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Asymptotic Densities of Stopping Times Associated with Tests of Power One*

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Summary. An asymptotic formula (as $t \to \infty$) is derived for the density of the first-exit time of the Brownian motion over certain upper class functions. This result is applied to the study of the performance of tests of power one as the drift goes to zero.

1. Introduction

Let $\psi(t)$ be a positive increasing upper class function of the standard Brownian motion $W(t)$ at infinity, so that $P(W(t) < \psi(t)$ for all large $t) = 1$, and let $\psi(t) = o(t)$ as $t \to \infty$. We define the first-exit time of $W(t)$ over $\psi(t)$ by

$$T = \inf\{t > 0 \mid W(t) \geqq \psi(t)\} \tag{1}$$

with $T = \infty$ when the infimum is taken over the empty set.

We assume that $P(T < \infty) < 1$. The present paper deals with the asymptotic form of the density f of T as $t \to \infty$. We show that under certain regularity conditions on ψ

$$f(t) \doteq P(T = \infty) \frac{\psi(t) - t\psi'(t)}{t^{3/2}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \tag{2}$$

holds as $t \to \infty$. Here φ denotes the density of the standard normal distribution and the symbol $\doteq$ means that the ratio of the two expressions converges to one.

The situation considered here, with a fixed boundary curve ψ and $t \to \infty$, is different from that considered in [8], where one has a family of boundary curves ψ_a; $a > 0$ receding to infinity. In that case the global factor $P(T = \infty)$ reduces to 1 and the asymptotic density is purely local (see Theorem 4 in [8]).

From (2) we derive several consequences for tests of power one. For

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example, we evaluate the asymptotic moments of the stopping time under the alternative hypothesis when the drift tends to zero.

The assumptions under which (2) holds are similar to those of Strassen's Theorem 3.6 in [14], which gives the asymptotic density of the last time of Brownian motion above an upper class function. It is possible to prove (2) using Strassen's method, as in the proof of Theorem 1 in [8]. Here we take another approach based on evaluating asymptotically certain integral equations for the first-exit density. To our knowledge this idea was first employed by Daniels in [1]. The method is developed further by Ferebee [5, 6] and Jennen [9].

2. Density Approximations

We assume for the paper that ψ is increasing and continuously differentiable on $(0,\infty)$. Thus according to Strassen (cf. [14]) the stopping time T defined in (1) has a continuous density f on $(0,\infty)$. Let $\Lambda(t)=\psi(t)-t\psi'(t)$ denote the intercept of the tangent at t to the curve ψ on the vertical axis.

Theorem 1. *Assume that*

(I) $\psi(t)/t^{\alpha}$ *is decreasing for some* $\alpha\in(1/2,1)$,

(II) $\psi(t)/t^{\beta}$ *is ultimately increasing for some* $\beta\in(2\alpha-1,\alpha)$,

(III) *for every* $\varepsilon>0$ *there exist* $\delta>0$ *and* $t_1>0$ *such that* $|s/t-1|<\delta$ *implies*

$$|\psi'(s)/\psi'(t)-1|<\varepsilon \quad \text{when } t\geqq t_1,$$

(IV) $P(T<\infty)<1$.

Then

$$f(t)\doteq P(T=\infty)\frac{\Lambda(t)}{t^{3/2}}\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \quad \text{as } t\to\infty. \tag{3}$$

The asymptotic density (3) consists of two factors, one global and one local. The global factor $P(T=\infty)$ depends on the whole course of the boundary curve. The local factor is simply the density at t of the first exit-time over the tangent to the curve at t. Since $P(T\geqq t)\to P(T=\infty)$ statement (3) can be rewritten as $f(t)/P(T\geqq t)\doteq\frac{\Lambda(t)}{t^{3/2}}\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)$ as $t\to\infty$, which means that the conditional density given that $T\geqq t$ is asymptotically a purely local quantity, the tangent approximation.

