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Multiple Points of a Gaussian Vector Field

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Introduction

In this note the methods of Cuzick (1978) are used to extend the results of Kôno (1978) who studied the existence of double points for a special class of Gaussian fields, and Goldman (1981) who studied higher order multiplicities for this class of processes. Let $\mathbf{X}(\mathbf{t})$ be a mean zero Gaussian vector field with domain $\mathbb{R}^N$ (or $\mathbb{R}^N_+$) and taking values in $\mathbb{R}^d$, written briefly as an (N, d) field. We say $\mathbf{z}$ is a point of multiplicity n for $\mathbf{X}$ on the set $B \subseteq \mathbb{R}^N$ if there exist distinct (random) times $\mathbf{t}_j \in B$, $j=1, \ldots, n$ such that $\mathbf{X}(\mathbf{t}_1)=\mathbf{X}(\mathbf{t}_2)=\ldots=\mathbf{X}(\mathbf{t}_n)=\mathbf{z}$. If we let $\mathbf{T}=(\mathbf{t}_1, \ldots, \mathbf{t}_n)$ and define $\mathbf{Y}(\mathbf{T})$ as the vector $(\mathbf{X}(\mathbf{t}_1)-\mathbf{X}(\mathbf{t}_2), \ldots, \mathbf{X}(\mathbf{t}_{n-1}) -\mathbf{X}(\mathbf{t}_n))$ then $\mathbf{Y}$ is an $(nN, (n-1)d)$ field and $\mathbf{X}$ has points of multiplicity n on the set B iff $\mathbf{Y}$ hits the origin for some $\mathbf{T} \in B^n \cap \{\mathbf{t}_j \neq \mathbf{t}_k, j \neq k\}$. This latter phenomenon has been studied in [2] and here the results are applied to the present problem. For the cases studied by Kôno (1978) and Goldman (1981), where the coordinate processes are independent with incremental variance $\operatorname{Var}(\mathbf{X}(\mathbf{s})-\mathbf{X}(\mathbf{t}))=|\mathbf{t}-\mathbf{s}|^{2\alpha}$, $0<\alpha<1$, it was shown that $\mathbf{X}$ has points of multiplicity n if $d\alpha < \frac{n}{n-1} N$, but not when $d\alpha > \frac{n}{n-1} N$. More general results require a few definitions.

Notation, Definitions and Statement of Results

Let $|\cdot|$ denote Euclidean distance. Denote the components of the centred Gaussian field $\mathbf{X}$ by X_i, so that $\mathbf{X}(\mathbf{t})=(X_1(\mathbf{t}), \ldots, X_d(\mathbf{t}))$. Although it is not essential for many of the results, to simplify notation we shall always assume that $\mathbf{X}$ has *homogeneous increments*, i.e. for any $k \geqq 2$ the distribution of

$$(\mathbf{X}(\mathbf{t}_2+\mathbf{t})-\mathbf{X}(\mathbf{t}_1+\mathbf{t}), \ldots, \mathbf{X}(\mathbf{t}_k+\mathbf{t})-\mathbf{X}(\mathbf{t}_{k-1}+\mathbf{t}))$$

does not depend on $\mathbf{t}$. Denote the incremental variances of the coordinate processes by $\sigma_i^2(\mathbf{t}-\mathbf{s})=E(X_i(\mathbf{t})-X_i(\mathbf{s}))^2$ and let $\det \operatorname{Cov}(\mathbf{Z})$ be the determinant

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of the covariance matrix of the vector of random variables $\mathbf{Z}$. We say that $\mathbf{X}$ has *approximately independent components* on B if there exists an $\varepsilon>0$ such that

$$\det \operatorname{Cov}(\mathbf{X}(\mathbf{t})-\mathbf{X}(\mathbf{s})) \geqq \varepsilon \prod_{i=1}^{d} \sigma_i^2(\mathbf{t}-\mathbf{s})$$

for all $(\mathbf{s},\mathbf{t})\in B\times B$. If B is replaced by some neighbourhood of the diagonal $\{\mathbf{s}=\mathbf{t}\}$, then $\mathbf{X}$ is said to have *locally approximately independent components.* The coordinate X_i is said to have *index* α_i if

$$\begin{aligned}\alpha_i &= \sup\{\alpha\colon \limsup_{|\mathbf{t}|\to 0} |\mathbf{t}|^{-\alpha}\sigma_i(\mathbf{t})=0\}\\ &= \inf\{\alpha\colon \liminf_{|\mathbf{t}|\to 0} |\mathbf{t}|^{-\alpha}\sigma_i(\mathbf{t})=\infty\}.\end{aligned}$$

It is well known that $0\leqq\alpha_i\leqq 1$.

