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Two-Parameter Filtrations with Respect to Which all Martingales are Strong

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In this paper necessary and sufficient conditions are given, so that all the martingales attached to a two-parameter filtration are strong. These filtrations have the conditional independence property (F4) of Cairoli and Walsh [1]. Using a counter-example it is emphasized that if G_n and G are separations and $G_n \searrow G$, it should not necessarily be inferred that $\mathscr{F}_{G_n} \searrow \mathscr{F}_G$.

§ 1. Preliminaries and General Notations

Let $T=R_+^2$; the points of T are denoted by $z, z', \ldots, z_i, \ldots$ or using the coordinates: $z=(s,t)$, $z'=(s',t'), \ldots, z_i=(s_i,t_i)$ a.s.o. T will be endowed with the trace of the usual topology on R^2. If $z_1, z_2 \in T$, we write: $z_1 \leqq z_2$ iff $s_1 \leqq s_2$, $t_1 \leqq t_2$; $z_1 < z_2$ iff $s_1 < s_2, t_1 < t_2$; $z_1 \prec z_2$ iff $s_1 \geqq s_2$, $t_1 \leqq t_2$. If $z_1 < z_2$, $(z_1, z_2]$ means the set of those z from T such that $z_1 < z \leqq z_2$; $[z_1, z_2]$ is the set $\{z \in T / z_1 \leqq z \leqq z_2\}$; R_z is the interval $[0, z]$ and if $A \subset T$, $R_A = \bigcup_{z \in A} R_z$.

A set $G \subset T$ is called a separation iff $G = \partial R_G$. The separation ∂R_z is denoted by $\underline{z}$. If G_1 and G_2 are two separations, $G_1 \leqq G_2$ means that $R_{G_1} \subset R_{G_2}$ and $G_1 < G_2$ denotes the fact that $R_{G_1} \subset \operatorname{Int}(R_{G_2})$. If G_n is a decreasing sequence of separations, we write $G_n \searrow G$ iff $R_G = \bigcap_{n=1}^{\infty} R_{G_n}$.

Let $(\Omega, \mathscr{K}, P)$ be a complete probability space and $\mathscr{F} \subset \mathscr{K}$ be a complete σ-algebra. We shall write $f \in \mathscr{F}$ iff $f: \Omega \to R$ is a bounded $\mathscr{F}$-measurable function. The conditional expectation operator will be denoted sometimes $E^{\mathscr{F}}$ instead of $E(./\mathscr{F})$.

If A is an arbitrary set belonging to $\mathscr{K}$ and i_A: $A \to \Omega$ is the canonical injection, the σ-algebra $i_A^{-1}(\mathscr{F})$ will be also denoted by $\mathscr{F}|_A$. It is obvious that if $A \in \mathscr{F}$, then $\mathscr{F}|_A = \{CA / C \in \mathscr{F}\}$ (we shall systematically omit the sign of intersection "$\bigcap$" between two sets).

Let I be an arbitrary index set and for every $\alpha \in I$ a $\mathscr{F}$-measurable real-valued mapping f_α. Then $\operatorname{ess\,sup}_{\alpha \in I} f_\alpha$ is a $\mathscr{F}$-measurable function f satisfying the following two assumptions:

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(i) $f \geqq f_\alpha$ a.s. for every $\alpha \in I$ and

(ii) If $g \geqq f_\alpha$ a.s. for every $\alpha \in I$ and g is $\mathscr{F}$-measurable, then $g \geqq f$ a.s.

One defines by symmetry $\operatorname{ess\,inf}_{\alpha \in I} f_\alpha$. For any set A, its indicator function is denoted by 1_A.

If $(A_\alpha)_{\alpha \in I}$ are sets belonging to $\mathscr{F}$, we prefer to write $A = \operatorname{ess\,sup}_{\alpha \in I} A_\alpha$ instead of $1_A = \operatorname{ess\,sup}_{\alpha \in I} 1_{A_\alpha}$. It is obvious that $\operatorname{ess\,sup}_{\alpha \in I} A_\alpha^c = (\operatorname{ess\,inf}_{\alpha \in I} A_\alpha)^c$. It is well known that ess sup and ess inf can be attained after countable subsets of I (see e.g. [4]).

Throughout the paper, all the relations between random variables and sets must be interpreted as occurring almost surely, if not stated otherwise. For instance $A \subset B$ means that $1_A \leqq 1_B$ a.s.

A family $(\mathscr{H}_t)_{t \geqq 0}$ of complete σ-algebras included in K is called a standard filtration (or, in short, a filtration, because we shall not deal with not-standard ones) iff $s < t \Rightarrow \mathscr{H}_s \subset \mathscr{H}_t$ and $\mathscr{H}_t = \bigcap_{s > t} \mathscr{H}_s$. The right side σ-algebra will be denoted by $\mathscr{H}_{t^+}$.

A family $(\mathscr{F}_z)_{z \in T}$ of σ-algebras contained in $\mathscr{K}$ is called a two-parameter standard filtration (or, in short, a filtration if no confusions occur) iff $z < z' \Rightarrow \mathscr{F}_z \subset \mathscr{F}_{z'}$ and $\mathscr{F}_z = \mathscr{F}_{z^+} = \bigcap_{z' > z} \mathscr{F}_{z'}$. In this case $\mathscr{F}_{s,\infty}$ means $\bigvee_{t' \geqq 0} \mathscr{F}_{s,t'}$, and $\mathscr{F}_{\infty,t} (\mathscr{F}_z^{\bullet}, \mathscr{F}_{\infty,\infty})$ denote the σ-algebras $\bigvee_{s' \geqq 0} \mathscr{F}_{s',t} (\mathscr{F}_{s,\infty} \vee \mathscr{F}_{\infty,t}, \bigvee_{z \in T} \mathscr{F}_z)$. We shall suppose in the sequel that $\mathscr{F}_{\infty,\infty} = \mathscr{K}$.

If G is a separation, $\mathscr{F}_G$ denotes the σ-algebra $\bigvee_{z \in G} \mathscr{F}_z = \bigvee_{z \in R_G} \mathscr{F}_z$.

The conditional expectation operators which will appear are: E_z, $E_{s,\infty}$, $E_{\infty,t}$ and $E_z^{\bullet}$ denoting respectively $E^{\mathscr{F}_z}$, $E^{\mathscr{F}_{s,\infty}}$, $E^{\mathscr{F}_{\infty,t}}$, $E^{\mathscr{F}_z^{\bullet}}$.

We say that the filtration satisfies the (F4)-hypothesis of Cairoli and Walsh [1] iff $E_{s,\infty} E_{\infty,t} = E_{\infty,t} E_{s,\infty} = E_z$ for every $z = (s,t)$ from T. In this case we say that the filtration has (F4), or merely say (F4). Of course (F4) $\Rightarrow \mathscr{F}_z = \mathscr{F}_{s,\infty} \cap \mathscr{F}_{\infty,t}$.

As usual, a process $x_z : \Omega \to R$ is said to be adapted to the filtration $(\mathscr{F}_z)_z$ iff x_z is $\mathscr{F}_z$-measurable for every $z \in T$.

A process x such that $x_z \in L^1(\mathscr{F}_z)$ for every z is said to be a martingale (respectively 1-martingale, 2-martingale) iff $z \leqq z' \Rightarrow E_z(x_{z'}) = x_z$ (respectively $E_{s,\infty}(x_{s+h,t}) = x_z$, $E_{\infty,t}(x_{s,t+h}) = x_z$ for every $h > 0$).

It is obvious that (F4) $\Leftrightarrow$ every martingale is an i-martingale ($i = 1, 2$).

Given a process x, we define a finitely-additive signed measure on rectangles by the equality $x(z, z'] = x_{z'} - x_{s',t} - x_{s,t'} + x_z$.

A martingale x is called a strong martingale iff $z < z' \Rightarrow E_z^{\bullet}(x(z, z']) = 0$.

The question that prompted this study is: given a filtration $(\mathscr{F}_z)_{z \in T}$, what supplementary conditions should be added in order that every martingale be a strong one? For reasons of commodity we shall say that these filtrations have the (F5)-property; in short, (F5).

§ 2. Local Comparability

Proposition 1. *Let* $(\Omega, \mathscr{F}, P, (\mathscr{F}_z)_{z \in T})$ *be a standard filtration. Then* (F5) $\Leftrightarrow$ (F4) *and* $L^2(\mathscr{F}_z^{\bullet}) = L^2(\mathscr{F}_{s,\infty}) + L^2(\mathscr{F}_{\infty,t})$ *for every* $z \in T$.

Proof. "$\Rightarrow$".

Every strong martingale is both a 1- and a 2-martingale (see [5], Proposition 1.1). Therefore every martingale is an i-martingale ($i=1,2$) and (F4) follows.

