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Local Times for Markov Processes*

By

R. M. BLUMENTHAL and R. K. GETOOR

Introduction and preliminaries

The present paper is devoted to the construction and study of a special class of additive functionals of a Markov process. These additive functionals measure, in a certain sense, the amount of time the process X spends at a particular point x_0, and hence the additive functional, A_{x_0} associated with x_0 is called the *local time* of X at x_0. The local time will be constructed under the assumptions that X satisfies HUNT's hypothesis (A) and that there is positive probability of reaching $\{x_0\}$. More precisely let $T = \inf\{t > 0 : X(t) = x_0\}$ be the hitting time of $\{x_0\}$, then we assume that there is at least one x in the state space such that $P^x[T < \infty] > 0$. In the terminology of [*13*] the set $\{x_0\}$ is not polar. The most interesting situation is that in which x_0 is regular for $\{x_0\}$, that is, $P^{x_0}[T = 0] = 1$. This is easily seen to be equivalent to requiring that $\{x_0\}$ is not semi-polar, again using the terminology of [*13*]. We will call this last case the regular case and the majority (sections 1—5) of the paper is devoted to it. The non-regular case in which $\{x_0\}$ is semi-polar but not polar is essentially trivial, and we postpone its consideration until section 6.

Local times for regular linear diffusions have been constructed by TROTTER [*16*] and by MCKEAN and ITO [*12*]. In Chapter 6 of [*12*] MCKEAN and ITO give a detailed discussion of local times for this situation. More recently BOYLAN [*6*] has constructed local times for a fairly extensive class of one dimensional Markov processes and STONE [*14*] has given a number of applications of BOYLAN's local times similar to those in [*12*] and in section 4 of this paper. Both TROTTER and BOYLAN are concerned with the dependence of the local time on x_0 and they show that $A_{x_0}(t)$ is jointly continuous in t and x_0 in the cases they are considering. In this paper we will not concern ourselves with investigating the continuity in x_0. For x_0 fixed A is continuous in t if and only if $\{x_0\}$ is not semi-polar, i. e. in the regular case.

Section 1 contains the construction of the local time A and the study of some of its properties; in particular the relationship between A and $\{t : X(t) = x_0\}$. In section 2 we obtain the stochastic structure of the function inverse to A. Section 3 contains further properties of the local time when X satisfies HUNT's hypothesis (F) in addition to (A). In this case it is shown that the local time is a density for the occupation time relative to the basic measure ξ. In addition a simple sufficient condition is given for the regular case when (F) holds. Section 4 gives a number of applications of these results similar to those in Chapter 6 of [*12*] and in [*14*]. The

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results obtained for processes with stationary independent increments on the real line and for symmetric stable processes in R^N seem to be new. In section 5 we show that under a mild restriction the Lévy measure corresponding to the inverse of A is absolutely continuous. Finally the non-regular case is discussed in section 6.

Throughout this paper X will denote a temporally homogeneous Markov process satisfying Hunt's hypothesis (A) with a locally compact separable metric space E as state space. We refer the reader to [3] or [13] for a summary of the relevant properties of X, or, of course, to [9, section 1]. We will adopt for the most part the notation of [13], but for the convenience of the reader we repeat some of the basic definitions. Let $\bar{E} = E \cup \{\Delta\}$ where Δ is a point adjoined to E as the point at infinity if E is non-compact and as an isolated point if E is compact. Let Ω denote the space of all paths w, that is, all maps w from $[0, \infty]$ to $\bar{E}$ that are right continuous and have left hand limits on $[0, \infty)$ with $w(\infty) = \Delta$ and if $w(t) = \Delta$ then $w(s) = \Delta$ for all $s \geqq t$. As usual $X_t(w) = X(t, w) = w(t)$. We let $\mathscr{F}_t^0(\mathscr{F}^0)$ denote the σ-algebra of subsets of Ω generated by sets of the form $X_s^{-1}(B)$ for $s \leqq t\,(s < \infty)$ where B is in $\mathscr{B}$, the Borel sets of E. Also $\mathscr{F}_t(\mathscr{F})$ denotes the intersection of the P^μ completions of $\mathscr{F}_t^0(\mathscr{F}^0)$ taken over all finite measures μ on $\mathscr{B}$. We let $\sigma = \inf\{t : X(t) = \Delta\}$ denote the lifetime of the process. The reader is referred to [13] for definitions and terminology not explicitly mentioned in this paper.

An additive functional $A = \{A(t);\ 0 \leqq t \leqq \infty\}$ of X is a family of random variables such that

(i) $A(0) = 0$, $t \to A(t)$ is right continuous and nondecreasing, and $A(\infty) = \lim_{t \uparrow \sigma} A(t)$; each of these statements holding almost surely. Here and henceforth almost surely means almost surely P^x for all x in E.

(ii) $A(t)$ is $\mathscr{F}_t$ measurable for each t.

(iii) For each t, s one has almost surely

$$A(t + s, w) = A(t, w) + A(s, \Theta_t w). \tag{1}$$

Meyer proved in [13] that A has the strong Markov property, that is t may be replaced by any stopping time T and s by any non-negative random variable S in (1). If A is an additive functional, $u(x) = E^x[A(\infty)]$ is an excessive function called the potential of A. We refer the reader to [13] for further properties of additive functionals.

It is often convenient when terminating a process at an independent exponentially distributed time to use a representation of the terminated process that is different from its canonical representation on its path space described above. When convenient we will do so without particular mention. The reader is referred to [7] or [13] for the appropriate discussion.

1. Existence of local times: Regular case

Let x_0 be a fixed point of E and let $T = \inf\{t > 0 : X_t = x_0\}$ be the hitting time of $\{x_0\}$. We assume throughout this section that x_0 is regular for $\{x_0\}$, that is $P^{x_0}[T = 0] = 1$. For $\lambda, \mu > 0$ define

$$\Phi^\lambda(x) = E^x(e^{-\lambda T})$$

$$\Psi^\lambda_\mu = \Phi^\mu - (\lambda - \mu) U^\lambda \Phi^\mu . \tag{1.1}$$

Here U^λ is the λ-potential operator (or resolvent) for X, that is

$$U^\lambda f(x) = E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt .$$

Lemma 1.1. *Φ^λ is uniformly λ-excessive and $\Psi^\lambda_\mu = b^\lambda_\mu \Phi^\lambda$ where b^λ_μ is a positive constant.*

Proof. Introducing the operator $P^\lambda_T f(x) = E^x\{e^{-\lambda T} f(X_T)\}$ where T is defined above we have

$$\begin{aligned} P^\lambda_T \Phi^\lambda(x) &= E^x\{e^{-\lambda T} E^{X(T)}(e^{-\lambda T})\} \\ &= E^x\{e^{-\lambda T}\} E^{x_0}\{e^{-\lambda T}\} \\ &= \Phi^\lambda(x) , \end{aligned}$$

since $X(T) = x_0$ almost surely on $\{T < \infty\}$ and $P^{x_0}(T = 0) = 1$. But Φ^λ is clearly λ-excessive and so given $\varepsilon > 0$ there exists $\delta > 0$ such that $P^\lambda_t \Phi^\lambda(x_0) \leqq \Phi^\lambda(x_0) \leqq P^\lambda_t \Phi^\lambda(x_0) + \varepsilon$ whenever $t \leqq \delta$. Moreover $P^\lambda_T(x, dy)$ is concentrated on $\{x_0\}$ and hence if $t \leqq \delta$

$$\Phi^\lambda = P^\lambda_T \Phi^\lambda \leqq P^\lambda_T [P^\lambda_t \Phi^\lambda + \varepsilon] \leqq P^\lambda_t \Phi^\lambda + \varepsilon .$$

Thus $P^\lambda_t \Phi^\lambda \to \Phi^\lambda$ as $t \to 0$ uniformly on E, that is Φ^λ is uniformly λ-excessive.

Regarding the second statement of the lemma we compute

$$\begin{aligned} U^\lambda \Phi^\mu(x) &= E^x \int_0^\infty e^{-\lambda t} E^{X(t)}(e^{-\mu T})\, dt \\ &= E^x \int_0^\infty e^{-\lambda t} e^{-\mu T(\Theta_t)}\, dt . \end{aligned}$$

Since $T(\Theta_t) = T - t$ almost surely on $\{T > t\}$ we may write

$$\begin{aligned} I_1 &= E^x \int_0^T e^{-\lambda t} e^{-\mu T(\Theta_t)}\, dt \\ &= E^x\{e^{-\mu T} \int_0^T e^{-(\lambda - \mu) t}\, dt\} \\ &= (\lambda - \mu)^{-1} [\Phi^\mu(x) - \Phi^\lambda(x)] . \end{aligned}$$

On the other hand since $\{T < t\} \in \mathscr{F}_t$ we have

$$\begin{aligned} I_2 &= E^x \int_T^\infty e^{-\lambda t} e^{-\mu T(\Theta_t)}\, dt = E^x \int_T^\infty e^{-\lambda t} \Phi^\mu(X_t)\, dt \\ &= E^x \int_0^\infty e^{-\lambda(t+T)} \Phi^\mu(X_{t+T})\, dt = E^x\{e^{-\lambda T} U^\lambda \Phi^\mu(X_T)\} \\ &= P^\lambda_T U^\lambda \Phi^\mu(x) . \end{aligned}$$

Combining these computations we find

$$\Psi_\mu^\lambda = \Phi^\lambda - (\lambda - \mu)\, P_T^\lambda U^\lambda \Phi^\mu . \tag{1.2}$$

But $P_T^\lambda U^\lambda \Phi^\mu(x) = E^x(e^{-\lambda T})\, U^\lambda \Phi^\mu(x_0)$, and so $\Psi_\mu^\lambda = b_\mu^\lambda \Phi^\lambda$ with $b_\mu^\lambda = 1 - (\lambda - \mu)\, U^\lambda \Phi^\mu(x_0)$.