Finally $\mathbf{X}$ is *locally nondeterministic* on B if for each $k>0$ there exist $\varepsilon_k>0$ and $\delta_k>0$ (depending *only* on k) such that for any $\mathbf{t}_j\in B$ with $|\mathbf{t}_j-\mathbf{t}_k|<\delta_k$, $j=1,\ldots,k$, the conditional vector $\mathbf{X}(\mathbf{t}_k)$ given $\mathbf{X}(\mathbf{t}_j)$, $j=1,\ldots,k-1$ satisfies

$$\begin{aligned}&\det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k)\,|\,\mathbf{X}(\mathbf{t}_j),\ j=1,\ldots,k-1)\\ &\quad\geqq \varepsilon_k \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k)-\mathbf{X}(\mathbf{t}_k^*))\end{aligned} \tag{1}$$

where $\mathbf{t}_k^*=\mathbf{t}_i$, $i<k$ if $|\mathbf{t}_i-\mathbf{t}_k|=\inf_{j<k}|\mathbf{t}_j-\mathbf{t}_k|$.

Berman (1973) introduced local nondeterminism for Gaussian processes ($N=d=1$) and Pitt (1978) extended the definition to include processes with multidimensional time parameter ($(N,1)$ fields). Pitt's condition is equivalent to requiring that

$$\begin{aligned}&\operatorname{Var}(X(\mathbf{t}_k)-X(\mathbf{t}_{k-1})\,|\,X(\mathbf{t}_j)-X(\mathbf{t}_{j-1}),\ 2\leqq j<k)\\ &\quad\geqq \varepsilon_k \operatorname{Var}(X(\mathbf{t}_k)-X(\mathbf{t}_{k-1}))\end{aligned}$$

where $\mathbf{t}_1,\ldots,\mathbf{t}_k$ are ordered such that $|\mathbf{t}_j-\mathbf{t}_{j-1}|\leqq|\mathbf{t}_j-\mathbf{t}_i|$, for $1\leqq i<j\leqq k$. In this case our definition differs slightly from his in that we do not preorder the $\{\mathbf{t}_j\}$ and our conditioning set has been augmented by $X(\mathbf{t}_1)$. However Pitt was interested primarily in fields with $X(\mathbf{0})=0$, where this last alteration is of no consequence. In the case of Gaussian processes, our concept reduces to a 2-sided local nondeterminism, as the conditioning set is not restricted to times either strictly larger or smaller than t_k. (See Cuzick and Du Preez (1982) for more details.) However, this change does not invalidate the spectral methods used by Berman (1973) and Cuzick (1978a) for determining when processes are locally nondeterministic (2-sided). Pitt (1978) has shown how these methods can be adapted for use in the case of fields and further modification when considering vector fields is straightforward. In particular for isotropic vector fields with independent components, the radial spectral measure for each component can be studied in the same manner as for Gaussian processes (see Cuzick (1978) for an example).

Theorem 1. *Let* $\mathbf{X}(\mathbf{t})$ *be an* (N,d) *field. Assume that* $\mathbf{X}$ *has locally approximately independent coordinates which have index* $\alpha_i>0$, $i=1,\dots,d$ *and that* $\mathbf{X}$ *is locally nondeterministic. Then for any open set* $B\subseteq\mathbb{R}^N$, $\mathbf{X}$ *has a point of multiplicity* n *on* B *with positive probability if*

$$N>\frac{n-1}{n}\sum_{i=1}^{d}\alpha_i. \tag{2}$$

If $N<\frac{n-1}{n}\sum_{i=1}^{d}\alpha_i$, *there are no points of multiplicity* n *with probability one.*

Remarks. (i) When all $\alpha_i\equiv\alpha$, then $\mathbf{X}$ has points of multiplicity n when $N>\frac{n-1}{n}\alpha d$. When $N>\alpha d$, $\mathbf{X}$ has points of arbitrarily high multiplicity, which agrees with the known result [2] that in this case $\mathbf{X}$ hits any fixed point uncountably often with positive probability.

