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Fictitious States, Coupled Laws and Local Time

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Summary. The work of Ray and Neveu has established that, for any transition function P on a countable set E, (i) there exists a best possible entrance boundary E^+ supporting a right continuous, strong Markov process X with transition function P and that (ii) the points y of E^+ are in one-one correspondence with the extremal entrance laws g^y of P. Here, it is shown that, if a point y of E^+ is regular for itself, then the derived characteristic f^y of the local time at y is a regular extremal entrance law "coupled" with g^y in the sense of Neveu. Further, coupled laws arise only in this fashion. By using excursion theory, a simple explicit formula for f^y in terms of g^y may be obtained. The paper contains a conjecture about the intrinsic character of the Ray-Neveu topology and an example which shows emphatically that, in general, local time is not a derivative of occupation time.

§ 1. Prerequisites

1.1. Basic Notation

Throughout the paper, unexplained terminology and notation are exactly as in Dynkin's book [3].

Let E be a countable set and let P be a conservative transition function on E:

$$P_t(i,\Gamma)=\sum_{j\in\Gamma}P_t(i,j)\qquad (i\in E;\Gamma\subseteq E;t\geqq 0);$$

$$P_t(i,j)\geqq 0;\quad P_t(i,E)=1\qquad (i,j\in E;t\geqq 0);$$

$$\sum_{j\in E}P_s(i,j)\,P_t(j,k)=P_{s+t}(i,k)\qquad (i,k\in E;s,t\geqq 0);$$

such that

$$\lim_{u\downarrow 0}P_u(i,i)=P_0(i,i)=1\qquad (i\in E).$$

To simplify the notation, we adopt the following

Conventions. 1. The Laplace transform

$$\int_{(0,\infty)}\exp(-\lambda t)\,h_t\,dt$$

of a function $h_.$ defined on $(0,\infty)$ will be denoted by $h(\lambda)$. Thus, for example,

$$P(\lambda;i,j)\equiv\int_{(0,\infty)}\exp(-\lambda t)\,P_t(i,j)\,dt\,.$$

(Throughout the paper, the symbol "$\equiv$" will mean "(which) is defined to be equal to".) For all the functions with which we shall be concerned, $h(\lambda)$ will be finite for $\lambda>0$ but $h(0)$ may be infinite.

2. The integral over $(0,t)$ of a function on $(0,\infty)$ represented by a lower case letter will be denoted by using the corresponding upper case letter, e.g.,

$$H_t\equiv\int_{(0,t)}h_s\,ds\,.$$

3. The scalar product $\sum a(i)\, b(i)$ of two vectors on E will be denoted by $\langle a, b\rangle$ and the tensor product symbol $a \otimes b$ will denote the matrix C with (i, j)-th component

$$C(i, j) = a(i)\, b(j).$$

4. The unit vector on E will be denoted by 1.

1.2. Entrance and Exit Laws

In the next three subsections, we recall some of Neveu's theory [12, 14, 15] of entrance and exit laws. Much of the early part of the paper will be found to read almost as a commentary on Neveu's work from a probabilistic standpoint but we shall see later the probabilistic theory allows a considerable strengthening of the analytic results.

The reader familiar with Neveu's notation will notice that we have interchanged the roles of f and g making f the typical *exit* and g the typical *entrance* law. This is basically because the f-functions and g-functions in Chung [2] and in many other works on Markov chains are exit, entrance laws respectively (though not relative to P).

An *entrance law* relative to P is a family g_t $(t > 0)$ of non-negative measures on E which is such that

$$G_s(E) < \infty, \qquad g_s P_t = g_{s+t} \qquad (s > 0; t \geqq 0).$$

Dually, an *exit law* relative to P is a family f_t $(t > 0)$ of non-negative functions on E which is such that

$$\sup_{j \in E} F_t(j) < \infty, \qquad P_s f_t = f_{s+t} \qquad (s \geqq 0; t > 0).$$

We shall *not* consider the trivial law $g[f]$ with

$$g_t(j) \equiv 0 \qquad (t > 0; j \in E)$$
$$f_t(j) \equiv 0 \qquad (t > 0; j \in E)$$

as an entrance [exit] law.

Because P is conservative, the condition

$$G_s(E) < \infty \qquad (s > 0)$$

in the definition of an entrance law may be replaced by the condition

$$g_s(E) < \infty \qquad (s > 0)$$

and we note that

$$g_s(E) = \sum_{j \in E} g_s(j)$$

is independent of s for $s > 0$. We denote the common value of $g_s(E)$ by $g(E)$.

Neveu showed that there is a canonical Choquet representation of the cone of entrance laws (the *entrance cone*). The basic facts are these.

The map of E into the closed unit sphere of $l_1(E)$ defined by

$$i \to P(1; i, \cdot)$$

is one-one (see Proposition 1 of [14]) and so *we may, and shall from now on, identify the point i of E with the point $P(1; i, \cdot)$ in $l_1(E)$*. Let E^+ denote the set of extremal points of the (strongly) closed convex hull of E in $l_1(E)$. Unless the contrary is explicitly stated, we shall work with the Ray-Neveu topology of E^+, namely, the topology induced by the $l_1(E)$ norm. At a later stage, it will be convenient to use

Proposition N 1. *E is a dense subset of E^+ and there is a metric ϱ on E^+ defining the same topology as the $l_1(E)$ norm and such that (E^+, ϱ) is a complete (separable) metric space.*

This proposition, which asserts that E^+ is a "Polish space", is contained in Proposition 3 of NEVEU [14].

For each $y = y(\cdot)$ in E^+, there is a unique entrance law g^y such that

$$g^y(1;j) \equiv \int_{(0,\infty)} e^{-s} g_s^y(j)\, ds = y(j) \qquad (j \in E).$$

Note that

$$g_j^i(t) = P_t(i,j) \qquad (i,j \in E;\, t > 0).$$

The entrance laws $g^y (y \in E^+)$ are extremal. An *extremal* entrance law g is an entrance law with the following property: every entrance law g' such that $g_t'(j) < g_t(j)$ $(j \in E;\, t > 0)$ is a scalar multiple of g.

An entrance law g is extremal if and only if there is a point y of E^+ such that

$$g = g(E)\, g^y.$$

The points of E^+ are therefore in one-one correspondence with the extremal rays of the entrance cone.

Every entrance law g has a unique Choquet representation of the form

$$g = g(E) \int_{E^+} g^y\, \mu(dy),$$

where μ is a probability measure on the Borel σ-algebra of E^+. NEVEU sketches the Choquet theory of the exit cone but we shall not make use of this.

1.3. Coupling

NEVEU [14] called an entrance law g and an exit law f *coupled* if the relation

$$P(\lambda) - [a(\lambda)]^{-1} f(\lambda) \otimes g(\lambda) \geqq 0 \qquad (\lambda > 0) \tag{1.1}$$

holds, where a_t is defined (independently of $s < t$) by the equation

$$a_t \equiv \langle g_s, f_{t-s} \rangle. \tag{1.2}$$

(When we say, for example, "suppose g and f are coupled", it is to be understood that g is the entrance and f the exit law.)

Proposition 4 of NEVEU [14], which will be particularly important for our treatment, divides into the following two statemants.

Proposition N 2. *If g and f are coupled, then each is extremal.*

Proposition N 3. *If two entrance laws g^1 and g^2 are each coupled with the same exit law f, then g^1 and g^2 differ only by a scalar factor. The dual result is also valid.*

We see that coupling is really a property of *rays*.

