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On the "Zero-Two" Law for Conservative Markov Processes

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Summary. Let $P=T^*$ be a conservative Markov operator on $L_\infty(X,\Sigma,m)$, and let $h(x)=\limsup_{n\to\infty}\{P^n(I-P)f\colon \|f\|_\infty\leqq 1\}$. Then $h(x)$ is zero or two a.e. The sets $E_0=\{h=0\}$ and $E_1=\{h=2\}$ are invariant, and we have:

(a) $\| |T^n(I-T)|u|\|_1\to 0$ for $u\in L_1(E_0)$,

(b) $\| |T^n(I-T)|u|\|_1=2\|u\|$ for every n, $0\leqq u\in L_1(E_1)$.

If Σ is countably generated and P is given by $P(x,A)$, we have

(a) $\|P^n(x,\cdot)-P^{n+1}(x,\cdot)\|\to 0$ a.e. on E_0,

(b) $\|P^n(x,\cdot)-P^{n+1}(x,\cdot)\|=2$ a.e. on E_1, for every n.

A sufficient (but not necessary) condition for $m(E_1)=0$ is that $\sigma(P)\cap\{|\lambda|=1\}=\{1\}$.

If $\{P_t\}$ is a conservative semi-group given by $P_t(x,A)$ bi-measurable, there are invariant sets E_0 and E_1 such that:

(a) $\forall\alpha\in\mathbb{R}$, $\lim_{t\to\infty}\|P_t(x,\cdot)-P_{t+\alpha}(x,\cdot)\|=0$ a.e. on E_0,

(b) for a.e. $\alpha\in\mathbb{R}$, $\lim\|P_t(x,\cdot)-P_{t+\alpha}(x,\cdot)\|=2$ a.e. on E_1.

1. Introduction

Let (X,Σ) be a measurable space, and let $P(x,A)$ be a transition probability. For every bounded measurable function we set $Pf(x)=\int f(y)\,P(x,dy)$, and for a finite signed measure μ we define $\mu P(A)=\int P(x,A)\,d\mu(x)$. It is well known that if $mP\ll m$, $(m>0)$ then $T\left(\frac{d\mu}{dm}\right)=\frac{d(\mu P)}{dm}$ defines a positive linear contraction on $L_1(m)$, with adjoint $T^*=P$ (i.e., Pf is in the class of T^*f). For the ergedic theory of positive ontractions of $L_1(m)$ we refer to [4] (see also [7]).

Harris [10] introduced the following recurrence condition: *If* $m(A)>0$, *then* $\sum_{n=0}^{\infty}P^{(n)}(x,A)=\infty$ *for every* $x\in X$. Jamison and Orey [12] proved that if all P^j

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satisfy Harris' condition (aperiodic case), then for any two probabilities μ, ν we have $\|(\mu-\nu)P^n\|\to 0$. This clearly implies $\|(\delta_x-\delta_x P)P^n\|\to 0$ for every $x\in X$. Also, the Harris condition implies that $Pf=f$ for f bounded implies $f\equiv$constant.

By using results of Derriennic [2], it can be shown that for any transition probability, the two conditions:

(i) $Pf=f$, f bounded $\Rightarrow f\equiv$constant, and

(ii) $\lim_{n\to\infty}\|(\delta_x-\delta_x P)P^n\|=0$ for every $x\in X$, are equivalent to the convergence

(iii) for any two probabilities μ and ν, $\|(\mu-\nu)P^n\|\to 0$.

Ornstein and Sucheston [15] proved the following "zero-two" theorem: *If $Pf\leqq f$ for $0\leqq f$ bounded implies that f is m-a.e. constant* ($mP\ll m$), *then either,* $\|\delta_x(P^{n+1}-P^n)\|\xrightarrow[n\to\infty]{}0$ a.e., *or for every n,* $\|\delta_x(P^{n+1}-P^n)\|=2$ *a.e.* (They assume that Σ is countably generated.) This yields that $\|T^n u\|_1\to 0$ for $u\in L_1(m)$ with $\int u\,dm=0$, if the first alternative holds. They deduced from this the above mentioned Jamison-Orey theorem.

Foguel [5-7] eliminated the separability of the σ-algebra, and proved that if P is ergodic and conservative on $L_\infty(m)$, then $h(x)\equiv\lim_{n\to\infty}\sup\{(P^{n+1}-P^n)f(x):\|f\|_\infty\leqq 1\}$ is constant a.e., the constant being zero or two. (The sup in the above limit is the essential sup in L_∞.) In this context, there is no need to have P defined by a transition probability anymore.

Derriennic [2] looked at the problem of convergence without the ergodicity assumption. In that case, one would like to know when $\left\|N^{-1}\sum_{k=0}^{n-1}\mu P^k\right\|\to 0\Rightarrow\|\mu P^n\|\to 0$. His result is:

$$\sup\{\lim_{n\to\infty}\|(\mu-\mu P)P^n\|:0\leqq\mu,\ \mu(X)=1\}=\sup_x\{\lim\|(\delta_x-\delta_x P)P^n\|\}=\begin{cases}0\\2\end{cases}.$$

For contractions in $L_1(m)$, Ornstein and Sucheston [15] had previously proved that

$$\sup\{\lim_{n\to\infty}\|T^n(u-Tu)\|_1:0\leqq u,\ \|u\|_1=1\}=\begin{cases}0\\2\end{cases}.$$

(We note that Derriennic's result can be proved by using the result of Ornstein and Sucheston.) Derriennic also studied the relationships of these results with the tail σ-algebras of the Markov chains associated with $P(x,A)$.