The assumption that $P(T<\infty)<1$ implies that ψ is an upper class function and that $\lim_{t\to\infty}\psi(t)/\sqrt{t}=\infty$. The Kolmogorov-Petrovsky-Erdös test (cf. [7]) establishes that a function ψ for which (I′) $\psi(t)/t$ is decreasing and (II′) $\psi(t)/\sqrt{t}$ is increasing, is an upper class function if and only if

$$\int_1^{\infty}\frac{\psi(t)}{t^{3/2}}\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)dt<\infty. \tag{4}$$

The reader should note that assumption (I) of Theorem 1 is stronger than (I′) and (II) is weaker than (II′) at least when $\alpha\in(1/2, 3/4)$.

Proof. We need the following integral equations for the first-exit densities. For the first equation see Durbin [2], for the second one Durbin [3] and Ferebee [5]. Both equations hold when ψ is continuously differentiable on $(0,\infty)$ (cf. [5]).

$$\frac{1}{\sqrt{t}}\,\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)=\int_0^t \frac{1}{\sqrt{t-s}}\,\varphi\left(\frac{\psi(t)-\psi(s)}{\sqrt{t-s}}\right) f(s)\,ds, \tag{5}$$

$$f(t)=\frac{\Lambda(t)}{t^{3/2}}\,\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)-\int_0^t \frac{\psi(t)-\psi(s)-(t-s)\,\psi'(t)}{(t-s)^{3/2}}\,\varphi\left(\frac{\psi(t)-\psi(s)}{\sqrt{t-s}}\right) f(s)\,ds. \tag{6}$$

We begin by rewriting (5) and (6) in a more convenient form. Let us define h and Δ by

$$f(u)=\frac{1}{\sqrt{u}}\,\varphi\left(\frac{\psi(u)}{\sqrt{u}}\right) h(u)$$

and

$$\Delta(t,u)=\frac{(\psi(t)-\psi(u))^2}{t-u}+\frac{\psi(u)^2}{u}-\frac{\psi(t)^2}{t}.$$

Completing the square in Δ we get $\Delta(t,u)=\frac{t}{u(t-u)}\left(\psi(t)\frac{u}{t}-\psi(u)\right)^2$. Then the Eqs. (5) and (6) become

$$\begin{aligned}1&=\int_0^t \sqrt{\frac{t}{u(t-u)}}\,\varphi\left(\frac{\psi(t)-\psi(u)}{\sqrt{t-u}}\right)\varphi\left(\frac{\psi(u)}{\sqrt{u}}\right)\varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)^{-1} h(u)\,du\\ &=\int_0^t \sqrt{\frac{t}{2\pi\,u(t-u)}}\,e^{-\Delta(t,u)/2}\,h(u)\,du,\end{aligned} \tag{7}$$

$$h(t)=\frac{\Lambda(t)}{t}-\int_0^t \frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{t-u}\sqrt{\frac{t}{2\pi\,u(t-u)}}\,e^{-\Delta(t,u)/2}\,h(u)\,du. \tag{8}$$

We decompose the integral on the right-hand side of Eq. (8) into three parts:

$$\int_0^t=\int_0^r+\int_r^s+\int_s^t=T_1+T_2+T_3.$$

We choose

$$r(t)=t(t/\psi(t)^2)^{\gamma} \quad \text{with} \quad \beta^{-1}<\gamma<(2\alpha-1)^{-1}$$

and

$$s(t)=t(1-(t/\psi(t)^2)^{\delta}) \quad \text{with} \quad 1/2<\delta<1.$$

Note that $r(t)=o(t)$ and $r(t)\to\infty$. This follows by condition (I) since $\psi(t)\leqq\psi(1)\,t^{\alpha}$ and therefore $r(t)=t(t/\psi(t)^2)^{\gamma}\geqq t(t/\psi(1)^2\,t^{2\alpha})^{\gamma}=\psi(1)^{-2\gamma}\,t^{1-\gamma(2\alpha-1)}$, which tends to infinity as $t\to\infty$.

We prove first that $T_2+T_3=o\left(\frac{\Lambda(t)}{t}\right)$ and then $T_1=P(T<\infty)\frac{\Lambda(t)}{t}(1+o(1))$ as $t\to\infty$.