Let now $f \in L^2(\mathscr{F}^{\bullet}_{z_0})$ with z_0 fixed. Let also $x_z = E_z(f)$. Being a martingale, x is a strong one; so that if $z < z_0$, we have

$$E^{\bullet}_{z_0}(x_z - x_{s_0,t} - x_{s,t_0} + x_{z_0}) = 0. \tag{1}$$

Let $z \to (\infty, \infty)$ and take into account that then $x_z \to f$, $x_{s_0,t} \to x_{s_0,\infty}$, $x_{s,t_0} \to x_{\infty,t_0}$ (all these convergences are in L^2) and that E_{z_0} is a continuous operator from L^2 into L^2. Then we can take limits in (1) and obtain

$$E^{\bullet}_{z_0}(f - x_{s_0,\infty} - x_{\infty,t_0} + x_{z_0}) = 0 \Rightarrow f = x_{s_0,\infty} + x_{\infty,t_0} - x_{z_0} \tag{2}$$

and (2) implies exactly that the function f belongs to $L^2(\mathscr{F}_{s_0,\infty}) + L^2(\mathscr{F}_{\infty,t_0})$. Remark that (2) and (F4) also imply the equalities

$$E^{\bullet}_z = E_{s,\infty} + E_{\infty,t} - E_{s,\infty} E_{\infty,t} = E_{s,\infty} + E_{\infty,t} - E_{\infty,t} E_{s,\infty}. \tag{3}$$

The converse inclusion $L^2(\mathscr{F}_{s,\infty}) + L^2(\mathscr{F}_{\infty,t}) \subset L^2(\mathscr{F}^{\bullet}_z)$ for every z is trivial.

"$\Leftarrow$"

In general, if X is a Hilbert space and H, K are two Hilbert subspaces of X so that their orthogonal projectors P_H and P_K commute ((F4)!) then

$$P_{H+K} = P_H + P_K - P_H P_K. \tag{4}$$

Indeed, let Q be the right member of the above equality. It is an easy calculus to check that $Q(X) \subset H+K$. Conversely, if x belongs to $H+K$ then there exists $y \in H$ and $z \in K$ such that $x = y + z$. Then

$$\begin{aligned} Qx &= P_H x + P_K x - P_H P_K x = y + P_H z + z + P_K y - P_H P_K y - P_K P_H z \\ &= y + z = x \Rightarrow x \in Q(X) \Rightarrow H + K = Q(X). \end{aligned}$$

In our case $X = L^2(\mathscr{F})$, $H = L^2(\mathscr{F}_{s,\infty})$, $K = L^2(\mathscr{F}_{\infty,t})$ and the equality (4) reduces to (3) which, corroborated with (F4) put as "every martingale is an i-martingale" gives quickly (F5).

The following proposition has been implicitly used from the very beginning of the theory of martingales with two indices.

Proposition 2. *Let $(\mathscr{F}_z)_{z \in T}$ be a standard filtration. Then* (F4) *implies the fact that the one-parameter filtrations $(\mathscr{F}_{s,\infty})_{s \geqq 0}$ and $(\mathscr{F}_{\infty,t})_{t \geqq 0}$ are right-continuous.*

Proof. For reasons of symmetry it is enough to check only one from the two assertions, say the second. One must verify that $f \in \bigcap_n \mathscr{F}_{\infty,t+1/n} \Rightarrow f \in \mathscr{F}_{\infty,t}$. But (F4) implies that $\bigcap_{n \geqq 1} \bigvee_{k \geqq 1} \mathscr{F}_{k,t+1/n} = \bigvee_{k \geqq 1} \bigcap_{n \geqq 1} \mathscr{F}_{k,t+1/n}$. To see this take $f \in L^2(\mathscr{F}_{\infty,t^+})$ and set $x_{-n,k} = E_{k,t+1/n}(f)$. Then the following equalities hold because there is convergence in one parameter uniformly with respect to the other one (Doob's maximal inequality!):

$$f = L^2 - \lim_{n,k} x_{-n,k} = L^2 - \lim_{n} x_{-n,\infty} = L^2 - \lim_{k} x_{-\infty,k}$$

and the last term is measurable with respect to the σ-algebra $\bigvee_{k \geqq 1} \bigcap_{n \geqq 1} \mathscr{F}_{k,t+1/n}$. QED.

Proposition 3. *Let $(\Omega, \mathscr{K}, P)$ be a complete probability space and $\mathscr{F}$, $\mathscr{G}$ be two complete σ-algebras contained in K. Then the following two assertions are equivalent:*

(i) $L^2(\mathscr{F} \vee \mathscr{G}) = L^2(\mathscr{F}) + L^2(\mathscr{G})$ *and* $E^{\mathscr{F}} E^{\mathscr{G}} = E^{\mathscr{G}} E^{\mathscr{F}}$.

(ii) *There exists a set* $A \in \mathscr{F} \cap \mathscr{G}$ *such that* $\mathscr{F}|_A \subset \mathscr{G}|_A$ *and* $\mathscr{F}|_{A^c} \supset \mathscr{G}|_{A^c}$.

Remark. Two σ-algebras satisfying (ii) are called locally comparable. It is clear that if $\mathscr{F}$ and $\mathscr{G}$ are comparable (i.e. $\mathscr{F} \subset \mathscr{G}$ or $\mathscr{F} \supset \mathscr{G}$) they are also locally comparable.

Proof. (i)$\Rightarrow$(ii).

The equality (4) gives

$$E^{\mathscr{F} \vee \mathscr{G}} = E^{\mathscr{F}} + E^{\mathscr{G}} - E^{\mathscr{F}} E^{\mathscr{G}} = E^{\mathscr{F}} + E^{\mathscr{G}} - E^{\mathscr{G}} E^{\mathscr{F}}. \tag{5}$$

Let $f \in \mathscr{F}$, $g \in \mathscr{G}$. Then $fg \in \mathscr{F} \vee \mathscr{G}$ and $fg = E^{\mathscr{F} \vee \mathscr{G}}(fg) = f E^{\mathscr{F}}(g) + g E^{\mathscr{G}}(f) - E^{\mathscr{F}}(g) E^{\mathscr{G}}(f)$ or

$$(f - E^{\mathscr{G}}(f))(g - E^{\mathscr{F}}(g)) = 0. \tag{6}$$

Set $\mathscr{H} = \mathscr{F} \cap \mathscr{G}$. Since $E^{\mathscr{F}}$ and $E^{\mathscr{G}}$ commute, (6) may also be written as

$$(f - E^{\mathscr{H}}(f))(g - E^{\mathscr{H}}(g)) = 0 \qquad \text{for every } f \in \mathscr{F},\ g \in \mathscr{G}. \tag{7}$$

Let $D_f = (f \neq E^{\mathscr{H}}(f))$ and $F_g = (g \neq E^{\mathscr{H}}(g))$. Then (7) implies the fact that that $D_f \cap F_g = \emptyset$ for every $f \in \mathscr{F}$, $g \in \mathscr{G}$. From the definition of D_f and F_g it follows

$$f\, 1_{D_f^c} = (E^{\mathscr{H}} f)\, 1_{D_f^c} \quad \text{and} \quad g\, 1_{F_g^c} = (E^{\mathscr{H}} g)\, 1_{F_g^c}. \tag{8}$$

Let $D = \operatorname{ess\,sup}_{f \in \mathscr{F}} D_f$ and $F = \operatorname{ess\,sup}_{g \in \mathscr{G}} F_g$; then $D^c = \operatorname{ess\,inf}_{f \in \mathscr{F}} D_f^c$ and $F^c = \operatorname{ess\,inf}_{g \in \mathscr{G}} F_g^c$. Since $D^c \subset D_f^c$, $F^c \subset F_g^c$ for every $f \in \mathscr{F}$ and $g \in \mathscr{G}$, (8) implies that for every $f \in \mathscr{F}$, $g \in \mathscr{G}$ we have $f\, 1_{D^c} = (E^{\mathscr{H}} f)\, 1_{D^c}$ and $g\, 1_{F^c} = (E^{\mathscr{H}} g)\, 1_{F^c}$. But $\mathscr{F}$ and $\mathscr{G}$ belong even to $\mathscr{H}$. (We check the assertion only for D: it is obvious that $D \in \mathscr{F} \Rightarrow 1_{D^c} \in \mathscr{F}$. Set $f = 1_{D^c}$. Then we have $1_{D^c} = E^{\mathscr{H}}(1_{D^c})\, 1_{D^c}$ hence $P(D^c) = E(1_{D^c}) = E(E^{\mathscr{H}}(1_{D^c})\, 1_{D^c}) \leqq$

$$\leqq E(E^{\mathscr{H}}(1_{D^c})) = P(D^c) \quad \text{therefore} \quad f = E^{\mathscr{H}}(f) \Rightarrow f = 1_{D^c} \in \mathscr{H} \Rightarrow D \in \mathscr{H}).$$

Now the above equalities become $f\, 1_{D^c} = E^{\mathscr{H}}(f\, 1_{D^c})$, $g\, 1_{F^c} = E^{\mathscr{H}}(g\, 1_{F^c})$ for $f \in \mathscr{F}$, $g \in \mathscr{G}$, or, otherwise written

$$\begin{aligned} f \in \mathscr{F} &\Rightarrow f\, 1_{D^c} \in \mathscr{H} \subset \mathscr{G} \Rightarrow \mathscr{F}|_{D^c} \subset \mathscr{G}|_{D^c}, \\ g \in \mathscr{G} &\Rightarrow g\, 1_{F^c} \in \mathscr{H} \subset \mathscr{F} \Rightarrow \mathscr{G}|_{F^c} \subset \mathscr{F}|_{F^c}. \end{aligned} \tag{9}$$

Since $D \cap F = \emptyset \Rightarrow D \subset F^c \Rightarrow \mathscr{G}|_D \subset \mathscr{F}|_D$. Set $A = D^c$ and (ii) follows.