To complete the proof we must show that $b_\mu^\lambda > 0$. Since Φ^μ is μ-excessive and $\mu > 0$, there exists a sequence $\{f_n\}$ of non-negative functions such that $U^\mu f_n \uparrow \Phi^\mu$. Making use of the resolvent equation and the fact that $\Phi^\mu(x_0) = 1$, one easily obtains $b_\mu^\lambda \geqq 0$. However if $b_\mu^\lambda = 0$, then $\Psi_\mu^\lambda = 0$ or equivalently $\Phi^\mu = (\lambda - \mu)\, U^\lambda \Phi^\mu$. Clearly this implies $\lambda > \mu$. Also $\Phi^\mu(x) \leqq \Phi^\mu(x_0) = 1$, and so $1 \leqq (\lambda - \mu)\, U^\lambda 1 \leqq 1 - (\mu/\lambda)$ which is a contradiction. Thus Lemma 1.1 is established.

We now define for $\lambda > 0$

$$\Psi^\lambda = \Psi_1^\lambda = \Phi^1 - (\lambda - 1)\, U^\lambda \Phi^1 , \tag{1.3}$$

and it follows from Lemma 1.1 that Ψ^λ is uniformly λ-excessive for each λ.

Theorem 1.2. *There exists a continuous additive functional A of X such that $A(t)$ is almost surely finite for each $t \geqq 0$ and for each $\lambda > 0$*

$$E^x \int_0^\infty e^{-\lambda t}\, dA(t) = \Psi^\lambda(x) .$$

A is called the *local time* of X at x_0 and is unique up to the equivalence of additive functionals.

Proof. Let S^λ be a positive random variable which is independent of the process X and with distribution function $1 - e^{-\lambda t}$. Let $X^\lambda = (X, S^\lambda)$ be the process X "killed at time S^λ", that is $X^\lambda(t) = X(t)$ if $t < S^\lambda$ and $X^\lambda(t) = \Delta$ if $t \geqq S^\lambda$. Of course, S^λ must in general be defined on a larger spac than Ω and the representation of X^λ as (X, S^λ) is not the canonical representation of this process on its path space, but we will not dwell here on these familiar technical points. Since Ψ^λ is uniformly excessive relative to X^λ it follows from a theorem of Volkonski [7] that for each $\lambda > 0$ there exists a continuous additive functional A^λ of the process X^λ such that

$$E^x A^\lambda(\infty) = E^x A^\lambda(S^\lambda) = \Psi^\lambda(x) .$$

If λ and μ are positive and S^λ and S^μ are independent, then letting $a \wedge b = \min(a, b)$ we have

$$E^x A^\lambda(S^\lambda \wedge S^\mu) = E^x A^\lambda(S^\lambda) - E^x[A^\lambda(S^\lambda) - A^\lambda(S^\mu);\, S^\mu < S^\lambda) . \tag{1.4}$$

The first term on the right side of (1.4) is $\Psi^\lambda(x)$, while the second term equals

$$\begin{aligned} &E^x[A^\lambda(\infty) - A^\lambda(S^\mu);\quad S^\mu < S^\lambda] = E^x\{E^{X(S^\mu)}[A^\lambda(\infty)];\quad S^\mu < S^\lambda]\} \\ &= \mu\, E^x \int_0^\infty e^{-(\lambda+\mu)t}\, \Psi^\lambda(X_t)\, dt = \mu\, U^{\lambda+\mu} \Psi^\lambda . \end{aligned}$$

Therefore using the resolvent equation

$$E^x A^\lambda (S^\lambda \wedge S^\mu) = \Psi^{\lambda+\mu}(x).$$

Since this last expression is symmetric in λ and μ, Meyer's uniqueness theorem [13] implies that $A^\lambda(t) = A^\mu(t)$ almost surely on $\{t < S^\lambda \wedge S^\mu\}$. It now follows by standard arguments from the independence of X, S^λ, and S^μ that there exists a continuous additive functional A of the process X such that for each $\lambda > 0$, $A^\lambda(t) = A(t)$ if $t < S^\lambda$ and $A^\lambda(t) = A(S^\lambda)$ if $t \geqq S^\lambda$. Hence

$$E^x \int_0^\infty e^{-\lambda t}\, dA(t) = E^x A(S^\lambda) = E^x A^\lambda(\infty) = \Psi^\lambda(x).$$

The uniqueness of A follows from the fact that A^λ is uniquely determined (up to equivalence) by Ψ^λ. This completes the proof of Theorem 1.2.

Let us emphasize that A is an additive functional of X and does not depend on the auxiliary variables S^λ introduced in the construction. Moreover it is not too difficult to see using Lemma 1.1 that if we had used Ψ^λ_μ in place of $\Psi^\lambda = \Psi^\lambda_1$ in the construction, then the resulting additive functional of X would differ from A only by a multiplicative constant.

We next examine in more detail the relationship between A and X. The following result is the key step. As usual T is the hitting time of $\{x_0\}$.

Theorem 1.3. *Let* $R = \inf\{t: A(t) > 0\} = \sup\{t: A(t) = 0\}$, *then* $P^x(R = T) = 1$ *for all* x.

Proof. Clearly R is a stopping time and the continuity of A implies that $A(R) = 0$ almost surely. We observed at the beginning of the proof of Lemma 1.1 that $P^\lambda_T \Phi^\lambda = \Phi^\lambda$, and hence $P^\lambda_T \Psi^\lambda = \Psi^\lambda$. Thus

$$\begin{aligned} 0 &= \Psi^\lambda(x) - P^\lambda_T \Psi^\lambda(x) \\ &= E^x \int_0^T e^{-\lambda t}\, dA(t). \end{aligned}$$

But this implies that $P^x[A(T) = 0] = 1$ and hence $P^x[T \leqq R] = 1$.

To complete the proof of the theorem we will show that $P^x(T < R) = 0$. To this end we first observe that $R = T + R(\Theta_T)$ almost surely on $\{T < R\}$. Therefore

$$\begin{aligned} P^x[T < R] &= P^x[R(\Theta_T) > 0;\, T < R] \\ &\leqq E^x\{P^{X(T)}(R > 0)\} \\ &\leqq P^{x_0}(R > 0), \end{aligned}$$

and so it will suffice to show $P^{x_0}(R > 0) = 0$. Since $\Psi^1 = \Phi^1 \leqq 1$, we have

$$\begin{aligned} 1 = \Phi^1(x_0) &= E^{x_0} \int_0^\infty e^{-t}\, dA(t) = E^{x_0} \int_R^\infty e^{-t}\, dA(t) \\ &= E^{x_0}\{e^{-R}\Phi^1(X_R)\} \leqq E^{x_0}(e^{-R}). \end{aligned}$$

But clearly this implies that $P^x(R > 0) = 0$, and so Theorem 1.3 is established.

We define the following sets each of which depends on w.

$$Z = \{t : X_t = x_0\}$$

$$I = \{t : A(t+\varepsilon) - A(t) > 0 \quad \text{for all} \quad \varepsilon > 0\}$$

$$I' = \{t : A(t+\varepsilon) - A(t-\varepsilon) > 0 \quad \text{for all} \quad \varepsilon > 0\}$$

More descriptively Z consists of the x_0 points of X, I the points of right increase of A, and I' the points of increase of A. Since A is almost surley continuous it is clear that $I' = \bar{I}$ (the closure of I).

Lemma 1.4. *Almost surely $Z \subset I'$ and $\bar{Z} = I'$.*

Proof. As usual T is the hitting time of $\{x_0\}$, and so $t + T(\Theta_t) = \inf\{s > t : X_s = x_0\}$. Therefore

$$\{w : Z \not\subset I'\} \subset \bigcup_{r,s} \{A(s) - A(r) = 0; \quad r + T(\Theta_r) < s\}$$

where the union is taken over all rationals $s > r \geqq 0$. But for any x

$$\begin{aligned} P^x[A(s) - A(r) = 0; \quad r + T(\Theta_r) < s] \\ &= E^x\{P^{X(r)}[A(s-r) = 0; T < s - r]\} \\ &\leqq E^x\{P^{X(r)}(T < R)\} \\ &= 0, \end{aligned}$$

by Theorem 1.3. Thus $Z \subset I'$ almost surely.

To prove the second relationship it now suffices to show $I' \subset \bar{Z}$ almost surely. For a fixed w suppose $t_0 \in I'$ and $t_0 \notin \bar{Z}$, then since $\bar{Z}$ is closed there exist rationals r and s such that $r < t_0 < s$, $A(s) - A(r) > 0$, and $[r, s] \cap \bar{Z}$ is empty. Thus

$$\{w : I' \not\subset \bar{Z}\} \subset \bigcup_{r,s} \{A(s) - A(r) > 0; X_t \neq x_0 \text{ for all } t \in [r, s]\}$$

where again the union is over all rationals $s > r \geqq 0$. But

$$\begin{aligned} P^x\{A(s) - A(r) > 0; \quad X_t \neq x_0 \quad \text{for all} \quad t \in [r, s]\} \\ &= E^x\{P^{X(r)}[A(s-r) > 0; \quad T > s - r]\} \\ &\leqq E^x\{P^{X(r)}(T > R)\} = 0. \end{aligned}$$

Thus Lemma 1.4 is established.

Theorem 1.5. *Let f be a bounded Borel measurable function, then for each $\lambda > 0$*

$$E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dA(t) = f(x_0)\, \Psi^\lambda(x).$$

Proof. It suffices to consider f non-negative and continuous. Let w be fixed and such that $I'(w) = \bar{Z}(w)$, $t \to X_t(w)$ is right continuous, and $t \to A(t, w)$ is continuous. In view of Lemma 1.4 the set of w's not having all three of these properties is almost surely null. For such a w the measure $dA(t, w)$ is supported by $I'(w) = \bar{Z}(w)$. Since f is continuous $t \to f[X_t(w)]$ is right continuous and hence has at most countably many discontinuities, and if N denotes this set of discontinuities then N has $dA(t, w)$ measure zero. Therefore

$$\int_0^\infty e^{-\lambda t} f[X_t(w)]\, dA(t, w) = \int_{\bar{Z}-N} e^{-\lambda t} f[X_t(w)]\, dA(t, w).$$

But if $t \in \bar{Z} - N$ then $f[X_t(w)] = f(x_0)$, and so

$$\int_0^\infty e^{-\lambda t} f[X_t(w)]\, dA(t, w) = f(x_0) \int_0^\infty e^{-\lambda t}\, dA(t, w). \tag{1.5}$$

Thus (1.5) holds almost surely and integrating with respect to P^x yields Theorem 1.5.