Suppose now that g and f are coupled. Then g, being extremal, is of the form $g = g(E)\, g^y$ for some y in E^+. NEVEU establishes the following results in the course of his proof of the above propositions.

There is a unique transition function P^- on $E \backslash y$ such that

$$P^-(\lambda) = P(\lambda) - [a(\lambda)]^{-1} f(\lambda) \otimes g(\lambda) \tag{1.3}$$

and this transition function satisfies

$$\lim_{u \downarrow 0} P_u^-(i, i) = 1 \qquad ((i \in E \backslash y).$$

Further, there is a unique entrance law g^- [exit law f^-] relative to P^- on $E \backslash y$ such that

$$g(\lambda) = a(\lambda) g^-(\lambda), \qquad [f(\lambda) = f^-(\lambda)\, a(\lambda)]. \tag{1.4}$$

Set

$$g_s^-(y) = f_s^-(y) = 0 \qquad s > 0$$

in the case when $y \in E$ and, in all cases, define

$$a_t^- = \langle g_s^-, f_{t-s}^- \rangle \qquad (0 < s < t)$$

independently of s in $(0, t)$. Then

$$[a(\lambda)]^{-1} = \int_{(0,\infty)} (1 - e^{-\lambda t})\, a_t^-\, dt + [a(0)]^{-1}. \tag{1.5}$$

1.4. Extension of P to E^+

If we write

$$P_t(y, j) \quad \text{for} \quad g_t^y(j) \qquad (j \in E;\, y \in E^+;\, t > 0)$$

and define (for Γ a Borel subset of E^+, $t > 0$ and $y \in E^+$)

$$P_t(y, \Gamma) \equiv \sum_{j \in E \cap \Gamma} P_t(y, j),$$

then the extended P is a transition function on E^+ such that

$$P_t(y, E) = 1 \qquad (t > 0;\, y \in E^+).$$

Of considerable importance is NEVEU's result that *the extended P is strong Feller and stochastically continuous on E^+*. We recall that the statement that P is *strong Feller* means that for every $t > 0$ and every bounded function ζ on E^+, the function $P_t\zeta$ defined by the equation

$$(P_t\zeta)(y) \equiv \sum_{j \in E} P_t(y, j)\, \zeta(j)$$

is continuous on E^+. The strong Feller property of P is an immediate consequence of Proposition 2 of NEVEU [14]. That P is *stochastically continuous* means that, for every bounded continuous function ζ on E^+,

$$\lim_{t \downarrow 0} (P_t\zeta)(y) = \zeta(y) \qquad (y \in E^+).$$

This property of stochastic continuity is established at a key stage in the proof of Proposition 3 of NEVEU [14].

1.5. *The Strong Markov Theorem of* RAY

In [16], RAY derived some of the results of subsection 1.2 and proved a powerful strong Markov theorem. Unfortunately, there were some errors in RAY's original treatment but a recent paper of KUNITA and WATANABE [8] provides a corrected and elegantly streamlined version of RAY's argument. In our situation, these papers guarantee the existence of a right continuous strong Markov process $X = (x_t, \mathscr{M}_t, \mathsf{P}_x)$ with values in E^+ and with transition function P on E^+:

$$\mathsf{P}_y\{x_t = j\} = P_t(y, j), \qquad (y \in E^+; j \in E; t > 0).$$

The reader will find that the space E^+ coincides exactly with the space $\mathscr{X}_R \backslash \mathscr{X}_b$ of RAY's paper. If, with the notation of KUNITA and WATANABE [8], we choose for the 1-excessive functions of C_1 the functions

$$P(1; \cdot, j) \qquad (j \in E)$$

on E (and this is the only natural choice) and if we then use KNIGHT's completion, we find that our space E^+ is exactly the space denoted by $\bar{E}_R \cap (\bar{E} - \bar{E}_b)$ in KUNITA and WATANABE [8]. As stated there, there is no doubt that E^+ is the best possible "entrance boundary".

For a point y of E^+, define the hitting time:

$$T^y \equiv \inf\{s: s > 0, x_s = y\}.$$

By the very useful lemma on page 138 of KUNITA and WATANABE's paper, T^y is a Markov time. Hence, according to BLUMENTHAL's zero-one law, either

$$\mathsf{P}_y\{T^y = 0\} = 1 \text{ in which case } y \text{ is called } \textit{regular}$$

or

$$\mathsf{P}_y\{T^y = 0\} = 0 \text{ when we call } y \textit{ semi-polar}.$$

Because

$$\lim_{u \downarrow 0} P_u(i, i) = 1 \qquad (i \in E)$$

all points of E are regular.

1.6. *Local Time*

We now recall some results of BLUMENTHAL and GETOOR [1].

Let y be a regular point of E^+. Then there exists a continuous (non-negative, homogeneous) additive functional $\psi(y)$, the *local time* at y, such that

$$\mathsf{M}_x \int_{(0,\infty)} e^{-t}\, d\psi_t(y) = \mathsf{M}_x \exp(-T^y) \qquad (x \in E^+). \tag{1.6}$$

Two continuous additive functionals ψ^1 and ψ^2 satisfying (1.6) are *equivalent:*

$$\mathsf{P}_x\{\psi_t^1 = \psi_t^2\} = 1, \qquad (x \in E^+; t > 0).$$

The aptness of the term *local time* derives from the fact that $\psi(y)$ *grows only when X is at y:*

$$\psi_t(y) = \int_{(0,t)} \chi^y(x_s)\, d\psi_s(y), \qquad (t > 0).$$

χ^y being the characteristic function of the set $\{y\}$, and that, conversely, any continuous additive functional with this property is equivalent to a scalar multiple of $\psi(y)$.

1.7. Normalisation of Local Time

In general, there is no canonical normalisation of local time. In our situation, the *actual time* at j:

$$\beta_t(j) \equiv m\{s: 0 \leqq s \leqq t, \quad x_s = j\}$$

m denoting Lebesgue measure, is a more natural measure of local time at a point j of E than is $\psi(j)$ which is equivalent to

$$\beta(j)/P(1; j, j)$$

We therefore introduce the additive functional $\varphi(y)$ defined as follows.

Definition. *For a point j of E, define*

$$\varphi_t(j) \equiv \beta_t(j).$$

For a regular point y of $E^+ \setminus E$, we define $\varphi(y)$ (up to equivalence) by the equation

$$\varphi_t(y) \equiv \psi_t(y).$$

For a semi-polar point y of $E^+ \setminus E$, we define

$$\varphi_t(y) \equiv \textit{number of visits by } X \textit{ to } y \textit{ during time } [0, t].$$

In § 5 of their paper, Blumenthal and Getoor show that for a semi-polar point y, $\varphi_t(y)$ is a *finite* (though, of course, discontinuous) additive functional

§ 2. Summary of Results

2.1. Regular Laws

As already explained, an extremal entrance law g is necessarily of the form

$$g = K P_t(y, j) = K \frac{d}{dt} \mathsf{M}_y \varphi_t(j) \tag{2.1}$$

where $y \in E^+$. The partial dual result provided by Lemma 2.2 is important. First we make a

Definition. Call an extremal entrance law g *regular* if the point y of E^+ associated with g as in Eq. (2.1) is regular. Call an exit law f (extremal or not) *regular* if

$$\limsup_{t \downarrow 0} \sup_{j \in E} F_t(j) = 0.$$

The definition of regularity is transferred from extremal laws to extremal rays in the obvious manner.

From Proposition N 2, we know that, if g and f are coupled, then each is extremal. It is also true that each is regular but we shall prove this in several steps. We indicate some of these here.

