Z. Wahrscheinlichkeitstheorie 3, 32—49 (1964)

Ratio Limit Theorems for Cascade Processes *

By

P. E. Ney

1. Introduction and summary

In this paper we study the evolution of a family of particles each of which is characterized by a real valued quantity which for convenience, we call its energy. We assume that the process originates at "time" $t = 0$ with a single "parent" particle of energy X_0, which after a time T splits into N other particles of energies $X_1, \ldots, X_N$ respectively. Each of these "offspring" particles then behaves as a parent; its behavior depending only on its energy, and being independent of any other existing particles or of the history of the process. The quantities T, N, $X_1, \ldots, X_N$ are random variables. Let $G(t) = P\{T \leqq t\}$ be the distribution function of T; $q_j = P\{N = j\}$ be the probability function of N; and $\Phi_j(x_1, \ldots, x_j \mid x_0) = P\{X_1 \leqq x_1, \ldots, X_j \leqq x_j \mid x_0\}$ be the conditional joint distribution function of $X_1, \ldots, X_j$, given that a parent of energy x_0 has given rise to j offspring.

Let $N(x, t \mid x_0)$ be the number of particles existing at time t which have energy equal to at least x, given that the process started at $t = 0$ with one particle of energy x_0. Write $N(x, t \mid 1) = N(x, t)$; and let $p_n(x, t \mid x_0) = P\{N(x, t \mid x_0) = n\}$ and $p_n(x, t) = P\{N(x, t) = n\}$.

The process $N(0, t \mid x_0)$ is simply the total number of particles at t, and is called a branching process. These processes have been extensively studied by Harris [*4*], [*5*], Bellman and Harris [*1*], Levinson [*7*], and others.

In the case when $q_2 = 1$ the process is called binary. If in addition $\Phi_2(x_1, x_2 \mid x_0)$ admits a density function, which further satisfies certain homogeneity and "conservation of energy" requirements, and if G is the exponential distribution, then the process becomes what is usually called the binary nucleon cascade. In this setting the parameter t usually plays the role of the depth of an absorber, rather than time, and the fact that G is exponential is expressed by saying that the cascade is in homogeneous matter. There is an extensive literature on these cascades which may be found summarized in Bharucha-Reid [*2*].

Clearly the language of time and energy is not essential, and the general cascade model defined in the first paragraph can be applied to a variety of problems of population growth in the physical and biological sciences. Another aspect of the general process, namely the total energy of all particles existing at t, was studied by the author in [*10*] and [*11*]. These papers contain further references to treatments of other aspects of population processes. The most comprehensive work on this field is likely to be the book of T. E. Harris on "Branching Processes", publication of which is expected soon.

---

* Research sponsored by the Office of Naval Research under contract number Nonr-401 (43).

The main purpose here is to study $p_n(x, t)$ for large t. Under an assumption that no new energy can be created upon collision, i. e., that $P\{X_1 + \cdots + X_N \leqq X_0\} = 1$, one would expect that for sufficiently large t all particles will have energy $< x$. This means $p_0(x, t) \to 1$, and $p_n(x, t) \to 0$ for $n \geqq 1$. A more refined question concerns the relative magnitude of $p_n(x, t)$ and $p_m(x, t)$. The main result of this paper is to show that for $n > m$

$$p_n(x, t)/p_m(x, t) \to 0 \quad \text{as} \quad t \to \infty . \tag{1.1}$$

On the way to proving (1.1) we shall also show that (under suitable conditions on G and Φ_j), for any $x' < x''$ and any $n \geqq 1$

$$p_n(x'', t)/p_n(x', t) \to 0 \quad \text{as} \quad t \to \infty . \tag{1.2}$$

Both (1.1) and (1.2) are easily shown to be equivalent to the following statements about the factorial moments of the process. Let

$$\mu_k(x, t) = \sum_{n=k}^{\infty} \binom{n}{k} k!\, p_n(x, t) . \tag{1.3}$$

Then we shall see that the assertion (1.1) for all $n > m$ is equivalent to the assertion that for all $n > m$

$$\mu_n(x, t)/\mu_m(x, t) \to 0 ; \tag{1.4}$$

and that (1.2) for all n is equivalent to

$$\mu_n(x'', t)/\mu_n(x', t) \to 0 . \tag{1.5}$$

Results of the form (1.1), (1.4), and (1.5) were first obtained by J. LOPUSZANSKI for the special case of the binary cascade in homogeneous matter (exponential G) (see URBANIK [*13*], and LOPUSZANSKI [*8*], [*9*]). The arguments in these papers use closely the properties of the exponential distribution. The distribution Φ_2 is assumed to admit a density Φ_2, but no regularity conditions on the latter are explicitly stated. There are some gaps in the argument, and in fact (1.2) and (1.5) are not correct without further restrictions on φ_2. This can be seen by the following counterexample. Let $q_2 = 1$, $G(t) = 1 - e^{-t}$ for $t \geqq 0$, $= 0$ elsewhere. Take any $0 < \alpha < \frac{1}{4}$. Let $F(x) = 0$ for $x \leqq \frac{1}{2} - \alpha$, $= (2x + 2\alpha - 1)/2\alpha$ for $\frac{1}{2} - \alpha < x \leqq \frac{1}{2}$, $= 1$ for $x > \frac{1}{2}$. Let $\Phi_2(x_1, x_2) = F(x_1) F(x_2)$. Then an elementary calculation shows that for sufficiently small α, $p_1(0.40, t) = p_1(0.41, t) = 2te^{-t} - e^{-t} + 2e^{-2t}$; and $p_2(0.40, t) = p_2(0.41, t) = e^{-t} - e^{-2t}$; $p_n = 0$ for $n \geqq 3$. This contradicts (1.2) and (1.5). The proofs of (1.1), and (1.4) in [*8*], [*9*] rest on (1.5), and hence it is not clear if these results are true for arbitrary φ_2, even for the special case when G is exponential. (We have not, however, been able to construct a counterexample.) A summary of the URBANIK-LOPUSZANSKI results may also be found in BARUCHA-REID's book ([*2*], pages 273–4).

Since in the present paper we deal with a general class of distributions G, our methods are entirely different from those of [*8*], [*9*]. We also, however, show (1.5) first, and go from there to the remaining results.

Another question of interest is the following. We have remarked that $p_n(x, t) \to 0$ for $n \geqq 1$, but if we take $x = x_t$ a function of t which is decreasing at a suitable rate, then one might hope to get more refined results on the asymptotic form of $p_n(x_t, t)$. Results of this character will be given in part II of this paper.

2. The integral equation of the process

Our formal starting point is an integral equation in the probability function $p_n(x, t \mid x_0)$. We shall show that this equation has a unique bounded solution which is a probability function, and thence all results will be derived from the equation.

First, however, we give a brief heuristic derivation of the equation from the physical situation described in section 1. First take $n > 1$. Starting with a single particle of energy x_0, there are several mutually exclusive ways in which the process may, at time t, arrive in a state in which exactly n particles have energy at least equal to x_0. Each way involves the creation, at some time $y < t$, of some number j of offspring of energies $x_1, \ldots, x_j$, as a result of the splitting of the initial particle; and the subsequent creation, respectively, of $n_1, \ldots, n_j$ particles of energies at least x during the remaining time $t - y$ by the j new cascades originating at time y. The numbers $n_1, \ldots, n_j$ are subject to the restriction $n_1 + \cdots + n_j = n$. Since each of the j new cascades stemming from the particles created at y now act as new independent processes, we may multiply the probabilities of the above events and then sum over all possible values of $j, y, x_1, \ldots, x_j, n_1, \ldots, n_j$. This yields (for $n > 1$)

$$(2.1) \quad p_n(x, t \mid x_0) = \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int_0^{\infty} \cdots \int_0^{\infty} \Phi_j(dx_1, \ldots, dx_j \mid x_0) \sum_{B_{j,n}} \prod_{i=1}^{j} p_{n_h}(x, t - y \mid x_i),$$

where $B_{j,n} = \{(n_1, \ldots, n_j) : 0 \leqq n_h,\ h = 1, \ldots, j;\ n_1 + \cdots + n_j = n\}$.
(The symbol $\Phi_j(dx_1, \ldots, dx_j \mid x_0)$ means that the integral is the Stieltjes integral with respect to the distribution function $\Phi_j(x_1, \ldots, x_j \mid x_0)$.)