The next theorem is the main result of this section.

Theorem 1.6. *Almost surely* $I \subset Z \subset I'$.

Proof. We have already shown in Lemma 1.4 that $Z \subset I'$ almost surely. It follows from Theorem 1.5 and the uniqueness theorem for continuous additive functionals that if f is the characteristic function of $\{x_0\}$, then

$$\int_0^t f(X_s)\, dA(s) = A(t) \qquad \text{a. s.} \tag{1.6}$$

Let w be such that (1.6) holds, $t \to X_t(w)$ is right continuous, and $t \to A(t, w)$ is continuous. For such a w suppose $t_0 \notin Z(w)$, then there exists an $\varepsilon > 0$ such that $X_t(w) \neq x_0$ for $t_0 \leqq t \leqq t_0 + \varepsilon$. But (1.6) then implies $A(t_0 + \varepsilon, w) = A(t_0, w)$ so that $t_0 \notin I(w)$. Therefore $I \subset Z$ almost surely; thus the proof of Theorem 1.6 is complete.

In general Theorem 1.6 can not be improved. Namely if x_0 is a holding point then $Z = I$, while if X has continuous paths $Z = \bar{Z} = I'$. But $I \neq I'$ unless x_0 is a trap. Lamperti [11] has shown that $Z = \bar{Z}$ almost surely for an extensive class of processes with independent increments.

2. The inverse of the local time

In this section x_0 will be a fixed point of E such that x_0 is regular for $\{x_0\}$, or, equivalently, $\{x_0\}$ is not semi-polar. Let A be the local time of X at x_0 so that A is a *continuous* additive functional with $A(t)$ a. s. finite. Define

$$\tau(t, w) = \inf\{s : A(s, w) > t\} \tag{2.1}$$

where it is understood that $\tau(t, w) = \infty$ if the set in braces is empty. Since A is continuous and non-decreasing it is immediate that τ is right continuous and *strictly* increasing on $\{t : \tau(t) < \infty\}$, these statements holding almost surely. The following facts are well known and easy to check, see [13, part II, sec. 7]:

(i) Each $\tau(t)$ is a stopping time and

$$\tau(t+s, w) = \tau(t, w) + \tau(s, \Theta_{\tau(t)} w), \quad \text{a. s.} \tag{2.2}$$

(ii) If f is a measurable function from $[0, \infty]$ to $[0, \infty]$ with $f(\infty) = 0$, then

$$\int_0^\infty f(t)\, dA(t) = \int_0^\infty f[\tau(t)]\, dt \quad \text{a. s.}$$

We will now compute the distribution of $\tau(t)$. The notation is that of section 1.

Theorem 2.1. For each $\lambda > 0$

$$E^x[e^{-\lambda\tau(t)}] = \Phi^\lambda(x) \exp[-t/\Psi^\lambda(x_0)].$$

Proof. We define $e^{-\lambda\tau(t)}$ to be zero on the set $\{\tau(t) = \infty\}$ and adopt the convention that any function f on E is automatically extended to $\bar{E}$ by setting $f(\Delta) = 0$ unless explicitly stated otherwise. If f is bounded and measurable define

$$Q_t^\lambda f(x) = E^x\{e^{-\lambda\tau(t)} f[X_{\tau(t)}]\}.$$

It is easily checked using (2.2) and the strong Markov property that for each $\lambda > 0$ the family $\{Q_t^\lambda; t \geqq 0\}$ is a semi-group on the space of bounded (universally) measurable functions with $||Q_t^\lambda|| \leqq 1$ for all $t \geqq 0$ and $\lambda > 0$. (Note that Q_0^λ is not in general the identity since $\tau(0)$ need not be zero.) Let

$$\begin{aligned} J^\mu f(x) &= \int_0^\infty e^{-\mu t}\, Q_t^\lambda f(x)\, dt \\ &= E^x \int_0^\infty e^{-\mu t}\, e^{-\lambda\tau(t)} f[X_{\tau(t)}]\, dt \end{aligned}$$

be the resolvent of $\{Q_t^\lambda\}$. Note that

$$\begin{aligned} J^0 f(x) &= E^x \int_0^\infty e^{-\lambda\tau(t)} f[X_{\tau(t)}]\, dt \\ &= E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dA(t) \\ &= f(x_0)\, \Psi^\lambda(x) \end{aligned}$$

by Theorem 1.5. Therefore J^0 is bounded with $||J^0|| = \sup \Psi^\lambda(x) = \Psi^\lambda(x_0)$. The resolvent equation implies $J^\mu[I + \mu J^0] = J^0$ and so if $0 \leqq \mu < ||J^0||^{-1}$ we may write

$$\begin{aligned} J^\mu &= J^0[I + \mu J^0]^{-1} \\ &= \sum_{k=0}^\infty (-\mu)^k (J^0)^{k+1}. \end{aligned}$$

But $J^0 1(x) = \Psi^\lambda(x)$ and hence $(J^0)^{k+1} 1(x) = \Psi^\lambda(x)\, [\Psi^\lambda(x_0)]^k$. Thus

$$J^\mu 1(x) = \Psi^\lambda(x)\, [1 + \mu\, \Psi^\lambda(x_0)]^{-1}; \quad 0 \leqq \mu < [\Psi^\lambda(x_0)]^{-1}. \tag{2.3}$$

However $\mu \to J^\mu 1(x)$ is the Laplace transform of $t \to E^x[e^{-\lambda\tau(t)}]$ which is right continuous, and so inverting (2.3) (both sides of which are analytic in $Re(\mu) > 0$) we obtain

$$E^x(e^{-\lambda\tau(t)}) = \frac{\Psi^\lambda(x)}{\Psi^\lambda(x_0)} \exp[-t/\Psi^\lambda(x_0)].$$

Finally Lemma 1.1 implies that $\Psi^\lambda(x) = \Psi^\lambda(x_0)\Phi^\lambda(x)$, and thus the proof of Theorem 2.1 is complete.

If $Q = \{t < \infty : \tau(s) = t \text{ for some } s, 0 \leqq s \leqq \infty\}$ is the range of τ, then it follows easily from the continuity of A that $Q = I$ almost surely. Here I is the set of points of right increase of A which was defined in Section 2. Important consequences of this and Theorem 1.6 are the facts that

$$\begin{aligned} &Q = I \subset Z \subset I' = \bar{Z} = \bar{Q} \quad \text{a. s.,} \\ &X[\tau(t)] = x_0 \quad \text{a. s.} \quad \text{on } \{\tau(t) < \infty\}. \end{aligned} \tag{2.4}$$

We will study the process $\tau(t)$ with probability law P^{x_0}. Since $\tau(0) = R$ it follows from Theorem 1.3 that $P^{x_0}[\tau(0) = 0] = 1$.

Theorem 2.2. *If $0 = t_0 < t_1 < \cdots < t_n$ and if $B_1, \ldots, B_n$ are Borel subsets of $[0, \infty)$, then*

$$P^{x_0}[\tau(t_j) - \tau(t_{j-1}) \in B_j, j = 1, \ldots, n;\ \tau(t_n) < \infty] = \prod_{j=1}^{n} P^{x_0}[\tau(t_j - t_{j-1}) \in B_j].$$

Proof. This is obvious if $n = 1$ since B_1 is a subset of $[0, \infty)$. Let $\Lambda_n = \bigcap_{j=1}^{n} \{\tau(t_j) - \tau(t_{j-1}) \in B_j\}$; then using (2.2) and (2.4) we have

$$\begin{aligned} &P^{x_0}(\Lambda_n;\ \tau(t_n) < \infty) \\ &= P^{x_0}[\Lambda_{n-1};\ \tau(t_n - t_{n-1}, \Theta_{\tau(t_{n-1})} w) \in B_n;\ \tau(t_{n-1}) < \infty] \\ &= E^{x_0}(P^{X[\tau(t_{n-1})]}[\tau(t_n - t_{n-1}) \in B_n];\ \Lambda_{n-1};\ \tau(t_{n-1}) < \infty) \\ &= P^{x_0}[\tau(t_n - t_{n-1}) \in B_n]\, P^{x_0}[\Lambda_{n-1};\ \tau(t_{n-1}) < \infty], \end{aligned}$$

and so Theorem 2.2 follows by induction.

Roughly speaking Theorem 2.2 states that τ has stationary independent increments on $\{\tau(t) < \infty\}$. Let us recall that if $\{Y(t);\ t \geqq 0\}$ is a real-valued process with stationary, independent, and *non-negative* increments, $Y(0) = 0$, and right continuous paths, defined on some probability space $(\Omega^*, \mathscr{G}, P)$ then

$$E[e^{-\lambda Y(t)}] = e^{-tg(\lambda)}$$

where

$$g(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda u})\, \nu(dt). \tag{2.5}$$

In (2.5), $b \geqq 0$ and ν is a Borel measure on $(0, \infty)$ satisfying $\int_0^\infty u(1+u)^{-1}\nu(du) < \infty$. A simple calculation shows that $b = \lim_{\lambda\to\infty} \lambda^{-1} g(\lambda)$. Using the terminology of [2] we will call Y a *subordinator*; g is called the *exponent* of Y and ν the LEVY *measure* of Y. See [2] and [5] for a discussion of the above facts.

The following theorem gives the stochastic structure of $\tau(t)$ under P^{x_0}.

Theorem 2.3. (i) $C = \lim_{\lambda\to 0} [\Psi^\lambda(x_0)]^{-1}$ exists and $0 \leqq C < \infty$.

(ii) There exists a subordinator Y such that if S^C is an exponentially distributed random variable with parameter C that is independent of τ, and if $\tau^*(t) = Y(t)$ for $t < S^C$ and $\tau^*(t) = \infty$ for $t \geqq S^C$, then τ^* and τ under P^{x_0} are stochastically equivalent. Moreover the exponent g of Y is given by $g(\lambda) = [\Psi^\lambda(x_0)]^{-1} - C$.