Lemma 2.1. *If g and f are coupled, then f is regular.*

Lemma 2.2. *If y is any point of E^+, then the (componentwise) derivative*

$$f_t^y(j) \equiv \frac{d}{dt} \mathsf{M}_j \varphi_t(y) \qquad (t > 0) \tag{2.2}$$

exists and, if non-trivial, defines an exit law f^y relative to P. An exit law f is extremal *and regular if and only if it is of the form Kf^y where K is a positive constant and y is a regular point of E^+.*

Notes. 1. For a point i of E, we have

$$f_t^i(j) = P_t(j, i).$$

2. For a semi-polar point z of E^+, f_z need not be extremal and hence there are (non-regular) extremal exit laws which are not of the form Kf^y ($y \in E^+$).

We remark that a slight rearrangement of our discussion would make it independent of the work of BLUMENTHAL and GETOOR. It is possible to construct f^y by a procedure independent of the theory of additive functionals and then to deduce the existence of local time at y from the following lemma.

Lemma 2.3. *Let y be a regular point of E^+ and set*

$$\varphi_s(y;\delta) \equiv \sum_{j \in E} f_\delta^y(j)\,\beta_s(j). \tag{2.3}$$

Then, for every $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \mathsf{P}_x \left\{ \sup_{0 \leq s \leq t} |\varphi_s(y;\delta) - \varphi_s(y)| > \varepsilon \right\} = 0 \qquad (x \in E^+,\ t > 0).$$

We express the conclusion of this lemma by writing:

$$\varphi(y;\delta) \to \varphi(y) \quad \textit{uniformly in probability}.$$

The next theorem and (especially) its corollary are two of our main results.

Theorem A. *If y is a regular point of E^+, then the laws g^y and f^y are coupled.*

Conversely, if g and f are coupled, then there exist a regular point y of E^+ and constants K_1 and K_2 such that

$$g = K_1 g^y, \quad f = K_2 f^y.$$

The transition function P^- corresponding to the couple (g, f) (see § 1.3) is then the transition function of the process X killed at time T^y.

Corollary. *The coupling relation sets up a one-one onto map between the regular extremal rays of the entrance and exit cones.*

A regular point y of E^+ determines and is determined by a unique pair of coupled rays, namely, the rays thorugh g^y and f^y. NEVEU called a pair of coupled rays a *fictitious state*. We prefer to follow CHUNG's general formulation [2; § II.4] according to which each point of $E^+ \backslash E$ is a fictitious state. We shall call a regular point of $E^+ \backslash E$ a *regular* fictitious state.

2.2. Characterisation of the Topology of E^+

In the course of proving Lemma 2.2, we shall obtain a probabilistic characterisation of the RAY-NEVEU topology of the *regular* part of E^+.

Theorem B. *Let y be a regular point of E^+ and let $\{y(n): n = 1, 2, \ldots\}$ be any sequence of points of E^+. Then $y(n) \to y$ if and only if*

$$\lim_{n \to \infty} \mathsf{P}_{y(n)} \{T^y < t\} = 1 \qquad (t > 0). \tag{2.4}$$

In general, the "only if" part of Theorem B is false without the assumption that y is regular. (See the example in § 7.2.) An extension of Theorem B valid for all points y would establish that the RAY-NEVEU topology is truly "intrinsic". I make the following conjecture.

Conjecture. *Let y be any point of E^+ and let $\{y(n): n = 1, 2, \ldots\}$ be any sequence of points of E^+ such that $y(n) \neq y$ $(n = 1, 2, \ldots)$. Then a necessary and sufficient condition that $y(n) \to y$ is that*

$$\lim_{n\to\infty} \max[\mathsf{P}_y\{T^{y(n)} < t\}, \mathsf{P}_{y(n)}\{T^y < t\}] = 1 \quad (t > 0). \tag{2.5}$$

The sufficiency of condition (2.5) is easily established. What remains undecided is the necessity of the condition in the case when y is semi-polar.

2.3. Preparatory Remarks

Throughout the remainder of Section 2, y will denote a regular point of $E^+\backslash E$. (We assume that there is one.)

Our aim now is to find further formulae for $\varphi(y)$ which will enable us to express f^y explicitly in terms of g^y and the elements $P_t(i, j)$ of the original transition on E. We shall be able to express g^y in terms of f^y in the case when y is recurrent but not when y is transient.

The distinction between the recurrent and transient cases is important. As usual, we call y *recurrent* if

$$\sup\{s: x_s = y\} = \infty \qquad \text{a.s. } \mathsf{P}_y.$$

otherwise, we call y *transient.*

The following lemma, which is quite trivial, allows us to decide whether or not y is transient from a knowledge of either g^y or f^y.

Lemma 2.4. *The following statements are equivalent:*

(i) *y is recurrent;*

(ii) *as i ranges over E,*

$$g^y(0) \equiv \int_{(0,\infty)} g^y_u(i)\, du$$

takes no value other than 0 or ∞;

(iii) *as i ranges over E,*

$$f^y(0) \equiv \int_{(0,\infty)} f^y_u(i)\, du$$

takes no value other than 0 or ∞.

2.4. The Recurrent Case

When y is recurrent, we shall be interested solely in those states of E which can be reached from y and from which therefore y can be reached.

Let us therefore assume throughout this subsection that P is irreducible recurrent and let us denote by π the strictly positive invariant measure (unique but for a normalisation factor) on E:

$$0 < \pi(j) = \sum_{i\in E} \pi(i)\, P_t(i, j) \qquad (t \geqq 0;\ j \in E).$$

We denote by P^* the dual transition function on E defined as follows:

$$P_t^*(i, j) \equiv \pi(j) P_t(j, i)/\pi(i).$$

The transition function P^* is, of course, the transition function on E of the time reversal of the Markov random function $(x_t, \mathscr{M}_t, \mathsf{P}_\pi)$ (provided that we allow infinite probabilities in the null recurrent case).

We introduce the normalised local time:

$$L_t(i) \equiv \beta_t(i)/\pi(i) \qquad (i \in E)$$

and we shall show that the limit

$$L_t(y) \equiv \lim_{\delta \downarrow 0} \sum_{j \in E} P_\delta(y, j) L_t(j) \tag{2.6}$$

exists uniformly in probability and defines the canonical normalisation of local time at y.

(Note. The normalisation of π is, of course, arbitrary except in the positive recurrent case when it is natural to have $\pi(E) = 1$. However, once the normalisation of π is chosen, $L_t(y)$ is defined for all regular states y.)

Let us therefore redefine f^y by the equation

$$f_t^y(i) \equiv \frac{d}{dt} \mathsf{M}_i L_t(y). \tag{2.7}$$

Let us also set (compare definition (2.2))

$$f_t^{*y}(i) \equiv \frac{d}{dt} \mathsf{M}_y L_t(i) = g_t^y(i)/\pi(i). \tag{2.8}$$

Then f^{*y} is an exit law relative to P^*, the boundedness of f^{*y} being a consequence of the equation

$$\sup_{i \in E} \int_{(0,\infty)} f_t^{*y}(i)\, dt = \sup_{i \in E} \int_{(0,\infty)} f_t^y(i)\, dt. \tag{2.9}$$

Dually, we define

$$g_t^{*y}(i) \equiv \pi(i) f_t^y(i).$$

Then

$$\sum_{i \in E} g_t^{*y}(i) = 1 \qquad (t > 0)$$

and g^{*y} is an entrance law for P^*. It is trivial that (g^{*y}, f^{*y}) is a couple for P^* and so defines a fictitious state y^* of P^*. It is natural to identify y^* and y:

$$g_t^{*y}(i) = P_t^*(y, i).$$

We now have two topologies on the regular part of E^+: the original RAY-NEVEU topology induced by P, the *P-topology*, and the corresponding P^*-topology induced by P^*.