In this paper we are interested in the function

$$h(x)=\lim_{n\to\infty}\sup\{P^n(I-P)f:\|f\|_\infty\leqq 1\},$$

for a conservative Markov operator P on $L_\infty(m)$. It turns out that $h(x)$ is 0 or 2 a.e., which yields an interesting decomposition of the space. Though our theorem may fail for P non-conservative, we conjecture that $\|h\|_\infty$ is zero or two in any case.

Using the notion of the linear modulus of a bounded linear operator in L_1 (see [1], [17]), we can obtain that $h = \lim |P^n(I-P)|\,1$, and

$$\|h\|_\infty = \sup\{\lim_{n\to\infty} \|\,|T^n(I-T)|\,u\|_1 \colon\ 0 \leqq u \in L_1,\ \|u\|_1 = 1\}.$$

Using that approach, our decomposition theorem was proved by Greiner and Nagel [8] in the particular case that T has an equivalent invariant probability, and $P^j f = f \Rightarrow Pf = f$ a.e.

In Sect. 3 we look at some properties of the peripheral spectrum of a conservative P on $L_\infty(m)$ (which is extended to the complex $L_\infty(m)$).

Our decomposition theorem is extended to the continuous parameter case in Sect. 4, and generalizes Winkler's result [20], by dropping all ergodicity assumptions (Winkler needed that each P_t be ergodic, and not only the semigroup, which is a stringent condition). (Derriennic's results were extended to the continuous parameter case by Revuz [16], while the result of Jamison and Orey was extended by Duflo and Revuz [3].)

2. The "Zero-Two" Decomposition for a Conservative Operator

In this section we obtain a "zero-two" theorem for a conservative Markov operator without ergodicity assumptions.

Theorem 2.1. *Let P be a conservative Markov operator on $L_\infty(m)$. Then*

$$h = \lim_{n\to\infty} |P^n(I-P)|\,1 = \lim_{n\to\infty} \text{ess-sup}\,\{P^n(I-P)f\colon\ \|f\|_\infty \leqq 1\}$$

is an invariant function, $0 \leqq h \leqq 2$, and $m(\{0<h<2\})=0$. Let $E_1=\{h=2\}$, $E_0=\{h=0\}$. Then $|P^n(I-P)|\,1_{E_0} \to 0$ a.e., and $|P^n(I-P)|\,1_{E_1} = 2 1_{E_1}$ a.e.

Proof. Recall that $|P^n(I-P)|\,1_E = \text{ess-sup}\,\{P^n(I-P)f\colon -1_E \leqq f \leqq 1_E\}$. Let $h_n = |P^n(I-P)|\,1 = \text{ess-sup}\,\{P^n(I-P)f\colon |f| \leqq 1\}$. Then $0 \leqq h_{n+1} \leqq h_n \leqq 2$, so $h_n \to h$. Also $Ph_n \geqq h_{n+1}$, so $Ph = \lim Ph_n \geqq \lim h_n = h$, and since P is conservative, $Ph = h$. Thus E_0 and E_1 are invariant sets [4, p. 21]. Let $E = E_1^c = \{h<2\}$. Then

$$|P^n(I-P)|\,1 = |P^n(I-P)|(1_{E_1}+1_E) = |P^n(I-P)|\,1_{E_1} + |P^n(I-P)|\,1_E.$$

Each term converges a.e. since the restriction of P to each invariant set is also conservative. Hence on E_1 we have $|P^n(I-P)1_{E_1} \to 2$, since the other term is zero on E_1. Hence we have (since $h_n \downarrow h$) $|P^n(I-P)|\,1_{E_1} = 2$ a.e. on E_1.

We may and do assume that $h<2$ a.e., by restricting ourselves to E. Since $A_k = \left\{h \leqq 2 - \frac{1}{k}\right\}$ is invariant, and $E = \bigcup A_k$, we may restrict ourselves to A_k, and so we assume $h \leqq 2 - \frac{1}{k}$, and have to show $h=0$.

We need now the following lemma.

Lemma 2.2. *Let* $S_n = P^n \wedge P^{n+1}$. *If* $g \in L_\infty$ *is invariant, then:*

(a) $S_n(gf) = g S_n f$ *for* $f \in L_\infty$.

(b) $S_n g$ *converges to the invariant function* $Sg = (1 - \frac{1}{2}h)g$.

(c) $Sg \not\equiv 0$ *for* $0 \leqq g$, *with* $\|g\|_\infty > 0$, *when* $\|h\|_\infty < 2$.

Proof. (a) It is well known that $P^n(gf) = gP^n f$. Now assume $g \geqq 0$, $f \geqq 0$. Then [7, Def. 4.1] we have

$$\begin{aligned} P^n \wedge P^{n+1}(gf) &= \inf\{P^n \varphi + P^{n+1}(gf - \varphi): 0 \leqq \varphi \leqq gf\} \\ &= \inf\{P^n(\psi gf) + P^{n+1}(gf - \psi gf): 0 \leqq \psi \leqq 1\} \\ &= g \inf\{P^n(\psi f) + P^{n+1}(f - \psi f): 0 \leqq \psi \leqq 1\} = g S_n f. \end{aligned}$$

(b) $S_n g = g S_n 1$, and $S_n 1 = 1 - \frac{1}{2}|P^n(I-P)|\, 1 \to 1 - \frac{1}{2}h$.

(c) follows immediately from (b).