(ii) In the cases considered by Kôno, when (2) holds with $n=2$, $\mathbf{X}$ was shown to have double points with probability one. When $\mathbf{X}$ is homogeneous and ergodic and $B=\mathbb{R}_+^N$, the results of Theorem 1 can also be strengthened to hold with probability 1.

(iii) When (2) holds, the results of Cuzick (1980) show that with positive probability the Hausdorff dimension of

$$\begin{aligned}A&=\{(\mathbf{t}_1,\dots,\mathbf{t}_n)\colon \mathbf{X}(\mathbf{t}_1)=\dots=\mathbf{X}(\mathbf{t}_n), \mathbf{t}_i\neq\mathbf{t}_j \quad \text{for } i\neq j\}\\ &=\{\mathbf{Y}^{-1}(\mathbf{0})\}\cap\{\mathbf{t}_i\neq\mathbf{t}_j \quad \text{for } i\neq j\}\end{aligned}$$

is given by $\dim A=Nn-(n-1)\sum_{i=1}^{d}\alpha_i$. However these methods do not appear to be adequate to determine the dimension of the set B of points of multiplicity n or the set $\mathbf{X}^{-1}(B)$ of times associated with such points. However for the special case of planar Brownian motion, Taylor (1966) has shown that $\dim B=2$ for all n.

The critical case of equality in (2) can be resolved somewhat more finely as follows. We say that $\mathbf{X}$ is *nearly isotropic* if there exist functions σ_i^* and positive constants K_1 and K_2 such that

$$K_1\,\sigma_i^*(|\mathbf{t}-\mathbf{s}|)\leqq\sigma_i(\mathbf{t}-\mathbf{s})\leqq K_2\,\sigma_i^*(|\mathbf{t}-\mathbf{s}|),\, i=1,\dots,d$$

for $\mathbf{t}-\mathbf{s}$ in some neighbourhood of the origin.

Theorem 2. *Let* $\mathbf{X}(\mathbf{t})$ *be a nearly isotropic, locally nondeterministic* (N,d) *field with locally approximately independent coordinates. Then* $\mathbf{X}$ *has points of multiplicity* n *with positive probability if, for some* $\varepsilon>0$,

$$\int_0^{\varepsilon}\left[\int_t^{\varepsilon}\left\{\prod_{i=1}^{d}\sigma_i^*(s)\right\}^{-1}ds\right]^{n-1}dt<\infty.$$

Recall that $\omega(h)$ is a *local modulus* for a scalar field $X_i(\mathbf{t})$ if for each $\mathbf{t}$ there exists a finite random constant C and a neighbourhood of the origin A such that

$$|X_i(\mathbf{t}+\mathbf{s})-X_i(\mathbf{t})|\leqq C\,\omega(|s|), \qquad \text{for all } \mathbf{s}\in A.$$

Remark. Yadrenko (1971) has shown that when X_i has index α_i, then h^β is a local modulus for all $\beta<\alpha_i$. Dudley (1973) has shown that when X_i is nearly isotropic and certain other weak conditions are satisfied, then $\sigma_i^*(h)\,|\log\sigma_i^*(h)|^{\frac{1}{2}}$ is a uniform modulus and thus *a fortiori* it is a local modulus. When $N=1$, it is well known that $|\log\sigma_i^*(h)|^{\frac{1}{2}}$ can be replaced by $(\log|\log\sigma_i^*(h)|)^{\frac{1}{2}}$ under weak conditions (Sirao and Watanabe, 1970).