Theorem C. *The following relations hold on E:*

$$g_t^{*y} = \lim_{\delta \downarrow 0} g_\delta^y P_t^*; \quad g_t^y = \lim_{\delta \downarrow 0} g_\delta^{*y} P_t;$$
$$f_t^{*y} = \lim_{\delta \downarrow 0} P_t^* f_\delta^y; \quad f_t^y = \lim_{\delta \downarrow 0} P_t f_\delta^{*y}.$$

A statement equivalent to Theorem C is the following.

Theorem C′. *The transition function P^* is stochastically continuous at y in the P-topology, i.e.,*

$$\lim_{\delta\downarrow 0} \sum_{j\in E} P^*_\delta(y,j)\,\zeta(j) = \zeta(y)$$

whenever ζ is bounded and continuous in the P-topology of E^+.

Corollary to Theorem C. *If an entrance law g and an exit law f are coupled, then there exist constants K_1 and K_2 such that the equations*

$$f_t(i) = K_1 \lim_{\delta\downarrow 0} \sum_{j\in E} P_t(i,j)\, g_\delta(j)/\pi(j),$$
$$g_t(i) = K_2 \lim_{\delta\downarrow 0} \sum_{j\in E} \pi(j)\, f_\delta(j)\, P_t(j,i)$$

hold for i in E and $t > 0$.

Theorem C and C′ would be trivial if the P- and P^*-topologies of the regular part of E^+ were identical. However, the one-sided nature of the characterisation of the Ray-Neveu topology afforded by Theorem B makes it clear that this need not be so. We shall give a concrete example in § 7.2. The important point is that, *whereas the property of stochastic continuity is preserved at regular points by "time-reversal", the Feller property is, in general, lost.* That stochastic continuity is preversed at regular points is perhaps not surprising because, at a fixed time, x. is almost surely left continuous.

Eq. (2.6) shows that $L(y)$ is an "average" of $L(j)$ over states j near y. More generally, we have the following result.

Theorem D. *Suppose that h_δ $(\delta \in \Delta)$ is a sequence of probability measures on E such that*

$$\lim_{\delta} \mathsf{M}_{h_\delta} \langle L_T, h_\delta \rangle = 0 \tag{2.10}$$

where $T \equiv T^y$. Then

$$\lim_{\delta} \langle L, h_\delta \rangle = L(y) \quad \textit{uniformly in probability}.$$

The peculiar significance of the expression

$$\mathsf{M}_{h_\delta} \langle L_T, h_\delta \rangle$$

will be explained in § 6.2 but Eq. (2.10) is clearly a reasonable condition to impose on the averaging process h.

In general, it is not true that there is a sequence $\{i(n) : n = 1, 2, \ldots\}$ of points of E such that

$$\lim_{n\to\infty} L(i(n)) = L(y) \quad \text{uniformly in probability}.$$

Indeed, we shall give in Section 7 a rather striking example to show that, in general, it is *not* possible to define $L_t(y)$ by a formula of type

$$L_t(y) = \lim_{\Delta\ni\delta\downarrow 0} \frac{m\{s : 0 \leqq s \leqq t,\ \varrho(x_s, y) < \delta\}}{h_\delta}.$$

In other words, it is *not* possible to express local time as a derivative of occupation time.

2.5. *The Transient Case*

The case when y is transient is more difficult to study. It is natural to look for an analogue of Eq. (2.6) of the form

$$\varphi_t(y) = K \lim_{\delta \downarrow 0} \sum_j P_\delta(y, j)\, \beta_t(j)/\pi(j) \tag{2.11}$$

where π is a subinvariant (i. e. superregular) measure for P. The problem is: which subinvariant measures may be used? We shall content ourselves here with establishing one version of Eq. (2.11):

$$\varphi_t(y) = K \lim_{\delta \downarrow 0} \sum_j P_\delta(y, j)\, \beta_t(j)/g^y(0; j). \tag{2.12}$$

Integration of Eq. (2.12) with respect to P_i yields the formula

$$f^y(i) = K \lim_{\delta \downarrow 0} \sum_{j \in E} g^y_\delta(j)\, \beta_t(j)/g^y(0; j) \tag{2.13}$$

which expresses f^y explicitly in terms of g^y. The analogue of Theorem D with

$$g^y(0; j) \quad \text{for} \quad \pi(j)$$

is valid.

2.6. *Instantaneous States*

This paper provides rigorous confirmation of an idea implicit in much of LEVY's work [9, 10], namely, that the study of a regular fictitious state is exactly equivalent to that of a non-fictitious instantaneous state.

If b (in E) is an instantaneous state of X, i.e., if

$$P_0'(b, b) = -\infty,$$

then, in terms of the new time parameter $t - \beta_t(b)$, X behaves as a (strong Markov) process for which b is a regular fictitious state. Conversely, if y is a regular point of $E^+ \backslash E$, then, in terms of the new time parameter $t + \varphi_t(y)$, X behaves as a (strong Markov) process X^+ for which

$$\mathsf{P}_x\{x_t^+ \in E \cup y\} = 1 \qquad (x \in E^+,\ t > 0)$$

and y is an instantaneous state of X^+.

Details of the LÉVY measure for $\varphi(y)$ and of the theory of excursions from y are given in Section 5.

2.7

Some of the results of this paper were announced in WILLIAMS [19, 20]. The proof of RAY's theorem promised in [20] was similar to, but less general and less elegantly expressed than, that in KUNITA and WATANABE [8]. It is not worth publishing now.

I wish to thank Professors J. LAMPERTI, D. G. KENDALL and, especially, G. E. H. REUTER for their helpful comments on this work.

§ 3. The Direct Part of Theorem A

3.1

First, we prove two analytic results.

Proof of Lemma 2.1. Suppose that g and f are coupled. Then, with the notation of § 1.3,

$$F_t^- = (1 - P_t^- 1)/g(E) \leqq 1/g(E) \qquad (t > 0).$$

Hence, from Eq. (1.4),

$$F_t = \int_{(0,t)} a_s F_{t-s}^- ds \leqq \{A_t/g(E)\}\,1\,.$$

Since $a(\lambda)$ exists for $\lambda > 0$, A_t is finite for finite t and hence

$$\lim_{t\downarrow 0} A_t = 0\,.$$

The regularity of f follows.

Lemma 3.1. *Let f be any regular exit law. For each $t > 0$, extend $f_t(\cdot)$ to E^+ by the following definition which is independent of $s < t$:*

$$f_t(z) \equiv \sum_{j\in E} P_s(z, j)\, f_{t-s}(j) \qquad (z \in E^+\backslash E)\,.$$

Then, for each $t > 0$, $F_t(\cdot)$ is continuous on E^+.

Proof. Since P is strong Feller on E^+, the function $P_\delta F_t$ is continuous on E^+ when $\delta > 0$. However, the equation

$$P_\delta F_t - F_t = P_t F_\delta - F_\delta$$

implies that

$$s\text{-}\lim_{\delta\downarrow 0} P_\delta F_t = F_t$$

on E^+. Lemma 3.1 follows.

3.2

Suppose throughout the remainder of Section 3 that y is a regular point of E^+. The simple Markov property implies that the characteristic

$$F_t^y(j) \equiv \mathsf{M}_j \varphi_t(y)$$

of $\varphi(y)$ satisfies the equation

$$P_s F_t^y = F_{s+t}^y - F_s^y$$

on E. That the derivative f^y of F^y therefore exists and is an exit law is well known. (See Theorem 2.2.3 of Neveu [12].)

