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Commuting Point Transformations*

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Summary. The question considered is the following: If two invertible measure preserving point transformations commute, in what sense is one a function of the other? The main theorem is the following: If two invertible measure preserving transformations commute, and if the first admits of approximation by periodic transformations, then the second transformation is a piecewise power of the first, where we say that σ is *a piecewise power* of τ if there exists a sequence $[j(n)]$ of positive integers such that for each measurable set A the limit of the measure of the symmetric difference of $\tau(A)$ and $\sigma^{j(n)}(A)$ tends to zero.

1. Introduction

The general problem we consider is the following: If two invertible measure preserving point transformations commute, in what sense of special interest from the point of view of point transformations is one a function of the other? As is well known, the spectral theorem provides a general answer to this question, and our principal result may be regarded in a sense as a specialization of it.

As an application, we mention that it yields a simplification of the constructions given in [2] and [3] of transformations having continuous spectrum and no roots. We omit details since giving them would take us too far from the problem we are considering in this note. Certain related questions, for transformations having discrete spectrum, are considered in [1].

The main theorem is the following: If two invertible measure preserving transformations commute, and if the first admits of approximation by periodic transformations, then the second transformation is a piecewise power of the first (see § 2 for definitions). Our methods are related principally to those of [2] and [3].

2. Definitions

Let $(X, \mathscr{F}, \mu)$ be a normalized non-atomic Lebesgue space (i.e. isomorphic to the unit interval), and let σ and τ be invertible measure preserving transformations of X. All sets that are referred to are understood to be in $\mathscr{F}$ even if this is not explicitly stated.

Definition 2.1. We say that σ is a *piecewise power* of τ if there exists a sequence $\{k(n)\}$ such that

$$\lim_{n\to\infty} \mu(\sigma(A)\,\Delta\,\tau^{k(n)}(A)) = 0\,, \quad \text{for all} \quad A\in\mathscr{F}\,.$$

We say that σ is a *weak piecewise power* of τ if for each $A\in\mathscr{F}$ and $\varepsilon > 0$ there exists an integer k such that $\mu(\sigma(A)\,\Delta\,\tau^{k}(A)) \leqq \varepsilon$. (By $A\,\Delta\,B$ we mean the symmetric difference of the sets A and B.)

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Definition 2.2. A collection ξ of sets having union $X_\xi \subseteqq X$ will be called *a partition* if the sets are in $\mathscr{F}$ and are pairwise disjoint. If $X_\xi = X$ we call ξ *a partition of* X.

We remark that ROHLIN [5] has defined a (measurable) partition in a different and deeper way which, however, coincides with our definition if ξ is denumerable and $X_\xi = X$. Since we will make no use of the deeper concept, we choose to give the simpler definition.

Definition 2.3. Let $\xi = \{A_1, \ldots, A_n, \ldots\}$ be a finite or denumerable partition and let $A \in \mathscr{F}$. Among the unions of sets of ξ there is at least one whose symmetric difference with A has minimal measure. We denote by $A(\xi)$ any one of these sets. The resulting ambiguity is deliberate and simplifies the notation considerably. We define $1 \cdot B = B$ and $0 \cdot B = \emptyset$ so that we can write

$$A(\xi) = \sum_{i=1}^{\infty} a_i A_i$$

where a_i, $i = 1, 2, \ldots$, is equal to 0 or 1 depending (not uniquely) on A. Note that $A(\xi) = A$ if $A \subset X_\xi$ and $A \in \mathscr{F}(A_1, A_2, \ldots)$, where $\mathscr{F}(A_1, A_2, \ldots)$ is the Borel field generated by the sets $A_1, A_2, \ldots$.

Definition 2.4. τ is *an automorphism* (of $(X, \mathscr{F}, \mu)$) if τ is an invertible measure preserving transformation of X.

