-\delta_k)}\right\}\right\}\\
&\leqq\exp\left\{C k^N c_k^{-1}(L(k))^{-N}\exp\left\{-\frac{c_k^2-2c_k\,\varepsilon+2A\,c_k\,\delta_k^{1/2}}{2(1-\delta_k)}\right\}\right\}\\
&=\exp\left\{C c_k^{-1}\exp\left\{\frac{c_k^2}{2}-N\log L(k)-\frac{c_k^2-2c_k\,\varepsilon+2A\,c_k\,\delta_k^{1/2}}{2(1-\delta_k)}\right\}\right\}\\
&=\exp\left\{C c_k^{-1}\exp\left\{-\frac{\delta_k\,c_k^2}{2(1-\delta_k)}+\frac{\varepsilon\,c_k}{(1-\delta_k)}-\frac{\varepsilon\,c_k}{4}-o(1)\,c_k\right\}\right\}\\
&=\exp\left\{C c_k^{-1}\exp\left\{-o(1)\,c_k-\frac{\varepsilon\,c_k}{4}+\frac{\varepsilon\,c_k}{2(1-\delta_k)}\right\}\right\}\\
&\leqq\exp\{C_1\,c_k^{-1}\exp\{C_2\,c_k\}\}
\end{aligned}$$

since $\delta_k\to 0$ as $k\to\infty$. Hence

$$(\log k)^{\beta}\exp\left\{C m_k\,\psi\left(\frac{c_k-\varepsilon+A\,\delta_k^{1/2}}{(1-\delta_k)^{1/2}}\right)\right\}\to 0 \qquad \text{as } k\to\infty. \tag{3.4}$$

Combining (3.3) and (3.4) we get (3.2). □

Note. The mixing condition used in Theorem 3, $|r(t)|=o(1/\log|t|)$ as $|t|\to\infty$ has recently been weakened by Mittal [7] for the case $N=1$.

A similar result can be obtained for any set D_1 contained in a compact set and containing a neighboorhood of the origin.

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Received June 3, 1980; in revised form February 4, 1981