We now prove that g^y and f^y are coupled. Set

$$a_t^y \equiv \langle g_s^y, f_{t-s}^y\rangle \qquad (s < t)$$

(compare definition (1.2)). Then

$$A_{t+h}^y - A_h^y = \sum_{j\in E} g_h^y(j)\, F_t^y(j) = \sum_{j\in E} P_h(y, j)\, \mathsf{M}_j \varphi_t(y) = \mathsf{M}_y[\varphi_{t+h}(y) - \varphi_h(y)]\,.$$

Hence, letting h tend to zero,

$$A_t^y = \mathsf{M}_y \varphi_t(y)\,.$$

The Strong Markov Theorem applied at time T^y yields

$$f^y(\lambda) = a^y(\lambda) \int_{(0,\infty)} e^{-\lambda t} d\mathsf{P}\{T^y \leqq t\}$$

and so we have

$$P(\lambda) - f^y(\lambda) \otimes g^y(\lambda)/a^y(\lambda) = P^{-y}(\lambda)$$

where

$$P_t^{-y}(i, j) = \mathsf{P}_i\{T^y > t,\, x_t = j\}\,.$$

Hence g^y and f^y are coupled. We let f^{-y}, g^{-y}, a^{-y} be the appropriate functions f^-, g^-, a^- for the couple (g^y, f^y) (see § 1.3). Note that

$$f_t^{-y}(i) = \frac{d}{dt}\mathsf{P}_i\{T^y \leqq t\}.$$

3.3. Proof of Theorem B

Since f^y is regular, it follows from Lemma 3.1 that

$$f^y(\lambda; z) \quad \text{is continuous in } z \text{ on } E^+.$$

Since

$$f^y(\lambda; z) = f^{-y}(\lambda; z)\, a^y(\lambda) \qquad (z \in E^+)$$

where

$$f^{-y}(\lambda; z) \equiv \int_{(0,\infty)} e^{-\lambda t}\, d\mathsf{P}_z\{T^y \leqq t\}$$

we have

$$\lim_{z \to y} f^{-y}(\lambda; z) = 1.$$

By the (inverse) continuity theorem for Laplace-Stieltjes transforms.

$$\lim_{z \to y} \mathsf{P}_z\{T^y \leqq t\} = 1 \qquad (t > 0).$$

The proof of the statement that condition (2.4) implies that $y(n) \to y$ (whether or not y is regular) is easy and is left to the reader.

§ 4. Exit Laws and Additive Functionals

4.1

In Section 4, we shall prove, among other results, Lemmas 2.2 and 2.3 and the converse part of Theorem A. Let us examine the converse part of Theorem A first.

Suppose that g and f are coupled. Then (Proposition N 2) g is extremal so that $g = g(E) g^y$ for some point y of E^+. *If* we knew that g is regular, i.e., that y is regular, then we could conclude from Proposition N 3 and the direct part of Theorem A that $f = K f^y$. Actually, we do the reverse. We *do* know (Lemma 2.1) that f is regular and (Proposition N 2) that f is extremal. We shall establish Lemma 2.2 which will imply that f is of the form $K f^z$ for a regular point z of E^+. But then the converse part of Theorem A, i.e., the fact that $y = z$, will follow from the direct part of Theorem A and Proposition N 3.

4.2

We require a generalisation of Lemma 2.3. For the terminology used in this subsection, the reader is referred to Chapter VI of DYNKIN [3].

We shall call a functional φ of X *regular* if φ is non-negative and *(strictly)* homogeneous, continuous and additive and also such that

$$\lim_{t \downarrow 0} \sup_{x \in E} \mathsf{M}_x \varphi_t = 0.$$

In particular, a regular functional is a W-functional.

Theorem 4.1. *If f is a regular exit law, then there exist a sequence $\Delta = \{\delta_n\}$ with $\delta_n \downarrow 0$ and a set $\tilde{\Omega}$ of full measure such that, whenever $\omega \in \tilde{\Omega}$, the limit*

$$\varphi_t(\omega) \equiv \lim_{\Delta \ni \delta \downarrow 0} \sum_{j \in E} \beta_t(j; \omega)\, f_\delta(j)$$

exists uniformly on compact intervals. If we set

$$\varphi_t(\omega) \equiv 0 \qquad (t \geqq 0,\ \omega \in \Omega \backslash \tilde{\Omega}),$$

then φ is a regular functional and

$$F_t(i) = \mathsf{M}_i \varphi_t \qquad (i \in E;\ t \geqq 0).$$

Theorem 4.1 is very similar to Theorem 6.6 in Dynkin [3] and may be deduced from that theorem with the aid of earlier results, especially Lemma 6.2 and Proposition 6.9 B, from Dynkin's book. It should be noted that we require φ to be *strictly* and not merely *almost* homogeneous. It is possible to prove the theorem by a more direct argument involving a use of martingale theory similar to that on page 306 of Meyer [11].

Sketch of proof of Theorem 4.1. Set

$$\varphi_t(\delta) = \langle \beta_t, f_\delta \rangle.$$

Then, for $0 \leqq s \leqq t$,

$$\mathsf{M}_x\{\varphi_t(\delta) \,|\, \mathscr{M}_s\} = \varphi_s(\delta) + F_{t-s+\delta}(x_s) - F_\delta(x_s).$$

Hence

$$\begin{aligned} \psi_s(\delta_1, \delta_2) &\equiv \varphi_s(\delta_1) - \varphi_s(\delta_2) \\ &\quad + [F_{t-s+\delta_1}(x_s) - F_{t-s+\delta_2}(x_s) - F_{\delta_1}(x_s) - F_{\delta_2}(x_s)] \end{aligned} \tag{4.1}$$

is a right-continuous martingale in s on $[0, t]$ and we note that the bracketed expression on the right hand side of Eq. (4.1) is small with $\delta \equiv \max(\delta_1, \delta_2)$ *uniformly* in s, x_s and t. (The absolute value of the bracketed term is not greater than $2\|F_\delta\|$.) By Doob's Martingale Inequality,

$$\mathsf{M}_x\left[\sup_{s \leqq t} \psi_s(\delta_1, \delta_2)\right]^2 \leqq 4\,\mathsf{M}_x[\varphi_t(\delta_1) - \varphi_t(\delta_2)]^2$$

and, if $\delta < t$,

$$\mathsf{M}_x[\varphi_t(\delta_1) - \varphi_t(\delta_2)]^2 \leqq 4\,\|F_{2t}\|\,\|F_\delta\|.$$

(Compare Lemma 6.5 of Dynkin [3].) If we choose the sequence Δ such that

$$\sum_{\delta \in \Delta} \|F_\delta\| < \infty,$$

then Theorem 4.1 follows from the Borel-Cantelli lemma.

Corollary. *The equation*

$$F_t(i) = \mathsf{M}_i \varphi_t$$

exhibits a one-one correspondence between the set of regular exit laws and the set of equivalence classes of regular functionals.

Definition. A regular functional will be called *extremal* if the associated regular exit law is extremal.

The part of Lemma 2.2 not already proved is contained in the following result.