(ii)$\Rightarrow$(i).

First check that $E^{\mathscr{F}} E^{\mathscr{G}} = E^{\mathscr{G}} E^{\mathscr{F}}$; it would be enough to prove that $f \in \mathscr{F}$ implies that $E^{\mathscr{G}}(f) \in \mathscr{F} \cap \mathscr{G}$. But this is clear: $f = f 1_A + f 1_{A^c}$ and $f 1_A \in \mathscr{G}$, hence

$$E^{\mathscr{G}}(f) = f 1_A + E^{\mathscr{G}}(f) 1_{A^c} \Rightarrow E^{\mathscr{G}}(f) 1_{A^c} \in \mathscr{F}$$

because $\mathscr{G}|_{A^c} \subset \mathscr{F}|_{A^c}$.

Then it is an easy thing to see that

$$\begin{aligned} &E^{\mathscr{F} \vee \mathscr{G}}(f) = E^{\mathscr{F}}(f) 1_{A^c} + E^{\mathscr{G}}(f) 1_A \\ \text{and} \quad &E^{\mathscr{F} \cap \mathscr{G}}(f) = E^{\mathscr{F}}(f) 1_A + E^{\mathscr{G}}(f) 1_{A^c} \quad \text{for } f \in \mathscr{K}. \end{aligned} \tag{10}$$

Adding the two equalities we obtain $E^{\mathscr{F} \vee \mathscr{G}} + E^{\mathscr{F} \cap \mathscr{G}} = E^{\mathscr{F}} + E^{\mathscr{G}}$, fact that completes the proof.

Remark. Looking to the proof of the first implication one can observe that there is no unicity of the set A. Another set could be F. But $A = D^c$ has the following maximality property: if $B \in \mathscr{H}$ is another set such that $\mathscr{F}|_B \subset \mathscr{G}|_B$, then $B \subset A$. Indeed, $\mathscr{F}|_B \subset \mathscr{G}|_B \Rightarrow \mathscr{F}|_B \subset \mathscr{H}|_B$. Therefore for every

$$\begin{aligned} f \in \mathscr{F} \Rightarrow f 1_B \in \mathscr{H} \Rightarrow E^{\mathscr{H}}(f 1_B) = f 1_B \Rightarrow (f - E^{\mathscr{H}}(f)) 1_B \Rightarrow B \subset D_f^c \\ \Rightarrow B \subset \operatorname*{ess\,inf}_{f \in F} D_f^c \Rightarrow B \subset D^c = A. \end{aligned}$$

Remark. We can write the equalities (10) in the form

$$\mathscr{F} \vee \mathscr{G} = \mathscr{F}|_{A^c} + \mathscr{G}|_A \quad \text{and} \quad \mathscr{F} \cap \mathscr{G} = \mathscr{F}|_A + \mathscr{G}|_{A^c}. \tag{11}$$

Proposition 4. *Let $(\Omega, \mathscr{K}, P)$ be a complete probability space.*

(i) *Let $(\mathscr{F}_i)_{i \geqq 0}$ and $(\mathscr{G}_j)_{j \geqq 0}$ be two discrete filtrations having the property that for every i and j, $\mathscr{F}_i$ and $\mathscr{G}_j$ are locally comparable. Let also*

$$A_{i,j} = \operatorname*{ess\,inf}_{f \in F_i} (f = E^{\mathscr{F}_i \cap \mathscr{G}_j}(f)) = \operatorname*{ess\,inf}_{f \in F_i} (f = E^{\mathscr{G}_j}(f))$$

(the last equality is due to (F4)!*). Then the following inclusions hold for every $i, j \geqq 0$:*

$$A_{i+1,j} \subset A_{i,j} \subset A_{i,j+1}. \tag{12}$$

(ii) *Let $(\mathscr{F}_s)_{s \geqq 0}$ and $(\mathscr{G}_t)_{t \geqq 0}$ be two standard filtrations. Suppose that $\mathscr{F}_s$ and $\mathscr{G}_t$ are locally comparable for every $s, t \geqq 0$. Set*

$$A_z = A_{s,t} = \operatorname*{ess\,inf}_{f \in \mathscr{F}_s} (f = E^{\mathscr{F}_s \cap \mathscr{G}_t}(f)) = \operatorname*{ess\,inf}_{f \in \mathscr{F}_s} (f = E^{\mathscr{G}_t}(f)).$$

Then

$$z_1 \prec z_2 \Rightarrow A_{z_1} \subset A_{z_2} \quad \textit{and} \quad A_{s,t} = A_{s,t^+} := \operatorname*{ess\,inf}_{t' \quad t} A_{s,t'}. \tag{13}$$

(iii) *If, in addition, $(\mathscr{F}_s)_{s \geqq 0}$ is also left-continuous, then $\operatorname*{ess\,inf}_{s' < s} A_{s',t} = A_{s,t}$ (the left-side set is denoted by $A_{s^-,t}$).*

Proof. (i) We shall use the first from the above remarks. For every $f \in \mathscr{F}_i$, we have

$$f\,1_{A_{i,j}} \in \mathscr{G}_j \subset \mathscr{G}_{j+1} \Rightarrow \mathscr{F}_i|_{A_{i,j}} \subset \mathscr{G}_{j+1}|_{A_{i,j}} \Rightarrow A_{i,j} \subset A_{i,j+1}$$

and

$$A_{i+1,j} = \operatorname*{ess\,inf}_{f \in F_{i+1}} (f = E^{\mathscr{G}_j}(f)) \subset \operatorname*{ess\,inf}_{f \in F_i} (f = E^{\mathscr{G}_j}(f)) = A_{i,j}.$$

(ii) The first relation is proved in the same way as (i). Remark that $A_{s,t^+} = \bigcap_{n \geq 1} A_{s,t_n}$ for every sequence $t_n \searrow t$ and that A_{s,t^+} belongs to $\mathscr{F}_s \cap \mathscr{G}_t$ due to the right-continuity of the filtrations. We only must check that $A_{s,t^+} \subset A_{s,t}$, the other inclusion being obvious. To this end, let $f \in \mathscr{F}_s$. Then

$$f\,1_{A_{s,t^+}} = \lim_n f\,1_{A_{s,t_n}} \in \bigcap_{n \geq 1} \mathscr{G}_{t_n} = \mathscr{G}_t \Rightarrow \mathscr{F}_s|_{A_{s,t^+}} \subset \mathscr{G}_t|_{A_{s,t^+}} \Rightarrow A_{s,t^+} \subset A_{s,t}.$$

We used once again the first remark made after Proposition 4.

(iii) Identifying the sets with their indicators and taking a sequence $s_n \nearrow s$, $s_n < s$, we have

$$\begin{aligned} A_{s^-,t} = \bigcap_{n \geq 1} A_{s_n,t} &= \inf_n \operatorname*{ess\,inf}_{f \in \mathscr{F}_{s_n}} (f = E^{\mathscr{G}_t}(f)) \\ &= \operatorname*{ess\,inf}_{f \in \mathscr{F}_{s^-}} (f = E^{\mathscr{G}_t}(f)) = A_{s,t}. \quad \text{QED.} \end{aligned}$$

Remark. For two locally comparable standard filtrations one cannot in general infere neither that $A_{s^-,t} = A_{s,t}$ nor that $A_{s^+,t} = A_{s,t}$. Counterexamples are readily available. Let, for instance $(\Omega, \mathscr{K}, P)$ be a complete probability space. For an arbitrary set A, not necessary measurable, denote by $\mathscr{K}_A$ the σ-algebra $\{C \in \mathscr{K} / C \subset A \text{ or } C \cap A = \emptyset\}$. Let now $A_s \searrow \emptyset$ be a right-continuous family of sets belonging to $\mathscr{K}$. Set $\mathscr{F}_s = \mathscr{K}_{A_s}$ and suppose $0 < P(A_0) < 1$. Clearly $(\mathscr{F}_s)_{s \geq 0}$ is a standard filtration. Let $\mathscr{G}_t = \mathscr{F}_{s_0}$ for every t with some fixed s_0. Then it is obvious that $(\mathscr{F}_s)_{s \geq 0}$ and $(\mathscr{G}_t)_{t \geq 0}$ are locally comparable and $A_{s,t} = \begin{cases} \Omega & \text{if } s \leq s_0 \\ A_s & \text{if } s > s_0 \end{cases}$. But then $A_{s_0^-,t} = A_{s_0,t} = \Omega$ and $A_{s_0^+,t} = A_{s_0}$.