If $n = 0$ or 1, then it is also possible to arrive at a state of n particles of energy at least x by having no collisions up to time t. If $n = 0$, this is the case for $x > x_0$; if $n = 1$, for $x \leqq x_0$. Letting $\delta_{ij} = 1$ if $i = j$, and zero otherwise; and $D(x) = 1$ if $x \geqq 0$, and zero otherwise; we may incorporate these boundary cases in (2.1) and obtain for all $n \geqq 0$

$$(2.2) \quad \begin{aligned} p_n(x, t \mid x_0) = {} & [\delta_{0n} D(x - x_0) + \delta_{1n} D(x_0 - x)]\,[1 - G(t)] + \delta_{0n}\, q_0\, G(t) + \\ & + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \sum_{B_{j,n}} \prod_{i=1}^{j} p_{n_h}(x, t - y \mid x_i). \end{aligned}$$

Let $Q(s, x, t \mid x_0) = \sum_{n=0}^{\infty} p_n(x, t \mid x_0) s^n$, $(|s| \leqq 1)$. Then multiplying (2.2) thru by s^n and summing over n yields

$$(2.3) \quad \begin{aligned} Q(s, x, t \mid x_0) = {} & [D(x - x_0) + s\, D(x_0 - x)]\,[1 - G(t)] + q_0\, G(t) + \\ & + \sum_{j=1}^{\infty} q_j \int_0^t d\,G(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \prod_{i=1}^{j} Q(s, x, t - y \mid x_i). \end{aligned}$$

Various special cases of (2.3) are well known and have been studied extensively. For example, if $G(t) = 1 - e^{-t}$ for $t \geqq 0$, $= 0$ otherwise, then differentiation of (2.3) with respect to t yields

$$(2.4) \quad \left[\frac{\partial}{\partial t} + 1\right] Q(s, x, t \mid x_0) = q_0 e^{-t} + \sum_{i=1}^{\infty} q_j \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \prod_{i=1}^{j} Q(s, x, t \mid x_i).$$

If it is further assumed that $q_2 = 1$, that Φ_2 satisfies the homogeneity condition $\Phi_2(kx_1, kx_2 \mid kx_0) = \Phi_2(x_1, x_2 \mid x_0)$, and if $\Phi_2(x_1, x_2 \mid 1)$ admits a density $\varphi_2(x_1, x_2)$, then we shall see below that (2.4) can be reduced to

$$(2.5)\qquad \left[\frac{\partial}{\partial t} + 1\right] Q(s, x, t) = \int\int \varphi_2(x_1, x_2)\, Q\left(s, \frac{x}{x_1}, t\right) Q\left(s, \frac{x}{x_2}, t\right) dx_1\, dx_2\,,$$

where $Q(s, x, t) = Q(s, x, t \mid 1)$. Equation (2.5) is usually called the JANOSSY G-equation (see L. JANOSSY [6]). For a summary of work relating to (2.5) see Chapter 5 of BARUCHA-REID [2]. We shall now prove:

Theorem 1. *If* $\nu = \sum_{n=0}^{\infty} n q_n < \infty$, *and* $G(0) = 0$, *then the set of equations* (2.2) *have a unique bounded solution* $p_n(x, t \mid x_0)$. *This solution is a probability function, i. e.,* $p_n(x, t \mid x_0) \geqq 0$ *and* $\sum_{n=0}^{\infty} p_n(x, t \mid x_0) = 1$. $Q(s, x, t \mid x_0) = \sum p_n(x, t \mid x_0) s^n$ *is the unique bounded* (for $|s| \leqq 1$) *solution of* (2.3).

Proof. We proceed by induction on n. First take $n = 0$. Then (2.2) reads

$$(2.6)\qquad p_0(x, t \mid x_0) = D(x - x_0)[1 - G(t)] + q_0 G(t) + \\ + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \prod_{i=1}^{j} p_0(x, t - y \mid x_i)\,.$$

Define $p_0^{(0)}(x, t \mid x_0) \equiv 0$, and for $k \geqq 0$ define

$$(2.7)\qquad p_0^{(k+1)}(x, t \mid x_0) = D(x - x_0)[1 - G(t)] + q_0 G(t) + \\ + \sum_{j=1}^{n} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \prod_{i=1}^{j} p_0^{(k)}(x, t - y \mid x_i)\,.$$

Then $0 \leqq p_0^{(1)}(x, t \mid x_0) \leqq 1 - G(t) + q_0 G(t) \leqq 1$, and if $0 \leqq p_0^{(k)}(x, t \mid x_0) \leqq 1$, then $0 \leqq p_0^{(k+1)}(x, t \mid x_0) \leqq 1 - G(t) + \sum_{j=0}^{\infty} q_j G(t) \leqq 1$. Hence by induction on k

$$0 \leqq p_0^{(k)}(x, t \mid x_0) \leqq 1 \quad \text{for} \quad k = 0, 1, 2, \ldots.$$

But now it is straight forward to show that

$$\left| \prod_{i=1}^{j} p_0^{(k)}(x, t \mid x_i) - \prod_{i=1}^{j} p_0^{(k-1)}(x, t \mid x_i) \right| \\ (2.8)\qquad \leqq \sum_{i=1}^{j} |p_0^{(k)}(x, t \mid x_i) - p_0^{(k-1)}(x, t \mid x_i)| \cdot \left| \prod_{h=1}^{i-1} p_0^{(k-1)}(x, t \mid x_h) \prod_{h=i+1}^{j} p_0^{(k)}(x, t \mid x_h) \right| \\ \leqq \sum_{i=1}^{j} |p_0^{(k)}(x, t \mid x_i) - p_0^{(k-1)}(x, t \mid x_i)|\,.$$

Hence

$$|p_0^{(k+1)}(x, t \mid x_0) - p_0^{(k)}(x, t \mid x_0)| \\ (2.9)\qquad \leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \left| \prod_{i=1}^{j} p_0^{(k)}(x, t - y \mid x_i) \right. \\ \left. - \prod_{i=1}^{j} p_0^{(k-1)}(x, t - y \mid x_1) \right|$$

$$\leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j | x_0) \sum_{i=1}^{j} | p_0^{(k)}(x, t-y | x_i) - p_0^{(k-1)}(x, t-y | x_i) | .$$

But $|p_0^{(1)}(x, t | x_0) - p_0^{(0)}(x, t | x_0)| \leqq 1$, and hence by induction on k in (2.9) we see that

$$|p_0^{(k+1)}(x, t | x_0) - p_0^{(k)}(x, t | x_0)| \leqq \nu^k G_k(t),$$

where $\nu = \sum j q_j < \infty$ by hypothesis.

Now $H(t) = \sum_{n=1}^{\infty} \nu^n G_n(t)$ satisfies the well known renewal equation

$$H(t) = \nu G(t) + \nu \int_0^t H(t-y) dG(y), \tag{2.10}$$

and hence $H(t) < \infty$ for any $t < \infty$ (see e.g. Feller [3]). Since $G_k(t)$ is non-decreasing, so is $H(t)$. Thus $|p_0^{(k+m)}(x, t | x_0) - p_0^{(k)}(x, t | x_0)| \leqq \sum_{i=k}^{\infty} \nu^i G_i(t)$ for all $m \geqq 0$ and all $x, x_0, 0 \leqq t \leqq t' < \infty$. Hence there is a $p_0(x, t | x_0) \leqq 1$ such that

$$p_0^{(k)}(x, t | x_0) \to p_0(x, t | x_0) \tag{2.11}$$

uniformly in x, x_0, and $0 \leqq t \leqq t'$. From (2.7) it also follows that $p_0(x, t | x_0)$ satisfies (2.3).