Definition 2.5. Let ε denote the partition of X into single points and let $\{\xi(n)\}$ be a sequence of denumerable partitions. We write $\xi(n) \to \varepsilon$ if for ech $A \in \mathscr{F}$ we have that

$$\lim_{n\to\infty} \mu(A \,\Delta\, A(\xi(n))) = 0 .$$

Definition 2.6. We say that an automorphism τ *admits of approximation by periodic automorphisms (transformations)* if there exists a sequence $\{\xi(n)\}$ of partitions with $\xi(n) = \{A_i(n), i = 1, 2, \ldots, q(n)\}$ such that

a) $\tau(A_i(n)) = A_{i+1}(n)$, $i = 1, 2, \ldots, q(n) - 1$,

b) $\xi(n) \to \varepsilon$, and

c) if $B(n) = c\left(\bigcup_{i=1}^{q(n)} A_i(n)\right)$ then $\lim_{n\to\infty} q(n)\, \mu(B(n)) = 0$,

where $c(A)$ is the complement of A relative to the whole space.

3. Approximation by Periodic Automorphisms

Several different notions of approximation have been given by various authors. KATOK and STEPIN in [4] defined a metric in terms of which an automorphism can be approximated by periodic automorphisms. The main result of this section is to relate this concept of approximation to that given in § 2.

Definition 3.1. An automorphism τ admits of *a KS-approximation* by periodic automorphisms with speed $f(n)$ if for each $n = 1, 2, \ldots$ there exists a partition of X, $\xi(n) = \{C_i(n), i = 1, 2, \ldots, q(n)\}$, and an automorphism τ_n such that:

1) $\xi(n) \to \varepsilon$,

2) $\tau_n \xi(n) = \xi(n)$ for each n and τ_n permutes the elements of $\xi(n)$ cyclically, and

3) $\sum_{i=1}^{q(n)} \mu(\tau_n C_i(n) \Delta \tau C_i(n)) < f(q(n))$.

Further if $f(n) = 0(1/n^j)$ we say that τ admits of a $KS(j)$ approximation.

Theorem 3.1. *If an automorphism τ admits of a $KS(2)$ approximation, then τ admits of approximation by periodic automorphisms. If τ admits of approximation by periodic automorphisms, then τ admits of a $KS(1)$ approximation.*

Proof. (1) Approximation by periodic automorphisms implies $KS(1)$ approximated by periodic automorphisms then for each n there exists a partition $\xi(n) = \{A_i(n), i = 1, 2, \ldots, q(n)\}$ satisfying the conditions of definition 2.6, where $B(n) = c\left(\bigcup_{i=1}^{q(n)} A_i(n)\right)$. Write $B(n)$ as the disjoint union of $q(n)$ sets of equal measure, so that $B(n) = F_1(n) \cup \cdots \cup F_{q(n)}(n)$ where $\mu(F_i(n)) = \mu(F_j(n))$ for $1 \leqq i$, $j \leqq q(n)$, and define a partition $\eta(n)$ of X by:

$$\eta(n) = \{A_i(n) \cup F_i(n), \quad i = 1, 2, \ldots, q(n)\}.$$

Define the automorphism σ_n arbitrarily on $F_i(n)$ but so that

$$\sigma_n F_i(n) = F_{i+1}(n), 1 \leqq i < q(n) \quad \text{and} \quad \sigma_n F_{q(n)}(n) = F_1(n).$$

Let $C_i(n) = A_i(n) \cup F_i(n)$ for $1 \leqq i \leqq q(n)$, and let

$$\tau_n = \begin{cases} \tau & \text{on} \quad A_i(n) \\ \sigma_n & \text{on} \quad F_i(n) \quad \text{for} \quad 1 \leqq i < q(n). \end{cases}$$

Let $G(n)$ be the subset of $A_{q(n)}(n)$ such that $\tau G(n) \subset B(n)$, and let $H(n)$ be the subset of $A_1(n)$ such that $\tau^{-1}(H(n)) \subset B(n)$ (then $\mu(G(n)) = \mu(H(n))$). Let ϱ_n be any automorphism such that $\varrho_n(G(n)) = H(n)$, and let

$$\tau_n = \begin{cases} \tau & \text{on} \quad A_{q(n)}(n) - G(n) \\ \sigma_n & \text{on} \quad F_{q(n)}(n) \\ \varrho_n & \text{on} \quad G(n). \end{cases}$$

Clearly $\tau_n C_i(n) = C_{i+1}(n)$ for $1 \leqq i < q(n)$, and $\tau_n C_{q(n)}(n) = C_1(n)$, so that τ_n permutes the elements of $\eta(n)$ cyclically. Since $\xi(n) \to \varepsilon$, we have

$$\lim_{n \to \infty} \mu(A \Delta A(\xi(n))) = 0 \quad \text{for} \quad A \in \mathscr{F}.$$

Given $\delta > 0$ choose N such that $n > N$ implies $\mu(A \Delta A(\xi(n))) < \delta/2$ and $\mu(B(n)) < \delta/2$. (Note that from the conditions of definition 2.6 $\lim_{n \to \infty} \mu(B(n)) = 0$.)