Theorem 4.2. *If φ is an extremal regular functional, then φ is equivalent to a regular functional of the type $K\varphi(y)$ where K is a constant and y is a regular point of E^+. The regular exit law f associated with φ is therefore equal to Kf^y.*

Proof of Theorem 4.2. Suppose that φ is extremal regular. Then, for every Borel subset A of E^+, the equation

$$\varphi_t(A) \equiv \int_{(0,t)} \chi_A(x_s)\,\varphi(ds),$$

χ_A denoting the characteristic function of A, defines a regular functional $\varphi(A)$. Since

$$\varphi(A) + \varphi(E^+\backslash A) = \varphi$$

and φ is extremal, it follows that

$$\varphi(A) = c(A)\,\varphi$$

where c is clearly a probability measure on the Borel subsets of E^+. But, for almost every ω.

$$\frac{d\varphi_\cdot(A)}{d\varphi_\cdot} = \chi_A(x_\cdot)$$

almost surely in φ-measure and so the only values which c may take are 0 and 1. It follows easily that c must be concentrated at a single point y of E^+:

$$c(y) = 1.$$

(Recall that, by Proposition N 1, (E^+, ϱ) is a complete separable metric space.) Hence

$$\varphi_t = \int_{(0,t)} \chi_{\{y\}}(x_s)\,d\varphi_s. \tag{4.2}$$

Since φ is continuous, the point y must be regular (for a semipolar point is visited only finitely often). We already know that Eq. (4.2) characterises multiples of $\varphi(y)$ (see § 1.6). Theorem 4.2 follows.

The results of this section suggest that Choquet methods may provide a natural for proving theorems which assert that time substitutions must arise from local time integrals. See Theorem 8.4 of DYNKIN [3] and § 5.9 of ITO and MCKEAN [6].

§ 5. Inverse Local Time. Excursions

5.1

Throughout Sections 5 and 6, y will denote a fixed regular point of $E^+\backslash E$ (we again assume that there is one) and we shall simplify the notation by writing

$$\varphi, g, a, f^-, g^-, a^-, P^-$$

for

$$\varphi(y), g^y, a^y, f^{-y}, g^{-y}, a^{-y}, P^{-y}.$$

Set

$$\varrho_t \equiv \infty \quad \text{if} \quad \varphi_\infty \equiv \lim_{u\uparrow\infty} \varphi_u \leqq t,$$
$$\equiv \inf\{s\colon \varphi_s > t\} \quad \text{otherwise.}$$

The Strong Markov Theorem implies that $(\varrho_t, \mathscr{X}^0, \mathsf{P}_y)$ is a process with independent increments.

Theorem 5.1.

$$\mathsf{M}_y \exp[-\lambda\varrho_t] = \exp[-t\Psi(\lambda)] \tag{5.1}$$

where $\Psi(\lambda)$ *has the following explicit Lévy-Khinchin decomposition:*

$$\Psi(\lambda) = [a(\lambda)]^{-1} = \int_{(0,\infty)} (1 - e^{-\lambda t})\, a_t^- \, dt + [a(0)]^{-1}. \tag{5.2}$$

Proof (see NEVEU [14, 15]). On integrating Eq. (5.1) with respect to t, we obtain

$$[\Psi(\lambda)]^{-1} = \mathsf{M}_y \int_{(0,\infty)} \exp[-\lambda \varrho_t]\, dt = \mathsf{M}_y \int_{(0,\infty)} \exp(-\lambda t)\, d\varphi_t = a(\lambda)$$

The remaining assertion of Eq. (5.2) is simply a restatement of NEVEU's formula at (1.5).

For completeness, we state without proof

Lemma 5.1. *Let*

$$\tau_t \equiv \inf\{s\colon\ s + \varphi_s(y) > t\},$$
$$x_t^+ \equiv x_{\tau_t}, \quad \mathscr{M}_t^+ \equiv \mathscr{M}_{\tau_t}.$$

Then $X^+ \equiv (x_t^+, \mathscr{M}_t^+, \mathsf{P}_x)$ *is a strong Markov process such that*

$$\mathsf{P}_x\{x_t^+ \in E \cup y\} = 1 \qquad (x \in E^+, t > 0).$$

The transition function P^+ *of* X^+ *satisfies:*

$$\begin{aligned} P^+(\lambda; y, y) &= [\lambda + \Psi(\lambda)]^{-1}; \\ P^+(\lambda; y, j) &= [\lambda + \Psi(\lambda)]^{-1} g^-(\lambda; j); \\ P^+(\lambda; i, y) &= f^-(\lambda; i)\, [\lambda + \Psi(\lambda)]^{-1}; \\ P^+(\lambda) &= P^-(\lambda) + [\lambda + \Psi(\lambda)]^{-1} f^-(\lambda) \times g^-(\lambda) \\ &\quad \text{on} \quad E \times E. \end{aligned}$$

5.2

Definition. For any additive functional φ, write

$$\tilde{\varphi}_t \equiv \varphi_{\varrho_t}.$$

The remainder of our treatment hinges on the following result.

Theorem 5.2. *For* $t > 0$, $i \in E$ *and* $j \in E$,

$$\mathsf{M}_y\{\tilde{\beta}_t(i) \,|\, \varrho_t < \infty\} = t\, g^-(0; i)\, f^-(0; i), \tag{5.3}$$

$$\begin{aligned} &\mathrm{Cov}_y\{\tilde{\beta}_t(i), \tilde{\beta}_t(j) \,|\, \varrho_t < \infty\} \\ &\quad = t[g^-(0; i)\, P^-(0; i, j)\, f^-(0; j) + g^-(0; j)\, P^-(0; j, i)\, f^-(0; i)]. \end{aligned} \tag{5.4}$$

There is no doubt that the "right" way to prove this theorem is by the use of excursion theory. The formulae required for the present situation have been known for some time, thanks to NEVEU [13, 14], but no full probabilistic treatment from "strong Markov" arguments has been given. However, NEVEU provides very convincing evidence for his formulae at the end of his paper [14].

The structure of the random set $\{t\colon x_t = y\}$ is completely described by Theorem 5.1. The interior of the set $\{t\colon x_t \neq y\}$ is, of course, the union of a countable number of disjoint open intervals. The statement that the behaviour of X inside one of these intervals depends only on the length of that particular interval may be made precise as follows.

Take a fixed $t > 0$. Let

$$\sigma \equiv \sup\{s:\ s \leqq t, x_s = y\}$$
$$\eta \equiv \min\{u:\ u \geqq t, x_u = y\}$$

with $\eta \equiv \infty$ if no u satisfies the bracketed condition. (Note that η is a Markov time but that σ is not.) Let $\mathscr{M}_\eta^*$ be the Borel field generated by the random variables $x_{\eta+u}$ $(u \geqq 0)$ and let $\mathscr{M}_\sigma$ be the smallest complete Borel field containing the sets

$$\{x_s = j; \sigma > s\} \qquad (j \in E; 0 \leqq s \leqq t).$$

Then, on the set $\{\tau < \infty\}$,

$$\mathsf{P}_y\{x_t = i \,|\, \sigma, \eta, \mathscr{M}_\eta^*, \mathscr{M}_\sigma\} = \frac{g^-_{t-\sigma}(i)\, f^-_{\eta-t}(i)}{a^-_{\eta-\sigma}} \tag{5.5}$$

Here is an extremely informal proof of Eq. (5.3) based on Eq. (5.5). Suppose that $x_0 = y$, i.e., use the P_y measure. Given that an excursion interval is of length l, it follows from Eq. (5.5) that the expected time spent in i during that interval is

$$\frac{1}{a^-(l)} \int_0^l g_s^-(i)\, f_{l-s}^-(i)\, ds\,.$$

However, the content of the Lévy-Khinchin formula (5.2) is that the number of excursion intervals of lengths between l and $l + dl$ made by X before time ϱ_t is Poisson parameter $a^-(l)\, dl$. Hence

$$\begin{aligned}\mathsf{M}_y\{\tilde{\beta}_t(i) \,|\, \varrho_t < \infty\} &= \int_{l=0}^{\infty} dl \int_{s=0}^{l} g_s^-(i)\, f_{l-s}^-(i)\, ds \\ &= g^-(0; i)\, f^-(0; i)\,.\end{aligned}$$

as required. Eq. (5.4) may be "proved" similarly.