We need also the following claim. It can be proved using the result of [15] (as was done in [14]). The following simpler proof is due to S.R. Foguel.

Claim. *If* $\|h\|_\infty < 2$, *then* $P^r g = g \Rightarrow Pg = g$.

Proof. Since P is conservative, so is P^r, so we have to show only $P^r 1_A = 1_A \Rightarrow P 1_A = 1_A$. Apply Lemma 3.3 of [7] successively and obtain $P 1_A = 1_B$. Hence, for every integer n we have

$$1_B - 1_A = P 1_A - 1_A = P^{nr}(P - I) 1_A = \tfrac{1}{2} P^{nr}(I - P)(1 - 2 1_A) \leqq \tfrac{1}{2} h_{nr} \to \tfrac{1}{2} h.$$

Hence $1_B - 1_A \leqq \frac{1}{2} h \leqq 1 - \varepsilon$, which yields $B \subset A$, and $P 1_A \geqq 1_A$ implies $P 1_A = 1_A$, since P is conservative.

Proof of the Theorem. Given $m > 0$, and $0 \leqq g \not\equiv 0$ invariant, $S^m g \not\equiv 0$ by the above lemma. S is pointwise continuous, so for some $n_1 > 0$, $S_{n_1} S^{m-1} g \not\equiv 0$. But $S_{n_1} S^{m-1} g = \lim_{n \to \infty} S_{n_1} S_n S^{m-2} g$, so $S_{n_1} S_{n_2} S^{m-2} g \not\equiv 0$. Hence we can find $n_1, n_2, \ldots, n_m$ with $S_{n_1} S_{n_2} \ldots S_{n_m} g \not\equiv 0$.

In order to show that $h = 0$, define $g = (h - 2/\sqrt{m})^+$, which is Σ_i-measurable, hence invariant. ($m > 0$ is a fixed integer.)

Let $n_1, n_2, \ldots, n_m$ be as above, and $n_0 = \sum_{i=1}^m n_i$. Define $Q = \prod_{i=1}^m S_{n_i}$, and $U = P^{mn_0 + m} - Q\left(\frac{I+P}{2}\right)^m$. By definition, $S_{n_i} \leqq P^{n_i}$ and $S_{n_i} \leqq P^{n_i + 1}$, so $S_{n_i}(I + P) \leqq P^{n_i+1} + S_{n_i} P \leqq 2P^{n_i+1}$. Hence

$$Q\left(\frac{I+P}{2}\right)^m = \left(\prod_{i=1}^m S_{n_i}\right)\left(\frac{I+P}{2}\right)^m \leqq \prod_{i=1}^m P^{n_i+1} = P^{mn_0+m},$$

so U is a positive linear operator. If $r = mn_0 + m$, then

$$P^r = U + Q\left(\frac{I+P}{2}\right)^m; \quad P^{2r} = U^2 + (UQ + QP^r)\left(\frac{I+P}{2}\right)^m,$$

and $P^{jr} = U^j + R_j \left(\frac{I+P}{2}\right)^m$ (with $R_{j+1} = U^j Q + R_j P^r$).

$$\begin{aligned} P^{jr}(I-P) &= U^j(I-P) + R_j \left(\frac{I+P}{2}\right)^m (I-P) \\ &= U^j(I-P) + 2^{-m} R_j \sum_{k=0}^{m} \binom{m}{k} (P^k - P^{k+1}) \\ &= U^j(I-P) + 2^{-m} R_j \left\{ \sum_{k=1}^{m} \left[\binom{m}{k} - \binom{m}{k-1}\right] P^k + I - P^{m+1} \right\}. \end{aligned}$$

For $\|f\|_\infty \leqq 1$, we have (since $R_j 1 = 2^{-m} R_j (I+P)^{-m} 1 \leqq 1$),

$$P^{jr}(I-P) f \leqq 2U^j 1 + 2^{-m} \sum_{k=0}^{m+1} \left| \binom{m}{k} - \binom{m}{k-1} \right| \leqq 2U^j 1 + \frac{2}{\sqrt{m}}.$$

Now $U1 \leqq P^r 1 \leqq 1$, so $U^j 1$ decreases, and $\bar{h} = \lim_{j\to\infty} U^j 1$ is U-invariant, so $P^r \bar{h} \geqq \bar{h}$, hence $P^r \bar{h} = \bar{h}$. Now, by the claim, $P\bar{h} = \bar{h}$.

But $U\bar{h} = \bar{h}$ and $P\bar{h} = \bar{h}$, so $Q\bar{h} = 0$ by the definition. Taking limits we have

$$h = \lim_j |P^{jr}(I-P)| \, 1 \leqq 2\bar{h} + \frac{2}{\sqrt{m}}.$$

Hence $g = \left(h - \frac{2}{\sqrt{m}}\right)^+ \leqq 2\bar{h}$, and $Qg \leqq 2Q\bar{h} = 0$. Hence $(h - 2/\sqrt{m})^+ = g = 0$ (since $g \not\equiv 0$ implies $Qg \not\equiv 0$), yielding $h \leqq 2/\sqrt{m}$. Since $m > 0$ is arbitrary, $h = 0$.

Corollary 2.3. *Let P be a conservative Markov operator,* E_0 and E_1 *as in Theorem* 2.1. *Then*

(a) $0 \leqq u \in L_1(E_0) \Rightarrow \| |T^n(I-T)| u \|_1 \to 0$.

(b) $0 \leqq u \in L_1(E_1)$, $\|u\|_1 = 1 \Rightarrow \| |T^n(I-T)| u \|_1 = 2$ *for* $\forall n$.