Proof. Theorem 2.1 with $x = x_0$ implies that $[\Psi^\lambda(x_0)]^{-1}$ is an increasing function of λ on $0 < \lambda < \infty$ and hence statement (i) is obvious. Let $R^+ = [0, \infty)$ and for each $t > 0$ define a measure μ_t on R^+ by $\mu_t(B) = P^{x_0}[\tau(t) \in B]$. An immediate consequence of Theorem 2.1 is that $\mu_{t+s} = \mu_t \star \mu_s$ for all $t, s > 0$ where "$\star$" denotes convolution. Also since $P^{x_0}[\tau(t) \to 0 \text{ as } t \downarrow 0] = 1$ it is easy to see that $\mu_t \to \varepsilon_0$ weakly as $t \downarrow 0$. Finally using Theorem 2.1 we see that

$$\begin{aligned} \mu_t(R^+) &= P^{x_0}[\tau(t) < \infty] \\ &= \lim_{\lambda\to 0} E^{x_0}(e^{-\lambda\tau(t)}) = e^{-Ct}. \end{aligned}$$

Thus if we define $\nu_t = e^{Ct}\mu_t$, the family $\{\nu_t; t > 0\}$ is a semi-group of *probability* measures on R^+ such that $\nu_t \to \varepsilon_0$ weakly as $t \downarrow 0$. Under these circumstances it is well known that there exists a subordinator Y defined on some probability space $(\Omega^*, \mathcal{G}, P)$ with stationary, independent, and non-negative increments, right continuous sample functions, and $Y(0) = 0$ such that ν_t is the distribution of $Y(t)$. Moreover

$$\begin{aligned} E(e^{-\lambda Y(t)}) &= \int_0^\infty e^{-\lambda u}\, \nu_t(du) \\ &= e^{Ct} E^{x_0}(e^{-\lambda\tau(t)}) \\ &= \exp\{-t([\Psi^\lambda(x_0)]^{-1} - C)\}. \end{aligned}$$

Thus the exponent of Y is $g(\lambda) = [\Psi^\lambda(x_0)]^{-1} - C$.

If B is a Borel set of R^+, then

$$\begin{aligned} P[\tau^*(t) \in B] &= P[Y(t) \in B;\, t < S^C] \\ &= e^{-Ct} P[Y(t) \in B] \\ &= e^{-Ct} \nu_t(B) \\ &= P^{x_0}[\tau(t) \in B]. \end{aligned}$$

It is now an easy consequence of Theorem 2.2 and the definition of τ^*, that τ and τ^* are stochastically equivalent. This completes the proof of Theorem 2.3.

Theorem 2.3 gives the stochastic structure of τ under P^{x_0}. Under P^x for an arbitrary x the process $\tau(t) - \tau(0)$ on $\{\tau(0) < \infty\}$ has the stochastic structure indicated by Theorem 2.3. More precisely let $\Omega_0 = \{\tau(0) < \infty\}$ and suppose $P^x(\Omega_0) > 0$. Define Q^x on Ω_0 by $Q^x(\Lambda) = P^x(\Lambda)/P^x(\Omega_0)$. Then $\tau(t) - \tau(0)$ on (Ω_0, Q^x) is stochastically equivalent to τ^*. Moreover it is not difficult to see that $\tau(t) - \tau(0)$ and $\tau(0)$ are independent on (Ω_0, Q^x).

Remark. $\nu(R^+) = \infty$ if and only if x_0 is not a holding point.

Recall that $\nu[(a, \infty)]$ is the expected number of jumps of Y of magnitude exceeding a in unit time, so that $\nu(R^+) < \infty$ if and only if the expected number of jumps of Y in unit time is finite. But this last statement is easily seen to be equivalent to

$$\lim_{\varepsilon \downarrow 0} P[Y(t) - bt = 0 \text{ for all } t \leqq \varepsilon] = 1 \tag{2.6}$$

where b is the constant appearing in (2.5). Thus if $\nu(R^+) < \infty$ there exists an $\varepsilon > 0$ such that $P^{x_0}\{\tau(t) = bt \text{ on } [0, \varepsilon]\} > 0$. In this situation $b > 0$ since $\tau(t)$ is strictly increasing. Hence $P^{x_0}\{A(t) = b^{-1}t \text{ on } [0, \varepsilon]\} > 0$ and thus Theorem 1.6 implies that x_0 is a holding point. Conversely if x_0 is a holding point the first jump of τ is positive with P^{x_0} probability one and hence (2.6) must be valid. Therefore $\nu(R^+) < \infty$.

We will prove in section 5 that under a mild additional assumption ν is absolutely continuous.

3. Hypothesis (F)

In this section we will investigate certain special facts which hold when our process X satisfies HUNT's hypothesis (F) [9, III] in addition to hypothesis (A). This is of some importance since most familiar processes satisfy (F). We refer the reader to [4, sec. 2] or [13, pt. II, sec. 6] for a summary of the relevant consequences of (F), or to [9, III] for a complete discussion. The basic fact is the existence of a Radon measure ξ on E and point kernels $U^\lambda(x, y)$ such that the potential kernels $U^\lambda(x, dy)$ of X are given by

$$U^\lambda(x, dy) = U^\lambda(x, y)\, \xi(dy). \tag{3.1}$$

Of course (3.1) is equivalent to the statement that for a bounded measurable f

$$E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt = \int U^\lambda(x, y) f(y)\, \xi(dy).$$

U^0 may be identically infinite. We will write dx for $\xi(dx)$ in the sequel.

As in the previous sections T will denote the hitting time of $\{x_0\}$, $\Phi^\lambda(x) = E^x(e^{-\lambda T})$; and Ψ^λ was defined in (1.3). Applying HUNT's general theory of capacity [9, sec. 19] to the singleton $\{x_0\}$ yields the existence for each $\lambda > 0$ of a unique measure π^λ concentrated on $\{x_0\}$ such that $\Phi^\lambda = U^\lambda \pi^\lambda$. The λ-capacity, $C(\lambda)$, of $\{x_0\}$ is defined to be the total mass of π^λ. Since π^λ is concentrated at x_0 we have in this simple case $\pi^\lambda = C(\lambda)\, \varepsilon_{x_0}$, and hence $\Phi^\lambda(x) = C(\lambda)\, U^\lambda(x, x_0)$. The resolvent equation and the properties of $U^\lambda(x, y)$ imply that

$$\Psi^\lambda = U^\lambda \pi^1 = C(1)\, U^\lambda(\cdot, x_0). \tag{3.2}$$

Finally $C(\lambda) = 0$ if and only if $\Phi^\lambda = 0$, that is if and only if $\{x_0\}$ is polar.

The following result is a useful sufficient condition for x_0 to be regular for $\{x_0\}$.

Theorem 3.1. *Suppose* (F) *holds and that for some* $\lambda > 0$ *the function* $x \to U^\lambda(x, x_0)$ *is bounded and continuous at* $x = x_0$. *Then* x_0 *is regular for* $\{x_0\}$.

Proof. Let us suppose that x_0 is not regular for $\{x_0\}$. The fact that $x \to U^\lambda(x, x_0)$ is bounded implies that $\{x_0\}$ is not polar, that is $C(\lambda) > 0$. (See formula (19.6) of [*9*, III].) Therefore there exists an $x \neq x_0$ such that $P^x(T < \infty) > 0$. Let $\{G_n\}$ be a decreasing sequence of open sets with $x \notin G_1$ and $\bigcap_n \bar{G}_n = \{x_0\}$. Let T_n be the first hitting time of G_n. Since no point is regular for $\{x_0\}$, Proposition 18.5 of [*9*, III], implies that $T_n \uparrow T$ and $T_n < T$ for all n almost surely P^x on $\{T < \infty\}$. But now Theorem 6.2 of [*9*, I] implies that $\Phi^\lambda[X(T_n)] \to 1$ almost surely P^x on $\{T < \infty\}$. By hypothesis $\Phi^\lambda(x) = C(\lambda)\, U^\lambda(x, x_0)$ is continuous at x_0 and since $X(T_n) \to x_0$ almost surely on $\{T < \infty\}$, it follows that $\Phi^\lambda(x_0) = 1$. However this contradicts the assumption that x_0 is not regular for $\{x_0\}$, and so Theorem 3.1 is established.

Inasmuch as the local time at x_0 is only determined up to a multiplicative constant it will be convenient to absorb the constant $C(1)$ in (3.2) into the local time. Thus whenever (F) holds and x_0 is regular for $\{x_0\}$ the local time at x_0 will be understood to be the *unique continuous additive functional* A of X satisfying for all $\lambda > 0$

$$E^x \int_0^\infty e^{-\lambda t}\, dA(t) = U^\lambda(x, x_0)\,. \tag{3.3}$$

Since $C(\lambda) = [U^\lambda(x_0, x_0)]^{-1}$ we have in this case

$$E^x(e^{-\lambda \tau(t)}) = \Phi^\lambda(x)\, e^{-t C(\lambda)}\,. \tag{3.4}$$

Also in Theorem 2.3 the exponent g of the subordinator Y becomes $g(\lambda) = C(\lambda) - C$ where now $C = \lim_{\lambda \to 0} C(\lambda)$. Moreover we can identify the linear component $b\lambda$ of $g(\lambda)$ in the present situation. Recall that $b = \lim_{\lambda \to \infty} \lambda^{-1} g(\lambda) = \lim_{\lambda \to \infty} \lambda^{-1} C(\lambda)$. If we fix $\lambda_0 > 0$ then specializing equation (19.7) of [*9*, III] to the present case, we see that for $\lambda > \lambda_0$

$$C(\lambda) = C(\lambda_0) + (\lambda - \lambda_0) \int \hat{\Phi}^\lambda(x)\, \Phi^{\lambda_0}(x)\, dx$$

where $\hat{\Phi}^\lambda(x) = \hat{E}^x(e^{-\lambda T})$ and $\hat{E}^x$ is the expectation operator relative to the dual process $\hat{X}$ and T is the first hitting time of $\{x_0\}$. Since semi-polar and cosemi-polar sets are the same, x_0 is coregular for $\{x_0\}$. Hence $\hat{\Phi}^\lambda(x) \to \hat{P}^x(T = 0) = I(x)$ as $\lambda \to \infty$ where I is the characteristic function of $\{x_0\}$. Also

$$\int \Phi^{\lambda_0}(x)\, dx = C(\lambda_0) \int dx\, U^{\lambda_0}(x, x_0) \leqq C(\lambda_0)/\lambda_0\,.$$

Combining these facts with the dominated convergence theorem yields

$$\begin{aligned} b = \lim_{\lambda \to \infty} C(\lambda)/\lambda &= \int I(x)\, \Phi^{\lambda_0}(x)\, dx \\ &= \xi(\{x_0\})\,. \end{aligned}$$

Thus writing ξ_0 for $\xi(\{X_0\})$ we have under (F)

$$g(\lambda) = C(\lambda) - C;\ C = \lim_{\lambda \to 0} C(\lambda);\ b = \lim_{\lambda \to \infty} \lambda^{-1} C(\lambda) = \xi_0 . \tag{3.5}$$

We will now show that if the local time exists at all points of E then it is a density with respect to ξ for the occupation times. To this end the following measurability result is basic.