Suppose $A(\xi(n)) = \sum_{i=1}^{q(n)} a_i(n) A_i(n)$, and define $\tilde{A}(\xi(n)) = \sum_{i=1}^{q(n)} a_i(n) C_i(n)$. We note the inequality

$$|\mu(A \Delta (B \cup C)) - \mu(A \Delta B)| \leqq \mu(C) \quad \text{for} \quad A, B, \text{ and } C \in \mathscr{F},$$

from which follows

$$| \mu(A \mathbin{\Delta} \tilde{A}(\xi(n))) - \mu(A \mathbin{\Delta} A(\xi(n))) | \leqq \mu\left(\sum_{i=1}^{q(n)} a_i(n) F_i(n)\right) \leqq \mu(B(n)) \leqq \delta/2 \quad \text{for} \quad n > N .$$

This implies that $\mu(A \mathbin{\Delta} A(\eta(n))) < \delta$ for $n > N$, so that $\eta(n) \to \varepsilon$. Lastly we have

$$\sum_{i=1}^{q(n)} \mu(\tau_n C_i(n) \mathbin{\Delta} \tau C_i(n)) \leqq 2\mu(G), + 2\sum_{i=1}^{q(n)} \mu(F_i(n)) \leqq 4\mu(B(n)) . \text{ Let } f(q(n)) = 4\mu(B(n)) .$$

Then since $\lim_{n\to\infty} q(n)\,\mu(B(n)) = 0$, we have $\lim_{n\to\infty} q(n)\, f(q(n)) = 0$, and τ admits of a $KS(1)$ approximation.

(2) $KS(2)$ approximation implies approximation by periodic automorphisms.

Assume that τ admits of a $KS(2)$ approximation, then for each n there exists an automorphism τ_n and a partition of X, $\xi(n) = \{C_i(n),\ i = 1, 2, \ldots, q(n)\}$, satisfying the conditions of definition 3.1 with $f(n) = 0(1/n^2)$. For each i, $1 \leqq i \leqq q(n)$, we can write $C_i(n) = C_i(n) = G_i(n) \cup F_i(n)$ where $\tau G_i(n) \subset C_{i+1}(n)$ and $\tau F_i(n) \cap C_{i+1}(n) = \emptyset$ for $1 \leqq i < q(n)$, and where $\tau G_{q(n)}(n) \subset C_1(n)$ and $\tau F_{q(n)}(n) \cap C_1(n) = \emptyset$. Clearly we have

$$\mu(\tau_n C_i(n) \vartriangle \tau C_i(n)) = \mu(\tau F_i(n)) + \mu\, C_{i+1}(n) - \tau G_i(n)) = 2\mu(F_i(n))$$

for $1 \leqq i < q(n)$, and the analogous equation holds for $i = q(n)$.

Let $A_1(n) = C_1(n) \cap G_1(n) \cap \tau^{-1} G_2(n) \cap \cdots \cap \tau^{-q(n)+1} G_{q(n)}(n)$, and define $A_i(n) = \tau^{i-1} A_1(n)$, $1 \leqq i \leqq q(n)$. The we have

$$B(n) = \bigcup_{i=1}^{q(n)} (C_i(n) - A_i(n)), \quad \text{and}$$

$$\mu(B(n)) = \sum_{i=1}^{q(n)} \mu(C_i(n) - A_i(n)) = q(n)\,\mu(C_1(n) - A_1(n)) .$$