To estimate T_3 we show that

$$\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{t-u}=o\left(\frac{\Lambda(t)}{t}\right)\quad\text{uniformly for } u\in[s,t]. \tag{9}$$

From the mean value theorem we get

$$\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{t-u}=\psi'(\xi_u)-\psi'(t)\quad\text{for some } \xi_u\in[u,t].$$

Since by condition (I) $(\psi(t)/t^{\alpha})'\leqq 0$ and therefore $\Lambda(t)\geqq t\psi'(t)\left(\frac{1-\alpha}{\alpha}\right)$ we obtain

$$|\psi'(\xi_u)-\psi'(t)|\,t/\Lambda(t)\leqq\left|\frac{\psi'(\xi_u)}{\psi'(t)}-1\right|\frac{\alpha}{1-\alpha}\quad,$$

which tends to zero as $s/t\to 1$ by condition (III). Using (9) and the integral equation (7) we get

$$\begin{aligned}|T_3|&\leqq\int_s^t\left|\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{t-u}\right|\sqrt{\frac{t}{2\pi\,u(t-u)}}\,e^{-\Delta(t,u)/2}\,h(u)\,du\\&\leqq o\left(\frac{\Lambda(t)}{t}\right)\int_0^t\sqrt{\frac{t}{2\pi\,u(t-u)}}\,e^{-\Delta(t,u)/2}\,h(u)\,du\\&=o\left(\frac{\Lambda(t)}{t}\right).\end{aligned}\tag{10}$$

To estimate T_2 we show that for large t

$$\exp(-\Delta(t,u)/2)\leqq\exp(-(\psi(t)/\sqrt{t})^{\eta})\quad\text{with}\quad \eta>0. \tag{11}$$

Since

$$\Delta(t,u)=\frac{t}{u(t-u)}\left(\psi(t)\frac{u}{t}-\psi(u)\right)^2$$

condition (I) yields for $u\leqq t$ $\psi(u)\geqq\psi(t)\left(\frac{u}{t}\right)^{\alpha}$ and therefore

$$\Delta(t,u)\geqq\frac{\psi(t)^2}{t}\,\frac{u}{t-u}\left(\left(\frac{t}{u}\right)^{1-\alpha}-1\right)^2. \tag{12}$$

For sufficiently large t the right-hand side of (12) assumes its minimum at one of the endpoints $r(t)$ or $s(t)$. Since $r(t)/t\to 0$ for $t\to\infty$ the definition of r yields

$$\frac{r}{t-r}\left(\left(\frac{t}{r}\right)^{1-\alpha}-1\right)^2 \geqq \frac{1}{2}\left(\frac{t}{r}\right)^{1-2\alpha}=\frac{1}{2}(t/\psi(t)^2)^{(2\alpha-1)\gamma}. \tag{13}$$

Since $s/t\to 1$ for $t\to\infty$ we obtain for large t

$$\frac{s}{t-s}\left(\left(\frac{t}{s}\right)^{1-\alpha}-1\right)^2 \geqq \frac{(1-\alpha)^2}{2}\,\frac{(t-s)}{t}=\frac{(1-\alpha)^2}{2}\left(\frac{t}{\psi(t)^2}\right)^{\delta}. \tag{14}$$

Since $(2\alpha-1)\gamma<1$ and $\delta<1$ from (12), (13) and (14) for all $u\in[r,s]$ and all large t we get statement (11).

We estimate now the other parts of the integrand of (8). Since ψ is monotone increasing and $\psi'(t)\leqq\alpha\psi(t)/t$ by condition (I) we get

$$\begin{aligned}\left|\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{(t-u)}\right| &\leqq (1+\alpha)\frac{\psi(t)}{t-u}\\ &\leqq (1+\alpha)\frac{\psi(t)}{t}\,\frac{t}{t-s}=(1+\alpha)\frac{\psi(t)}{t}\left(\frac{\psi(t)^2}{t}\right)^{\delta}.\end{aligned} \tag{15}$$

By Lemma 1 $h(u)\leqq u^{-1}\,\psi(u)$ but $\psi(u)\leqq\psi(t)$ and thus for $u\in[r,s]$

$$h(u)\leqq\psi(t)/u\leqq\frac{\psi(t)}{\sqrt{ru}}=\left(\frac{\psi(t)}{\sqrt{t}}\right)^{1+\gamma}\frac{1}{\sqrt{u}}. \tag{16}$$