Now suppose that we have shown for $n = 0, 1, \ldots, N-1$, $(N \geqq 2)$, that there is a function $p_n(x, t | x_0)$ satisfying (2.2) and such that $0 \leqq p_n(x, t | x_0) \leqq 1$. Define $p_N^{(0)}(x, t | x_0) = 0$, and for $k \geqq 0$

$$p_N^{(k+1)}(x, t | x_0) = W_N(x, t | x_0) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j | x_0) \sum_{\alpha=i}^{j} \left\{ p_N^{(k)}(x, t-y | x_\alpha) \prod_{\substack{i=1 \\ i \neq \alpha}}^{j} p_0(x, t-y | x_i) \right\}, \tag{2.12}$$

where

$$W_N(x, t | x_0) = [\delta_{0n} D(x - x_0) + \delta_{1n} D(x_0 - x)][1 - G(t)] + \delta_{0n} g_0 G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j | x_0) \sum_{B'_{j,N}} \prod_{i=1}^{j} p_{n_h}(x, t-y | x_i),$$

and where

$$B'_{jN} = \{(n_1, \ldots, n_j) : (n_1, \ldots, n_j) \in B_{jn}, \quad n_1 \neq N, \ldots, n_j \neq N\}.$$

We shall need to carry along with the induction the fact that $\sum_{i=0}^{n} p_i(x, t | x_0) \leqq 1$. This has been shown for $n = 0$. Suppose it true for $n = 1, \ldots, N-1$. Let

$$\varrho_n(x, t | x_0) = \sum_{i=0}^{n} p_i(x, t | x_0), \quad \text{and} \quad \varrho_n^{(k)} = \sum_{i=0}^{n-1} p_i(x, t | x_0) + p_n^{(k)}(x, t | x_0).$$

Then applying (2.2) for $n = 0, 1, \ldots, N-1$, and (2.12), yields

$$\varrho_N^{(k+1)}(x, t | x_0) \leqq [1 - G(t)] + q_0 G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j | x_0) \prod_{i=1}^{j} \varrho_N^{(k)}(x, t | x_0).$$

Now induction on k yields at once that

$$\varrho_N^{(k)}(x, t \mid x_0) \leqq 1 \tag{2.13}$$

for all $k = 0, 1, 2, \ldots$. From (2.12) it is clear that $p_N^{(k)}(x, t \mid x_0) \geqq 0$, and hence for $k = 0, 1, \ldots$

$$0 \leqq p_N^{(k)}(x, t \mid x_0) \leqq 1 . \tag{2.14}$$

Now by (2.12)

$$\begin{aligned} (2.15)\quad & |p_N^{(k+1)}(x, t \mid x_0) - p_N^{(k)}(x, t \mid x_0)| \\ & \leqq \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, d_j \mid x_0) \sum_{\alpha=1}^{j} |p_N^{(k)}(x, t-y \mid x_\alpha) - p_N^{(k-1)}(x, t-y \mid x_\alpha)| . \end{aligned}$$

But (2.15) is of exactly the same form as (2.9), and hence the same argument as before yields that there is a function $0 \leqq p_N(x, t \mid x_0) \leqq 1$, such that

$$p_N^{(k)}(x, t \mid x_0) \to p_N(x, t \mid x_0) \tag{2.16}$$

uniformly in x, x_0 and $0 \leqq t \leqq t' < \infty$, where $p_N(x, t \mid x_0)$ satisfies (2.2). This completes the induction for the first part of the argument. For the second we need only note that (2.13) and (2.16) imply

$$\varrho_N(x, t \mid x_0) \leqq 1 .$$

To show that these are the unique bounded solutions, suppose that there is another set, say $r_n(x, t \mid x_0)$. Let $u_n(x, t \mid x_0) = e^{-\alpha_n t} |p_n(x, t \mid x_0) - r_n(x, t \mid x_0)|$, where $\alpha_n \geqq 0$ is to be chosen later. Then the same kinds inequalities that led to (2.8) yield

$$u_0(x, t \mid x_0) \leqq \sum_{j=1}^{\infty} q_j \int_0^t e^{-\alpha_0 y} dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j \mid x_0) \sum_{i=1}^{j} u_0(x, t-y \mid x_i). \tag{2.17}$$

Since p_0 and r_0 are bounded (uniformly in x, x_0), it follows that $\sup \{u_0(x, t \mid x_0)\colon x, x_0, 0 \leqq t \leqq t'\} = u(t') < \infty$. Hence taking the supremum of both sides of (2.17) over $x, x_0, 0 \leqq t \leqq t'$, we get

$$1 \leqq \sum j q_j \int_0^{t'} e^{-\alpha_0 y} dG(y) . \tag{2.18}$$

Since by assumption $G(0) = 0$, (2.18) is contradicted by taking α_0 sufficiently large. Writing

$$\begin{aligned} & p_n(x, t \mid x_0) = W_n(x, t \mid x_0) \\ & + \sum q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_i \mid x_0) \sum_{\alpha=1}^{j} \left\{ p_n(x, t-y \mid x_\alpha) \prod_{\substack{i=1 \\ i \neq \alpha}}^{j} p_0(x, t-y \mid x_i) \right\}, \end{aligned}$$

proceeding by induction on n, and supposing that uniqueness has been proved for $n = 1, \ldots, N-1$, we are led to an equation of the form (2.17) for $u_N(x, t \mid x_0)$, and thence to the uniqueness for $n = N$.

We have seen that $p_n(x, t \mid x_0) \geqq 0$, and hence to show that this is a probability function we need only show that $\varrho(x, t \mid x_0) = \sum_{n=0}^{\infty} p_n(x, t \mid x_0) = 1$. But ϱ satisfies

$$(2.19)\quad \varrho(x,t\,|\,x_0) = [1-G(t)] + q_0\,G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int\cdots\int \Phi_j(dx_1,\ldots,dx_j\,|\,x_0) \prod_{i=1}^{j} \varrho(x,t-y\,|\,x_i)\,.$$

An almost direct copy of the uniqueness proof just given shows that (2.19) also has a unique bounded solution. But $\varrho(x,\,t\,|\,x_0) = 1$ is a solution, and hence the only bounded solution.

Turning to (2.3), it follows from the definition of Q that for $|s| \leqq 1$ the latter is a bounded solution of this equation. The uniqueness proof again proceeds exactly as above. This completes the proof.

During the remainder of the paper we shall need an essential homogeneity assumption on the Φ_j-distributions, namely that for all constants $k \geqq 0$, and all $j = 1, 2, \ldots$

$$(\text{A-1})\qquad \Phi_j(kx_1,\ldots,kx_j\,|\,kx_0) = \Phi_j(x_1,\ldots,x_j\,|\,x_0)\,.$$

If this condition is satisfied, then by replacing $p_n(x,\,t\,|\,x_0)$ by $p_n(kx,\,t\,|\,kx_0)$ one may easily show that the latter satisfies (2.2). Hence by the uniqueness part of theorem 1 we have

$$(2.20)\qquad p_n(x,t\,|\,x_0) = p_n(kx,t\,|\,kx_0)\,.$$

Thus $p_n(x,\,t\,|\,x_0) = p_n(x/x_0,\,t\,|\,1)$, and writing $p_n(x,\,t\,|\,1) = p_n(x,\,t)$, $Q(s,\,x,\,t\,|\,1) = Q(s,x,\,t)$, and $\Phi_j(x_1,\,\ldots\,,x_j\,|\,1) = \Phi_j(x_1,\,\ldots\,,x_j)$, we see from (2.2) and (2.3) that these functions satisfy

$$(2.21)\quad p_n(x,t) = [\delta_{0n}D(x-1) + \delta_{1n}D(1-x)][1-G(t)] + \delta_{0n}q_0\,G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int\cdots\int \Phi_j(dx_1,\ldots,dx_j) \sum_{B_{j,n}} \prod_{i=1}^{j} p_{n_i}\left(\frac{x}{x_1},t-y\right)$$

and

$$(2.22)\quad Q(s,x,t) = [D(x-1) + s\,D(1-x)][1-G(t)] + q_0\,G(t) + \sum_{j=1}^{\infty} q_j \int_0^t dG(y) \int\cdots\int \Phi_j(dx_1,\ldots,dx_j) \prod_{i=1}^{j} Q\left(s,\,\frac{x}{x_i},t-y\right).$$

That these equations have unique bounded solutions can be argued as before. We thus have

Corollary 1. *If* $\nu = \sum_{n=0}^{\infty} n q_n < \infty$, $G(0) = 0$, *and* (A-1) *is satisfied, then the equations* (2.21) *have a unique bounded solution* $p_n(x,\,t) = p_n(x,\,t\,|\,1)$, *which is a probability function.* $Q(s,\,x,\,t) = \sum_{n=0}^{\infty} p_n(x,\,t)s^n$ *is the unique bounded* (*for* $|s| \leqq 1$) *solution of* (2.22).