Since $C_1(n) - A_1(n) = C_1(n) \cap (F_1(n) \cup \tau^{-1} F_2(n) \cup \cdots \cup \tau^{-q(n)+1} F_{q(n)}(n))$, it follows that

$$\mu(C_1(n) - A_1(n)) \leqq \sum_{i=1}^{q(n)} \mu(\tau^{-i+1} F_i(n)) = \sum_{i=1}^{q(n)} \mu(F_i(n)) = \tfrac{1}{2} \sum_{i=1}^{q(n)} \mu(\tau_n C_i(n) \mathbin{\Delta} \tau C_i(n)) \leqq \tfrac{1}{2} f(q(n)) ,$$

and we have $q(n)\,\mu(B(n)) \leqq \frac{1}{2} q(n)^2 f(q(n))$, so that

$$\lim_{n\to\infty} q(n)\,\mu(B(n)) = 0 \quad \text{since} \quad f(n) = 0\left(\frac{1}{n^2}\right).$$

Let $\eta(n) = \{A_i(n),\ i = 1, 2, \ldots, q(n)\}$. By the definition of $A_i(n)$, it is clear that $\tau A_i(n) = A_{i+1}(f)$ for $1 \leqq i < q(n)$. The proof that $\eta(n) \to \varepsilon$ is exactly the same as the proof used to show the corresponding fact in the first part of this proof, and τ admits of approximation by periodic automorphisms.

4. Commuting Transformations

It will be convenient to introduce some notation first. Throughout this section, we will assume that τ and σ are automorphisms and that τ admits of approximation by periodic automorphisms and that $\{\xi(n)\}, \xi(n) = \{A_i(n), i = 1, \ldots, q(n)\}$ is the associated sequence of partitions. For each n, we divide $A_1(n)$ into $q(n)+1$ pairwise disjoint sets $B_1(n), B_2(n), \ldots, B_{q(n)+1}(n)$ where $B_i(n)$ consists of those points in $A_1(n)$ which are mapped into $A_i(n)$ by σ. That is $A_1(n) = \bigcup_{i=1}^{q(n)+1} B_i(n)$, $B_i(n) \cap B_j(n) = \emptyset$ for $i \neq j$, $\sigma(B_i(n)) \subset A_i(n)$ for $i = 1, 2, \ldots, q(n)$, and

$$B_{q(n)+1}(n) = A_1(n) - \bigcup_{i=1}^{q(n)} B_i(n).$$

We will also have occasion to consider transformations which are not necessarily defined everywhere, and if τ is such a transformation, by $\tau(A)$ we mean $\{y: y = \tau(x), x \in A \cap D_\tau\}$ where D_τ is the domain of definition of τ.

Remark. If σ and τ commute, then $\sigma(x) = y$ implies $\sigma(\tau^k(x)) = \tau^k(y)$.

Definition 4.1. For each n, we set ${}_n\tau = \tau$ on $X_{\xi(n)} = \bigcup_{i=1}^{q(n)} A_i(n)$ and leave it undefined elsewhere.

Lemma 4.1. *For any positive integer k,*

$${}_n\tau^k(\sigma(B_i(n))) \cap \sigma(B_j(n)) = \emptyset \quad \text{if} \quad i \neq j\,.$$

Proof. Note that

$${}_n\tau^k(\sigma(B_i(n))) \cap \sigma(B_j(n)) = \sigma(({}_n\tau^k(B_i(n))) \cap B_j(n))\,,$$

and that

$${}_n\tau^k(B_i(n)) \cap B_j(n) = \emptyset$$

since if k is not a multiple of $q(n)$, ${}_n\tau^k(B_i(n)) \not\subset A_1(n)$, and if k is a multiple of $q(n)$ it is easy to check that ${}_n\tau^k(B_i(n)) \cap A_1(n) \subset B_i(n)$.

Definition 4.2. We say that a set A approximates a set B with an error of δ if the measure of the symmetric difference of A and B is no more than δ, and we write in this case $A = B + E(\delta)$.

We note also the following combinatorial lemma given in [2]. We include a shorter proof for the sake of completeness.

Lemma 4.2. *Let $\alpha_j = (x_{j1}, \ldots, x_{jn})$, $j = 1, \ldots, k$ be k sequences of zeros and ones of length n. Let $\{b_j, j = 1, \ldots, k\}$ be non-negative numbers having sum equal to one, and suppose that there is a subset H of $\{1, \ldots, n\}$ such that*

$$\sum_{j=1}^{k} b_j x_{ji} \geqq 1 - \eta \quad \textit{for all} \quad i \in H\,.$$

Then there exists an integer ω, $1 \leqq \omega \leqq k$, such that

$$\sum_{i \in H} \sum_{j=1}^{k} b_j x_{ji} x_{\omega i} \geqq N(H)(1 - 2\eta)$$

where $N(H)$ is the number of elements in H.