Proof. (a) We restrict ourselves to E_0, so we may assume $h \equiv 0$. Then

$$\| |T^n(I-T)| u \|_1 = \langle |T^n(I-T)| u, 1 \rangle = \langle u, |P^n(I-P)| 1 \rangle \to 0$$

by the bounded convergence theorem.

(b) Restricting ourselves to E_1, we obtain $h = 2$, or $|P^n(I-P)| 1 = 2$ a.e. Then for $0 \leqq u \in L_1(E_1)$ with $\int u \, dm = 1$ we have

$$\| |T^n(I-T)| u \|_1 = \langle u, |P^n(I-P)| 1 \rangle = 2.$$

Theorem 2.4. *Let P have a finite invariant measure μ equivalent to m, and let $\hat{P}$ be the dual Markov operator. Then P and $\hat{P}$ have the same decomposition.*

Proof. We may assume $\mu = m$. Since $\hat{P} = T$ on $L_\infty(\mu)$, and P-invariant sets are $\hat{P}$-invariant, $\lim |\hat{P}^n(I-\hat{P})| 1_{E_0}$ exists a.e., and is 0 by Corollary 2.3 (a) and Lebesgue's theorem. Also $\| |\hat{P}^n(I-\hat{P})| 1_{E_1} \|_1 = 2\mu(E_1)$ for every n, and since $0 \leqq |\hat{P}^n(I-\hat{P})| 1_{E_1} \leqq 2 1_{E_1}$ a.e., it is 2 a.e. on E_1.

Remarks. 1. The proof of Theorem 2.1 uses some ideas of G. Greiner and R. Nagel [8]. However, their Banach lattice approach requires that the norm be order continuous (which does not apply to L_∞), and the existence of a positive fixed point (which is not always available for T, the L_1 pre-dual of P). Hence their result implies Theorem 2.1 only for P with an equivalent finite invariant measure (more or less in the form of Corollary 2.3).

(The extra step involved in our proof is the existence of $n_1, \ldots, n_m$ such that $S_{n_1} \ldots S_{n_m} g \not\equiv 0$.) Convergence in L_∞ norm is treated by Foguel [6], [7].

2. The claim in the proof eliminated the need for the assumption that all P^k have the same invariant sets as P, needed in the general set-up in [8].

3. Theorem 2.4 is new even in the ergodic case, and its proof is evident by the use of the modulus operator. It is not known if it is true if μ is σ-finite (and P conservative).

4. In the form of the results of Derriennic [2], Corollary 2.3 can be written as the following "zero-two" law:

$$\sup\{\lim_{n\to\infty} \| |T^n(I-T)| u\|_1 : 0 \leqq u \in L_1,\ \|u\|_1 = 1\} = \|h\|_\infty \in \{0, 2\}.$$

5. If P is not conservative Theorem 2.1 may fail, even if h *is* invariant. E.g., let P be given on $\{1, 2, 3, 4\}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

6. The zero alternative (i.e., $m(E_1)=0$) implies that for every $u \in L_1$ we have $\|T^n(I-T)u\|_1 \to 0$. However, the next lemma shows that for a Markov operator $Pf(x) = f(\theta x)$, obtained from a conservative non-singular transformation θ, we have $m(E_0)=0$ (unless θ is the identity). If θ is exact and conservative, we see that the zero alternative is a strictly stronger property.

Lemma 2.5. *Let θ be a conservative transformation. If* $\lim (P^{n+1} - P^n) 1_A = 0$ a.e., *then* $P1_A = 1_A$.

Proof. The condition is $1_A(\theta^{n+1}x) - 1_A(\theta^n x) \to 0$ a.e. Hence either $\theta^n x \in A$ for all $n \geqq n_0(x)$, or $\theta^n x \notin A$ for $n \geqq n_0(x)$. Hence $B = \{x\colon \theta^n x \in A\ \forall n \geqq n_0(x)\}$ is an invariant set, and contains A. If $m(B-A) > 0$, $x \in B - A \Rightarrow \theta^n x \notin B - A$ for $n \geqq n_0(x)$, contradicting the recurrence. Hence $A = B$ is invariant.

We now turn to the probabilistic interpretation of the results.

Lemma 2.6. *Let $P(x, A)$ and $Q(x, A)$ be transition probabilities, $mP \ll m$, $mQ \ll m$. If Σ is countably generated, then $\|P(x,\cdot) - Q(x,\cdot)\|$ is measurable, and $\|P(x,\cdot) - Q(x,\cdot)\| = \operatorname{ess\,sup}\{(P-Q)f\colon \|f\|_\infty \leqq 1\}$* a.e.

Proof. The measurability is from [15]. Let $h(x) = \operatorname{ess\,sup}\{(P-Q)f\colon \|f_\infty \leqq 1\}$. Then for f bounded measurable with $\sup|f(x)| \leqq 1$ we have

$$|(P-Q)f(x)| \leqq \|P(x,\cdot) - Q(x,\cdot)\|,$$

hence $h(x) \leqq \|P(x,\cdot) - Q(x,\cdot)\|$.

Let Σ_k be the finite σ-algebra generated by the first k generators of Σ. Then $L_\infty(\Sigma_k)$ is finite dimensional (and thus separable), so a.e. we have

$$\begin{aligned} h(x) &\geqq \sup\{(P-Q)f(x)\colon |f|\leqq 1, f \text{ is } \Sigma_k\text{-measurable}\} \\ &= \|P(x,\cdot)-Q(x,\cdot)\|_{M(\Sigma_k)} \xrightarrow[k\to\infty]{} \|P(x,\cdot)-Q(x,\cdot)\|. \end{aligned}$$

(The convergence is proved in [15]).