Theorem 3. *Let* $\mathbf{X}(\mathbf{t})$ *be an* (N,d) *field and assume* ω_i *is a local modulus for* X_i, $i=1,\ldots,d$. *If*

$$\overline{\lim_{h\downarrow 0}}\, h^{-nN}\left\{\prod_{i=1}^{d}\omega_i(h)\right\}^{(n-1)}=0,$$

then $\mathbf{X}$ *does not have a point of multiplicity* n *with probability one.*

3. Proofs

We shall only give a proof of Theorem 1. The arguments are easily adapted to establish Theorems 2 and 3 also.

Proof of Theorem 1. Given any open set $B\subseteq\mathbb{R}^N$, let A be a ball in $B^n\subseteq\mathbb{R}^{Nn}$ of radius δ and centre chosen so that for all $\mathbf{T}=(\mathbf{t}_1,\ldots,\mathbf{t}_n)\in A$. $|\mathbf{t}_k-\mathbf{t}_j|>2\delta$ for $j\neq k$, $j,k=1,\ldots,n$ and (1) holds.

If $\mathbf{Y}(\mathbf{T})=(\mathbf{X}(\mathbf{t}_2)-\mathbf{X}(\mathbf{t}_1),\ldots,\mathbf{X}(\mathbf{t}_n)-\mathbf{X}(\mathbf{t}_{n-1}))$, then to show that $\mathbf{X}$ has a point of multiplicity n on B, it is enough to show that $\mathbf{Y}$ hits the origin for some $\mathbf{T}\in A$. It follows from Theorem 3 of [2] that Y hits zero with positive probability for some $\mathbf{T}\in A$ if

$$\iint_{A\times A}\det\mathrm{Cov}(\mathbf{Y}(\mathbf{T})-\mathbf{Y}(\mathbf{S}))^{-\frac{1}{2}}\,d\mathbf{T}\,d\mathbf{S}<\infty. \tag{3}$$

Now if $\mathbf{T}=(\mathbf{t}_1,\ldots,\mathbf{t}_n)$, $\mathbf{S}=(\mathbf{s}_1,\ldots,\mathbf{s}_n)$, $\mathbf{Z}_k=\mathbf{X}(\mathbf{t}_k)-\mathbf{X}(\mathbf{s}_k)$, $k=1,\ldots n$ then

$$\begin{aligned}&\det\mathrm{Cov}(\mathbf{Y}(\mathbf{T})-\mathbf{Y}(\mathbf{S}))\\&=\det\mathrm{Cov}(\mathbf{Z}_k-\mathbf{Z}_{k-1},\,k=2,\ldots,n)\\&=\prod_{k=2}^{n}\det\mathrm{Cov}(\mathbf{Z}_k-\mathbf{Z}_{k-1}\,|\,\mathbf{Z}_j-\mathbf{Z}_{j-1},\,2\leqq j<k)\end{aligned} \tag{4}$$

where the last equality follows from repeated use of the fact that for any Gaussian vectors $\mathbf{X},\mathbf{Y}$,

$$\det\mathrm{Cov}(\mathbf{X},\mathbf{Y})=\det\mathrm{Cov}(\mathbf{X}\,|\,\mathbf{Y})\det\mathrm{Cov}(\mathbf{Y}).$$

Adding further conditioning variables only reduces the magnitude of the terms in (4), so that (4) is greater than or equal to

$$\prod_{k=2}^{n}\det\mathrm{Cov}(\mathbf{Z}_k-\mathbf{Z}_{k-1}\,|\,\mathbf{Z}_j-\mathbf{Z}_{j-1},\,j\neq k). \tag{5}$$

By adding $\mathbf{X}(\mathbf{s}_{k-1})$, $\mathbf{X}(\mathbf{t}_{k-1})$ and $\mathbf{X}(\mathbf{s}_k)$ to the conditioning set it follows that

$$\begin{aligned} &\det \operatorname{Cov}(\mathbf{Z}_k - \mathbf{Z}_{k-1} \mid \mathbf{Z}_j - \mathbf{Z}_{j-1}, j \neq k) \\ &\quad \geqq \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) \mid \mathbf{X}(\mathbf{t}_j), j \neq k, \mathbf{X}(\mathbf{s}_l), l=1, \ldots, n). \end{aligned} \tag{6}$$