Now (11), (15) and (16) yield for large t

$$\begin{aligned}|T_2| &\leqq \int_r^s \left|\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{t-u}\right| \sqrt{\frac{t}{2\pi\, u(t-u)}}\, e^{-\Delta(t,u)/2}\, h(u)\, du\\ &\leqq \frac{\psi(t)}{t}\,\frac{1+\alpha}{\sqrt{2\pi}}\left(\frac{\psi(t)}{\sqrt{t}}\right)^{1+3\delta+\gamma} e^{-(\psi(t)/\sqrt{t})^{\eta}}\int_r^t u^{-1}\,du.\end{aligned}$$

As $\int_r^t u^{-1}\,du=\gamma\log\left(\frac{\psi(t)^2}{t}\right)$ we get for an appropriate $\eta'>0$ and for t large enough

$$|T_2|\leqq\frac{\psi(t)}{t}\exp(-(\psi(t)/\sqrt{t})^{\eta'})=o\left(\frac{\psi(t)}{t}\right). \tag{17}$$

It remains to estimate T_1. Assumption (II), the monotonicity of ψ and the fact that $r\to\infty$ imply that for $u\in(0,r)$

$$\psi(u)\leqq\psi(r)\leqq\psi(t)\left(\frac{r}{t}\right)^{\beta}.$$

As $r/t\to 0$ we get for $u\in(0,r)$

$$\frac{\psi(t)-\psi(u)-(t-u)\,\psi'(t)}{(t-u)}=\frac{\Lambda(t)}{t}(1+o(1)). \tag{18}$$

Thus

$$
\begin{aligned}
T_1 &= \frac{\Lambda(t)}{t}(1+o(1)) \int_0^r \sqrt{\frac{1}{2\pi u}}\, e^{-\Delta(t,u)/2}\, h(u)\, du \\
&= \frac{\Lambda(t)}{t}(1+o(1))\, P(T<r \mid W(t)=\psi(t)).
\end{aligned} \tag{19}
$$

We prove now

$$
P(T<r(t) \mid W(t)=\psi(t)) = P(T<r(t))\,(1+o(1)). \tag{20}
$$

For this we show

$$
\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Delta(t,u)\right) \Big/ \varphi\left(\frac{\psi(u)}{\sqrt{u}}\right) \to 1 \tag{21}
$$

uniformly on $[0, r]$.

$$
\begin{aligned}
&|(\psi(t)-\psi(u))^2/(t-u) - \psi(t)^2/t| \\
&\leqq 2\left(\frac{\psi(t)^2}{t}\frac{r}{t} + \frac{\psi(u)^2}{t} + 2\frac{\psi(t)\,\psi(u)}{t}\right) \\
&\leqq 8\frac{\psi(t)^2}{t}\left(\frac{r}{t}\right)^{\beta} = 8\left(\frac{\psi(t)^2}{t}\right)^{1-\beta\gamma}
\end{aligned} \tag{22}
$$

for large t. The first line follows because $t/(t-u) \to 1$ and the second because $\psi(u) \leqq \psi(t)\,(u/t)^{\beta}$. Thus $\gamma > \beta^{-1}$ implies (21). From (19) and (20) we get

$$
T_1 = \frac{\Lambda(t)}{t} P(T<r)\,(1+o(1)). \tag{23}
$$

Since $P(T<r) \to P(T<\infty)$ the theorem follows from (10), (17) and (23).

Lemma 1.

$$
f(t) \leqq \frac{\psi(t)}{t^{3/2}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) \quad \textit{for all } t \in (0, \infty).
$$

Proof. This follows from Lemma 3.1 of [14]. Consider the functions $\psi_1(s) = \psi(s)$ and $\psi_2(s) = \psi(t)$ for a fixed t. Then $\psi_1 \leqq \psi_2$ for $s<t$ and $\psi_1 \geqq \psi_2$ for $s \geqq t$. Thus the related first-exit densities satisfy $f_{\psi_1}(t) \leqq f_{\psi_2}(t) = \frac{\psi(t)}{t^{3/2}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right)$.

Theorem 1 can also be proved by the method used in [8] and conversely the results in [8] can be proved by the present method.