Remark. Lemma 2.6 shows that [5] is a generalization of [15]. Though implicit there, it was not proved. Thus, Theorem 2.1 shows that $\|\delta_x P^n - \delta_x P^{n+1}\| \to 0$ for a.e. $x \in E_0$. Since $h(x) = \lim_{n\to\infty} \|\delta_x P^n - \delta_x P^{n+1}\|$ exists everywhere and is measurable, we have that $F_0 = \{x\colon h(x) = 0\}$ is measurable. (We still assume that Σ is countably generated.)

Proposition 2.7. *Let μ be a probability measure. If $\mu(X - F_0) = 0$, then*

$$\|\mu(I-P)P^n\| \to 0.$$

Proof. Let $h_n(x) = \|\delta_x(I-P)P^n\|$, which is measurable by [15]. Since $h_n(x)\downarrow 0$ on F_0, by Egorov's theorem there is a set A with $\mu(A^c) < \varepsilon$ such that $h_n(x) \to 0$ uniformly on A. Hence, for $n > n_0$, $h_n(x) < \varepsilon$ on A. For $|f| \leqq 1$ measurable we have

$$\begin{aligned} |\langle \mu(I-P)P^n, f\rangle| &\leqq \int_A |P^n(I-P)f(x)|\,d\mu + \int_{A^c} |P^n(I-P)f(x)|\,d\mu \\ &\leqq \int_A h_n(x)\,d\mu + 2\mu(A^c) < 3\varepsilon. \end{aligned}$$

Hence $\|\mu(I-P)P^n\| < 3\varepsilon$ for $n > n_0$.

Derriennic [2] studied the relationship between the tail σ-field and the convergence $\|\mu(I-P)P^n \to 0$. Thus, the zero alternative yields that the tail σ-field equals the invariant σ-field (for the shift) P_μ a.e., for every μ as in Proposition 2.7. This is stronger then having the equality of these σ-fields P_m a.e. (which is equivalent to their equality P_μ, for $\mu \ll m$).

Remark. The proof of proposition 2.7 can be adapted to show that

$$\sup\{\lim_{n\to\infty} \|\mu(I-P)P^n\|\colon \mu \geqq 0, \|\mu\| = 1\} = \sup\{\lim_{n\to\infty} \|\delta_x(I-P)P^n\|\colon x \in X\}.$$

(In the case that Σ is not countably generated, it is necessary, for given μ, to look at the admissible σ-algebra Σ' generated by the Hahn sets of $\{\mu(I-P)P^n\}$.) This is another proof of the first equality of [2, Th. 3]. The supremum is 0 or 2 by reduction, in $L_1(\Sigma 2^{-n-1}\mu P^n)$, to the result of [15].

3. On the Peripheral Spectrum of a Conservative Markov Operator

In this section we give a spectral condition for the zero alternative in Theorem 2.1 to hold, extending the result of [14] to the non-ergodic case. We look at the connection between $\sigma(P)$ and $\sigma(\hat{P})$ when P has a σ-finite invariant measure.

We are interested in the peripheral spectrum $\sigma(P) \cap \{\lambda: |\lambda|=1\}$. ($P$ is extended to the complex L_∞.)

Theorem 3.1. *Let P be a conservative Markov operator such that $\sigma(P) \cap \{|\lambda|=1\} = \{1\}$. Then* $\lim_{n\to\infty} \sup\{P^n(I-P)f: |f| \leqq 1\} = 0$ a.e.

Proof. Clearly $P^k f = f \in L_\infty \Rightarrow Pf = f$, since k-th roots of unity are not in $\sigma(P)$. By Theorem 2.1 for fixed k, there is an invariant set A_k with $\lim |P^{nk}(I-P^k)| 1 = 0$ on A_k, 2 on $X - A_k$. We restrict ourselves to A_k. Let $Q = I + P + \ldots + P^{k-1}$. Q is invertible since k-th roots of unity are not in $\sigma(P)$ (spectral mapping theorem).

$$\sup\{P^{nk}(I-P)f: \|f\|_\infty \leqq 1\} = \sup\{P^{nk}(I-P)QQ^{-1}f\}$$
$$\leqq \|Q^{-1}\| \sup\{P^{nk}(I-P^k)g: \|g\|_\infty \leqq 1\} \to 0 \text{ a.e.}$$

Hence $|P^n(I-P)|1_{A_k} \to 0$ a.e. Hence on $\bigcup_{k=1}^\infty A_k$, $|P^n(I-P)|1 \to 0$ a.e. But $X = \bigcup_{k=1}^\infty A_k$, as is proved in [14], because $I+P$ is invertible.

Theorem 3.2. *Let θ be a conservative non-singular transformation. Then either $\theta^k =$ Identity for some $k>0$, or $\sigma(P) \supset \{|\lambda|=1\}$.*

Proof. It is shown in Schaefer [17, p. 326] that $\sigma(P) \cap \{|\lambda|=1\}$ is cyclic. Hence, if it is not the full unit circle, it is a discrete subgroup of the circle, so for some $k>0$, $\sigma(P^k) \cap \{|\lambda|=1\} = \{1\}$. Hence P^k satisfies the conditions of the previous theorem, and for every $A \in \Sigma$, $\lim P^{nk}(I-P^k)1_A = 0$ a.e. By Lemma 2.5 $P^k = I$.