We remark that if $G(t) = 1 - e^{-t}$ then (2.22) can easily be reduced to the form (2.5).

For one of our subsequent results we shall also need a smoothness condition in x_0. For any $\Delta > 0$, let $\Delta\Phi_j(x_1,\,\ldots\,,x_j\,|\,x_0)$ denote the probability (with respect to Φ_j) of the j-cube whose sides are $(x_i,\,x_i+\Delta)$, $i = 1,\,\ldots\,,j$. Then the requirement is that there is an $\varepsilon > 0$ and an $\alpha < \infty$ such that for all $\Delta > 0$, and all k satisfying $1 < k < 1+\varepsilon$, we have

$$(\text{A-2})\qquad \frac{\Delta\Phi_j(x_1,\ldots,x_j\,|\,kx_0)}{\Delta\Phi_j(x_1,\ldots,x_j\,|\,x_0)} \leqq \alpha\,.$$

(Note that qualitatively, the physical condition that higher energy parents tend to have higher energy offspring is consistent with (A-2) with $\alpha = 1$.)

Most of our subsequent analysis will be carried out in terms of the moments of the process, and we hence proceed to write down the equations satisfied by these functions. It will be convenient to work with the factorial moments. The k-th factorial moment is defined to be

$$\mu_k(x, t) = k! \sum_{n=0}^{\infty} \binom{n}{k} p_n(x, t),$$

where we adopt the usual convention that $\binom{n}{k} = 0$ for $n < k$. We introduce our next main assumption. Note that until now there has been no restriction on the range of Φ_j (only that it be zero for negative arguments). We now assume that

$$\text{(A-3)} \qquad \int_{A_j} \cdots \int \Phi_j(dx_1, \ldots, dx_j) = 1 \quad \text{for} \quad j = 1, 2, \ldots,$$

where $A_j = \left\{(x_1, \ldots, x_j) : 0 \leqq x_i,\, i = 1, \ldots, j; \sum_{i=1}^{j} x_i \leqq 1\right\}$. In words this says that the total energy of a family of offspring particles does not exceed the energy of their parent particle (with probability one).

It will become apparent later that this assumption is crucial for our main results to hold. An immediate consequence of (A-3) is the fact that

$$\text{(2.23)} \qquad p_n(x, t) = 0 \quad \text{for} \quad n > \left[\frac{1}{x}\right],$$

where $[x]$ is the largest integer smaller than or equal to x. Heuristically, one may argue this fact by saying that due to (A-3) the total energy of the process at any time t is always at most one, since this is the initial energy, and no new energy is ever created. But then it is impossible to have more than $[1/x]$ particles of energy at least x. Analytically, (2.23) can be easily deduced from (2.21) by induction on n, and using the fact that

$$\text{(2.24)} \qquad p_0(x, t) = 1 \quad \text{for} \quad x > 1.$$

The latter follows from the fact that for $x > 1$, the function 1 satisfies (2.21) for $n = 0$, and from the uniqueness of the solution.

An immediate consequence of (2.23) is that

$$\text{(2.25)} \qquad \mu_k(x, t) = k! \sum_{n=0}^{[1/x]} \binom{n}{k} p_n(x, t)$$

is a finite sum, and hence all moments exist. Another trivial but useful consequence is that

$$\text{(2.26)} \qquad \mu_k(x, t) = 0 \quad \text{for} \quad k > \left[\frac{1}{x}\right].$$

The existence of moments thus being guaranteed, we may multiply both sides of (2.21) by $k! \binom{n}{k}$ and add over n, or differentiate (2.22) k times with respect to s. This yields

$$\text{(2.27)} \qquad \mu_k(x, t) = \nu_k(x, t) + \nu \int_0^t dG(y) \int_0^1 dF(\eta)\, \mu_k\left(\frac{x}{\eta}, t - y\right),$$

where

(2.28) $$F(x) = \nu^{-1} \sum_{j=1}^{\infty} \sum_{i=1}^{j} q_j \Phi_{ij}(x),$$

and

(2.29) $\Phi_{ij}(x) = \Phi_j(1, \ldots, 1, x, 1, \ldots, 1)$, the x being the i-th component;

and where $v_1(x, t) = [1 - G(t)] D(1 - x)$, and for $k \geqq 2$

(2.30) $$v_k(x, t) = \sum_{j=2}^{\infty} q_j \int_0^t dG(y) \int \cdots \int \Phi_j(dx_1, \ldots, dx_j) \sum_{B'_{j,k}} \binom{k}{k_1, \ldots, k_j} \prod_{i=1}^{j} \mu_{k_i}\left(\frac{x}{x_i}, t - y\right).$$

(B'_{jk} was defined below (2.12).) Note that F is in fact a distribution function and that $F(1) = 1$.

It will be convenient to adopt the following notation for convolutions:

$$\int_0^\infty g(t - y)\, dG(y) \equiv g * G$$

$$\int_0^1 f\left(\frac{x}{\eta}\right) dF(\eta) \equiv f \circ F$$

$$\int_0^\infty \int_0^1 h\left(\frac{x}{\eta}, t - y\right) dF(\eta)\, dG(y) \equiv h \circledast [GF],$$

where g, f, and h are any functions of one and two variables respectively for which the above integrals converge. We shall also write $G = G_1$, $G_n = G_{n-1} * G$, $F = F_1$, $F_n = F_{n-1} \circ F$, $F_0 = G_0 = 1$, $F_n^* = 1 - F_n$. Finally set

$$v_k^{(0)}(x, t) = v_k(x, t), \quad \text{and for } n \geqq 1 \text{ set}$$

(2.31) $$v_k^{(n)}(x, t) = v_k^{(n-1)}(x, t) \circledast [GF] = v_k(x, t) \circledast [G_n F_n].$$

Corollary 2. *For all* $k \geqq 1$, $\mu_k(x, t) = \sum_{n=0}^{\infty} \nu^n v_k^{(n)}(x, t)$. *The series converges uniformly in* t *and for* $x \geqq x_0 > 0$, *and is the unique bounded solution of* (2.27) (*bounded in* $0 \leqq t \leqq t' < \infty$, $0 < x_0 \leqq x \leqq 1$).

Proof. It is easy to show that there is a $K < \infty$ such that $v_k(x, t) < K$ for all t and all $x \geqq x_0 > 0$, and hence that $v_k^{(n)}(x, t) < K\, G_n(t) F_n^*(x) \leqq K F_n^*(x_0)$. The fact that $\sum_{i=0}^{\infty} \nu^i F_i^*(x_0) < \infty$ for any $\nu < \infty$ then implies the first part of the corollary. The uniqueness follows by an argument very similar to that of theorem 1.

3. Main results

We shall need regularity assumption on F and G. Let X be the random variable with distribution F. We shall require a smoothness condition on F in the neighborhood of 1. To be precise, we assume that there exists a $\gamma \geqq 0$ such that

(A-4) $$\lim_{x \to 0+} x^{-\gamma} P\{e^{-x} < X < 1\} \quad \text{exists.}$$

It is easy to find various reasonable sufficient conditions for (A-4) to hold; for example that X should have a positive continuous density in an interval $(\varepsilon, 1]$ for some $0 \leqq \varepsilon < 1$, or that X should have a density with a positive continuous derivative of some order in $(\varepsilon, 1]$. That some condition such as (A-4) is needed can be seen from counter-examples of the kind given in section 1.