Because of the absolute continuity of Brownian motion with drift with respect to Brownian motion without drift, Theorem 1 can be restated in the following form, which will be used in the next section. Let $f_\theta(t)$ denote the first-exit density of T (defined in (1)) for Brownian motion with drift θ. Then $f_\theta(t) = \exp(\psi(t)\,\theta - t\theta^2/2)\, f(t)$. Thus multiplying both sides of (3) by the factor $\exp(\psi(t)\,\theta - t\theta^2/2)$ yields an analogous asymptotic formula for Brownian motion with drift.

Theorem 2. *Let the conditions* (I)–(IV) *hold. Then*

$$f_\theta(t) \doteq P_0(T=\infty)\frac{\Lambda(t)}{t^{3/2}}\,\varphi\left(\frac{\psi(t)-\theta t}{\sqrt{t}}\right) \tag{24}$$

uniformly in θ *as* $t\to\infty$.

3. Conclusions for Tests of Power One

We now discuss the implications of Theorem 2 for tests of power one for the drift θ of a Brownian motion.

Let P_θ denote the measure of Brownian motion with drift θ. A level-α test of power one for the one-sided testing problem $H_0: \theta \leqq 0$ against $H_1: \theta > 0$ is given by a stopping time R of Brownian motion which satisfies

(i) $P_\theta(R<\infty)\leqq\alpha$ for $\theta\leqq 0$,
(ii) $P_\theta(R<\infty)=1$ for $\theta>0$.

Stopping means rejection of the null hypothesis. The stopping times considered in Theorem 1 define tests of power one. We now apply Theorem 2 to study the behaviour of the distribution $\mathscr{L}_\theta(T)$ as θ tends to zero. This situation has already been studied by several authors. See for instance [4, 10, 12] and [13]. All these results deal only with the asymptotic behaviour of the expected sample size.

The asymptotic behaviour of the distribution $\mathscr{L}_\theta(T)$ is based on the fact, first noticed by T.L. Lai and Lerche, that as $\theta\searrow 0$ the distribution $\mathscr{L}_\theta(T)$ splits up into two parts. One part of the mass consists of paths which still cross the curve when $\theta=0$. The other part of the mass flows out to infinity as $\theta\searrow 0$ and vanishes when $\theta=0$. More precisely, let a_θ be the time when the ray from the origin with slope θ crosses the curve, i.e. the solution of the equation $\psi(a_\theta) = \theta a_\theta$. By condition (I) a_θ exists and is unique for every $\theta>0$. When θ is small the distribution $\mathscr{L}_\theta(T|T>t_0)$ (with t_0 large but $t_0 \ll a_\theta$) is very heavily concentrated around a_θ. The portion of the mass of $\mathscr{L}_\theta(T)$ which flows out to infinity as $\theta\searrow 0$ is asymptotically $P_0(T=\infty)$ (see Theorem 3). The concentration of $\mathscr{L}_\theta(T|T>t_0)$ around a_θ is so strong that the moments of T also degenerate (Theorem 4). The distribution $\mathscr{L}_\theta(T|T>t_0)$ rescaled in the right way becomes asymptotically normal (Theorem 7). In the case of Brownian motion considered here, all these results are derived from Theorem 2, which contains much detailed information. There are more general situations in which Theorems 3–6 still hold, for instance they are true for exponential families. But the proofs are different since a result similar to Theorem 2 is not known. This will be discussed in a paper which will appear elsewhere.

For all the subsequent theorems we assume the conditions of Sect. 2 and of Theorem 1.

Theorem 3. *Let* $\{s_\theta\}$ *be a sequence of real numbers which tends to infinity and satisfies* $s_\theta = o(a_\theta)$ *as* $\theta\searrow 0$. *Then*

$$\lim_{\theta\searrow 0}\mathscr{L}\left(\frac{T}{a_\theta}\,\middle|\, T>s_\theta\right)=\delta_1, \tag{25}$$

where δ_1 *is the point mass in the point* 1.

