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On the Structure of Stationary Flat Processes. II

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Summary. The present paper continues the work by Davidson, Krickeberg, Papangelou, and the author on proving, under weakest possible assumptions, that a stationary random measure η or a simple point process ξ on the space of k-flats in R^d is a.s. invariant or a Cox process respectively. The problems for ξ and η are related by the fact that ξ is Cox whenever the Papangelou conditional intensity measure ζ of (a thinning of) ξ is a.s. invariant. In particular, η is shown to be a.s. invariant, whenever it is absolutely continuous with respect to some fixed measure μ and has no (so called) outer degeneracies. When $k=d-2\geqq 2$, no absolute continuity is needed, provided that the first moments exist and that η has no inner degeneracies either. Under a certain regularity condition on ξ, it is further shown that ξ and ζ are simultaneously non-degenerate in either sense.

1. Introduction

The present paper continues the work by Davidson, Krickenberg, Papangelou, and myself (see e.g. [1, 3, 6, 9–11]) on proving, under weakest possible assumptions, that a stationary random measure η or a simple point process ξ on the space of k-flats in R^d is a.s. invariant or a Cox process (i.e. a mixture of Poisson processes) respectively. A *k-flat* is a k-dimensional affine subspace of R^d, and the notions of *stationarity* and a.s. *invariance* are defined with respect to the group of translations in R^d. (Note that stationary refers to the probability distributions whereas invariance refers to the sample realizations.) The problems for ξ and η are related by the fact that ζ is Cox whenever the Papangelou conditional intensity measure [5] of ξ (or of a homogeneous thinning of ξ) is a.s. invariant.

The most complete results so far have been obtained for $k=d-1$. In this case, a stationary *first order* (i.e. such that first order moments exist) random measure η is a.s. invariant, provided that it a.s. gives mass zero to all sets of parallel flats [1, 3]. It follows that a stationary first order point process ξ is Cox whenever it is *regular*, in the sense that the conditional intensity of a thinning of ξ has the above-mentioned property.

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For arbitrary k and d, it was shown in [3, 11] that η is a.s. invariant, provided that $(B\eta)\pi^{-1} \ll \mu$ a.s. for all bounded sets B, where μ is the homogeneous measure on the space of directions. (Here π denotes projection onto that space, whereas $B\eta$ means the restriction of η to B.) This result was extended in [6] to arbitrary locally invariant measures μ. Below we prove (in Theorem 4.3) that μ can be taken to be any measure with no *outer degeneracies*. By this we mean that μ assigns zero mass to any set of directions lying in a common proper subspace of R^d. (As in [3], the *direction* of a flat is identified with the parallel flat going through the origin.) We conjecture that it is enough, at least under moment restrictions, that all projections $(B\eta)\pi^{-1}$ have this property a.s.

In [3] it was further shown that a stationary first order random measure η is a.s. invariant, if a.s. the pairs of directions span R^d a.e. η^2. Note that this is only possible when $k \geqq d/2$. In Theorem 5.1 below, this result is used to prove that, for $k = d-2 \geqq 2$, η is a.s. invariant whenever its projections $(B\eta)\pi^{-1}$ have a.s. neither outer nor inner degeneracies. (A measure μ on the space of directions has no *inner degeneracies*, if it assigns mass zero to every set of directions containing a common line.)

When results like this are to be applied to point processes ξ, via the conditional intensity ζ of a thinning of ξ, the need arises to state the non-degeneracy conditions on ζ directly in terms of ξ. This turns out to be easy, since by Theorem 3.3 below, ξ and ζ are simultaneously a.s. non-degenerate in either sense, provided that ξ is regular. (In the point process case, non-degeneracy means by definition that $\xi\pi^{-1}$ should a.s. give *finite* mass to any set of directions lying in a common proper subspace or containing a common line, respectively.) Thus, in particular, the above conjecture for η implies the corresponding statement for all regular ξ. A counterexample in [4] shows that regularity is essential here.