The requirement on G involves the tails of the distribution. We shall assume that for any $n > 1$ there is an $\alpha_0 > 0$, such that

$$\frac{G(t+\alpha) - G(t)}{G_n(t+\alpha) - G_n(t)} \to 0 \quad \text{as} \quad t \to \infty, \tag{A-5}$$

uniformly for $0 < \alpha \leqq \alpha_0$. Note that a sufficient condition for (A-4) is that the associated densities (g, g_n) of (G, G_n) should satisfy $g(t)/g_n(t) \to 0$ as $t \to \infty$. This is satisfied by many common densities, such as the exponential, gamma, chi-square, truncated normal (truncated at zero), etc.

We are now in a position to state our main results.

Theorem 2. *If* $\nu < \infty$, $G(0) = 0$, *and* (A-1), (A-3), (A-4), *are satisfied, and* G *either satisfies* (A-5) *or is truncated* (*i.e.*, $G(t) = 1$ *for some* $t < \infty$), *then for any* $0 \leqq x' < x'' \leqq 1$, *and any integer* k *such that* $0 < k \leqq [1/x']$,

$$p_k(x'', t)/p_k(x', t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.1}$$

If $k > [1/x]$, *then* $p_k(x, t) = 0$.

Theorem 3. *Under the conditions of theorem 2 plus* (A-2) *we have for all integers* $m \leqq [1/x]$ *and all* $n > m$, *that*

$$p_n(x, t)/p_m(x, t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.2}$$

We break up the proofs into several lemmas. The first states a well known relation between the probability function $\{p_n(x, t)\}$ and the moments $\{\mu_k(x, t)\}$. It enables us to reduce the problem about the probabilities to one about the moments, a device previously used by Lopuszanski in [8] and [9].

Lemma 1. $$p_n(x, t) = \frac{1}{n!} \sum_{k=n}^{[1/x]} \frac{(-1)^{k+n}}{(k-n)!} \mu_k(x, t).$$

Proof. Substitute the definition of $\mu_k(x, t)$ and note (2.26).

In the next two lemmas we summarize some consequences of (A-4) and (A-5).

Lemma 2. *If* F *satisfies* (A-4) *and if* b, b', b'' *are any constants satisfying* $0 < b' \leqq b \leqq b'' < 1$, *then there exist constants* $A < \infty$ *and* $B < 1$, *such that for any integer* $n_0 \geqq 0$ *we have*

$$\frac{F_n^*(x'')}{F_{n_0+n}^*(x')} < A B^n \tag{3.3}$$

for all $n \geqq 1$, *and all* (x', x'') *satisfying* $b' \leqq x' \leqq b x'' \leqq b''$.

Proof. We need a theorem of the form of theorem 2 of [12]. Let $Z_i = -\log X_i$, where $\{X_i\}$ are independent identically distributed with distribution F. Let H denote the distribution function of Z_i. Then H satisfies the hypothesis of theorem 2 [12]. Then the latter theorem states that for $y_1 < y_2$, we have $H_n(y_1)/H_n(y_2) \to 0$ as $n \to \infty$. Although the convergence is not asserted to be uniform for any particular set of (y_1, y_2), the proof is actually strong enough to yield such uniformity, and we shall here require it. It is easily seen that from the definition of $\{B_n\}$ and $\{C_n\}$ in [*12*], it follows that given any $\varepsilon > 0$ and any $0 < y_1 \leqq y_2 < \infty$, there are sequences $\{B_{0n}\}$ and $\{C_{0n}\}$ such that $B_{0n} \to 0$, $C_{0n} = 0(n^c)$, $c < \infty$, and such that for *all* $y \,\varepsilon\, [y_1, y_2]$,

$$(3.4) \qquad (1-\varepsilon)^n (1 - B_{0n}) \leqq \frac{\Gamma(n\gamma + 1) H_n(y)}{[k\Gamma(\gamma+1) y^\gamma]^n} \leqq (1+\varepsilon)^n C_{0n}\,.$$

This implies that for any $0 < a' \leqq a \leqq a'' < \infty$ there are constants $A < \infty$, $B < 1$ such that

$$(3.5) \qquad \frac{H_n(y')}{H_{n_0+n}(y'')} \leqq A\,B^n$$

for all (y', y'') satisfying $a' \leqq y' + a \leqq y'' \leqq a''$. Setting $y' = -\log x''$, $y'' = -\log x'$, $a' = -\log b''$, $a'' = -\log b'$, and $a = -\log b$, we have our lemma.

Lemma 3. *If G satisfies* (A-5) *then for any integer n*

$$(3.6) \qquad \frac{G_{n-1}(t+\alpha) - G_{n-1}(t)}{G_n(t+\alpha) - G_n(t)} \to 0 \quad \text{as} \quad t \to \infty, \quad \textit{uniformly for} \quad 0 < \alpha \leqq \alpha_0\,;$$

and

$$(3.7) \qquad \frac{G_{n-1}(t) - G_n(t)}{G_n(t) - G_{n+1}(t)} \to 0\,.$$

Proof. This can be shown by induction on n. We omit the details of the calculation. (In the case of (3.7) take $\alpha = \infty$.)

The next two lemmas are the main ones in the proof of theorems 2 and 3.

Lemma 4. *If the conditions of theorem 2 are satisfied, and $0 < b' \leqq b \leqq b'' < 1$, then*

$$(3.8) \qquad \mu_k(x'', t)/\mu_k(x', t) \to 0 \quad \text{as} \quad t \to \infty$$

uniformly for $b' \leqq x' \leqq b\,x'' \leqq b''$.

Proof. Suppose that G satisfies (A-5). The proof is by induction on k. First take $k = 1$. By corollary 2 of theorem 1

$$\frac{\mu_1(x'',t)}{\mu_1(x',t)} = \frac{\sum\limits_{n=0}^{\infty} \nu^n v_1^{(k)}(x'',t)}{\sum\limits_{n=0}^{\infty} \nu^n v_1^{(k)}(x',t)} = \frac{\sum\limits_{n=0}^{\infty} \nu^n \{[1-G(t)] * G_n(t)\}\cdot\{D(1-x'') \bigcirc F_n(x'')\}}{\sum\limits_{n=0}^{\infty} \nu^n \{[1-G(t)] * G_n(t)\}\cdot\{D(1-x') \bigcirc F_n(x')\}}$$

$$(3.9) \qquad = \frac{\sum\limits_{n=0}^{\infty} \nu^n [G_n(t) - G_{n+1}(t)]\,[F_n^*(x'')]}{\sum\limits_{n=0}^{\infty} \nu^n [G_n(t) - G_{n+1}(t)]\,[F_n^*(x')]}$$

$$\leqq \frac{\sum\limits_{n=0}^{N} \nu^n [G_n(t) - G_{n+1}(t)]\,[F_n^*(x'')]}{\sum\limits_{n=0}^{N+1} \nu^n [G_n(t) - G_{n+1}(t)]\,[F_n^*(x')]} + \frac{\sum\limits_{n=N+1}^{\infty}}{\sum\limits_{n=N+1}^{\infty}}\,.$$

The first term on the right side of (3.9) goes to zero because of (3.7). On the other hand, by lemma 2, the second term can be made arbitrarily small (uniformly for (x', x'') in the required domain) for N sufficiently large. This takes care of $k = 1$.