Proof. To see that we may assume without loss of generality that $b_j = 1/k$, we proceed as follows. First note that we can certainly suppose that b_j's to be rational numbers with a common denominator so that $b_j = m_j/m$. Now consider the problem obtained by taking each $\alpha_j m_j$ times; this reduces the lemma to the case where the new k equals the sum of the m_j's and the new b_j's are all equal to $1/m$. So assuming $b_j = 1/k$, we have

$$\sum_{\omega=1}^{k} \sum_{i\in H} \sum_{j=1}^{k} b_j x_{ji} x_{\omega i} = \sum_{\omega=1}^{k} \sum_{i\in H} x_{\omega i} \left(\frac{1}{k} \sum_{j=1}^{k} x_{ji}\right)$$
$$\geqq (1-\eta) \sum_{\omega=1}^{k} \sum_{i\in H} x_{\omega i} = (1-\eta) \sum_{i\in H} \sum_{\omega=1}^{k} x_{\omega i}$$
$$\geqq (1-\eta) \sum_{i\in H} k(1-\eta) = k N(H) (1-\eta)^2 \geqq k N(H) (1-2\eta).$$

Therefore, for some ω, $1 \leqq \omega \leqq k$ we have

$$\sum_{i\in H} \sum_{j=1}^{k} b_j x_{ji} x_{\omega i} \geqq N(H)(1-2\eta).$$

We are now ready to prove our main result.

Theorem 4.1. *If $\sigma\tau = \tau\sigma$ and if τ admits of approximation by periodic automorphisms, then σ is a piecewise power of τ.*

We divide the proof into several lemmas.

Lemma 4.3. *Under the hypotheses of the theorem, given $A \in \mathscr{F}$ and $\delta > 0$, for each n there exist pairwise disjoint sets $C_1(n), \ldots, C_{q(n)}(n) \in \mathscr{F}$ such that for n sufficiently large $\sigma(A)$ is approximated with error δ by the set*

$$\sum_{i=1}^{q(n)} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n) \quad \textit{where} \quad a_{i-j+1} = 0 \quad \textit{or} \quad a_{i-j+1} = 1.$$

Further $\bigcup_{j=1}^{q(n)} C_j(n) \subset A_1(n)$ and $\lim_{n\to\infty} q(n)\, \mu(A_1(n) - \bigcup_{i=1}^{q(n)} C_j(n)) = 0$.

Proof. Recalling that $\{\xi(n)\}$, $\xi(n) = \{A_i(n),\ i = 1, 2, \ldots, q(n)\}$, is the sequence of partitions used in approximating τ, we choose n_0 large enough so that for $n > n_0$

$$A(\xi(n)) = \sum_{i=1}^{q(n)} a_i(n) A_i(n) = \sum_{i=1}^{q(n)} a_i(n)\, \tau^{i-1} A_1(n)$$

and

$$A = A(\xi(n)) + E(\delta/3).$$

It then follows that $\sigma(A) = \sigma(A(\xi(n))) + E(\delta/3)$ and

$$\sigma(A(\xi(n))) = \sum_{i=1}^{q(n)} a_i(n)\, \tau^{i-1} \sigma(A_1(n))$$
$$= \sum_{i=1}^{q(n)} a_i(n)\, \tau^{i-1} \sigma\left(\bigcup_{i=1}^{q(n)} B_j(n)\right) + \sum_{i=1}^{q(n)} a_i(n)\, \tau^{i-1} \sigma(B_{q(n)+1}(n)).$$

The measure of this last set is clearly less than or equal to $q(n)\,\mu(B_{q(n)+1}(n))$. Since τ admits of approximation by periodic transformations, by Theorem 3.1 it admits of a $KS(1)$ approximation where we can choose $f(q(n) = 4\mu(B(n))$. Since $B_{q(n)+1}(n) \subset B(n)$, it follows that

$$q(n)\,\mu(B_{q(n)+1}(n)) \leqq q(n)\,\mu(B(n)) \leqq \tfrac{1}{4}\, q(n)\, f(q(n))\,.$$