Remark. For θ having a σ-finite invariant measure and invertible, a similar result was obtained by different methods for the L_2 operator, by A. Bellow (Ionescu Tulcea) in [11], and (by another method) by R. Sine [18]. However, for a unitary operator it can be proved easily using the spectral theorem, and if θ is not invertible, the result for the L_2 isometry holds by the theorem of B. Sz.-Nagy and C. Foias [19, p. 85].

We now show that the converse of Theorem 3.1 is false, even if P is also ergodic. Our example will also show that the spectral assumption of Theorem 3.1 need not hold for a Harris recurrent Markov operator, even if it has a finite invariant measure (and even if the dual Markov operator is Doeblin).

Example 3.3. We let $X = \{0, 1, 2, \ldots\}$, and $m\{j\} = 2^{-j-1}$. Define $(Tu)(j) = (u_0 + u_{j+1})/2$. Then T is a contraction of $L_1(X, m)$, and since $T1 = 1$, it is conservative, and easily checked to be ergodic, and also aperiodic. $P = T^*$ is the Markov operator on $L_\infty(m)$, and, being Harris aperiodic, satisfies the zero alternative (i.e., $m(E_1)=0$). ([12, 15]) To compute $\sigma(P)$ we can compute $\sigma(T)$. We show that, for $|\lambda|=1$, $\lambda I - T$ is not onto L_1.

Let $v_k = \dfrac{(2\lambda)^k}{(k+1)(k+2)}$ for $k \geqq 0$. Then $\sum_{k=0}^\infty |v_k| m\{k\} < \infty$.

We try to solve $(\lambda I - T)u = v$, with $u \in L_1$. We get the equations

$$\lambda u_j - (u_0 + u_{j+1})/2 = v_j \qquad (j = 0, 1, 2, \ldots)$$

or

$$u_{j+1}=2\lambda u_j-u_0-2v_j.$$

Thus a finite solution can be obtained, given u_0.

Hence $u_1=(2\lambda-1)u_0-2v_0$, and by induction we have

$$u_j=\left[2^j\lambda^j-\sum_{k=0}^{j-1}(2\lambda)^k\right]u_0-2\sum_{k=0}^{j-1}(2\lambda)^k v_{j-k-1}.$$

To have $u\in L_1$ we check if $\sum_{j=0}^{\infty}|u_j|/2^j<\infty$.

$$\begin{aligned}\frac{u_j}{2^j}&=\left[\lambda^j-\frac{1-(2\lambda)^j}{(1-2\lambda)2^j}\right]u_0-\sum_{k=0}^{j-1}\lambda^k\frac{v_{j-k-1}}{2^{j-k-1}}\\&=\left[\lambda^j-\frac{1-(2\lambda)^j}{(1-2\lambda)2^j}\right]u_0-\lambda^{j-1}\sum_{k=0}^{j-1}v_{j-k-1}/(2\lambda)^{j-k-1}\\&=\left[\lambda^j-\frac{1-(2\lambda)^j}{(1-2\lambda)2^j}\right]u_0-\lambda^{j-1}\sum_{k=0}^{j-1}v_k/(2\lambda)^k.\end{aligned}$$

With our particular choice of v_k,

$$\sum_{k=0}^{j-1}v_k/(2\lambda)^k=\sum_{k=0}^{j-1}\frac{1}{(k+1)(k+2)}=\frac{j}{j+1}.$$

Hence

$$\frac{u^j}{2^j}=\frac{-u_0}{(1-2\lambda)2^j}+\frac{2(1-\lambda)}{1-2\lambda}\lambda^j u_0-\lambda^{j-1}\frac{j}{j+1}.$$

Since $\sum_j\left|\lambda^{j-1}\left(\frac{2\lambda(1-\lambda)u_0}{1-2\lambda}-\frac{j}{j+1}\right)\right|=\sum_j\left|\alpha-\frac{j}{j+1}\right|=\infty$ for any constant α, no choice of u_0 will yield $u\in L_1$. Hence $\lambda\in\sigma(T)$.

Remark. We note that the previous example showed $\sigma(P)\cap\{|\lambda|=1\}\neq\sigma(\hat{P})\cap\{|\lambda|=1\}$ (even for a finite invariant measure). However, we do have the following result.

Theorem 3.4. *Let P be conservative with σ-finite invariant measure, and $\hat{P}$ its dual operator. If $0\neq f\in L_\infty$ satisfies $Pf=\lambda f$, with $|\lambda|=1$, then* $\hat{P}f=\bar{\lambda}f$ (and $\hat{P}\bar{f}=\lambda\bar{f}$). *Hence P and $\hat{P}$ have the same unimodular eigenvalues.*

Proof. Let Σ_i be the σ-field of P-invariant sets. These sets are also invariant for $\hat{P}$. Now $Pf=\lambda f\Rightarrow P|f|\geq|Pf|=|f|$. By conservativity $P|f|=|f|$, and $|f|$ is Σ_i-measurable. We can therefore restrict ourselves to $\{|f|>0\}$. Hence, without loss of generality we may and do assume $|f|>0$ a.e. We also assume $\|f\|_\infty=1$.

$1/|f|$ is also Σ_i-measurable (though not necessarily bounded).

If $A\in\Sigma_i$, then $P(1_A g)=1_A Pg$, as is easily checked. By linearity and continuity, for each Σ_i-measurable $f_0\in L_\infty$, $P(f_0 g)=f_0 Pg$ for $g\in L_\infty$. By monotone continuity, also $P(g/|f|)=(Pg)/|f|$ for $g\in L_\infty$. The same holds for $\hat{P}$.