Now, because $\mathbf{s}_k$ is the point closest to $\mathbf{t}_k$ among those in the conditioning set, and $\mathbf{X}$ is locally nondeterministic, we can use (1) to see that (6) is bounded below by a constant times

$$\det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) - \mathbf{X}(\mathbf{s}_k)).$$

By adding $\mathbf{X}(\mathbf{s}_{k-1})$, $\mathbf{X}(\mathbf{s}_k)$, and $\mathbf{X}(\mathbf{t}_k)$ to the conditioning set instead of $\mathbf{X}(\mathbf{s}_{k-1})$, $\mathbf{X}(\mathbf{t}_{k-1})$, and $\mathbf{X}(\mathbf{s}_k)$ and repeating the argument, it follows that the left hand side of (6) is greater than a constant times

$$\max\{\det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) - \mathbf{X}(\mathbf{s}_k)), \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_{k-1}) - \mathbf{X}(\mathbf{s}_{k-1}))\}.$$

Since, for $x_k > 0$, $i=1, \ldots, n$

$$\prod_{k=2}^{n} \max(x_k, x_{k-1}) \geqq \frac{\prod_{k=1}^{n} x_k}{\min\{x_1, \ldots, x_n\}}$$

it follows that (4) is bounded below by a constant times

$$\frac{\prod_{k=1}^{n} \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) - \mathbf{X}(\mathbf{s}_k))}{\min\limits_{k=1, \ldots, n} \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) - \mathbf{X}(\mathbf{s}_k))}.$$

Since $\mathbf{X}$ has locally approximately independent coordinates of index α_i, $i=1, \ldots, d$, it follows that for $|t_k - s_k|$ sufficiently small and any $\beta_i > \alpha_i$, there exist positive constants K_1 and K_2 such that

$$\begin{aligned} \det \operatorname{Cov}(\mathbf{X}(\mathbf{t}_k) - \mathbf{X}(\mathbf{s}_k)) &\geqq K_1 \prod_{i=1}^{d} \sigma_i^2(\mathbf{t}_k - \mathbf{s}_k) \\ &\geqq K_2 |\mathbf{t}_k - \mathbf{s}_k|^{2\beta} \end{aligned}$$

where $\beta = \sum_{i=1}^{d} \beta_i$. Thus the left hand side of (3) is bounded above by a constant times

$$\iint\limits_{A \times A} \frac{\prod_{k=1}^{n} |\mathbf{t}_k - \mathbf{s}_k|^{-\beta}}{\min\limits_{k} (|\mathbf{t}_k - \mathbf{s}_k|^{-\beta})} d\mathbf{t}_1, \ldots, d\mathbf{t}_n \, d\mathbf{s}_1 \ldots d\mathbf{s}_n. \tag{7}$$

As the integrand is a function only of the increments $\mathbf{t}_k - \mathbf{s}_k$ and is also homogeneous in these increments, it is enough to consider the set on which $|\mathbf{t}_1 - \mathbf{s}_1| \leqq \min\limits_{2 \leqq k \leqq n} |\mathbf{t}_k - \mathbf{s}_k|$, and after a polar change of coordinates it follows that (7) is finite if

$$\int_0^\delta t^{N-1}\left[\int_t^\delta s^{-\beta}s^{N-1}\,ds\right]^{n-1}dt<\infty,$$

which is true when $N+(n-1)(N-\beta)>0$. Since β_i can be any value less than α_i, the first part of the theorem is proven.

To prove the second part (and also Theorem 3) let ω_i be a local modulus for $X_i(\mathbf{t})$. Then the proof of Theorem 2 in [2] is easily adapted to show that $\mathbf{Y(T)}$ does not hit any fixed point with probability one if

$$\overline{\lim_{h\to 0}}\, h^{-nN}\left\{\prod_{i=1}^{d}\omega_i(h)\right\}^{n-1}=0.$$

Since t^{β_i} is a local modulus for X_i for all $\beta_i>\alpha_i$, this part of the theorem is also established.

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