Suppose that the lemma is proved for $k = 1, \ldots, M$. By corollary 2

$$(3.10) \qquad \frac{\mu_{M+1}(x'', t)}{\mu_{M+1}(x', t)} = \frac{\sum_{n=0}^{\infty} \nu^n v_{M+1}(x'', t) \circledast [G_n(t) F_n(x'')]}{\sum_{n=0}^{\infty} \nu^n v_{M+1}(x', t) \circledast [G_n(t) F_n(x')]} .$$

Let
$$v_k(x; n; t_1, t_2) = \sum_{j=1}^{\infty} q_j \int_{t_1}^{t_2} dG_n(y) \int \cdots \int_{A_j} \Phi_j(dx_1, \ldots, dx_j) \cdot$$
$$\cdot \sum_{B'_{j,k}} \binom{k}{k_1, \ldots, k_j} \prod_{i=1}^{j} \mu_{k_i}\left(\frac{x}{x_i}, t - y\right).$$

Then for any $0 \leqq A_0 \leqq t$, (3.10)

$$(3.11) \qquad \leqq \frac{\sum_{n=0}^{\infty} \nu^n v_{M+1}(x''; n+1; 0, t-A_0) * F_n(x'')}{\sum_{n=0}^{\infty} \nu^n v_{M+1}(x'; n+1; 0, t-A_0) * F_n(x')} + \frac{\sum_{n=0}^{\infty} \nu^n v_{M+1}(x''; n+1; t-A_0, t) * F_n(x'')}{\sum_{n-0}^{\infty} \nu^n v_{M+1}(x'; n+1; t-A_0, t) * F_n(x')} .$$

We shall show that for suitable choice of A_0 and t, the right side of the above inequality can be made arbitrarily small. First we show that given ε there is $t_0 < \infty$ and an $A_0 \leqq t_0$ such that

$$(3.12) \qquad \frac{v_{M+1}(x''; n+1; 0, t-A_0) * F_n(x'')}{v_{M+1}(x'; n+1; 0, t-A_0) * F_n(x')} < \varepsilon \quad \text{for all} \quad t \geqq t_0 .$$

We start by finding an N_0 such that (3.12) holds for all $n \geqq N_0$. Divide the interval $[x'', 1]$ into m subintervals of equal length δ, where m is chosen large enough so that $\delta < \min\left\{\frac{b'}{2}, \frac{1}{4}(x'' - x')\right\}$. Then for any n the left side of (3.12) $\leqq$

$$(3.13) \qquad \frac{\sum_{k=0}^{m-1} \int_{x''+k\delta}^{x''+(k+1)\delta} v_{M+1}\left(\frac{x''}{\eta}; n+1; 0, t-A_0\right) dF_n(\eta)}{\sum_{k=0}^{m-1} \int_{x'+(k+1)\delta}^{x'+(k+2)\delta} v_{M+1}\left(\frac{x'}{\eta}; n+1; 0, t-A_0\right) dF_n(\eta)}$$

where the inequality may hold since the segments $[x', x' + \delta]$ and $[1 - (3/4) \cdot \cdot (x'' - x'), 1]$ have been deleted from the range of integration of the denominator. Then (3.13) $\leqq$

$$(3.14) \frac{\sum_{k=0}^{m-1} (F_n^*[x'' + k\delta] - F_n^*[x'' + (k+1)\delta]) v_{M+1}\left(\frac{x''}{x'' + (k+1)\delta}; n+1; 0, t-A_0\right)}{\sum_{k=0}^{m-1} (F_n^*[x' + (k+1)\delta] - F_n^*[x' + (k+2)\delta]) v_{M+1}\left(\frac{x'}{x' + (k+1)\delta}; n+1; 0, t-A_0\right)} .$$

Now clearly one can find a $b_0 < 1$ such that for all (x', x'') satisfying $b' \leqq x' \leqq$

$\leqq b x'' \leqq b''$, and $k = 0, \ldots, m-1$, we have

$$\frac{x' + (k+1)\delta}{x'' + k\delta} \leqq b_0; \quad \frac{x' + (k+1)\delta}{x'' + (k+1)\delta} \leqq b_0; \quad \frac{x' + (k+1)\delta}{x' + (k+2)\delta} \leqq b_0. \tag{3.15}$$

Hence by lemma 2

$$\frac{F_n^*[x'' + k\delta]}{F_n^*[x' + (k+1)\delta]}; \quad \frac{F_n^*[x'' + (k+1)\delta]}{F_n^*[x' + (k+1)\delta]}; \quad \frac{F_n^*[x' + (k+2)\delta]}{F_n^*[x' + (k+1)\delta]}$$

all $\to 0$ as $n \to \infty$, uniformly for k, x', x'' in the required ranges. Next note that $x''/[x'' + (k+1)\delta] \geqq x'/[x' + (k+1)\delta]$, and hence that

$$v_{M+1}\left(\frac{x''}{x'' + (k+1)\delta}; n+1; 0, t - A_0\right) \leqq v_{M+1}\left(\frac{x'}{x' + (k+1)\delta}\right); n+1; 0, t - A_0)$$

(since by its definition v_{M+1} is clearly a non-increasing function of its first argument). Hence, dividing the numerator and denominator of (3.14) by $F_n^*[x' + (k+1)\delta]$, we see that the left side of (3.12) goes to zero as $n \to \infty$, and thus is $< \varepsilon$ for $n >$ some sufficiently large N_0.

We now consider the ratio (3.12) for $n \leqq N_0$. By the induction hypothesis we have for $k = 1, \ldots, M$ that

$$\mu_k(x'', t)/\mu_k(x', t) \to 0 \quad \text{as} \quad t \to \infty,$$

uniformly for $b' \leqq x' \leqq b x'' \leqq b''$. Thus given any $A_0 < \infty$ and $\varepsilon > 0$, there is a $t_0 < \infty$ such that for $t \geqq t_0$ and $y \leqq A_0$

$$\mu_k\left(\frac{x''}{x_i}, t - y\right) \leqq \varepsilon \mu_k\left(\frac{x'}{x_i}, t - y\right) \tag{3.16}$$

for (x', x'') in the required range and $x_i \geqq x''$. Therefore by definition of $v_k(x; n; t_1, t_2)$

$$v_{M+1}(x'', n+1, 0, t - A_0) \leqq \varepsilon v_{M+1}(x'; n+1; 0, t - A_0)$$

for all $t \geqq t_0$, $b' \leqq x' \leqq b x'' \leqq b''$.
This in turn implies that for $t \geqq t_0$,

$$\begin{aligned} & v_{M+1}(x''; n+1; 0, t - A_0) * F_n(x'') = \\ & = \int_{x''}^{1} v_{M+1}\left(\frac{x''}{\eta}; n+1; 0, t - A_0\right) dF_n(\eta) \\ & \leqq \varepsilon \int_{x''}^{1} v_{M+1}\left(\frac{x'}{\eta}; n+1; 0, t - A_0\right) dF_n(\eta) \\ & \leqq \varepsilon \int_{x'}^{1} v_{M+1}\left(\frac{x'}{\eta}; n+1; 0, t - A_0)\right) dF_n(\eta) \\ & = \varepsilon v_{M+1}(x'; n+1; 0, t - A_0) * F_n(x'). \end{aligned} \tag{3.17}$$

Although in the above construction t_0 may depend on n, it is clear that one may choose it sufficiently large so that (3.17) holds for $n = 0, 1, \ldots, N_0$. Since the result has already been established for $n \geqq N_0$ we have (3.12).