Theorem 3.2. *Suppose that x is regular for $\{x\}$ for all x in E and let A_x denote the local time of X at x. Then assuming (F), the function $(x, w) \to A_x(t, w)$ is jointly measurable with respect to the $\nu \times P^\mu$ completion of $\mathscr{B} \times \mathscr{F}_t$ for all finite measures ν, μ on $\mathscr{B}$.*

Proof. Standard considerations show that it will suffice to prove that A_x^λ has the required measurability properties where A_x^λ is the additive functional of (X, S^λ) defined in the proof of Theorem 1.2. For typographical convenience we will suppress the λ in our notation for the remainder of this proof. We will need the following lemma which is a routine extension of an important result of Meyer [*13*]. Recall that a semi-group $\{Q_t\}$ is said to be subordinate to $\{P_t\}$, the semi-group of X, provided $Q_t f \leqq P_t f$ for all bounded, *non-negative*, measurable f and the function $t \to Q_t 1(x)$ is continuous at $t = 0$ for all x.

Lemma 3.3. *For each y in E let $\{Q_t^y\}$ be a semi-group subordinate to $\{P_t\}$, and suppose that $(y, x) \to Q_t^y f(x)$ is jointly (Borel) measurable in (y, x) for each t and bounded continuous f. Then there exists a multiplicative functional M_t^y such that $Q_t^y f(x) = E^x[f(X_t) M_t^y]$ for each bounded measurable f and $(y, w) \to M_t^y(w)$ is measurable with respect to the $\nu \times P^\mu$ completion of $\mathscr{B} \times \mathscr{F}_t$ for all ν, μ and t.*

The proof of the lemma is exactly the same as the proof of Theorem 2.2, Part I of [*13*] except that one must keep track of the parameter y at each step.

Returning to the proof of Theorem 3.2, let us define

$$g_n(x, y) = n[U(x, y) - P_{1/n} U(\cdot, y)(x)] .$$

Clearly g_n is $\mathscr{B} \times \mathscr{B}$ measurable for each n. Define

$$A_y^n(t, w) = \int_0^t g_n(X_u(w), y)\, du$$

so that $(y, w) \to A_y^n(t, w)$ is $\mathscr{B} \times \mathscr{F}_t$ measurable for each n and t. It is well known that in the present situation ($x \to U(x, y)$ is uniformly excessive)

$$E^\mu[(A_y^n(t) - A_y(t))^2] \to 0$$

for each μ and t. See [*17*] or [*15*]. Thus if we let

$${}_yQ_t^n f(x) = E^x\{f(X_t) \exp[-A_y^n(t)]\} ,$$

it follows that ${}_yQ_t^n f(x) \to Q_t^y f(x)$ as $n \to \infty$ for each x and bounded Borel measurable f, where $\{Q_t^y\}$ is a semi-group subordinate to $\{P_t\}$ and satisfying the measurability assumption of Lemma 3.3. See [*3*, pp. 411, 412]. Applying Lemma 3.3 and letting $\tilde{A}_y(t) = -\log M_t^y$ it is clear that $(y, w) \to \tilde{A}_y(t, w)$ is measurable with respect to the $\nu \times P^\mu$ completion of $\mathscr{B} \times \mathscr{F}_t$ for all ν, μ, and t, and that for each

y the additive functionals A_y and $\bar{A}_y$ are equivalent. Thus we may take $\bar{A}_y$ to be the local time of X at y and so the proof of Theorem 3.2 is complete.

Remark. Since the set of (x, w) for which $t \to A_x(t, w)$ is not continuous is $\nu \times P^\mu$ null, it is immediate that $(t, x, w) \to A_x(t, w)$ is measurable with respect to $\mathscr{T} \times (\mathscr{B} \times \mathscr{F})^{\nu,\mu}$ for all ν, μ where $\mathscr{T}$ is the Borel sets of $[0, \infty)$ and $(\mathscr{B} \times \mathscr{F})^{\nu,\mu}$ is the $\nu \times P^\mu$ completion of $\mathscr{B} \times \mathscr{F}$.

Corollary 3.4. *Under the assumptions of Theorem 3.2 we have for each $B \in \mathscr{B}$*

$$\int_B A_x(t)\,dx = \int_0^t I_B(X_u)\,du\,, \quad a.s.$$

where I_B is the characteristic function of B.

Proof. Let $A(t) = \int_B A_x(t)\,dx$. Then using Fubini's Theorem several times it is easy to check that $A(t)$ is a finite continuous additive functional of X. Also $L(t) = \int_0^t I_B(X_u)\,du$ is a continuous additive functional. If $\lambda > 0$, then

$$\begin{aligned} E^y \int_0^\infty e^{-\lambda t}\,dA(t) &= \int_B U^\lambda(y, x)\,dx \\ &= E^y \int_0^\infty e^{-\lambda t}\,dL(t)\,, \end{aligned}$$

and so Meyer's uniqueness theorem yields the equivalence of A and L.

It is perhaps worthwhile to note that if m is any positive measure on E such that $U^\lambda m$ is bounded for each $\lambda > 0$, then under the conditions of Theorem 3.2

$$A(t) = \int A_x(t)\,m(dx)$$

is a finite continuous additive functional whose λ-potential (i.e., $E^x A(S^\lambda)$) is $U^\lambda m$ for all $\lambda > 0$. The proof is similar to that of Corollary 3.4. We are, of course, still assuming that all points are non-semi-polar. It is also possible to prove the converse of the above statement under these assumptions by methods similar to those used in [*4*], but we will not enter into this.

4. Applications

In this section we will give some applications of the results of section 3. Many of these applications are the same as those given by McKean and Ito [*12*, Ch. *6*] for diffusion theory or by Stone [*14*] for semi-stable processes. We will not strive for the utmost generality, but rather we will be content to indicate the type of results that may be obtained.

We begin with some results on the Hausdorff dimension of $Z = \{t: X(t) = x_0\}$. We assume that $\{x_0\}$ is not semi-polar and we use the notation of the preceding sections.

Theorem 4.1. *Let* $\beta = \inf\{\alpha: \lambda^\alpha \Psi^\lambda(x_0) \to \infty$ *as* $\lambda \to \infty\}$ *and* $\sigma = \sup\{\alpha: \lambda^\alpha \Psi^\lambda(x_0) \to 0$ as $\lambda \to \infty\}$. *Then* $0 \leq \sigma \leqq \beta \leqq 1$ *and* $P^{x_0}[\sigma \leqq \dim Z \leqq \beta] = 1$.

Proof. It follows from Theorem 2.3 that $[\lambda \Psi^\lambda(x_0)]^{-1} \to b$ as $\lambda \to \infty$ where $0 \leqq b < \infty$. Hence if $\alpha > 1$ then $\lambda^\alpha \Psi^\lambda(x_0) \to \infty$ as $\lambda \to \infty$ and so $\beta \leqq 1$. Obviously $0 \leqq \sigma \leqq \beta$. In view of (2.4) and the fact that $\bar{Q}(w) - Q(w)$ is at most countable (Q is the range of τ as defined above (2.4)), it will suffice to prove $P^{x_0}[\sigma \leqq \dim Q \leqq \beta] = 1$. However, this follows from Theorem 2.3 and the known results on the dimension of the range of a subordinator ([2], Theorem 8.3 and Corollary 9.1). In [2] we assumed that $b = 0$, but the extension of the results of [2] to the case $b > 0$ is simple.

Suppose X is a real valued process with stationary independent increments and right continuous paths; then as is well known

$$E^x(e^{iyX(t)}) = e^{iyx} e^{-t\Psi(y)}$$

where

$$\Psi(y) = i m y + \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} \left(1 - e^{iyu} - \frac{iyu}{1+u^2}\right) \nu(du) \tag{4.1}$$

with ν satisfying $\int_{-\infty}^{\infty} x^2(1+x^2)^{-1}\nu(dx) < \infty$. For simplicity we assume $m = 0$ and $\sigma^2 = 0$. In addition, writing Ψ_R for the real part of Ψ, we assume

$$\int_{-\infty}^{\infty} [\lambda + \Psi_R(x)]^{-1} dx < \infty \tag{4.2}$$

for all positive λ. If we take Lebesgue measure as our basic measure then X satisfies hypotheses (A) and (F), and in fact

$$p(t, x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iy(z-x)} e^{-t\Psi(y)} dy \tag{4.3}$$

$$U^\lambda(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iy(z-x)}}{\lambda + \Psi(y)} dy \tag{4.4}$$

are the transition density and potential kernel for X, the integrals in question existing absolutely under the assumption (4.2). Moreover U^λ is bounded and continuous and so each $\{x\}$ is not semi-polar. Let us take $x_0 = 0$; then we have the following corollary to Theorem 4.1.