For every $\varepsilon > 0$

$$\lim_{\theta \searrow 0} P_\theta\{(1-\varepsilon)\, a_\theta \leqq T \leqq (1+\varepsilon)\, a_\theta\} = P_0(T=\infty). \tag{26}$$

The statements (25) and (26) are equivalent since $\lim_{\theta \searrow 0} P_\theta(T > s_\theta) = P_0(T=\infty)$. The proof of the last equation is a by-product of the proof of Theorem 4.

The following result states the asymptotic degeneracy of the moments.

Theorem 4. *Let* $\kappa > 0$. *As* $\theta \searrow 0$

$$E_\theta T^\kappa \doteq P_0(T=\infty)\, a_\theta^\kappa. \tag{27}$$

The special cases $\kappa = 1$, $\psi(t) \doteq \sqrt{t \log t}$ and $\psi(t) \doteq \sqrt{2t \log \log t}$ are contained in the literature [12]. A minimal asymptotic growth rate for the limes superior of the expected sample size as $\theta \searrow 0$ was found by Farrell in [4] for exponential families. We state an analogue in our context.

Theorem 5. *Let* $\kappa > 0$.

$$\limsup_{\theta \searrow 0} E_\theta T^\kappa \Big/ \left(\frac{2}{\theta^2} \log \log \left(\frac{1}{\theta^2}\right)\right)^\kappa \geqq P_0(T=\infty).$$

If additionally $\psi(t)/\sqrt{t}$ *is ultimately non-decreasing the statement also holds for the limes inferior.*

In [13], p. 425 an example is given which shows that the limes inferior and the limes superior can differ.

For the central part of the mass near a_θ we get an approximation result and a local central limit theorem.

Let $g_\theta(t) = \dfrac{A(a_\theta)}{t^{3/2}}\, \varphi\left(\dfrac{\psi(a_\theta) + (t - a_\theta)\, \psi'(a_\theta) - \theta t}{\sqrt{t}}\right)$. This is the first-exit density for the tangent of ψ at a_θ. Let us denote by $[x \pm a]$ the interval $[x-a, x+a]$.

Theorem 6. *For every* $b > 0$

$$\lim_{\theta \searrow 0} \sup_{t \in \left[a_\theta \pm \frac{b}{\theta} \sqrt{a_\theta}\right]} \left| \frac{f_\theta(t)}{g_\theta(t)} - P_0(T=\infty) \right| = 0.$$

Theorem 7. *Assume that* $\lim_{\theta \to 0} \psi'(a_\theta)/\theta = \gamma$. *Then for every* $b > 0$

$$\lim_{\theta \searrow 0} \sup_{t \in \left[a_\theta \pm \frac{b}{\theta} \sqrt{a_\theta}\right]} \left| f_\theta(t) \Big/ \frac{\theta(1-\gamma)}{\sqrt{a_\theta}}\, \varphi\left(\frac{\theta(t - a_\theta)}{\sqrt{a_\theta/(1-\gamma)^2}}\right) - P_0(T=\infty) \right| = 0.$$

We now prove Theorem 4. The proof of Theorem 3 is essentially a by-product. We leave the details to the reader.

Proof of Theorem 4. The assumptions imply that $\psi(t)/\sqrt{t} \to \infty$ and thus $\theta^2 a_\theta = \psi(a_\theta)^2/a_\theta \to \infty$. Let $1/2 < \mu < 1$ and let $r_\theta = a_\theta/(\theta^2 a_\theta)^{\mu - 1/2}$. Then $r_\theta = o(a_\theta)$ and $r_\theta \to \infty$ when $\theta \to 0$. Thus

$$E_\theta T^\kappa = E_\theta T^\kappa\, 1_{\{T > r_\theta\}} + o(a_\theta^\kappa).$$

By (24)

$$E_\theta T^\kappa 1_{\{T>r\theta\}} \doteq P_0(T=\infty) \int_{r\theta}^{\infty} t^{\kappa-3/2} A(t)\, \varphi\left(\frac{\psi(t)-\theta t}{\sqrt{t}}\right) dt \tag{28}$$

as $\theta \to 0$. Thus we have only to evaluate the right-hand side of (28). We split up the integral into three parts:

$$\int_{r\theta}^{\infty} = \int_{r\theta}^{b\theta} + \int_{b\theta}^{c\theta} + \int_{c\theta}^{\infty} = a_\theta^\kappa (T_1+T_2+T_3),$$