We turn to

$$\frac{v_{M+1}(x''; n+1; t - A_0, t) * F_n(x'')}{v_{M+1}(x'; n+1; t - A_0, t) * F_n(x')}. \tag{3.18}$$

Arguing as we did for (3.12), we show that given any $\varepsilon > 0$, there is an N_0 such that (3.18) $< \varepsilon$ for all $n > N_0$ and all (x', x'') satisfying $b' \leqq x' \leqq bx'' \leqq b''$. We complete the proof by showing that for any N_0 we may pick A_0, so that for sufficiently large t

$$\frac{\sum_{n=0}^{N_0} v^n v_{M+1}(x''; n+1; t-A_0, t) * F_n(x'')}{\sum_{n=0}^{N_0+1} v^n v_{M+1}(x'; n+1; t-A_0, t) * F_n(x')} < \varepsilon. \tag{3.19}$$

To show the latter, it is in turn sufficient to prove that for $n = 0, 1, \ldots, N_0$,

$$\frac{\int_{x''}^{1} v_{M+1}\left(\frac{x''}{\eta}; n+1; t-A_0, t\right) dF_n(\eta)}{\int_{x''}^{1} v_{M+1}\left(\frac{x''}{\eta}; n+2; t-A_0, t\right) dF_{n+1}(\eta)} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.20}$$

Let $\quad h(t) = \int_{x''}^{1} \sum_{j=1}^{\infty} q_j \int \cdots \int \Phi_j(dx_1, \ldots, dx_j) \sum_{B_{j,k}} \binom{k}{k_1, \ldots, k_j} \prod_{i=1}^{j} \mu_{k_i}\left(\frac{x''}{\eta x_i}, t\right) dF_n(\eta).$

From (3.6), lemma 3, it follows that for $n = 1, 2, \ldots$

$$\int_{t-A_0}^{t} h(t-y)\, dG_n(y) \Big/ \int_{t-A_0}^{t} h(t-y)\, dG_{n+1}(y) \to 0 \quad \text{as} \quad t \to \infty.$$

This implies (3.20).

It remains only to consider the case when G is truncated. Say $G(\tau) = 1$. Then for sufficiently large t, first term of (3.9) and the second term of (3.11) may be dropped; and those parts of the proof dealing with these terms deleted. The remainder of the proof goes through as before.

Lemma 5. *Under the conditions of theorem 3*

$$\mu_n(x, t)/\mu_m(x, t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.21}$$

Proof. It is sufficient to prove (3.21) for $n = m + 1$; and to show this it is in turn sufficient (by lemma 4) to prove the existence of an $x' > x$ and a $D_0 < \infty$, such that

$$\mu_{n+1}(x, t) < D_0 \mu_n(x', t). \tag{3.22}$$

The proof is again by induction. Take $n = 1$. Then

$$\mu_2(x, t) = v_2(x, t) + v \int_0^t dG(y) \int dF(\eta)\, \mu_2\left(\frac{x}{\eta}, t-y\right), \tag{3.23}$$

where

$$v_2(x, t) = \sum_{j=1}^{\infty} q_j \int dG(y) \int_x^{1-x} \cdots \int_x^{1-x} \Phi_j(dx_1, \ldots, dx_j) \sum_{\substack{i,k=1 \\ i \neq k}}^{j} 2\, \mu\left(\frac{x}{x_i}, t-y\right) \mu\left(\frac{x}{x_k}, t-y\right). \tag{3.24}$$

Note that the range of integration of Φ_j may be truncated above at $1 - x$ since if $x_i > 1 - x$, then $(x_i, x_k) \varepsilon A_j$ implies $x_i + x_k \leqq 1$, which in turn implies $x_k < x$ and $\mu\left(\frac{x}{x_k}, t\right) = 0$; similarly if $x_k > 1 - x$. Next note that $\mu\left(\frac{x}{x_i}, t\right) = 0$ for $x_i > x$ implies that $\sum_{i=1}^{j} \mu\left(\frac{x}{x_i}, t\right) \leqq \left[\frac{1}{x}\right]$. Hence

$$(3.25)\qquad v_2(x,t) \leqq 2[1/x]\sum q_j \int_0^t dG(y) \int_x^{1-x}\cdots\int_x^{1-x} \Phi_j(dx_1,\ldots,dx_j)\sum_{i=1}^{j}\mu\left(\frac{x}{x_i},t-y\right).$$

Now making a change of variable $x_i = (1-x)x_i'$ in the above integral and applying (A-1) and (A-2) we can show that there is an $\alpha < \infty$ and a $k > 1$ such that

$$(3.26)\qquad v_2(x,t) \leqq 2\,\alpha\left[\frac{1}{x}\right]\mu(kx,t).$$

But $\mu(x,t) = \sum_{n=0}^{\infty} \nu^n (1-G(t))\,D(1-x) \circledast [G_n F_n] = [1-G(t)] * \sum_{n=0}^{\infty} \nu^n G_n(t)\,F_n^*(x)$.

Hence, we have

$$v_2^{(0)}(x,t) = v_2(x,t) \leqq 2\,\alpha\left[\frac{1}{x}\right](1-G(t)) * \sum_{n=0}^{\infty} \nu^n G_n(t)\,F_n^*(kx),$$

and by (2.31)

$$(3.27)\qquad \nu^n v_2^{(n)}(x,t) = \nu^n v_2^{(0)}(x,t) * [G_n F_n] \leqq 2\,\alpha\left[\frac{1}{x}\right](1-G(t)) * \sum_{i=n}^{\infty} \nu^i G_i(t)\,F_i^*(kx).$$

Thus by corollary 2

$$(3.28)\qquad \mu_2(x,t) \leqq 2\,\alpha\left[\frac{1}{x}\right](1-G(t)) * \sum_{n=0}^{\infty}(n+1)\,\nu^n G_n(t)\,F_n^*(kx).$$

Now choose k' such that $1 < k' < k$. Then by lemma 2 there is an $A < \infty$ and a $B < 1$ such that $F_n^*(kx)/F_n^*(k'x) < A\,B^n$. Hence there is a $C < \infty$ such that

$$\mu_2(x,t) \leqq 2\,\alpha\left[\frac{1}{x}\right]C(1-G(t)) * \sum_{n=0}^{\infty} \nu^n G_n(t)\,F_n^*(k'x).$$

Setting $\alpha C = D_0$ and $k'x = x'$, we thus have (3.22) for $n = 1$.

Now assume (3.22) true for $n = 1,\ldots,m-1$. Consider $v_k(x,t)$ as defined in (2.30). We have

$$(3.29)\qquad v_{m+1}(x,t) = \sum_{j=2}^{\infty} q_j \int_0^t dG(y) \sum_{B_{j,m+1}'} \binom{m+1}{k_1,\ldots,k_j} \int\cdots\int_{A_{j,m+1}^{(x)}} \Phi_j\,(dx_1,\ldots,dx_j)\cdot\prod_{i=1}^{j}\mu_{k_i}\left(\frac{x}{x_i},t-y\right),$$

where $A_{j,m}(x) = \{(x_1,\ldots,x_j) : (x_1,\ldots,x_j)\,\varepsilon\,A\ ,\ x_i \leqq 1 - x(m-k_i),\ i = 1,\ldots,j\}$. The interchange of order of summation and integration is permissible since there is a finite number of terms in the sum. The restriction of the range of integration to $A_{j,m+1}^{(\alpha)}$ is legitimate for the following reasons. First, by (2.26), $k_i \frac{x}{x_i} \leqq 1$, for otherwise $\mu_{k_i}\left(\frac{x}{x_i},t-y\right) = 0$. Second $(x_1,\ldots,x_j)\,\varepsilon\,A_j$ implies $\sum x_i \leqq 1$. Hence for any $i_0 = 1,\ldots,j$ we have $x_{i_0} \leqq 1 - \sum_{i \neq i_0} x_i \leqq 1 - x\sum_{i\neq i_0} k_i = 1 - x(m+1-k_{i_0})$.

By rearranging some terms in (3.29) we can show that

$$(3.30)\qquad v_{m+1}(x,t) \leqq \sum_{j=2}^{\infty} \int_0^t dG(y) \sum_{B_{j,m}} \left(\frac{(m+1)!}{k_1!\cdots k_j!}\right) \int\cdots\int_{A_{j,m+1}^{(x)}} \Phi_j(dx_1,\ldots,dx_j)\,\cdot$$

$$\cdot\left\{\prod_{i=1}^{j}\mu_{k_i}\left(\frac{x}{x_i},t-y\right)\right\}\left\{\sum_{i=1}^{j}\frac{\mu_{k_i+1}\left(\frac{x}{x_i},t-y\right)}{(k_1+1)\,\mu_{k_i}\left(\frac{x}{x_i},t-y\right)}\right\}.$$

(There would be equality in (3.30) except that the terms

$$\sum_{j=2}^{\infty} q_j \int dG \sum_{i=1}^{j} \int_{A_{j,m+1}^{(x)}} \cdots \int \Phi_j(dx_1, \ldots, dx_j)\, \mu_{m+1}\left(\frac{x}{x_i}, t-y\right)$$

have been added to the right side.)