Since $\lim_{n\to\infty} q(n)\, f(q(n)) = 0$, we may assume that n_0 is large enough so that $n > n_0$ implies $q(n)\,\mu(B_{q(n)+1}(n)) < \delta/3$. If we define $C_j(n) = \tau^{-j+1}\sigma(B_j(n))$, $j = 1,\ldots,q(n)$, then by Lemma 4.1, the sets $C_1(n), \ldots, C_{q(n)}(n)$ are pairwise disjoint. Therefore letting $d_n = \mu(B(n))$ we have

$$\begin{aligned}
&\sum_{i=1}^{q(n)} a_i(n)\,\tau^{i-1}\sigma\left(\bigcup_{j=1}^{q(n)} B_j(n)\right)\\
&\quad= \sum_{i=1}^{q(n)} a_i(n)\,\tau^{i-1}\left(\bigcup_{j=1}^{q(n)} \tau^{j-1} C_j(n)\right)\\
&\quad= \sum_{j=1}^{q(n)}\sum_{i=1}^{q(n)} a_i(n)\,\tau^{i+j-2} C_j(n)\\
&\quad= \sum_{j=1}^{q(n)}\sum_{i=j}^{q(n)} a_{i-j+1}\,\tau^{i-1} C_j(n) + E(q(n)\,d_n)\\
&\quad= \sum_{j=1}^{q(n)}\sum_{i=1}^{q(n)} a_{i-j+1}\,\tau^{i-1} C_j(n) + E(q(n)\,d_n)
\end{aligned}$$

where $a_l(n) = a_{l+q(n)}(n)$ for $-q(n) + 1 < l \leqq 0$. As we have already noted, we can assume that $q(n)\,d_n < \delta/3$ and thus we have proved

$$\sigma(A) = \sum_{i=1}^{q(n)}\sum_{j=1}^{q(n)} a_{i-j+1}\,\tau^{i-1} C_j(n) + E(\delta)\,.$$

It is obvious that $\bigcup_{j=1}^{q(n)} C_j(n) \subset A_1(n)$, and since $\mu(A_1(n) - \bigcup_{j=1}^{q(n)} C_j(n)) = \mu(B_{q(n)+1}(n))$, it is clear that $\lim_{n\to\infty} q(n)\,\mu(A_1(n) - \bigcup_{i=1}^{q(n)} C_j(n)) = 0$.

Lemma 4.4. *Suppose that for each positive integer n the measurable sets A_n, B_n, C_n and D_n satisfy the following relations*

1) $A_n \cap B_n = \emptyset$,
2) $C_n \subset A_n$ *and* $D_n \subset B_n$, *and*
3) *there exists a set* $F \in \mathscr{F}$ *such that*

$$\lim_{n\to\infty} \mu((A_n \cup B_n)\,\Delta\, F) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mu((C_n \cup D_n)\,\Delta\, F) = 0\,,$$

then

$$\lim_{n\to\infty} \mu(A_n - C_n) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \mu(B_n - D_n) = 0\,.$$

Proof. The proof is straightforward and we shall omit it.

Let $A \in \mathscr{F}$, and given $\delta > 0$ let $\sum_{i=1}^{q(n)} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n)$ be the set constructed in Lemma 4.3 which approximates $\sigma(A)$ with error δ for $n > n_0$. We canalso assume that for $n > n_0$ $[\sigma(A)]\,(\xi(n)) = \sum_{i=1}^{q(n)} a_i'(n)\, A_i(n)$ approximates $\sigma(A)$ with error δ. Consider, then, the intersection of these two sets,

$$\sum_{j=1}^{q(n)} \sum_{i=1}^{q(n)} a_i'(n)\, a_{i-j+1}(n)\, \tau^{i-1} C_j(n) = \sum_{i=1}^{q(n)} a_i'(n) \left(\sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n) \right), \tag{1}$$

which approximates $\sigma(A)$ with error 2δ. Define

$$G_n = \{ i \mid a_i'(n) = 1,\ i = 1, 2, \ldots, q(n) \}.$$

Since $\sum_{j=1}^{q(n)} a_{i-j+1}(n)\, C_j(n) \subset A_i$ for each i, we have for $i \in G_n$ that

$$\sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n) = A_i(n) + E(\gamma_i(n)) \tag{2}$$

where

$$\sum_{i=1}^{q(n)} \gamma_i(n) \leqq 2\delta .$$

For each fixed n, define $b_j(n) = \mu(C_j(n)) / \mu\left(\bigcup_{k=1}^{q(n)} C_k(n) \right)$, for $j = 1, 2, \ldots, q(n)$.