5.3

Instead of justifying the above argument, we now describe an alternative use of excursion theory which yields a proof of Theorem 5.2 requiring less sophisticated methods.

Sketch of proof of Theorem 5.2. Take a fixed pair (i, j) of states of E. Set

$$\begin{aligned}\alpha_t &\equiv \infty \quad \text{if} \quad \varphi_\infty + \beta_\infty(i) + \beta_\infty(j) \leqq t\,, \\ &\equiv \inf\{s:\ \varphi_s + \beta_s(i) + \beta_s(j) > t\} \quad \text{otherwise.}\end{aligned}$$

Set

$$z_t \equiv x_{\alpha_t}\,, \qquad \mathscr{I}_t \equiv \mathscr{M}_{\alpha_t}\,.$$

Then $(z_t, \zeta, \mathscr{I}_t)$ is a right continuous strong Markov process with lifetime

$$\zeta \equiv \varphi_\infty + \beta_\infty(i) + \beta_\infty(j) \leqq \infty$$

taking only the values y, i and j. The transition function of $Z \equiv \{z_t\}$ may be calculated by standard methods (see NEVEU [13]) but it may be seen directly that the problem of proving Theorem 9.1 reduces to that of proving the *exactly* analogous problem for Z. This is much more elementary.

In a finite time interval, Z makes only finitely many excursions from y and the excursions may be labelled in the order in which they occur. That the excursions are "independent of their past and future" may be stated in conventional strong Markov terms and the probabilistic facts required are covered by Corollary 2 to Theorem II.15.2 of (the second edition of) Chung [2]. The algebra which leads to the formulae of Theorem 5.2 is somewhat involved but as the entire calculation has now been reduced to the handling of exponential variables, it does not seem worth spelling it out.

§ 6. Theorems C and D

6.1

We now assume that X is irreducible and recurrent.

Lemma 6.1.

$$g^-(0;j) = K\,\pi(j) \qquad (j \in E)$$

for some constant K.

Proof. Set

$$B_t^- \equiv \int_{(t,\infty)} a_s^-\,ds = \langle g^-(0), f_t^- \rangle .$$

That y is recurrent implies that $a(0) = \infty$. Hence, from Eq. (1.5),

$$[a(\lambda)]^{-1} = \int_{(0,\infty)} \lambda e^{-\lambda t} B_t^-\,dt = \lambda \langle g^-(0), f^-(\lambda)\rangle . \tag{6.1}$$

It now follows that

$$\begin{aligned} \lambda g^-(0)\,P(\lambda) &= \lambda g^-(0)\,[P^-(\lambda) + f^-(\lambda) \otimes g^-(\lambda)\,a(\lambda)] \\ &= \lambda g^-(0)\,P^-(\lambda) + \lambda \langle g^-(0), f^-(\lambda)\rangle\, a(\lambda)\, g^-(\lambda) \\ &= \lambda g^-(0)\,P^-(\lambda) + g^-(\lambda) = g^-(0). \end{aligned}$$

The measure $g^-(0)$ is therefore invariant.

We may and shall assume that the local time at y is normalised so that the constant K of Lemma 6.1 is equal to unity: $K = 1$. Theorem 5.2 then takes the following simple form.

Theorem 6.1. *For* $t > 0$, $i \in E$ *and* $j \in E$,

$$\mathsf{M}_y\{\tilde{L}_t(i)\} = t ,$$

$$\mathrm{Cov}_y\{\tilde{L}_t(i), \tilde{L}_t(j)\} = t\left[\frac{P^-(0;i,j)}{\pi(j)} + \frac{P^-(0;j,i)}{\pi(i)}\right].$$

6.2. Proof of Theorem C

From Theorem 6.1, we deduce that

$$\begin{aligned} &\mathsf{M}_y \langle \tilde{L}_t, g_\delta\rangle = t , \\ &\mathrm{Var}_y \langle \tilde{L}_t, g_\delta\rangle = 2t \sum_i \sum_j g_\delta(i)\,P^-(0;i,j)\,g_\delta(j)/\pi(j) = 2t\,\mathsf{M}_{g_\delta}\langle L_T, g_\delta\rangle . \end{aligned} \tag{6.2}$$

A calculation very similar to that used in the proof of Lemma 6.1 shows that

$$\mathsf{M}_{g_\delta}\langle L_T, g_\delta\rangle \leqq A_\delta .$$

Since

$$\lim_{\delta \downarrow 0} A_\delta = 0 ,$$

it follows from TCHEBYCHEFF's inequality that

$$\lim_{\delta\downarrow 0} \mathsf{P}_y\{|\langle\tilde{L}_t, g_\delta\rangle - t| > \varepsilon\} = 0.$$

However, $(\langle\tilde{L}_t, g_\delta\rangle - t, \mathsf{P}_y)$ is a centered process with independent increments and so a martingale. We may therefore deduce (as in the proof of Theorem 4.1) that

$$\lim_{\delta\downarrow 0} \mathsf{P}_y\left\{\sup_{0\leqq s\leqq t} |\langle\tilde{L}_s, g_\delta\rangle - s| > \varepsilon\right\} = 0.$$

But, if $x_0 = y$, then

$$\sup_{0\leqq s\leqq t} |\langle\tilde{L}_s, g_\delta\rangle - s| = \sup_{0\leqq s\leqq \varrho_t} |\langle L_s, g_\delta\rangle - \varphi_s|.$$

(A positive [negative] "error"

$$\langle L_s, g_\delta\rangle - \varphi_s$$

would imply a larger absolute "error" at the first [last] visit to y after [before] time s.)

If we now piece together the above results, we reach the following conclusion.

Lemma 6.2. *Let Δ be a sequence of positive reals such that*

$$\sum_{\Delta} \Delta(\delta) < \infty.$$

Let $\tilde{\Omega}$ be the set of ω in Ω such that

$$\lim_{\Delta\ni\delta\downarrow 0} \langle L_t(\omega), g_\delta\rangle = \varphi_t(\omega) \tag{6.3}$$

for every $t > 0$. Then

$$\mathsf{P}_y(\tilde{\Omega}) = 1.$$

Let Δ and $\tilde{\Omega}$ be as in Lemma 6.2. Let i be a point of E and let

$$\tau \equiv T^i \equiv \inf\{s\colon x_s = i\}.$$

Because of the irreducible recurrent character of X, we have

$$\mathsf{P}_y\{\tau < \infty\} = 1.$$

If $\omega \in \tilde{\Omega}$, then, for every t,

$$\lim_{\Delta\in\delta\downarrow 0} \langle L_{t+\tau}(\omega) - L_\tau(\omega), g_\delta\rangle = \varphi_{t+\tau}(\omega) - \varphi_\tau(\omega),$$

i.e. $\omega \in \theta_\tau\tilde{\Omega}$. Hence $\tilde{\Omega} \subseteqq \theta_\tau\tilde{\Omega}$ and the Strong Markov Theorem gives

$$1 = \mathsf{P}_y(\theta_\tau\tilde{\Omega}) = \mathsf{P}_{x_\tau}(\tilde{\Omega}) = \mathsf{P}_i(\tilde{\Omega}).$$

Lemma 6.3. *With $\tilde{\Omega}$ as in Lemma 6.2, we have*

$$\mathsf{P}_i(\tilde{\Omega}) = 1 \qquad (i \in E).$$

We may therefore integrate Eq. (6.3) with respect to P_i and apply FATOU's Lemma to obtain

$$\liminf_{\Delta\in\delta\downarrow 0} \sum_j g_\delta(j) \int_{(0,t)} P_s^*(j, i)\, ds \geqq G_t^*(i). \tag{6.4}$$

A second application of FATOU's Lemma shows that the sum over of the left hand side of inequality (6.4) is not greater than t.