Let $h \in L_1$ with $1 \geqq h > 0$ a.e. Then

$$\int T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} \, dm = \int \frac{fh}{|f|} P\left(\frac{\bar{f}}{|f|}\right) dm = \int \frac{fh}{|f|^2} P\bar{f} \, dm = \bar{\lambda} \int h \, dm.$$

Hence

$$\int h \, dm = \left| \int T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} \, dm \right| \leqq \|fh/|f|\|_1 = \int h \, dm.$$

We must therefore have $\left| \int T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} \, dm \right| = \int \left| T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} \right| dm$. Hence there exists a complex α, with $|\alpha| = 1$, such that $T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} = \alpha \left| T\left(\frac{fh}{|f|}\right) \cdot \frac{\bar{f}}{|f|} \right|$ a.e., and the first equality we obtained shows that $\alpha = \bar{\lambda}$.

$$T(fh/|f|) \cdot \bar{f}/|f| = \bar{\lambda} |T(fh/|f|) \cdot \bar{f}/|f|| = \bar{\lambda} |T(fh/|f|)|.$$

Clearly $|T(fh/|f|)| \leqq Th$ by positivity of T. Since we obtained before that

$$\int |T(fh/|f|)| \, dm = \int h \, dm = \int Th \, dm,$$

we obtain now $|T(fh/|f|)| = Th$, so that $T(fh/|f|) = \bar{\lambda}(|f|/\bar{f}) Th$. Now (using the above remark on multiplication by invariant functions), since $\hat{P} = T$ on $L_1 \cap L_\infty$, we have

$$\hat{P}(fh) = |f| \hat{P}(fh/|f|) = \bar{\lambda}(|f|^2/\bar{f}) \hat{P}h = \bar{\lambda} f \hat{P}h.$$

Taking a sequence $0 < h_n \leqq 1$ in L_1 with $h_n \uparrow 1$, monotone continuity of $\hat{P}$ yields $\hat{P}f = \bar{\lambda} f \hat{P}1 = \bar{\lambda} f$.

Remark. If P is not conservative, the theorem may fail. Let T on l_1 be defined by $T(u_1, u_2, \ldots) = (u_2, u_3, \ldots)$. Then $T1 = 1$ and $\|T\|_1 \leqq 1$. Since $T(\lambda, \lambda^2, \lambda^3, \ldots) = \lambda T(\lambda, \lambda^2, \lambda^3, \ldots)$, $\hat{P}$ has all the unit circle in its point spectrum, while $T^n \to 0$ strongly in L_1 shows that P has no unimodular eigenvalues.

4. A "Zero-Two" Decomposition for a Conservative Semi-Group

In this section we treat the continuous time case: We deal with a semi-group $\{T_t\}$ of positive contractions on $L_1(m)$, with dual semi-group $\{P_t\}$. We assume continuity at $t > 0$. It is shown in [13] that if T_{t_0} is conservative, so is every T_t, and this is equivalent to having the whole semi-group conservative.

For technical reasons, we assume that the σ-algebra Σ is countably generated (e.g., X is a separable locally compact metric space). We assume that $\{P_t\}$ is obtained from a transition probability semi-group $P_t(x, A)$ such that $\int f(y) P_t(x, dy)$ is (t, x) measurable, for each bounded measurable function. This implies weak-measurability of $\{T_t\}$, and, since $L_1(m)$ is separable, continuity at $t > 0$.

Theorem 4.1. *Let $P_t(x, A)$ be a semi-group of transition probabilities on (X, Σ), m a probability on Σ with $mP_t \ll m$ for every $t>0$. Assume:*

(1) *$P_t f(x) = \int f(y) P_t(x, dy)$ is (t, x) measurable.*

(2) *Σ is countably generated.*

(3) *$\{P_t\}$ is conservative on $L_\infty(m)$.*

Then there exist invariant sets E_0 and $E_1 = E_0^c$, such that:

(i) $\forall \alpha \in \mathbb{R}$, $\lim_{t\to\infty} \|P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)\| = 0$ a.e. *on* E_0

(ii) *For* a.e. $\alpha \in \mathbb{R}$, $\lim_{t\to\infty} \|P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)\| = 2$ a.e. *on* E_1

Proof. Let $h_t(\alpha, x) = P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)\|$, for $t>0$ and $t+\alpha>0$. Since the underlying σ-algebra is countably generated, an inspection of the proof of [15, Theorem 3.1] yields that $h_t(\alpha, x) = \lim_{k\to\infty} \|P_t(x, \cdot) - P_{t+\alpha}(x, \cdot)\|_{M(\Sigma_k)}$ where Σ_k is a finite σ-algebra (generated by the first k generators of Σ). Hence $h_t(\alpha, x)$ is measurable in (α, x), by hypothesis (1).

Since $\|P_t\| \leqq 1$, h_t is decreasing, and $\lim_{t\to\infty} h_t(\alpha, x) = h(\alpha, x)$ is measurable in (α, x).