Corollary 4.2. *Using the above notation define*

$$\beta(X) = \inf\{\alpha \geqq 0 : |x|^{-\alpha} \Psi_R(x) \to 0 \text{ as } |x| \to \infty\},$$
$$\beta''(X) = \sup\{\alpha \geqq 0 : |x|^{-\alpha} \Psi_R(x) \to \infty \text{ as } |x| \to \infty\}$$

where $\beta''(X) = 0$ if there are no such α. Then under the above assumptions

$$P^0[1 - 1/\beta''(X) \leqq \dim Z \leqq 1 - 1/\beta(X)] = 1 .$$

Proof. In light of Theorem 4.1 it suffices to prove $\sigma \geqq 1 - 1/\beta''(X)$ and $\beta \leqq 1 - 1/\beta(X)$. Clearly (4.2) implies $\beta(X) \geqq 1$. In [2, Th. 3.2] we showed that

$|x|^{-\alpha}|\Psi(x)| \to 0$ as $|x| \to \infty$ for any $\alpha > \beta(X)$. Choosing $\alpha > \beta(X)$ we have

$$\begin{aligned}2\pi U^{\lambda}(0,0) &= \int \frac{dy}{\lambda + \Psi(y)} \\ &= \lambda^{1/\alpha - 1}\left[\int_{-1}^{1} [1 + \lambda^{-1}\Psi(\lambda^{1/\alpha} y)]^{-1} dy \right. \\ &\qquad \left. + \int_{|y|>1} [1 + \lambda^{-1}\Psi(\lambda^{1/\alpha} y)]^{-1} dy\right] \\ &= \lambda^{1/\alpha - 1}[I_1(\lambda) + I_2(\lambda)].\end{aligned}$$

Since $\Psi_R \geqq 0$, $|1 + \lambda^{-1}\Psi(\lambda^{1/\alpha}y)|^{-1} \leqq 1$ and since $\alpha > \beta(X)$, $1 + \lambda^{-1}\Psi(\lambda^{1/\alpha}y) \to 1$ as $\lambda \to \infty$ provided $y \neq 0$. Thus by the bounded convergence theorem $I_1(\lambda) \to 2$ as $\lambda \to \infty$. Also $\mathrm{Re}([1 + \lambda^{-1}\Psi(\lambda^{1/\alpha}y)]^{-1}) \geqq 0$ and so $\mathrm{Re}\, I_2(\lambda) \geqq 0$. Since $U^{\lambda}(0,0)$ is real this shows that $U^{\lambda}(0,0) = \lambda^{1/\alpha - 1} A(\lambda)$ where $A(\lambda)$ is bounded away from zero for large λ. Thus $\lambda^{\alpha'} U^{\lambda}(0,0) \to \infty$ as $\lambda \to \infty$ provided $\alpha' > 1 - 1/\alpha$, and since $\alpha > \beta(X)$ was arbitrary it follows that $\beta \leqq 1 - 1/\beta(X)$. The fact that $\sigma \geqq 1 - 1/\beta''(X)$ can be proved by a similar argument which we omit.

As a final application of Theorem 4.1 let $X(t)$ be the symmetric stable process of index α in Euclidean N-space, R^N. If $R_\alpha(t) = |X(t)|$ is the radial part of X, then R_α is a Markov process with state space $[0, \infty)$ satisfying hypothesis (A). If $\alpha = 2$, then X is N-dimensional Brownian and it is known, see [*10*] or [*12*], that the transition function $P_2(t, x, A)$ for R_2 is given by

$$P_2(t,x,A) = \int_A p_2(t,x,y)\,\xi(dy)$$

where $\xi(dy) = 2^{-(\gamma + 1/2)}[\Gamma(\gamma + 3/2)]^{-1} y^{2\gamma+1} dy$, $\gamma = (N-1)/2$ and $p_2(t,x,y) =$

$$= \Gamma\left(\gamma + \frac{1}{2}\right)(2t)^{-1}\left(\frac{xy}{2}\right)^{1/2-\gamma} e^{-\frac{x^2+y^2}{4t}}\, I_{\gamma - 1/2}\left(\frac{xy}{2t}\right)$$

where I_ν is the usual modified Bessel function.

Let T_β be the stable subordinator of index β, $0 < \beta < 1$, with $T_\beta(0) = 0$. Then $R_\alpha(t)$ and $R_2[T_{\alpha/2}(t)]$ are stochastically equivalent provided $T_{\alpha/2}$ and R_2 are completely independent. Therefore if $g_\beta(t, u)$ is the probability density function of $T_\beta(t)$, then the transition function of R_α is given by

$$P_\alpha(t,x,A) = \int_A p_\alpha(t,x,y)\,\xi(dy)$$

where ξ was defined above and

$$p_\alpha(t,x,y) = \int_0^\infty g_{\alpha/2}(t,u)\,p_2(u,x,y)\,du\,.$$

It is now clear that for each α, R_α satisfies hypothesis (F) with ξ as basic measure and since p_α is symmetric in x and y, a set is polar if and only if it is semi-polar. See Hunt [*9*, Sec. 20].

Let $x_0 > 0$ and let $Z = \{t: R_\alpha(t) = x_0\} = \{t: |X(t)| = x_0\}$ be the x_0-points of R_α.

Corollary 4.3. *If* $0 < \alpha \leqq 1$ *then* $P^{x_0}[Z = \{0\}] = 1$, *while if* $1 < \alpha \leqq 2$ *then* $P^{x_0}[\dim Z = 1 - 1/\alpha] = 1$.

Proof. The asymptotic expansion of I_ν implies that $p_2(t, x_0, x_0) \sim a t^{-1/2}$ as $t \to 0$ where $a > 0$; of course, a depends on x_0. Since $g_\beta(t, u) = t^{-1/\beta} g_\beta(1, t^{-1/\beta} u)$ we obtain

$$\int_0^\infty g_{\alpha/2}(t, u)\, u^{-1/2}\, du = b\, t^{-1/\alpha}$$

where $b = \frac{2}{\sqrt{\pi}} \Gamma(1 + 1/\alpha)$, although the precise value of b is of no importance. Given $\varepsilon > 0$ there exists $u_0 > 0$ such that $(1 - \varepsilon) a u^{-1/2} \leqq p_2(u, x_0, x_0) \leqq$ $\leqq (1 + \varepsilon) a u^{-1/2}$ for $0 < u \leqq u_0$. Also

$$t^{1/\alpha} \int_{u_0}^\infty g_{\alpha/2}(t, u)\, p_2(u, x_0, x_0)\, du \to 0$$

as $t \to 0$ since $p_2(u, x_0, x_0)$ is bounded for $u \geqq u_0$. Similarly

$$t^{1/\alpha} \int_{u_0}^\infty g_{\alpha/2}(t, u)\, u^{-1/2}\, du \to 0 \quad \text{as} \quad t \to 0\,.$$

Combining these facts we easily see that

$$p_\alpha(t, x_0, x_0) \sim a b\, t^{-1/\alpha}$$

as $t \to 0$. Hence

$$U^\lambda(x_0, x_0) = \int_0^\infty e^{-\lambda t} p_\alpha(t, x_0, x_0)\, dt$$

is infinite if $0 < \alpha \leqq 1$. This implies that $\{x_0\}$ is polar for R_α and so the first statement follows. If $1 < \alpha \leqq 2$ an Abelian theorem for Laplace transforms implies that

$$U^\lambda(x_0, x_0) \sim d \lambda^{1/\alpha - 1}$$

as $\lambda \to \infty$ where d is a positive constant, and so Corollary 4.3 follows from Theorem 4.1.

Perhaps it is worthwhile to remark that if $\alpha > 1$, then for an arbitrary x, $\dim Z = 1 - 1/\alpha$ almost surely P^x on the set $\{T < \infty\}$, where T is the first hitting time of $\{x_0\}$.

We return now to a general process X on E and we will compute

$$P^{x_0}[X(s) = x_0 \quad \text{for some} \quad s, a \leqq s \leqq a + t]$$

under the following assumption:
(D) X satisfies hypothesis (F) and for each $t > 0$, $x \in E$ the transition function $P_t(x, dy)$ of X is absolutely continuous with respect to ξ, and there exists a density $p(t, x, y)$ jointly measurable in (t, x) such that $t \to p(t, x, x_0)$ is bounded on $[\varepsilon, 1/\varepsilon]$ for each $x \in E$, $\varepsilon > 0$, and satisfying

(i) $$U^\lambda(x, x_0) = \int_0^\infty e^{-\lambda t} p(t, x, x_0)\, dt; \qquad \lambda > 0\,, \quad x \in E\,,$$

(ii) $$p(t + u, x_0, x_0) = \int P_t(x_0, dy)\, p(u, y, x_0); \qquad t, u > 0\,.$$

Note that if X is a process on the real line with stationary independent increments satisfying (4.2), then (D) holds. If $\xi_0 > 0$ then, of course, $p(t, x, x_0) = \xi_0^{-1} P_t(x, \{x_0\})$ and hence (i) and (ii) are automatically satisfied for any density.

Theorem 4.4. *Assuming x_0 is regular for $\{x_0\}$ and (D), let $h(u) = \nu[(u, \infty)]$ where ν is the Lévy measure of τ, If T is the first hitting time of x_0, then for $t > 0$*

$$P^y[T \leqq t] = \xi_0 p(t, y, x_0) + \int_0^t [C + h(t-u)] p(u, y, x_0)\, du$$

where C was defined in Theorem 2.3. (See also (3.5).) Moreover for any $a > 0$

$$P^{x_0}[X(s) = x_0 \quad \text{for some} \quad s, a \leqq s \leqq a + t]$$
$$= \xi_0 p(a+t, x_0, x_0) + \int_0^t [C + h(t-u)] p(a+u, x_0, x_0)\, du\,.$$

Proof. Because of the right continuity of the paths and the fact that x_0 is regular for $\{x_0\}$, the probability appearing in the second statement is just $P^{x_0}[T(\Theta_a w) \leqq t]$. Thus the second statement follows from the first and condition (ii) of (D). To prove the first statement let $F(t) = P^y[T \leqq t]$ for a fixed y. Then

$$\Phi^\lambda(y) = P^y[T < S^\lambda] = P^y[T \leqq S^\lambda] = \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt\,.$$

But $\Phi^\lambda(y) = C(\lambda) U^\lambda(y, x_0)$ and from Theorem 2.5 and (3.5) $C(\lambda) = C + \xi_0 \lambda + {} + g^*(\lambda)$ where $g^*(\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \nu(du)$. It is easily seen that $u h(u) \to 0$ as $u \to 0$ and so we may integrate by parts obtaining

$$\frac{g^*(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda u} h(u)\, du\,.$$

Hence

$$\int_0^\infty e^{-\lambda t} F(t)\, dt = \left[\xi_0 + \frac{C}{\lambda} + \frac{g^*(\lambda)}{\lambda}\right] U^\lambda(y, x_0)$$

and it now follows from the uniqueness and convolution theorems for Laplace transforms that

$$F(t) = \xi_0 p(t, y, x_0) + \int_0^t [C + h(t-u)] p(u, y, x_0)\, du \tag{4.5}$$

for almost all (Lebesgue measure) t. But $F(t)$ is right continuous. On the other hand it is not difficult to see that the integral appearing in (4.5) is continuous in t on $\{t > 0\}$ by making use of the facts that h is decreasing (and hence has at most a countable number of discontinuities) and that p is bounded on $[\varepsilon, 1/\varepsilon]$ for every $\varepsilon > 0$. Thus (4.5) holds for $t > 0$ if $\xi_0 = 0$. If $\xi_0 > 0$ then as pointed out above $p(t, y, x_0) = \xi_0^{-1} P_t(y, \{x_0\})$, and it is known [*1*] that $t \to P_t(y, B)$ is continuous on $\{t > 0\}$ for any Borel set B under assumption (D). Thus the proof of Theorem 4.4 is complete.