Let $H_n = \left\{ i \in G_n \mid \sum_{j=1}^{q(n)} a_{i-j+1} b_j \geqq 1 - \eta \right\}$ where η is an arbitrary fixed number, $0 < \eta < 1$, and let $I_n = G_n - H_n$.

Lemma 4.5. *Given $\delta > 0$ $\sum_{i \in H_n} \sum_{j=1}^{q(n)} a_{i-j+1}\, \tau^{i-1} C_j(n)$ approximates $\sigma(A)$ with error δ for n sufficiently large. Furthermore,*

$$\lim_{n \to \infty} N(H_n)\, \mu(A_1(n)) = \lim_{n \to \infty} N(H_n)\, \mu\left(\bigcup_{j=1}^{q(n)} C_j(n) \right) = \mu(\sigma(A)) .$$

Proof. Assume that the lemma is not true, then since by (1)

$$\sum_{i \in G_n} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n), \quad n > n_0 ,$$

approximates $\sigma(A)$ with error 2δ we must have that for n arbitrarily large,

$$\sum_{i \in I_n} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n)$$

must be a set of substantial measure. That is, there must exist $\beta > 0$ such that we have $\mu\left(\sum_{i \in I_n} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n) \right) > \beta > 0$, for arbitrarily large n. This implies

$$0 < \beta < \mu\left(\sum_{i \in I_n} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\, \tau^{i-1} C_j(n) \right) < (1 - \eta)\, \mu\left(\bigcup_{i=1}^{q(n)} C_j(n) \right) N(I_n). \tag{3}$$

Since

$$\sum_{i\in H_n}\sum_{j=1)}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)\subset\sum_{i\in H_n} a_i'(n)\,A_i(n)$$

and

$$\sum_{i\in I_n}\sum_{j=1}^{q(n} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)\subset\sum_{i\in I_n} a_i'(n)\,A_n(n)\,,$$

these sets satisfy the relations given in Lemma 4.4 with $F=\sigma(A)$, and applying this lemma we get the following results:

Since

$$\mu\left(\sum_{i\in I_n} a_i'(n)\,A_i(n)\right)=N(I_n)\,\mu(A_1(n))$$

$$\lim_{n\to\infty}\left|N(I_n)\,\mu(A_1(n))-\mu\left(\sum_{i\in I_n}\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)\right)\right|=0\,.$$

Since

$$\lim_{n\to\infty} q(n)\,\mu(A_1(n)-\bigcup_{j=1}^{q(n)}C_j(n))=0\quad\text{and}\quad N(I_n)\leqq q(n)$$

this implies that

$$\lim_{n\to\infty}\left|N(I_n)\,\mu\left(\bigcup_{j=1}^{q(n)}C_j(n)\right)-\mu\left(\sum_{i\in I_n}\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)\right)\right|=0\,.$$

Since η and β are independent of n, this gives a contradiction in (3) proving the first part of the lemma, and proves further that we must have

$$\lim_{n\to\infty}\sum_{i\in I_n}\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)=0$$

so that

$$\lim_{n\to\infty}\mu\left(\sum_{i\in I_n} a_i'(n)\,A_i(n)\right)=\lim_{n\to\infty}N(I_n)\,\mu(A_1(n))=0\,,$$

which implies that

$$\lim_{n\to\infty}\mu\left(\sum_{i\in H_n} a_i'(n)\,A_i(n)\right)=\lim_{n\to\infty}N(H)\,\mu(A_1(n))=\lim_{n\to\infty}N(H_n)\,\mu\left(\bigcup_{j=1}^{q(n)}C_j(n)\right)=\mu(\sigma(A)).$$

Lemma 4.6. *Given $\delta>0$, for n sufficiently large there exists an integer ω depending on n, $1\leqq\omega\leqq q(n)$ such that*

$$\sum_{i\in G_n} a_{i-\omega+1}(n)\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1}C_j(n)$$

approximates $\sigma(A)$ with error 3δ.