where $b_\theta = a_\theta - \theta^{2(\mu-1)} a_\theta^\mu$ and $c_\theta = a_\theta + \theta^{2(\mu-1)} a_\theta^\mu$. The estimates for T_1 and T_3 are very similar. Therefore we give only that for T_1. First we bound the exponential term in T_1. Using assumption (I) and that $\psi(t)/t$ is decreasing we get

$$\begin{aligned}
(\psi(t)-\theta t)^2/t &= t\left(\frac{\psi(t)}{t}-\theta\right)^2 \\
&\geqq t\left(\frac{\psi(b_\theta)}{b_\theta}-\theta\right)^2 \\
&\geqq t\left(\frac{\psi(a_\theta)}{b_\theta}\left(\frac{b_\theta}{a_\theta}\right)^\alpha-\theta\right)^2 \\
&= t\theta^2\left(\left(\frac{a_\theta}{b_\theta}\right)^{1-\alpha}-1\right)^2 \\
&\geqq \eta t \theta^2 (\theta^2 a_\theta)^{2(\mu-1)} \\
&= \eta\left(\frac{t}{a_\theta}\right)(\theta^2 a_\theta)^{2\mu-1}
\end{aligned}$$

with $\eta < (1-\alpha)^2$. The last inequality follows from $\theta^2 a_\theta \to \infty$. We have further $A(t) \leqq \psi(t) \leqq \psi(a_\theta) = \theta a_\theta$. Thus

$$\begin{aligned}
T_1 &= \int_{r\theta}^{b\theta} \left(\frac{t}{a_\theta}\right)^\kappa \frac{A(t)}{t^{3/2}}\, \varphi\left(\frac{\psi(t)-\theta t}{\sqrt{t}}\right) dt \\
&\leqq \sqrt{\frac{\theta^2 a_\theta}{2\pi}} \int_{r\theta}^{b\theta} \left(\frac{t}{a_\theta}\right)^{\kappa-3/2} e^{-\frac{\eta}{2}(\theta^2 a_\theta)^\gamma\left(\frac{t}{a_\theta}\right)} \frac{dt}{a_\theta}
\end{aligned}$$

with $\gamma = 2\mu - 1$. Changing variables we get

$$\begin{aligned}
&= \sqrt{\frac{\theta^2 a_\theta}{2\pi}} \int_{r\theta/a\theta}^{1-(\theta^2 a\theta)^{\mu-1}} s^{k-3/2} e^{-\frac{\eta}{2}(\theta^2 a_\theta)^\gamma s}\, ds \\
&\leqq \left(\frac{r_\theta}{a_\theta}\right)^{\kappa-1/2} \sqrt{\frac{\theta^2 a_\theta}{2\pi}} \int_1^\infty v^{\kappa-3/2} e^{-\frac{\eta}{2}(\theta^2 a_\theta)^{\gamma/2} v}\, dv \\
&= 0(e^{-\delta(\theta^2 a_\theta)^{\mu-1/2}})
\end{aligned}$$

with $0 < \delta < \eta/2$. This shows that $T_1 = o(1)$. In the same way we get $T_3 = o(1)$. Now we show that $T_2 = 1 + o(1)$. We first estimate the exponential part of T_2.

The assumption (I) yields $\psi'(a_\theta) \leqq \alpha\theta$ and thus $\tilde{\theta} = \theta - \psi'(a_\theta) \geqq \theta(1-\alpha)$. By computation we get

$$(\psi(t) - \theta t)^2 = \left((a_\theta - t)\left(\tilde{\theta} + \psi'(a_\theta)\left(1 - \frac{\psi'(t_\theta)}{\psi'(a_\theta)}\right)\right)\right)^2$$

where t_θ and t are elements of $[b_\theta, c_\theta]$.

By assumption (III) and the fact that $b_\theta/c_\theta \to 1$, $\sup_{v \in [b_\theta, c_\theta]} \left|1 - \frac{\psi'(v)}{\psi'(a_\theta)}\right| \to 0$, and hence $(\psi(t) - \theta t)^2 = (a_\theta - t)^2 \tilde{\theta}^2 (1 + \varepsilon_\theta(t))^2$ where $\sup_{t \in [b_\theta, c_\theta]} |\varepsilon_\theta(t)| \to 0$ as $\theta \to 0$.