Since $\bar{Z}$ is closed $[0, u] - \bar{Z}$ consists of at most countably many disjoint (relatively) open intervals provided $u > 0$. Let $N(u, \varepsilon)$ be the number of such intervals which exceed ε in length.

Theorem 4.5. *If x_0 is not a holding point then as $\varepsilon \to 0$, $N(u, \varepsilon)/h(\varepsilon) \to A(u)$ with P^{x_0} probability one.*

Proof. If $C = 0$ and h is continuous this may be proved exactly as in [12, section 6.3]; $h(\varepsilon) \to \infty$ as $\varepsilon \to 0$ since x_0 is not a holding point. If h is not continuous the proof in [*12*] may still be carried through by decomposing h into a continuous component and a pure jump component. Finally if $C > 0$ standard considerations show that the conclusion of Theorem 4.5 is still valid. We will not enter into the details.

One final remark: Suppose (D) holds and $\xi_0 = C = 0$, then the distribution of $A(u)$ under P^{x_0} may be computed just as in section 3 of [*8*]. In particular the moments of $A(u)$ are given by

$$E^{x_0}[A(u)^k] = k! \int_0^u p_k(t)\, dt; \; k \geqq 1$$

where $p_1(t) = p(t, x_0, x_0)$ and

$$p_{k+1}(t) = \int_0^t p_k(t-u)\, p_1(u)\, du, \quad k \geqq 1 .$$

Thus $A(u)$ has a generalized "Mittag-Leffler" distribution based on the generalized "powers" p_k.

5. The absolute continuity of ν

In this section x_0 is a fixed point such that $\{x_0\}$ is not semi-polar. We use the same notation as in the previous sections. In particular T is the hitting time of $\{x_0\}$, ν the Lévy measure of τ (or more accurately of the corresponding subordinator), and $h(u) = \nu[(u, \infty)]$. We will prove the following two theorems.

Theorem 5.1. *Suppose that $P^y(T \in dt)$ is absolutely continuous on $(0, \infty)$ for all y. Then ν is absolutely continuous.*

Theorem 5.2. *Suppose condition (D) of section 4 holds with $\xi_0 = 0$, then $P^y(T \in dt)$ is absolutely continuous on $(0, \infty)$ for all y.*

We will give the proof of Theorem 5.2 first since it is simpler than the proof of Theorem 5.1.

Proof of Theorem 5.2. Let $F(t) = P^y(T \leqq t)$. Note that $dF = \varepsilon_0$ if $y = x_0$, while dF is concentrated on $(0, \infty)$ if $y \neq x_0$. Now according to Theorem 4.4 $(\xi_0 = 0)$ we have for $t > 0$

$$F(t) = \int_0^t [C + h(t-u)]\, p(u, y, x_0)\, du .$$

It will suffice to show $G(t) = \int_0^t h(t-u) p(u) du$ is absolutely continuous as a measure on $(0, \infty)$, where we have written $p(u)$ for $p(u, y, x_0)$. It will be convenient to define $p(u) = 0$ for $u \leqq 0$. Let Φ_0 be the characteristic function of $(1, \infty)$ and Φ_n the characteristic function of $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ for $n \geqq 1$. Define $h_n(t) = \Phi_n(t) h(t)$. Integrating by parts one obtains

$$G(t) = \sum_{n=0}^{\infty} \int_0^t h_n(t-u) p(u)\, du$$

$$= \int_0^t \left[-\sum_{n=0}^{\infty} \int_0^{\infty} p(s-u)\, dh_n(u) \right] ds,$$

and since all quantities are non-negative the expression in square brackets is finite for almost all s. Hence G is absolutely continuous on $(0, \infty)$.

Proof of Theorem 5.1. We begin by introducing some terminology. For a fixed w and $b > 0$ we say that $X(\cdot, w)$ has an *x_0-free interval of length exactly b* if there is a number $a > 0$ such that $X(t, w) \neq x_0$ for all t in $(a, a+b)$, $X(a+b, w) = x_0$, and for all $\delta > 0$ there exists t in $(a-\delta, a]$ such that $X(t, w) = x_0$. The terminology is not perfect since it rules out the situation in which $X(t, w) \neq x_0$ for all $t < b$ and $X(b, w) = x_0$. Given a set J we will say that $X(\cdot, w)$ has an *x_0-free interval of length in J* if it has an x_0-free interval of length exactly b for some b in J.

Now suppose that ν is not absolutely continuous with respect to Lebesgue measure (which we will denote by m in the present proof); then there exists a compact set J contained in $[\eta, \infty)$ for some $\eta > 0$ such that $\nu(J) > 0 = m(J)$. Since $\nu(J)$ is the expected number jumps of the subordinator $Y(t)$ (notation of Theorem 2.3) with magnitude in J, and since $\{Y(t), t \geqq 0\}$ and S^C are independent, it follows that the event

$$(5.1) \qquad \{\tau(t) - \tau(t-) \in J \text{ and } \tau(t) < \infty \text{ for some } t \geqq 0\}$$

has positive P^{x_0} measure. Recalling the relationship between A and τ, and making use of Theorem 1.6 and the fact that a point of increase of A is either a point of right increase or the limit from the left of points of right increase, one sees that the event in (5.1) is P^{x_0} almost surely contained in the set

$$(5.2) \qquad \{X \text{ has an } x_0\text{-free interval of length in } J\}.$$

Thus the proof of Theorem 5.1 is reduced to showing that the set in (5.2) is P^{x_0} null. It is necessary to treat certain measurability problems before proving this last statement.

As above J is a non-void compact set contained in $[\eta, \infty)$ for some $\eta > 0$. For a fixed w we say that $r > 0$ is a *J point for $X(\cdot, w)$* if;

(i) $X(r, w) = x_0$.

(ii) There exists t in J such that $X(s, w) \neq x_0$ for all s in $(r-t, r)$ and for every $\delta > 0$ there exists u in $(r-t-\delta, r-t]$ with $X(u, w) = x_0$.

If $\{r_n\}$ is a sequence of J points for $X(\cdot, w)$ which converges to r, then it is obvious that r is a J point for $X(\cdot, w)$. In fact, r_n must equal r for all n such that $|r_n - r| < \eta/2$.

We now define

$$T_J(w) = \inf\{r : r \text{ is a } J \text{ point for } X(\cdot, w)\}. \tag{5.3}$$

According to the above discussion $T_J(w)$ is a J point of $X(\cdot, w)$ provided it is finite. The next few paragraphs are devoted to showing that T_J is a stopping time.

Given positive numbers ε, r_1, r_2 with $r_1 < r_2$, let

$$\begin{aligned}\Delta(r_1, r_2, \varepsilon) = \{w : {} & X(t, w) = x_0 \text{ for some } t \text{ in } [r_2, r_2 + \varepsilon);\\ & X(t, w) \neq x_0 \text{ for all } t \text{ in } (r_1 + \varepsilon, r_2);\\ & X(t, w) = x_0 \text{ for some } t \text{ in } (r_1 - \varepsilon, r_1 + \varepsilon)\}.\end{aligned}$$

Clearly $\Delta(r_1, r_2, \varepsilon)$ is in $\mathscr{F}_{r_2+\varepsilon}$. Now let $\{t_k\}$ be a countable dense subset of J. Given $b > 0$ define

$$\Delta = \bigcup_m \bigcap_{n>m} \bigcup_{q \leqq b} \bigcup_k \Delta\left(q - t_k, q, \frac{1}{n}\right)$$

where q is rational. It is immediate that $\Delta \in \mathscr{F}_b$ and we will now show that $\Delta = \{T_J \leqq b\}$.

Suppose w is such that $T_J(w) \leqq b$; then there exist numbers $r \leqq b$ and t in J, which will be held fixed, satisfying conditions (i) and (ii) in the definition of a J point given above. Given a positive integer n choose a rational $q \leqq b$ such that $q < r < q + \frac{1}{n}$, and then choose t_k from our dense subset of J such that $|t - t_k| < \frac{1}{n} - (r - q)$. Elementary considerations now show that w is in $\Delta\left(q - t_k, q, \frac{1}{n}\right)$ with the above choice of q and t_k. Therefore $\{T_J \leqq b\} \subset \Delta$.

Conversely suppose w is in Δ; then there exists an n_0 such that for all $n > n_0$ there exist a rational $q_n \leqq b$ and a t_n in $\{t_k\}$ such that w is in $\Delta\left(q_n - t_n, q_n, \frac{1}{n}\right)$. By passing to a subsequence we may assume $q_n \to r \leqq b$ and $t_n \to t \in J$. In order to show $\Delta \subset \{T_J \leqq b\}$ it will suffice to show that r is a J point of $X(\cdot, w)$.

(i) If $q_n \geqq r$ for infinitely many n, then the right continuity of $X(\cdot, w)$ implies that $X(r, w) = x_0$. Suppose that $q_n < r$ for all large n and $X(r, w) \neq x_0$. From the right continuity one obtains a $\delta > 0$ such that $X(r + u, w) \neq x_0$ for all u in $[0, \delta]$. Choose N such that $1/N < \delta$ and $0 \leqq r - q_n < \eta$ for all $n \geqq N$; then there exists a point u in $[q_N, r)$ such that $X(u, w) = x_0$. But then u is in $\left(q_m - t_m + \frac{1}{m}, q_m\right)$ for all large m and this contradicts the fact that w is in $\Delta\left(q_m - t_m, q_m, \frac{1}{m}\right)$ for $m > n_0$. Thus $X(r, w) = x_0$.