Corollary 5. *Let* $(\mathscr{F}_z)_{z \in T}$ *be a standard filtration. Then* (F5) $\Leftrightarrow \mathscr{F}_z = \mathscr{F}_{s,\infty} \cap \mathscr{F}_{\infty,t}$ *and* $(\mathscr{F}_{s,\infty})_s$, $(\mathscr{F}_{\infty,t})_t$ *are locally comparable standard filtrations. Moreover, the sets of local comparability* A_z *can be chosen to satisfy the relations* (13).

Proof. To use Propositions 1, 2, 3 and 4.

Examples. If $\mathscr{H}_t$ is a one-parameter standard filtration and σ, τ two stopping times, then the σ-algebras $\mathscr{H}_\sigma$ and $\mathscr{H}_\tau$ are locally comparable. (We remind that $\mathscr{H}_\sigma = \{A \in \mathscr{H} / A(\sigma \leq t) \in \mathscr{H}_t \text{ for every } t \geq 0\}$.) Indeed, it is well-known that $A \in \mathscr{F}_\sigma \Rightarrow A(\sigma \leq \tau) \in \mathscr{F}_\tau$ and $B \in \mathscr{F}_\tau \Rightarrow B(\sigma > \tau) \in \mathscr{F}_\sigma$. Therefore, setting $A = (\sigma \leq \tau)$, we have the inclusions $\mathscr{H}_\sigma|_A \subset \mathscr{H}_\tau|_A$ and $\mathscr{H}_\sigma|_{A^c} \supset \mathscr{H}_\tau|_{A^c}$.

If $(\sigma_s)_{s \geq 0}$ and $(\tau_t)_{t \geq 0}$ are two increasing right-continuous families of stopping-times, then $(\mathscr{H}_{\sigma_s})_{s \geq 0}$ and $(\mathscr{H}_{\tau_t})_{t \geq 0}$ are two standard locally comparable filtrations. To see that fact, remark that $s_n \searrow s \Rightarrow \sigma_{s_n} \searrow \sigma_s$ hence $\mathscr{H}_{\sigma_{s_n}} \searrow \mathscr{H}_{\sigma_s}$ (see, for instance [3]). Therefore the filtration $\mathscr{F}_z = \mathscr{H}_{\sigma_s} \cap \mathscr{H}_{\tau_t} = \mathscr{H}_{\sigma_s \wedge \tau_t}$ has the property (F5) if we suppose in addition that $\sigma_\infty = \tau_\infty = \infty$. The sets $A_z = A_{s,t} = (\sigma_s \leqq \tau_t)$ satisfy the relations (13). For instance

$$A_{s,t^+} = \bigcap_{n \geqq 1} (\sigma_s \leqq \tau_{t_n}) \subset (\sigma_s \leqq \inf_n \tau_{t_n}) = (\sigma_s \leqq \tau_t) = A_{s,t}.$$

A natural problem arises: given two locally comparable standard filtrations $(\mathscr{F}_s)_{s \geqq 0}$ and $(\mathscr{G}_t)_{t \geqq 0}$ such that $\mathscr{F}_\infty = \mathscr{G}_\infty$, does there exist a standard filtration $(\mathscr{H}_t)_{t \geqq 0}$ and two increasing right-continuous families of stopping-times with respect to $(\mathscr{H}_t)_t$ denoted by $(\sigma_s)_s$ and $(\tau_t)_t$ such that $\mathscr{F}_s = \mathscr{H}_{\sigma_s}$ and $\mathscr{G}_t = \mathscr{H}_{\tau_t}$? The answer is affirmative.

§ 3. The Main Result

We begin with the discrete case.

Theorem 6. *Let $(\Omega, \mathscr{K}, P)$ be a complete probability space and $(\mathscr{F}_m)_{m \geqq 1}$, $(\mathscr{G}_n)_{n \geqq 1}$ be two locally comparable filtrations having the property that $\mathscr{F}_\infty = \mathscr{G}_\infty = \mathscr{F}$. Then there exists a filtration $(\mathscr{H}_k)_{k \geqq 1}$ and two increasing sequences of stopping times with respect to $(\mathscr{H}_k)_k$, σ_m and τ_n such that:*

(i) $\lim_m \sigma_m = \lim_n \tau_n = \infty$

(ii) $\mathscr{H}_{\sigma_m} = \mathscr{F}_m$, $\mathscr{H}_{\tau_n} = \mathscr{G}_n$

(iii) *$A_{m,n} = (\sigma_m \leqq \tau_n)$, where $A_{m,n}$ are the sets of local comparability of $\mathscr{F}_m$ and $\mathscr{G}_n$ from Proposition* 4(i).

Proof. According to Proposition 4(i) the following inclusions hold for every integers m, n: $A_{m+1,n} \subset A_{m,n} \subset A_{m,n+1}$. We make the convention that $A_{m,n} = \emptyset$ for $n \leqq 0$ and $A_{m,n} = \Omega$ if $m \leqq 0, n \geqq 1$. Let $C_i^k = A_{i,k-i+1} - A_{i,k-i}$ and $D_i^k = A_{k-i,i} - A_{k-i+1,i}$. Then it is obvious that $C_i^k \in \mathscr{F}_i \cap \mathscr{G}_{k-i+1}$ and that $D_i^k \in \mathscr{F}_{k-i+1} \cap \mathscr{G}_i$ and also that the sets $C_1^k, D_1^k, C_2^k, D_2^k, \ldots, C_k^k, D_k^k$ form a partition of Ω. Set

$$\mathscr{H}_k = \sum_{i=1}^{k} \mathscr{F}_i|_{C_i^k} + \sum_{j=1}^{k} \mathscr{G}_j|_{D_j^k} \tag{1}$$

(this merely means that $f \in \mathscr{H}_k \Leftrightarrow f 1_{C_i^k} \in \mathscr{F}_i$ and $f 1_{D_j^k} \in \mathscr{G}_j$ for every $i, j \leqq k$).

We are going to check that $\mathscr{H}_k$ is just the filtration that we need. To this end, let us define $T_k := \sum_{i=1}^{k} i 1_{(D_i^k + C_{k-i+1}^k)}$ and

$$S_k = k + 1 - T_k = \sum_{i=1}^{k} (k - i + 1) 1_{(D_i^k + C_{k-i+1}^k)}.$$

It is not hard to prove that S_k is a stopping-time with respect to $(\mathscr{F}_m)_m$ and that T_k is a stopping-time with respect to $(\mathscr{G}_n)_n$. We shall verify that

$$\mathscr{H}_k = \mathscr{F}_{S_k} \cap \mathscr{G}_{T_k} \tag{2}$$

and that implies that $(\mathscr{H}_k)_k$ is a filtration because S_k and T_k are increasing with respect to k.

Indeed, $A\in\mathscr{H}_k\Rightarrow AD_i^k\in\mathscr{G}_i,\ AC_{k-i+1}^k\in\mathscr{F}_{k-i+1}$ for every $i=1,2,\dots,k$. Using the properties of the sets $A_{m,n}$ and the definition of the sets C_i^k, D_i^k it results that

$$\mathscr{F}_{k-i+1}|_{C_{k-i+1}^k}\subset\mathscr{G}_i|_{C_{k-i+1}^k}\quad\text{and }\mathscr{G}_i|_{D_i^k}\subset\mathscr{F}_{k-i+1}|_{D_i^k} \tag{3}$$

hence

$$A(S_k=k-i+1)=A(T_k=i)=AD_i^k+AC_{k-i+1}^k\in\mathscr{F}_{k-i+1}\cap\mathscr{G}_i\Rightarrow A\in\mathscr{F}_{S_k}\cap\mathscr{G}_{T_k}.$$

Conversely

$$\begin{aligned}A\in\mathscr{F}_{S_k}\cap\mathscr{G}_{T_k}\Rightarrow A(S_k=k-i+1)=A(T_k=i)&\in\mathscr{F}_{k-i+1}\cap\mathscr{G}_i|_{(T_k=i)}\\&=\mathscr{F}_{k-i+1}\cap\mathscr{G}_i|_{D_i^k+C_{k-i+1}^k}\\&=\mathscr{F}_{k-i+1}|_{C_{k-i+1}^k}+\mathscr{G}_i|_{D_i^k}\end{aligned}$$

(for the last equality to use (3)). Therefore $AD_i^k\in\mathscr{G}_i$ and

$$AC_{k-i+1}^k\in\mathscr{F}_{k-i+1}\Rightarrow A\in\mathscr{H}_k.$$

We check that $T_k\leqq T_{k+1}$. Remark that $(T_k\leqq i)=A_{k-i,i}$ and that for every $i\leqq k$ we have:

$$\begin{aligned}(T_{k+1}<i)(T_k=i)&=(T_{k+1}\leqq i-1)(T_k=i)\\&=A_{k+1-(i-1),i-1}(A_{k-i,i}-A_{k-i+1,i-1})\\&\subset A_{k-i+2,i-1}-A_{k-i+1,i-1}=\emptyset\end{aligned}$$

hence $T_k\leqq T_{k+1}$. Taking into account that $(S_k\leqq i)=A_{i,k-i}^c$ one verifies in the same way that $S_k\leqq S_{k+1}$ for every k. Thus, $(\mathscr{H}_k)_k$ is a filtration.