But now $\mu_{k+1}(x, t) \leqq [1/x]\, \mu_k(x, t)$, and hence

$$\sum_{i=1}^{j} \{\mu_{k_i+1}(x/x_i, t)/(k_i+1)\, \mu_{k_i}(x/x_i, t\} \leqq [1/x]$$

Hence we deduce from (3.30) that

$$(3.31)\quad v_{m+1}(x, t) \leqq (m+1)\left[\frac{1}{x}\right] \sum_{j=2}^{\infty} q_j \int_0^t dG(y) \cdot \\ \cdot \sum_{B_{j,m}} \binom{m}{k_1, \ldots, k_j} \int_{A_{j,m+1}}\cdots\int_{(x)} \Phi_j(dx_1, \ldots, dx_j) \prod_{i=1}^{j} \mu_{k_i}\left(\frac{x}{x_i}, t-y\right).$$

Now let $x_i' = \frac{x_i}{k}$, where $k > 1$ is sufficiently close to 1 so that $1 - kx(m - k_i) > > \frac{1}{k} - \frac{x}{k}(m + 1 - k_i)$ and so that A-2 holds. Then using (A-1) and (A-2) one shows that

$$(3.32)\quad v_{m+1}(x, t) \leqq \alpha(m+1)\left[\frac{1}{x}\right] \int_0^t dG(y) \sum_{B_{j,m}} \binom{m}{k_1, \ldots, k_j} \int_{A_{j,m}^{(kx)}} \cdots \int \Phi_j(dx_1, \ldots, dx_j) \cdot \\ \cdot \prod_{i=1}^{j} \mu_{k_i}\left(\frac{kx}{x_i}, t-y\right) \leqq \alpha(m+1)\left[\frac{1}{x}\right] \mu_m(kx, t).$$

Now by corollary 2

$$(3.33)\qquad \mu_m(x, t) = \sum_{n=0}^{\infty} v_m(x, t) \circledast [\nu^n G_n F_n],$$

and thus

$$v_{m+1}(x, t) \leqq \alpha(m+1)\left[\frac{1}{x}\right] \sum_{n=0}^{\infty} v_m(kx, t) \circledast [\nu^n G_n F_n].$$

Thus

$$\nu^k v_{m+1}^{(k)}(x, t) \leqq \alpha(m+1)\left[\frac{1}{x}\right] \sum_{n=k}^{\infty} v_m(kx, t) \circledast [\nu^n G_n F_n],$$

and

$$(3.34)\qquad \mu_{m+1}(x, t) \leqq \alpha(m+1)\left[\frac{1}{x}\right] \sum_{n=0}^{\infty} (n+1)\, v_m(kx, t) \circledast [\nu^n G_n F_n].$$

Take any k' such that $1 < k' < k$. We have

$$(3.35)\qquad v_m(kx, t) \circledast [G_n F_n] = \int dG_n(y) \int_{kx}^{1} dF_n(\eta)\, v_m\left(\frac{kx}{\eta}, t-y\right).$$

Choose M such that $1/M\,(1 - kx) = \delta < \frac{1}{4}(k - k')x$. Then

$$(3.36)\quad v_m(kx, t) \circledast [G_n F_n] = \int_0^t dG_n(y) \sum_{i=0}^{M-1} \int_{kx+i\delta}^{kx+(i+1)\delta} dF_n(\eta)\, v_m\left(\frac{kx}{\eta}, t-y\right) \\ \leqq \int_0^t dG_n(y) \sum_{i=0}^{M-1} [F_n^*(kx + i\delta) - F_n^*(kx + (i+1)\delta)]\, v_m\left(\frac{kx}{kx + (i+1)\delta}, t-y\right);$$

and

$$v_m(k'x, t) \circledast [G_n F_n] \geqq \int_0^t dG_n(y) \sum_{i=0}^{M-1} \int_{k'x+(i+1)\delta}^{k'x+(i+2)\delta} dF_n(\eta)\, v_m\left(\frac{k'x}{\eta}, t-y\right)$$

(where $\neq$ may hold since part of the range of integration has been chopped off),

$$\text{(3.37)} \quad \geqq \int_0^t dG_n(y) \sum_{i=0}^{M-1} [F_n^*(k'x + (i+1)\delta) - F_n^*(k'x + (i+2)\delta]\, v_m\left(\frac{k'x}{k'x + (i+1)\delta}, t-y\right).$$

Now for all $i = 0, 1, \ldots, > -1$ we have

$$\frac{kx + i\delta}{k'x + (i+1)\delta} \geqq \beta_1 > 1, \quad \text{and} \quad \frac{k'x + (i+2)\delta}{k'x + (i+1)\delta} \geqq \beta_2 > 1\,.$$

Hence by lemma 2 there are $A < \infty$ and $B < 1$ such that

$$\text{(3.38)} \qquad \frac{F_n^*(kx + i\delta) - F_n^*(kx + (i+1)\delta)}{F_n^*(k'x + (i+1)\delta) - F_n^*(k'x + (i+2)\delta)} \leqq A\, B^n\,.$$

Furthermore $k' < k$ implies

$$\text{(3.39)} \qquad v_m\left(\frac{k'x}{k'x + (i+1)\delta}, t-y\right) \geqq v_m\left(\frac{kx}{kx + (i+1)\delta}, t-y\right).$$

Thus (3.36), (3.37), (3.38) and (3.39) imply

$$\text{(3.40)} \qquad v_m(kx, t) \circledast [F_n G_n] \leqq A \cdot B^n\, v_m(k'x, t) * [F_n G_n]\,,$$

which together with (3.34) implies (3.22) and hence the lemma.

Proof of Theorems 2 and 3. These results now follow at once from lemmas 1, 4, and 5.

References

[1] Bellman, R., and T. E. Harris: On age-dependent binary branching processes. Ann. of Math. **55**, 280—295 (1952).

[2] Bharucha-Reid, A. T.: Elements of the Theory of Markov Processes and their Applications. New York: McGraw-Hill 1960.

[3] Feller, W.: On the integral equation of renewal theory. Ann. math. Statistics **19**, 474—494 (1941).

[4] Harris, T. E.: Branching processes. Ann. math. Statistics **19**, 474—494 (1948).

[5] — Some mathematical models for branching processes. Proc. 2nd Berckley Sympos. math. Statistics Probability 305—328 (1951).

[6] Janossy, L.: Note on the fluctuation problem of cascades. Proc. phys. Soc. (London), Sect. A, **34**, 241—249 (1950).

[7] Levinson, N.: Limiting theorems for age-dependent branching processes. Illinois J. Math. **4**, 100—118 (1960).

[8] Lopuszanski, J.: Some remarks on the asymptotic behavior of the cosmic ray cascade for large depth of the absorber: II. Asymptotic behavior of the probability distribution function, Nuovo Cimento, X Ser. **2**, Suppl. 4, 1150—1160 (1955).

[9] — Some remarks on the asymptotic behavior of the cosmic ray cascade for large depth of the absorber: III. Evaluation of the distribution function, Nuovo Cimento, X. Ser. **2**, Suppl. 4, 1161—1167 (1955).

[10] Ney, P. E.: Generalized branching processes I: Existence and uniqueness theorems. To appear in Illinois J. Math. (1964).

[11] NEY, P. E.: Generalized branching processes II: Asymptotic theory. To appear in Illinois J. Math. (1964).

[12] — The limit of a ratio of convolutions. Ann. math. Statistics 34, 457—461 (1963).

[13] URBANIK, K.: Some remarks on the asymptotic behavior of the cosmic ray cascade for large depth of the absorber: I. Evaluation of the factorial moments. Nuovo Cimento, X. Ser. 2, Suppl. 4, 1147—1149 (1955).

Cornell University
Ithaca, New York, USA

(Received July 10, 1963)