Lemma 6.4.

$$\sum_i g_t^*(i) = 1 \qquad (t > 0).$$

Equality therefore holds at (6.4).

Proof. Since $\pi = g^-(0)$, we have

$$\sum_i \lambda g^*(\lambda; i) = \sum_i \lambda f(\lambda; i)\, g^-(0; i) = \lambda \langle g^-(0), f^-(\lambda)\rangle\, a(\lambda).$$

The proof is completed by quoting Eq. (6.1).

Because equality holds at (6.4) for any sequence Δ which tends to zero sufficiently rapidly, it follows that

$$\lim_{\delta \downarrow 0} \int_{(0,t)} \sum_j g_\delta(j)\, P_s^*(j, i)\, ds = G_t^*(i).$$

That we may differentiate formally to obtain the result

$$\lim_{\delta \downarrow 0} \sum_j g_\delta(j)\, P_t^*(j, i) = g_t^*(i) \tag{6.5}$$

follows from Proposition 2 of NEVEU [14]. (Note that, for each $\delta > 0$, $g_\delta P^*_\cdot$ is an entrance law relative to P^*.)

Eq. (6.5) is the first assertion of Theorem C and since the other assertions of Theorem C are easily seen to be equivalent to Eq. (6.5), Theorem C is proved.

Eq. (6.2) explains the significance of hypothesis (2.10) and Theorem D may be proved by an obvious modification of the methods used above.

6.3

That Theorem C′ would imply Theorem C is trivial. Consider the second assertion of Theorem C:

$$P_t(y, j) = \lim_{\delta \downarrow 0} \sum_{i \in E} P_\delta^*(y, i)\, P_t(i, j). \tag{6.6}$$

If we knew that P^* is stochastically continuous at y in the P-topology, then Eq. (6.6) would follow immediately from the fact that, for fixed t and j, $P_t(x, j)$ is continuous in x.

The converse proposition that Theorem C implies Theorem C′ is much less trivial. It follows from Eq. (6.6) that

$$\lim_{\delta \downarrow 0} \sum_{i \in E} P_\delta^*(y, i)\, P(1; i, j) = P(1; y, j)$$

for every j. This is precisely the condition which we need to deduce that the measures $P_\delta^*(y, \cdot)$ converge weak*-ly to the unit mass at y by the method explained on pages 328 and 329 of NEVEU [14].

The results of § 2.5 may be proved by methods very similar to those used above.

§ 7. Examples

7.1

We come now to the proof that, in general, local time at a fictitious state may not be expressed as a derivative of occupation time. In view of the essential equivalence of instantaneous states of E and regular fictitious states of $E^+\backslash E$, it will be sufficient for our purpose to establish that, in general, time in an instantaneous state may not be expressed by a formula of the type

$$\beta_t(b) = \lim_{\Delta \ni \delta \downarrow 0} \frac{m\{s: 0 \leqq s \leqq t;\, 0 \neq \varrho(x_s, b) < \delta\}}{h_\delta}.$$

Example 7.1. Consider the process X of Example 3 of § II.20 of CHUNG [2] with

$$q_j = 2^j \qquad (j = 1, 2, \ldots).$$

This is a special case of the process "K 1" which was constructed by KOLMOGOROV and exhaustively analysed in KENDALL and REUTER [7]. For this process, $E^+ = E$ and E^+ is a *compact* metric space with 0 as the (unique) limit point of the sequence 1, 2, 3, ...:

$$\lim_{i \to \infty} p_{ij}(t) = p_{0j}(t) \qquad (j \in E, t \geqq 0).$$

(This is Eq. (43) of KENDALL and REUTER [7].) We may therefore assume that X is a standard process.

We base our proof of the fact that *no formula of the type*

$$\beta_t(0) = \lim_{\Delta \ni n \to \infty} \frac{m\{s: 0 \leqq s \leqq t,\, x_s \geqq n\}}{h_n} \tag{7.1}$$

can hold on a remarkable "self-reproductive" property of X.

Theorem 7.1. *Let*

$$\tau_t \equiv \inf\left\{s:\ \beta_t(0) + 2\sum_{j \geqq 2} \beta_t(j) > t\right\},$$
$$z_t \equiv 0 \quad \text{if} \quad x_{\tau_t} = 0,$$
$$\equiv x_{\tau_t} - 1 \quad \text{otherwise.}$$

Then the strong Markov process Z has exactly the same transition function as X.

This is an intuitively obvious consequence of the analysis of K 1 given in KENDALL and REUTER [7] and CHUNG [2]. It may be proved easily by the methods of WILLIAMS [17].

Now set

$$\varrho_t \equiv \inf\{s:\ \beta_s(0) > t\},$$
$$\beta_s^z(0) \equiv m\{u:\ 0 \leqq u \leqq s;\, z_u = 0\},$$
$$\varrho_t^z \equiv \inf\{s:\ \beta_s^z(0) > t\}.$$

It is a simple exercise to show that

$$2m\{s:\ 0 \leqq s \leqq \varrho_t;\, x_s \geqq 2\} = m\{s:\ 0 \leqq s \leqq \varrho_t^z;\, z_s \geqq 1\}.$$

It therefore follows from Theorem 7.1 that

(Lemma 7.1.) *the* P_0*-distributions of the random variables*

$$2^k m\{s\colon 0 \leqq s \leqq \varrho_t; x_s \geqq k\} \qquad (k = 1, 2, \ldots) \tag{7.2}$$

are identical.

If Eq. (7.1) were true, then

$$\lim_{\Delta \ni k \to \infty} \frac{m\{s\colon 0 \leqq s \leqq \varrho_t; x_s \geqq k\}}{h_k} = t. \tag{7.3}$$

However, in view of Lemma 7.1, Eq. (7.3) could be true only if each of the random variables at (7.2) is constant almost surely (P_0) and this is obviously quite absurd.

Note. For conditions under which the local time at a point may be expressed as a derivative of occupation time, see GRIEGO [5]. Our example shows that GRIEGO's hypothesis that the convergence of $U^\lambda(x, y)$ to $U^\lambda(x, x_0)$ is *uniform in x* may not be relaxed completely. All of the other hypotheses of [5] are satisfied in our example.

7.2

The next example shows that the P and P^* topologies may differ at a regular point.

Example 7.2. We take Example 4 of § II.20 of CHUNG [2]. The following results may be verified easily.

In the P-topology, the limit

$$\lim_{n \to \infty} j_n$$

exists and is a semi-polar fictitious state j_∞ representing the top of the j-th escalator and

$$E^+ = E \cup \{1_\infty, 2_\infty, \ldots\}.$$

The process is irreducible positive recurrent and the measure π with

$$\pi(j_n) \equiv (q_{j_n})^{-1} \qquad (j \geqq 1, n \geqq 1)$$
$$\pi(0) \equiv 1$$

is invariant.

In the P^*-topology, there are no fictitious states and

$$\lim_{n \to \infty} j_n = 0.$$

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