We give an upper bound for T_2; the lower bound can be derived similarly. Let $\varepsilon > 0$. For θ sufficiently small

$$\begin{aligned} T_2 &= \int_{b_\theta}^{c_\theta} \left(\frac{t}{a_\theta}\right)^\kappa \frac{\Lambda(t)}{t^{3/2}} \varphi\left(\frac{\psi(t) - \theta t}{\sqrt{t}}\right) dt \\ &\leqq \int_{b_\theta}^{c_\theta} \left(\frac{t}{a_\theta}\right)^\kappa \frac{\Lambda(t)}{\sqrt{2\pi t^3}} e^{-\frac{\tilde{\theta}^2 (1-\varepsilon)(a_\theta - t)^2}{2t}} dt \\ &\leqq \int_{-(\tilde{\theta}^2 a_\theta)^{\mu-1}}^{(\tilde{\theta}^2 a_\theta)^{\mu-1}} (1+u)^{\kappa - 3/2} \frac{\Lambda(a_\theta(1+u))}{\sqrt{2\pi a_\theta}} e^{-\frac{\tilde{\theta}^2 a_\theta (1-\varepsilon) u^2}{2(1+u)}} du \end{aligned}$$

where we substituted $u = t/a_\theta - 1$. By computation we get for $-(\tilde{\theta}^2 a_\theta)^{\mu-1} \leqq u \leqq (\tilde{\theta}^2 a_\theta)^{\mu-1}$

$$\Lambda(a_\theta(1+u)) \leqq \tilde{\theta} a_\theta (1+\varepsilon).$$

This yields

$$\begin{aligned} T_2 &\leqq (1+2\varepsilon) \sqrt{\frac{\tilde{\theta}^2 a_\theta}{2\pi}} \int_{-(\tilde{\theta}^2 a_\theta)^{\mu-1}}^{(\tilde{\theta}^2 a_\theta)^{\mu-1}} e^{-\frac{(1-\varepsilon)\tilde{\theta}^2 a_\theta u^2}{2(1+\varepsilon)}} du \\ &\leqq (1+2\varepsilon) \frac{1}{\sqrt{2\pi}} \int_{-(\tilde{\theta}^2 a_\theta)^{\mu-1/2}}^{(\tilde{\theta}^2 a_\theta)^{\mu-1/2}} e^{-\frac{(1-\varepsilon) v^2}{2(1+\varepsilon)}} dv \leqq (1+2\varepsilon) \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}. \end{aligned}$$

Proof of Theorem 5. From condition (IV) we deduce that for every $\varepsilon > 0$ a sequence $\{t_n\}$ exists with

$$\lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \psi(t_n) \geqq \sqrt{2(1-\varepsilon) t_n \log\log t_n},$$

for otherwise $P_0(T < \infty) = 1$. For the sequence $\theta_n = \psi(t_n)/t_n$ we get

$$a_{\theta_n} = t_n \geqq \frac{2(1-\varepsilon)}{\theta_n^2} \log\log\left(\frac{1}{\theta_n^2}\right)(1 + o(1))$$

and thus by (27)

$$E_{\theta_n} T^\kappa \doteq P_0(T = \infty)\, a_{\theta_n}^\kappa \geqq P_0(T = \infty) \left(\frac{2(1-\varepsilon)}{\theta_n^2} \log\log\left(\frac{1}{\theta_n^2}\right)\right)^\kappa (1 + o(1))$$

for every $\varepsilon > 0$. This proves the first part of the theorem.

The assumption that $\psi(t)/\sqrt{t}$ is finally non-decreasing together with the other assumptions yields according to statement (4)

$$\int_1^\infty \frac{\psi(t)}{t^{3/2}} \varphi\left(\frac{\psi(t)}{\sqrt{t}}\right) dt < \infty.$$

Thus $\psi(t) \geqq \sqrt{2t \log\log t}$ at least for large t by the argument given in [11] p. 1421. This implies the second half of the theorem.

The proofs of Theorem 6 and 7 are obtained by straightforward calculation. We omit the details.

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