Moreover we have the following relations:

$$\mathscr{H}_\infty|_{A_{k,\infty}^c}=\mathscr{F}|_{A_{k,\infty}^c}=\mathscr{F}_k|_{A_{k,\infty}^c}\quad\text{and }\mathscr{H}_\infty|_{A_{\infty,j}}=\mathscr{F}|_{A_{\infty,j}}=\mathscr{G}_j|_{A_{\infty,j}}. \tag{4}$$

We shall only check the first set of relations. As

$$A_{k,n}^c=\sum_{i=0}^{k-1}(D_{n+i}^{n+k-1}+C_{k-i-1}^{n+k-1})$$

and

$$\mathscr{H}_{n+k-1}|_{A_{k,n}^c}=\sum_{i=0}^{k-1}(\mathscr{G}_{n+i}|_{D_{n+i}^{n+k-1}}+\mathscr{F}_{k-i-1}|_{C_{k-i-1}^{n+k-1}})$$

it follows that $\mathscr{H}_{n+k-1}|_{A_{k,n}^c}\supset\mathscr{G}_n|_{A_{k,n}^c}$. (To see the last inclusion remark that because $C_{k-i}^{n+k-1}=A_{k-i,n+i-1}-A_{k-i,n+i}$ we have $\mathscr{F}_{k-i}|_{C_{k-i}^{n+k-1}}\supset\mathscr{G}_{n+i}|_{C_{k-i}^{n+k-1}}\supset\mathscr{G}_n|_{C_{k-i}^{n+k-1}}$. As $A_{k,\infty}^c\subset A_{k,n}^c$ for every n it results that $\mathscr{H}_{n+k-1}|_{A_{k,\infty}^c}\supset\mathscr{G}_n|_{A_{k,\infty}^c}$ for every k hence $\mathscr{H}_\infty|_{A_{k,\infty}^c}\supset\mathscr{G}_n|_{A_{k,\infty}^c}$. Therefore $\mathscr{H}_\infty|_{A_{k,\infty}^c}\supset\mathscr{G}_\infty|_{A_{k,\infty}^c}=\mathscr{F}|_{A_{k,\infty}^c}$; the other inclusion be- obvious it follows that $\mathscr{H}_\infty|_{A_{k,\infty}^c}=\mathscr{F}|_{A_{k,\infty}^c}$. On the other hand, $\mathscr{F}_k|_{A_{k,\infty}^c}\supset\mathscr{G}_n|_{A_{k,\infty}^c}$ for every n. Thus $\mathscr{F}_k|_{A_{k,\infty}^c}=\mathscr{F}|_{A_{k,\infty}^c}$.

Now we shall construct the two sequences of stopping-times σ_k, τ_k with respect to the filtration $(\mathscr{H}_k)_k$. We define

$$\begin{aligned}\sigma_k &= \sum_{n=0}^{\infty} (k+n)\, 1_{C_k^{n+k}} + \infty 1_{A^c_{k,\infty}} \\ \tau_k &= \sum_{n=0}^{\infty} (k+n)\, 1_{D_k^{n+k}} + \infty 1_{A_{\infty,k}}\end{aligned} \qquad (5)$$

It is not hard to prove that the sets $(C_k^{n+k})_{n\geqq 0}$ and $A^c_{k,\infty}$ as well as the sets $(D_k^{n+k})_{n>0}$ and $A_{\infty,k}$ form partitions of Ω for every $k\geqq 1$ and that σ_k and τ_k are indeed stopping times with respect to $(\mathscr{H}_k)_k$. Moreover the following relations hold for every positive integers k, n:

$$(\sigma_k \leqq k+n) = A_{k,n+1} \quad \text{and} \quad (\tau_k \leqq k+n) = A^c_{n+1,k}. \qquad (6)$$

Therefore we have:

$$(\sigma_{k+1} \leqq k+n)\,(\sigma_k = k+n) = A^c_{n,k+1}(A_{k,n+1} - A_{k,n}) = \emptyset$$

which further implies that $(\sigma_k)_k$ is a strictly increasing sequence of stopping-times. The same thing is valid for the sequence $(\tau_k)_k$.

It remains only to check that $\mathscr{H}_{\sigma_k} = \mathscr{F}_k$ and $\mathscr{H}_{\tau_k} = \mathscr{G}_k$. In fact, it results:

$$\begin{aligned}\mathscr{H}_{\sigma_k} &= \sum_{k=0}^{\infty} \mathscr{H}_{k+n}|_{(\sigma_k = k+n)} + \mathscr{H}_{\infty}|_{(\sigma_k=\infty)} = \sum_{n=0}^{\infty} \mathscr{H}_{k+n}|_{C_k^{k+n}} + \mathscr{H}_{\infty}|_{A^c_{k,\infty}} \\ &= \sum_{n=0}^{\infty} \mathscr{F}_k|_{C_k^{k+n}} + \mathscr{F}_k|_{A^c_{k,\infty}} = \mathscr{F}_k.\end{aligned}$$

As about the second equality, the proof is the same. The checking of the point (iii) is a matter of easy calculus.

The proof of the theorem is complete.

Consider now the continuous case. First establish the following result:

Lemma 7. *Let $(\Omega, \mathscr{K}, P, (\mathscr{F}_t)_{t\geqq 0})$ be a standard filtration.*

(i) Let $(A_t)_{t\geqq 0}$ be an adapted family of sets satisfying the assumptions: $s<t \Rightarrow A_s \subset A_t$ (everywhere) and $A_t = \bigcap_{t'>t} A_{t'}$. Then there exists a stopping-time τ such that $(\tau \leqq t) = A_t$ for every t.

(ii) Let $(A_t)_{t\geqq 0}$ be an adapted family of sets satisfying the assumption $s<t \Rightarrow A_s \subset A_t$ (a.s.). Set $A_{t^+} = \operatorname{ess\,inf}_{t'>t} A_{t'} = \bigcap_{Q\ni q>t} A_q$. Then there exists a stopping-time τ such that $(\tau \leqq t) = A_{t^+}$ for every $t \geqq 0$.

Proof. (i) Define $\tau(\omega) = \begin{cases} \inf\{t \geqq 0/\omega \in A_t\} \\ \infty \quad \text{if } \omega \notin A_\infty \end{cases}$. Then $\tau(\omega) \leqq t \Leftrightarrow \omega \in A_{t'}$ for every $t'>t$ $\Rightarrow (\tau \leqq t) = \bigcap_{t'>t} A_{t'} = A_t$.

(ii) The sets A_{t^+} satisfy the assumptions from (i).

Theorem 8. *Let $(\Omega, \mathscr{K}, P)$ be a complete probability space. Let $(\mathscr{F}_s)_{s\geqq 0}$ and $(\mathscr{G}_t)_{t\geqq 0}$ be two locally comparable standard filtrations such that $\mathscr{F}_\infty = \mathscr{G}_\infty = \mathscr{F}$.*

Then there exists a standard filtration $(\mathscr{H}_t)_{t\geqq 0}$ *and two families* $(\sigma_s)_{s\geqq 0}$ *and* $(\tau_t)_{t\geqq 0}$ *of stopping-times with respect to* $(\mathscr{H}_t)_{t\geqq 0}$ *having the properties:*

(i) $s<t \Rightarrow \sigma_s<\sigma_t, \tau_s<\tau_t$ and $\sigma_\infty=\tau_\infty=\infty$.

(ii) $\mathscr{H}_{\sigma_s}=\mathscr{F}_s, \mathscr{H}_{\tau_t}=\mathscr{G}_t$,

(iii) *Set* $\sigma_{s^+}=\inf_{Q\ni q>s}\sigma_q, \tau_{t^+}=\inf_{Q\ni q>t}\tau_q$. Then $\sigma_s=\sigma_{s^+}, \tau_t=\tau_{t^+}$ for $s,t\geqq 0$.