Now $h(\alpha, x)$ is defined for every $\alpha \in \mathbb{R}$, $x \in X$, and we have [20] $h(-\alpha, x) = h(\alpha, x)$, $h(\alpha+\beta, x) \leqq h(\alpha, x) + h(\beta, x)$, for every $x \in X$. By Lemma 2.1 for fixed $\alpha > 0$ we have a.e.

$$\begin{aligned} h(\alpha, x) &= \lim_{n\to\infty} \|P_{n\alpha}(x, \cdot) - P_{n\alpha+\alpha}(x, \cdot)\| \\ &= \lim_{n\to\infty} \sup \{P_\alpha^n (I - P_\alpha) f \colon \|f\|_\infty \leqq 1\}. \end{aligned}$$

Since P_α is conservative, $h(\alpha, x)$ is 0 or 2 for a.e. x, by Theorem 2.1. For $\alpha < 0$ use $h(\alpha, x) = h(-\alpha, x)$ to obtain $h(\alpha, x)$ is 0 or 2 a.e.

Let μ be a probability measure on $\mathbb{R}$, equivalent to Lebesgue's measure.

Let $A = \{(\alpha, x) \colon 0 < h(\alpha, x) < 2\}$, and let $A_\alpha = \{x \colon 0 < h(\alpha, x) < 2\}$. We have just seen that $m(A_\alpha) = 0$ for every α, and $\mu \times m(A) = \int m(A_\alpha)\, d\mu = 0$.

Let $B = \{(\alpha, x) \colon h(\alpha, x) = 0\}$, and $B_x = \{\alpha \colon h(\alpha, x) = 0\}$. The properties $h(\alpha, x) = h(-\alpha, x)$ and $h(\alpha+\beta, x) \leqq h(\alpha, x) + h(\beta, x)$ imply that B_x is a subgroup of $\mathbb{R}$.

B is measurable in $\mathbb{R} \times X$, so $\int 1_B(\alpha, x)\, \mu(d\alpha)$ is measurable on X, and

$$E_1 = \{x \colon \int 1_B(\alpha, x)\, d\mu(\alpha) = 0\} = \{x \colon \mu(B_x) = 0\}$$

is measurable in X. Let $E_0 = E_1^c$.

Since B_x is a subgroup of $\mathbb{R}$, $\mu(B_x) > 0$ implies [9, p. 68] that B_x contains an interval around the origin, and therefore $B_x = \mathbb{R}$.

Now $x \in E_0 \Leftrightarrow \mu(B_x) > 0 \Leftrightarrow B_x = \mathbb{R} \Leftrightarrow h(\alpha, x) = 0 \,\forall \alpha \Leftrightarrow \mu(b_x) = 1$, and

$$\iint_{R\times E_0} h(\alpha, x)\, d(\mu \times m) = \int_{E_0} \left[\int_R h(\alpha, x)\, d\mu(\alpha)\right] dm(x) = 0.$$

Since $h(\alpha, x)$ is 0 or 2 $\mu \times m$-a.e., we have

$$\begin{aligned} 2m(E_1) &\geqq \iint_{R\times E_1} h(\alpha, x)\, d(\mu \times m) = \iint_{R\times X} h(\alpha, x)\, d(\mu \times m) = 2(\mu \times m)(B^c) \\ &= 2 - 2(\mu \times m)(B) = 2 - 2\int \mu(B_x)\, dm = 2 - 2m(E_0) = 2m(E_1). \end{aligned}$$

(We used the fact that $\mu(B_x)$ is 0 or 1.) Now equality in the previous inequality means $h(\alpha, x)=2$ a.e. on $R\times E_1$, and for μ a.e. α, $h(\alpha, x)=2$ a.e. on E_1.

It remains to prove the invariance (in $L_\infty(m)$) of the sets E_0 and E_1. Take $\alpha>0$ such that $h(\alpha, x)=2$ a.e. on E_1. Then $h(\alpha, x)=2 1_{E_1}$ a.e., so (by Lemma 4.1) E_0 and E_1 are the decomposition sets for P_α, given by Theorem 2.1. Hence $P_\alpha 1_{E_i}=1_{E_i}$. Weak-$*$ continuity of the semigroup yields the required result.

Remarks. 1. We may probably drop the assumptions that Σ is countably generated, and that $\{P_t\}$ is given by transition probabilities. We will need still a bi-measurable $g(t,x)$ such that $P_t f(x)=g(t,x)\,\mu\times m$ a.e., (g depends on $f\in L_\infty(m)$), in order to get [20, Lemma 3] $h(\alpha,x)$ measurable such that $h(\alpha,x) =\lim_{t\to\infty} |P_t(P_\alpha-I)|\,1(x)\,\mu\times m$ a.e. The limit is to be taken in L_∞ sense, or (equivalently) as $\lim_{n\to\infty} |P_{n\alpha}(P_\alpha-I)|\,1(x)$ (since $|P_t(P_\alpha-I)|\,1$ is decreasing, in L_∞). Lemma 2 of [20] needs the (simple) proof without transition probabilities, and then the version $h(\alpha,x)$ will have to satisfy *everywhere* $h(\alpha,x)=h(-\alpha,x)$; $h(\alpha+\beta,x) \leqq h(\alpha,x)+h(\beta,x)$ so that our proof will apply.

2. Winkler's proof [20] made use of the fact that for (almost) every α, $h(\alpha,x)$ is a.e. constant, which is not necessarily true without ergodicity of (almost) every P_α.

3. Revuz' remarks in [16] indicate that Theorem 4.1 (ii) cannot be improved to obtain $\|P_t(x,\cdot)-P_{t+\alpha}(x,\cdot)\|=2$ a.e. on E_1, for every $\alpha, t>0$. Let $P_t f(x) =f(e^{2\pi i t}x)$ on the unit circle, and let m be Lebesgue's measure. Using Lemma 2.5 for P_α (α not an integer), we obtain $m(E_0)=0$. But $P_{t+k}=P_t$.

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