(ii) Since $X(u, w) \neq x_0$ for any u in $\left(q_n - t_n + \frac{1}{n}, q_n\right)$ and $q_n \to r$, $q_n - t_n + \frac{1}{n} \to r - t$, it follows that $X(u, w) \neq x_0$ for any u in $(r - t, r)$. Finally given $\delta > 0$ we have $\left(q_n - t_n - \frac{1}{n}, q_n - t_n + \frac{1}{n}\right)$ contained in $(r - t - \delta, r - t + \delta)$ for all large n, and so it follows from the right continuity of $X(\cdot, w)$ that for every $\delta > 0$ there is a u in $(r - t - \delta, r - t]$ such that $X(u, w) = x_0$. Thus r is a J point for $X(\cdot, w)$, and hence the proof that T_J is a stopping time is complete.

Clearly

$$\{X \text{ has an } x_0\text{-free interval of length in } J\} = \{T_J < \infty\},$$

and so to complete the proof of Theorem 5.1, we must show that $P^{x_0}(T_J < \infty) = 0$.

We will need one more stopping time. If $q > 0$ let $T_q = \inf\{t \geqq q : X(u) \neq x_0$ for all u in $(t-q, t)\}$. An argument similar to (but much simpler than) that just given for T_J shows that T_q is a stopping time. We are finally ready to show that $P^{x_0}[T_J < \infty] = 0$ under the assumption of Theorem 5.1 and $m(J) = 0$.

Recall that J is a compact subset of $[\eta, \infty)$ with $\eta > 0$. Choose q such that $0 < q < \eta$ and let R be the stopping time T_q defined above. Define stopping times as follows: (here T is the first hit of $\{x_0\}$)

$$\begin{array}{ll} K_1 = T & L_1 = K_1 + R(\Theta_{K_1}) \\ \vdots & \vdots \\ K_{n+1} = L_n + T(\Theta_{L_n}) & L_{n+1} = K_{n+1} + R(\Theta_{K_{n+1}}) \\ \vdots & \vdots \end{array}$$

Note that $K_{n+1} \geqq L_n$ and that $L_n \geqq q + K_n$. Also $X(u, w) \neq x_0$ if $L_n(w) < u < K_{n+1}(w)$. Since $X(T_J) = x_0$ on $\{T_J < \infty\}$ the events $\{L_n < T_J < K_{n+1}\}$ are impossible for any n. It is easy to see that if Q is any stopping time such that $X(Q) = x_0$ on $\{Q < \infty\}$, then $T_J = Q + T_J(\Theta_Q)$ on $\{Q < T_J\}$. But $P^{x_0}[T_J \leqq R;\ T_J < \infty] = 0$ and so for any n and y

$$\begin{aligned} & P^y[K_n < T_J \leqq L_n;\ T_J < \infty] \\ & \leqq P^y[T_J(\Theta_{K_n}) \leqq R(\Theta_{K_n});\ T_J(\Theta_{K_n}) < \infty;\ K_n < \infty] \\ & \leqq P^{x_0}[T_J \leqq R;\ T_J < \infty] = 0 . \end{aligned}$$

Suppose that $P^{x_0}[T_J < \infty] > 0$. Clearly $T_J > K_1$ almost surely P^{x_0} and $K_n \to \infty$ as $n \to \infty$. Thus the above remarks imply that for some n

$$P^{x_0}[L_n < T_J = K_{n+1} < \infty] > 0 .$$

Using the fact that $K_{n+1} = L_n$ almost surely on $\{X(L_n) = x_0\}$ (since x_0 is regular for x_0) we see that $T_J - L_n$ is in $J - q$ almost surely on $\{L_n < T_J = K_{n+1} < \infty\}$. Hence

$$\begin{aligned} 0 & < P^{x_0}[L_n < T_J = K_{n+1} < \infty] \\ & \leqq P^{x_0}[K_{n+1} - L_n \in J - q;\ L_n < \infty] \\ & = E^{x_0}\{P^{X(L_n)}[T \in J - q];\ L_n < \infty\} = 0 , \end{aligned}$$

since $m(J - q) = m(J) = 0$ and T has an absolutely continuous distribution on $(0, \infty)$ under P^x for all x. This completes the proof of Theorem 5.1.

Similar, but simpler, considerations show that if $P^x(T \in dt)$ is continuous on $(0, \infty)$ for all x then ν is continuous.

6. Local Times: Non-Regular Case

In this section x_0 will denote a fixed point of E such that $\{x_0\}$ is not polar and x_0 is not regular for $\{x_0\}$, that is $\{x_0\}$ is semi-polar. As before T denotes the first hitting time of $\{x_0\}$, $\Phi^\lambda(x) = E^x(e^{-\lambda T})$, and $\Psi^\lambda = \Phi^1 - (\lambda - 1) U^\lambda \Phi^1$. Our hypotheses on x_0 are equivalent to $\Phi^\lambda(x_0) < 1$ and $\Phi^\lambda \neq 0$. One could use the method of section 1 to construct the local time of X at x_0. However, in the present case, the local time has a very simple structure and we will give an elementary construction of it.

Define $R_0 = 0$, $R_1 = T$, and $R_{n+1} = R_n + T(\Theta_{R_n})$ for $n \geqq 1$, so that R_n is the time of the n-th visit to x_0. Note that $R_n < R_{n+1}$ almost surely on $\{R_n < \infty\}$, since the fact that x_0 is not regular for $\{x_0\}$ implies that $P^{x_0}(T > 0) = 1$.

Lemma 6.1. *There are almost surely only a finite number of the R_n's contained in any finite interval $[0, t_0]$.*

Proof. It suffices to show that almost surely only a finite number of the events $\{R_n < S^\lambda\}$ ocur, where, as usual, S^λ is an exponentially distributed random variable independent of the process X. For each x we have

$$\begin{aligned} P^x[R_{n+1} < S^\lambda] &= P^x[R_n + T(\Theta_{R_n}) < S^\lambda] \\ &\leqq P^x[T(\Theta_{R_n}) < S^\lambda;\ R_n < S^\lambda] \\ &= P^{x_0}(T < S^\lambda)\, P^x(R_n < S^\lambda), \end{aligned}$$

and so

$$P^x(R_{n+1} < S^\lambda) \leqq [P^{x_0}(T < S^\lambda)]^n\, P^x(T < S^\lambda).$$

But $P^{x_0}(T < S^\lambda) = \Phi^\lambda(x_0) < 1$ and hence Lemma 6.1 follows from the Borel-Cantelli Lemma.

We now define $V(t)$ to be the number of visits to x_0 by X during the interval $[0, t]$. Clearly V is an additive functional of X which is constant except for jumps of magnitude one at the R_n's. Lemma 6.1 implies that $V(t)$ is almost surely finite for each t.

Theorem 6.2. *Let f be a bounded continuous function. Then for each $\lambda > 0$*

$$E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dV(t) = B\, \Psi^\lambda(x) f(x_0)$$

where $B = ([1 - \Phi^\lambda(x_0)]\,[1 - (\lambda - 1)\, U^\lambda\, \Phi^1(x_0)])^{-1}$ is positive and independent of λ.

Proof. The computations leading to formula (1.2) are independent of the assumption that x_0 is regular for $\{x_0\}$ and so we have

$$\Psi^\lambda = [1 - (\lambda - 1)\, U^\lambda \Phi^1(x_0)]\, \Phi^\lambda. \tag{6.1}$$

Under the present assumptions on x_0, $\Phi^1(x_0) < 1$, and so

$$1 - (\lambda - 1)\, U^\lambda \Phi^1(x_0) > \Phi^1(x_0) - (\lambda - 1)\, U^\lambda \Phi^1(x_0) \geqq 0.$$

The last inequality following as in the proof of Lemma 1.1.

From the definition of V and the fact that $X(R_n) = x_0$ almost surely on $\{R_n < \infty\}$ we obtain

$$E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dV(t) = \sum_{n=1}^\infty E^x[e^{-\lambda R_n} f(X_{R_n})]$$
$$= f(x_0) \sum_{n=1}^\infty \Phi^\lambda(x)\, [\Phi^\lambda(x_0)]^{n-1}$$
$$= f(x_0)\, [1 - \Phi^\lambda(x_0)]^{-1}\, \Phi^\lambda(x).$$

In view of (6.1) this proves Theorem 6.1 except for showing that B is independent of λ.

Define $B(\lambda) = ([1 - \Phi^\lambda(x_0)]\,[1 - (\lambda - 1)\, U^\lambda \Phi^1(x_0)])^{-1}$. If S^λ and S^μ are independent, a calculation similar to that used in the proof of Theorem 1.2 yields

$$E^x\, V(S^\lambda \wedge S^\mu) = E^x\, V(S^\lambda) - E^x\{V(S^\lambda) - V(S^\mu);\, S^\mu < S^\lambda\}$$
$$= B(\lambda)\, \Psi^{\lambda+\mu}(x).$$

Since $\Psi^{\lambda+\mu}$ is a non-zero multiple of $\Phi^{\lambda+\mu}$ there is at least one x for which $\Psi^{\lambda+\mu}(x) \neq 0$, and hence $B(\lambda) = B(\mu)$. This completes the proof of Theorem 6.2.

If we define $A(t) = B^{-1}\, V(t)$, then A is an additive functional of X that is uniquely determined by the relationship ([*13*, Pt. II, Th. 4.4])

$$E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dA(t) = \Psi^\lambda(x)\, f(x_0) \tag{6.2}$$

for all bounded continuous f and all x. Thus we may call A (or V) the local time at x_0 under the present assumptions.

If X satisfies (F), then using Proposition 18.5 of [*9*] one can show that the sample paths $t \to X_t(w)$ are almost surely continuous at T on $\{T < \infty\}$. It follows from this that V is a natural additive functional (that is, $t \to X_t(w)$ and $t \to A(t, w)$ have almost surely no common discontinuities). In the general case $t \to X_t(w)$ may be discontinuous at T and so V need not be natural.

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University of Washington
Seattle 5, Washington/USA

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