Proof. Let $A_{s,t}$ be the sets of local comparability given in Proposition 4(ii). So $\mathscr{F}_s|_{A_{s,t}}\subset\mathscr{G}_t|_{A_{s,t}}$ and $\mathscr{F}_s|_{A^c_{s,t}}\supset\mathscr{G}_t|_{A^c_{s,t}}$. Take $t>0$ and set $B_t(s)=A_{t-s,s}$. Using the order properties of the sets $A_{s,t}$ and the right continuity of the filtrations it is easy to see that $B_t(s^+)=\bigcap_{n=1}^{\infty}B_t(s+1/n)=A_{(t-s)^-,s}\in\mathscr{F}_{t-s}\cap\mathscr{G}_s$. Using Lemma 7 we define a stopping-time T_t with respect to $(\mathscr{G}_s)_s$ such that $(T_t\leqq s)=B_t(s^+)$ for $0<s<t$, $(T_t=0)=A_{t^-,0}$ and $T_t|_{A^c_{0^+,t^-}}=t$. Set also $S_t=t-T_t$. Then $(S_t<s)=A^c_{s^-,t-s}\in\mathscr{F}_s$ for every $s\leqq t$. Due to the right-continuity of the filtration $(\mathscr{F}_s)_s$ it follows that S_t is a stopping-time with respect to $(\mathscr{F}_s)_s$. Set

$$\mathscr{H}_t=\mathscr{F}_{S_t}\cap\mathscr{G}_{T_t} \tag{7}$$

and remark that:

1. $S_t\leqq t$, $T_t\leqq t$;
2. $t_1\leqq t_2\Rightarrow S_{t_1}\leqq S_{t_2}, T_{t_1}\leqq T_{t_2}$;
3. $h>0\Rightarrow(T_t\leqq s)\subset(T_{t+h}\leqq s+h)$ and $(S_t<s)\subset(S_{t+h}<s+h)$;
4. $T_{t+h}-T_t\leqq h$, $S_{t+h}-S_t\leqq h$ for every $h>0$;
5. $\mathscr{H}_{t^+}=\mathscr{H}_t$

The proof is easy. We check only 3. which, in fact, is the key of the sequel:

$$(T_{t+h}\leqq s+h)=A_{(t+h-s-h)^-,s+h}=A_{(t-s)^-,s+h}\supset A_{(t-s)^-,s}=(T_t\leqq s)$$

and

$$(S_{t+h}<s+h)=A^c_{(s+h)^-,t-s}\supset A^c_{s^-,t-s}=(S_t<s).$$

Now we shall define three families of stopping-times with respect to $(\mathscr{H}_t)_t$ denoted by $\sigma'_t, \sigma_t, \tau_t$ as follows:

$$(\sigma'_t\leqq t+s)=(T_{t+s}\leqq s)=(S_{t+s}\geqq t)=A_{t^-,s}\in\mathscr{F}_{S_{t+s}}\cap\mathscr{G}_{T_{t+s}}=\mathscr{H}_{t+s},$$
$$(\sigma'_t=t)=A_{t^-,0},\qquad(\sigma'_t=\infty)=A^c_{t^-,\infty}.$$

Then σ'_t is an increasing family of stopping times with respect to $(\mathscr{H}_t)_t$ and $\sigma'_t\geqq t$. Further set

$$\sigma_t=\operatorname*{ess\,inf}_{t'>t}\sigma'_{t'}=\inf_n\sigma'_{t+1/n}=\sigma'_{t^+}$$

and define τ_t by the relations

$$(\tau_t<t+s)=(T_{t+s}>t)=(S_{t+s}<s)=A^c_{s^-,t}\in\mathscr{H}_{t+s}$$
$$(\tau_t=\infty)=A_{\infty,t}\quad\text{ and }(\tau_t=t)=A^c_{0,t}.$$

(The last definition is good due to the fact that the family of sets $A^c_{s^-,t}$ is left-continuous in s.

First check that

$$\mathscr{F}_t \supset \mathscr{H}_{\sigma'_t} \supset \mathscr{F}_{t-}. \tag{8}$$

Let $A \in \mathscr{H}_{\sigma'_t}$. Then $A(\sigma'_t \leqq t+s) \in \mathscr{H}_{t+s}$ for every non-negative s. Therefore

$$A(\sigma'_t \leqq t+s)(\sigma'_t > t+s-h) = A(A_{t^-,s} - A_{t^-,s-h}) \in \mathscr{H}_{t+s} \subset \mathscr{F}_{S_{t+s}}$$

and, moreover, taking into account the remark 3. it follows that

$$A(\sigma'_t \leqq t+s)(\sigma'_t > t+s-h) = A(S_{t+s} \geqq t)(S_{t+s-h} < t)(S_{t+s} < t+h) \in \mathscr{F}_{t+h}.$$

On the other hand $A(\sigma'_t = t) = A A_{t^-,0} \in \mathscr{H}_t \subset \mathscr{F}_t$ and

$$A(\sigma'_t = \infty) = A A^c_{t^-,\infty} \in \mathscr{G}_\infty|_{A^c_{t^-,\infty}} \subset \mathscr{F}_t|_{A^c_{t^-,\infty}}$$

due to local comparability. Now partition the set A as follows:

$$A = A A_{t^-,0} + \sum_{i=0}^{\infty} A(A_{t^-,(i+1)/n} - A_{t^-,i/n}) + A A^c_{t^-,\infty} \in \mathscr{F}_{t+1/n}.$$

As n is arbitrary and $(\mathscr{F}_s)_s$ is right continuous the first inclusion from (8) follows. As about the second, one must check that $A \in \mathscr{F}_{t^-} \Rightarrow A(\sigma'_t \leqq t+s) \in \mathscr{H}_{t+s}$ for every $s>0$; or, otherwise written, that $A(T_{t+s} \leqq s) \in \mathscr{F}_{S_{t+s}} \cap \mathscr{G}_{T_{t+s}}$. But the last relation means that $A(S_{s+t} \geqq t)(S_{s+t} < u) \in \mathscr{F}_u$ for every nonnegative u and $A(T_{t+s} \leqq s)(T_{t+s} \leqq v) \in \mathscr{G}_v$ for every $v>0$. Only the second statement needs a proof. If $s \leqq v$ the second set becomes

$$A(T_{t+s} \leqq s) = A(S_{t+s} \geqq t)^c \in \mathscr{F}_{t^-}|_{A_{t^-,s}} \subset \mathscr{G}_s|_{A_{t^-,s}} \subset \mathscr{G}_v|_{A_{t^-,s}}$$

(here we used the local comparability). If $s>v$ then $t<t+s-v$ and

$$A(T_{t+s} \leqq s)(T_{t+s} \leqq v) = A(T_{t+s} \leqq v) = A A_{(t+s-v)^-,v}$$

and the last set belongs to $\mathscr{F}_{(t+s-v)^-}|_{A_{(t+s-v)^-,v}} \subset \mathscr{G}_v|_{A_{(t+s-v)^-,v}}$. Now (8) follows.

An immediate consequence of (8) is that $\mathscr{H}_{\sigma_s} = \mathscr{F}_s$.

Moreover, we claim that

$$\mathscr{H}_{\tau_t} = \mathscr{G}_t. \tag{9}$$

The proof follows the same way; we shall only sketch it.

"$\subset$": $A \in \mathscr{H}_{\tau_t} \Rightarrow A(\tau_t < t+s)(\tau_t \geqq t+s-h) \in \mathscr{G}_{t+h}$ for every $h>0$ and

$$A(\tau_t = t) = A A^c_{0,t} \in \mathscr{H}_t \subset \mathscr{G}_t, \qquad A(\tau_t = \infty) = A A_{\infty,t} \in \mathscr{H}_\infty|_{A_{\infty,t}} \subset \mathscr{G}_t|_{A_{\infty,t}};$$

then partition the set A in a similar manner as above.

"$\supset$": Let $A \in \mathscr{G}_t$. The problem is if $A(\tau_t < t+s) \in \mathscr{H}_{t+s} = \mathscr{F}_{S_{t+s}} \cap \mathscr{G}_{T_{t+s}}$ for $s>0$. But

$$\begin{aligned} &A(S_{t+s} < s)(S_{t+s} < u) \\ &= \begin{cases} A A^c_{s^-,t} & \text{for } s \leqq u \in \mathscr{G}_t|_{A^c_{s^-,t}} \subset \mathscr{F}_s|_{A^c_{s^-,t}} \subset \mathscr{F}_u|_{A^c_{s^-,t}} \\ A A^c_{u^-,t+s-u} & \text{for } s > u \in \mathscr{G}_{t+s-u}|_{A^c_{u^-,t+s-u}} \subset \mathscr{F}_u|_{A_{u^-,t+s-u}} \end{cases} \end{aligned}$$

and $A(T_{t+s} > t)(T_{t+s} \leqq u) \in \mathscr{G}_u$ for every u.