Proof. For $i\in H_n$ we have $\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,b_j(n)\geqq 1-\eta$. By applying Lemma 4.2 with $x_{ji}=a_{i-j+1}$ and $k=q(n)$ we get that there exists an integer ω, $1\leqq\omega\leqq q(n)$ such that

$$\sum_{i\in H_n} a_{i-\omega+1}(n)\sum_{j=1}^{q(n)} a_{i-j+1}(n)\,b_j(n)\geqq N(H_n)\,(1-2\eta)$$

which implies that

$$\sum_{i\in H_n} a_{i-\omega+1}(n) \sum_{j=1}^{q(n)} a_{i-j+1}\,\mu(\tau^{i-1} C_j(n)) \geqq N(H_n)\,\mu\left(\bigcup_{j=1}^{q(n)} C_1(n)\right)(1-2\eta).$$

Lemma 4.5 assures us that by taking n large we can assume that

$$N(H_n)\,\mu\left(\bigcup_{j=1}^{q(n)} C_j(n)\right) \geqq \mu(\sigma(A)) - \delta,$$

and letting $\eta = \delta/2(\mu(\sigma(A)) - \delta)$ we have that

$$\mu\left(\sum_{i\in H_n} a_{i-\omega+1}(n) \sum_{j=1}^{q(n)} a_{i-j+1}\,\tau^{i-1} C_j(n)\right) \geqq \mu(\sigma(A)) - 2\delta.$$

Obviously

$$\begin{aligned}&\sum_{i\in H_n} a_{i-\omega+1}(n) \sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1} C_j(n)\\ &\quad\subset \sum_{i\in G_n} a_{i-\omega+1}(n) \sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1} C_j(n)\\ &\quad\subset \sum_{i=1}^{q(n)} \sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1} C_j(n).\end{aligned}$$

By Lemma 4.3, this last set can be assumed to approximate $\sigma(A)$ with error δ. The measure of the first set is at least $\mu(\sigma(A)) - 2\delta$ therefore it approximates $\sigma(A)$ with error 3δ and the same holds true for the intermediate set, thereby proving the lemma.

Proof of Theorem 4.1. By taking n sufficiently large, we have

$$\sigma(A) = \sum_{i\in G_n} a_{i-\omega+1}(n) \sum_{j=1}^{q(n)} a_{i-j+1}(n)\,\tau^{i-1} C_j(n) + E(3\delta).$$

Using (2) we have (with all indices understood modulo $q(n)$)

$$\begin{aligned}\sigma(A) &= \sum_{i\in G_n} a_{i-\omega+1}(n)\,A_i(n) + E(5\delta)\\ &= \sum_{i+\omega-1\in G_n} a_i(n)\,\tau^{\omega-1} A_i(n) + E(6\delta)\\ &= \tau^{\omega-1} \sum_{i+\omega-1\in G_n} a_i(n)\,A_i(n) + E(6\delta).\end{aligned} \tag{4}$$

Since $\sum_{i+\omega-1\in G_n} a_i(n) A_i(n) \subset \sum_{i=1}^{q(n)} a_i(n)\,A_i(n)$, and this last set approximates A with error δ, it follows that $\mu\left(\sum_{i+\omega-1\in G_n} a_i(n)\,A_i(n)\right) \leqq \mu(A) + \delta$. Since τ and σ are measure preserving, however, it follows from (4) that
$\mu(A) - 6\delta = \mu(\sigma(A)) - 6\delta \leqq \mu\left(\sum_{i+\omega-1\in G_n} a_i(n)\,A_i(n)\right)$, so that $\sum_{i+\omega-1\in G_n} a_i(n)\,A_i(n)$ approximates A with error 7δ and therefore

$$\sigma(A) = \tau^{\omega-1} A + E(14\delta).$$

This shows that σ is a weak piecewise power of τ. To see that we may in fact obtain the stronger result, apply the previous result to $A_1(n)$ and note that since σ and τ commute and are measure preserving, the relation

$$\sigma(A_i(n)) = \tau^{k(n'\varepsilon)}(A_1(n)) + E(\varepsilon) \text{ implies } \sigma(A_i(n)) = \tau^{k(n,\varepsilon)}(A_i(n)) + E(\varepsilon), \quad i = 1, \ldots, q(n).$$

A straightforward argument completes the proof if we choose $\varepsilon = \varepsilon(n)$ so that $\varepsilon(n) \cdot q(n) \to 0$.

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