As for the right-continuity of τ_t it is enough to remark that

$$(\tau_{t^+}<t+s)=(\inf_n \tau_{t+1/n}<t+s)=\bigcup_n (\tau_{t+1/n}<t+s)$$
$$=\bigcup_n A^c_{s^-,t+1/n}=\bigcap_n (A_{s^-,t+1/n})^c=A^c_{s^-,t}=(\tau_t<t+s)$$

for every s; hence $\tau_t=\tau_{t^+}$ and we are done. QED

Corollary 9. *Let $(\Omega, \mathscr{K}, P, (\mathscr{F}_z)_{z\in T})$ be a standard filtration. Then*

(F5)$\Leftrightarrow$ *There exists a standard filtration $(\mathscr{H}_u)_{u\geqq 0}$ and two families $(\sigma_s)_s$ and $(\tau_t)_t$ of stopping-times with respect to $(\mathscr{H}_u)_u$ which are right-continuous such that $\mathscr{F}_z=\mathscr{H}_{\sigma_s\wedge\tau_t}$ for every $z=(s,t)$.*

Remark. There exist examples that point out that there is no unicity in choosing the filtration $(\mathscr{H}_u)_u$ and the two families of stopping-times. Anyhow, the set $(\sigma_s\leqq\tau_t)$ is included in $A_{s,t}$. The ones just constructed above satisfy the relations:

$$(\sigma_s\leqq s+t)=(\tau_t>s+t)=(\sigma_s<\tau_t)=(\sigma_s\leqq\tau_t)=A_{s^+,t}. \tag{11}$$

§ 4. Some Regularity Properties of the Filtrations Having (F5)

Let $(\Omega, \mathscr{K}, P, (\mathscr{H}_u)_{u\geqq 0})$ be a standard filtration, $(\sigma_s)_s$ and $(\tau_t)_t$ be two increasing right-continuous families of stopping-times such that $\sigma_\infty=\tau_\infty=\infty$. The right-continuity will be supposed to occur everywhere (if not, it is a matter of routine to find such good versions for the two families of increasing stopping-times). Set

$$T_z(\omega)=\sigma_s(\omega)\wedge\tau_t(\omega) \quad \text{and } \mathscr{F}_z=\mathscr{H}_{T_z}. \tag{1}$$

If G is a separation, let $T_G=\sup_{z\in G} T_z=\sup_{z\in R_G} T_z$. Let Z denote the set of integer numbers. We say that G is a simple separation iff there exists an interval $I\subset Z$ (i.e. $I=I'\cap Z$, I' being an interval of real numbers) and there exist some points $(z_i)_{i\in I}$ satisfying the asumptions that $i<j\Rightarrow z_i>z_j$ and $G=\bigvee_{i\in I} \underline{z}_i$. Admit the convention that if $s_i<t_i$ (i.e. z_i stays above the bisector of T), then $i<0$; otherwise $i\geqq 0$.

It is obvious that for every separation G there exists at least a sequence G_n of simple separations such that $G_n\searrow G$.

If G is an arbitrary separation, we shall also denote by ess T_G the stopping-time $\operatorname{ess\,sup}_{z\in G} T_z=\operatorname{ess\,sup}_{z\in R_G} T_z$.

Proposition 10. 1. $G<G'\Rightarrow T_G\leqq T_{G'}$.

2. *If G is a simple separation, $G=\bigvee_{i\in I} \underline{z}_i$ then $T_G=\text{ess } T_G=\sup_{i\in I} T_{z_i}$.*

3. *Set $A(\omega)=\{z\in T/\sigma_s(\omega)<\tau_t(\omega)\}$. Then $\partial A(\omega)$ is an increasing path with respect to the order "$\leqq$".*

4. *Let $z(\omega)=\sup(\partial A(\omega)\cap G)$. Then $T_G(\omega)=T_{z(\omega)}(\omega)$.*

5. *$G_n\searrow G\Rightarrow T_{G_n}\searrow T_G$; it follows that T_G is a stopping-time.*

6. *If G is a simple separation, then* $\mathcal{H}_{T_G} = \mathcal{F}_G$.

7. *For an arbitrary separation G, the following relations hold:*

$$\mathcal{F}_G \subset \mathcal{H}_{\text{ess}\, T_G} \subset \mathcal{H}_{T_G} = \bigcap_{\substack{G' > G \\ G' \text{simple}}} \mathcal{F}_{G'} = \bigcap_{G' > G} \mathcal{F}_{G'} = \mathcal{F}_{G^+}.$$

Proof. 2. If $z_i \succ z \succ z_{i+1}$ and $z \in G$, then $T_z \leqq T_{z_i} \vee T_{z_{i+1}}$.

3. $\partial A(\omega)$ is a totally ordered set. Indeed, suppose ad absurdum that there exist two points z, z' belonging to $\partial A(\omega)$ and $s < s'$, $t > t'$. Take an h small enough and $z_1, z_1' \in A(\omega)$, $z_2, z_2' \in A(\omega)^c$ such that $|z_1 - z| < h$, $|z_2 - z| < h$, $|z_1' - z'| < h$, $|z_2' - z'| < h$, $t_2 > t_1'$, $s_2 < s_1'$. Then $\sigma_{s_2}(\omega) \geqq \tau_{t_2}(\omega) \geqq \tau_{t_t'}(\omega) > \sigma_{s_1'}(\omega)$ contradicts the fact that $\sigma(\omega)$ is increasing.

On the other hand, $\partial A(\omega)$ is connected. Otherwise there exists a positive integer n such that $\partial A(\omega) \cap R_{n,n}$ be not connected. It follows that $\partial A(\omega) \cap R_{n,n} = K_1 \cup K_2$ with K_1, K_2 two disjoint compact sets. Then there exist two points $z_1 \in K_1$, $z_2 \in K_2$ such that $|z_1 - z_2| = \inf\{|z - z'| / z \in K_1, z' \in K_2$. Choose $z_1 \leqq z_2$. Then the rectangle $[z_1, z_2]$ also contains other points $z \in \partial A(\omega)$ which cannot belong neither to K_1 nor to K_2. But a connected totally ordered set is a path.

4. Remark that $z \in \partial A(\omega) \Rightarrow (s - 1/n, t + 1/n) \in A(\omega)$, $(s + 1/n, t - 1/n) \in A(\omega)^c$ hence

$$\sigma_{s^-}(\omega) \leqq \tau_{t^+}(\omega) = \tau_t(\omega)$$

and

$$\sigma_s(\omega) \geqq \tau_{t^-}(\omega) \Rightarrow \sigma_{s^-}(\omega) \vee \tau_{t^-}(\omega) \leqq T_z(\omega). \tag{2}$$

Let $z_0 = \inf(G \cap \partial A(\omega))$ and $z(\omega) = \sup(G \cap \partial A(\omega))$. Then

$$\begin{aligned} T_G(\omega) &= \sup_{z \in A(\omega) \cap G} T_z(\omega) \vee \sup_{z \in \partial A(\omega) \cap G} T_z(\omega) \vee \sup_{z \in G \cap (\overline{A(\omega)})^c} T_z(\omega) \\ &= \sigma_{s_0^-}(\omega) \vee T_{z(\omega)}(\omega) \vee \tau_{t_0^-}(\omega) = T_{z(\omega)}(\omega) \end{aligned}$$

due to (2). (Here some conventions are obvious: $G \cap A(\omega) = \emptyset \Rightarrow z_0 = z(\omega) = \infty$ and $s(\omega)$ or $t(\omega) = \infty \Rightarrow T_{z(\omega)}(\omega) = \infty$ for reasons of right-continuity of $\sigma(\omega)$ and $\tau(\omega)$).

5. $G_n \searrow G \Leftrightarrow R_G = \bigcap_n R_{G_n} \Rightarrow R_G \cap \partial A(\omega) = \bigcap_n (R_{G_n} \cap \partial A(\omega)) \Rightarrow \sup R_{G_n} \cap \partial A(\omega)$ converges to $\sup R_G \cap \partial A(\omega)$ due to the compacity and the total ordering of $\partial A(\omega)$. Thus $T_{G_n} \searrow T_G$. If G_n are simple separations, then T_{G_n} are indeed stopping-times due to the point 2. Every separation can be approximated from above with simple ones and it follows that T_G is a stopping-time being limit of stopping-times.

6. If τ_n are stopping-times and $\tau = \sup \tau_n$ has the property that for every ω there exists an $n(\omega)$ such that $\tau(\omega) = \tau_{n(\omega)}(\omega)$, then $\mathcal{H}_\tau = \bigvee_{n=1}^{\infty} \mathcal{H}_{\tau_n}$ (this is obvious: $A \in \mathcal{H}_\tau \Rightarrow A = \bigcup_{n=1}^{\infty} A(\tau = \tau_n) \in \bigvee_n \mathcal{H}_{\tau_n}$). In our case G is simple: $G = \bigvee_{i \in I} \underline{z_i} \Rightarrow T_G = \bigvee_{i \in I} T_{z_i}$ because of the point 2. and using the step 4., the supremum is attained for every ω. This fact implies the equalities

$$\mathcal{H}_{T_G} = \mathcal{H}_{\bigvee_{i \in I} T_{z_i}} = \bigvee_{i \in I} \mathcal{H}_{T_{z_i}} = \bigvee_{i \in I} \mathcal{F}_{z_i} = \bigvee_{z \in G} \mathcal{F}_z = \mathcal{F}_G.$$

7. Because $T_z \leqq \text{ess}\, T_G$, the first two inclusions are obvious. The first equality results from the fact that for a simple separation G' the equality $\mathscr{H}_{T_{G'}} = \mathscr{F}_{G'}$ holds and from the remark that if τ_n, τ are stopping-times and $\tau_n \searrow \tau$, then $\mathscr{H}_{\tau_n} \searrow \mathscr{H}_\tau$.

As for the second equality, we must only see that for every two separations $G_1 < G_2$, there exists a simple one, G', having the property that $G_1 < G' < G_2$.

The theorem is completely proved.

In the study of two-parameter filtrations, the following question is of interest: If $(\mathscr{F}_z)_{z\in T}$ is a standard filtration, and G_n, G are separations satisfying the assumption $G_n \searrow G$, does it result that $\mathscr{F}_{G_n} \searrow \mathscr{F}_G$? It is known that, in general, the answer is negative. But if we suppose that the filtration has (F4), we saw that the marginal filtrations $(\mathscr{F}_{s,\infty})_s$, $(\mathscr{G}_{\infty,t})_t$ remain right-continuous. Nevertheless, (F4) is not enough to assure the right-continuity of the filtration $\mathscr{F}_G$ considered upon all the separations of T. In fact, the answer to the question is negative even if the filtration has (F5). Indeed, all we can say is that $\mathscr{F}_{G_n} \searrow \mathscr{F}_{G^+}$ which has no reasons to be the same with $\mathscr{F}_G$. (If G is a simple separation, Proposition 10,7., points out that in this case $\mathscr{F}_G = \mathscr{F}_{G^+}$).

Counter-Example 11. Let $\Omega = [0,1]^2$; the points of Ω will be denoted by $x = (x_1, x_2)$. Let $\mathscr{B}$ be the σ-algebra of the Borel sets on $[0,1]$ and P be the Lebesgue measure on Ω. Let $(\mathscr{H}'_t)_{t\geqq 0}$ be a filtration on $[0,1]$ so that $\mathscr{H}'_0 \neq \mathscr{H}'_1$ and set $\mathscr{H}_t = \mathscr{B} \otimes \mathscr{H}'_t$ completed with respect to P. Let

$$\sigma_s(x) = \begin{cases} 1_{[0,s]}(x_1) & \text{if } s<1 \\ \infty & \text{if } s \geqq 1 \end{cases}$$

and

$$\tau_t(x) = \begin{cases} 1_{[1-t,1]}(x_1) & \text{if } t<1 \\ \infty & \text{if } t \geqq 1 \end{cases}.$$

One checks immediately that σ_s and τ_t are stopping-times (in fact they are $\mathscr{H}_0$-measurable) and that they are increasing in s (respectively in t) and also that they are right-continuous. Besides

$$T_z = T_{s,t} = \begin{cases} 1_{[1-t,s]}(x_1) & \text{if } s<1, t<1 \\ \sigma_s & \text{if } s<1, t\geqq 1 \\ \tau_t & \text{if } s\geqq 1, t<1 \\ \infty & \text{if } s\geqq 1, t\geqq 1 \end{cases}$$

and $T_{s,1-s} = 1_{[s]}(x_1) = 0 (\text{mod}\, P)$.

Then the filtration $\mathscr{F}_z = \mathscr{H}_{T_z}$ has (F5) (hence it has (F4)). Let

$$G = \{(s, 1-s)/0 \leqq s \leqq 1\}.$$

As $\mathscr{F}_{s,1-s} = \mathscr{H}_0$ for energy $0 \leqq s \leqq 1$ it results that $\mathscr{F}_G = \mathscr{H}_0$. But $T_G = 1$ implies that $\mathscr{F}_{G^+} = \mathscr{H}_{T_G} = \mathscr{H}_1$.

Remark. This example points out that if one wants to have good continuity properties for the filtration $(\mathscr{F}_G)_G$, $\mathscr{F}_G$ must be replaced with $\mathscr{F}_{G^+}$. See [2] in this respect.

§ 5. Properties of Martingales with Respect to Filtrations Having (F5)

Let $(\mathscr{H}_t)_{t\geqq 0}$ be a standard filtration and σ_s, τ_t be two increasing right-continuous families of stopping-times with respect to it, satisfying the assumptions $\sigma_\infty=\tau_\infty=\infty$. Let $(\mathscr{F}_z)_{z\in T}$ be the two-parameter filtration given by the relation $\mathscr{F}_z=\mathscr{H}_{\sigma_s}\cap\mathscr{H}_{\tau_t}=\mathscr{H}_{\sigma_s\cap\tau_t}$. We know now that all the filtrations having (F5) can be represented in this manner.

Let $(x_z, \mathscr{F}_z)_{z\in T}$ be a L^1-bounded martingale (hence a strong one) and $x=x_{\infty,\infty}$. Let y_t be a right-continuous left-limited version for the martingale $E(x/H_t)$. Then, using the optional sampling theorem, we have the representation

$$x_z=E(x/\mathscr{F}_z)=y_{\sigma_s\wedge\tau_t}=y_{\sigma_s}1_{(\sigma_s\leqq\tau_t)}+y_{\tau_t}1_{(\sigma_s>\tau_t)}.$$

Therefore x_z is a right-continuous left-limited version of the above martingale. Walsh proved in [5] the existence of such a version in the general case; but in our case the fact is obvious.

A process x_z is called a weak martingale if it is adapted and $E_z(x(z,z']))$ is equal to zero for every $z<z'$. An adapted process A_z is called increasing if $A((z,z'])\geqq 0$ for every $z<z'$.

Cairoli and Walsh proved in [1] that for every L^2-bounded martingale there exists an increasing process A'_z such that $(x^2-A_z, F_z)_z$ be a null-meaned weak martingale; moreover, if x_z is a strong martingale, there exists another increasing process A_z having the property that it is previsible and $(x_z^2-A_z,\mathscr{F}_z)$ is a null-meaned martingale (in our case even a strong one).

In the case when the filtration has (F5) an example of A may be easy computed. It is not necessary previsible, but it has a good property also posessed by the martingale x_z itself: namely, the measure $A_\omega(dz)$ is concentrated on a totally ordered set.

The process refered to is $A_z=\langle y\rangle_{\sigma_s\wedge\tau_t}$, where $\langle y\rangle$ signifies the natural increasing process attached to the martingale y. It is easy to check that A is indeed an increasing process and that it is concentrated on the intersection of the borders of the sets $(\sigma_s\leqq\tau_t)$ and $(\sigma_s>\tau_t)$.

Finally we want to give an example which points out that, unlike the situation in the one-parameter case, a martingale may have the property that A' is equal to zero. Or, otherwise speaking, that it is possible to exist martingales x_z with the property that x_z^2 is a weak martingale.

Example. Let $(\Omega,\mathscr{K},P)$ be a complete probability space and $\mathscr{G}_s,\mathscr{H}_t$ be two standard filtrations such that $\mathscr{G}_\infty=\mathscr{H}_\infty=\mathscr{K}$. Let A be a set belonging to $\mathscr{G}_0\cap\mathscr{H}_0$ chosen to satisfy $0<P(A)<1$. Let B be A^c. Set $\mathscr{F}_z=\mathscr{G}_s|_A+\mathscr{H}_t|_B$. Now $(\mathscr{F}_z)_z$ has of course (F5) and $E_z(f)=E(f/G_s)1_A+E(f/H_t)1_B$. Every L^1-bounded martingale $x_z=E_z(f)$ has the form $x_z=y_s 1_A+u_t 1_B$ with $y_s=E(f/\mathscr{G}_s)$ and $u_t=E(f/\mathscr{H}_t)$; hence $x_z^2=y_s^2 1_A+u_t^2 1_B$ has the property the $x^2((z,z'])=0$ for every $z<z'$. In other words, x^2 is a weak martingale and so, $A'=0$; the strong natural process is $A_z=\langle y\rangle_s 1_A+\langle u\rangle_t 1_B$.

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