Our main motivation for the present work is the interpretation of the results for $k=1$ in terms of systems of free particles, (cf. [6]). For almost periodic systems, related results were obtained in §4 of [7] by entirely different methods. In §6 below we treat the intermediate case, when the motion of the particles is free in one direction and otherwise almost periodic. Here absolute continuity turns out to hold automatically, so the analogue to our conjecture is true in this case.

As explained in §6 of [6], results in the stationary case may be used to prove statements about the asymptotic behavior of free particle systems. Note in particular that Theorem 6.1 in [6] extends to measures μ with no outer degeneracies, and that the conclusion of the corresponding point process version can be strengthened to mean convergence, in the sense of [7], (cf. [8]). No further remarks will be made on the non-stationary case.

As for the organization of the paper, our treatment of the main problem, that of proving a.s. invariance or Cox nature of a stationary flat process, will be postponed to §§4–6, since we shall first need to discuss the notions of degeneracies of a fixed measure on the space of flats (§2), and of a discrete flat process and its conditional intensity (§3). We conclude this introduction by introducing some general terminology and notation.

Let M_k^d be the space of k-flats in R^d, and let Φ_k^d be the subspace of k-flats through the origin, (i.e. of directions of k-flats). Write $M^d = \bigcup_k M_k^d$ and $\Phi^d = \bigcup_k \Phi_k^d$. If $u \in M^d$ with $\dim u \geqq k$ (dim being short for dimensionality), let $M_k^{(u)}$ and $\Phi_k^{(u)}$ be the classes of flats in M_k^d and Φ_k^d respectively which lie in u. For any set $A \subset R^d$, let $\mathscr{L}(A)$ denote the linear subspace spanned by A.

Given any $u \in M^d$, let σ_u and ρ_u denote the corresponding intersection and projection operators. More precisely, let $\sigma_u x$ be the intersection $u \cap x$, and let $\rho_u x$ be the orthogonal projection of x onto u. It is useful to observe that, when $u \in \Phi_{d-1}^d$, σ_u and ρ_u are *dual* in the following sense. Let $\tilde{x} \in \Phi_{d-k}^d$ be the dual to (or orthogonal complement of) $x \in \Phi_k^d$, and let $(\tilde{x})_u$ denote duality with respect to u. Then $(\widetilde{\sigma_u x})_u = \rho_u \tilde{x}$. To see this, show that $\rho_u \tilde{x} \perp \sigma_u x$ ($\perp$ denoting orthogonality), e.g. by noting that $\rho_u \tilde{x} = \sigma_u \mathscr{L}(\tilde{x}, \tilde{u})$, and check that $\dim \rho_u \tilde{x} + \dim \sigma_u x = d-1$. As in [6], $(\sigma_u x, \pi x)$ is a useful parametrization in $R^{2(d-1)}$ of lines $x \in M_1^d$ with $\pi x \not\subset u$.

In M_k^d we introduce the natural geometric topology, according to which $x_n \to x$ iff $\pi x_n \to \pi x$ and moreover $\sigma_u x_n \to \sigma_u x$ for every $u \in M_{d-k}^d$ intersecting x at a point. This clearly makes M_k^d locally compact and second countable (lcsc). (Note in particular that a set $B \subset M_k^d$ is bounded i.e. has compact closure iff all flats in B go through some fixed bounded region of R^d.) Thus the theory [2] of random measures on topological spaces applies to M_k^d.

We shall adhere to the general terminology and notation of [2]. Thus we mean by $\mathscr{B}(S)$ the class of bounded Borel sets in S, and by $\mathscr{F}_c(S)$ the class of continuous functions $S \to R_+$ with bounded support. By $\mathfrak{M}(S)$ and $\mathfrak{N}(S)$ we denote the spaces of R_+- and Z_+-valued Radon measures on S, being endowed with their respective vague topologies. The class of diffuse measures in $\mathfrak{M}(S)$ is denoted by $\mathfrak{M}_d(S)$. For measures μ and ν, $\mu \perp \nu$ means that μ and ν are mutually singular, and if f is a function, μf denotes the μ-integral of f whereas $f\mu$ denotes the measure $\ll \mu$ with μ-density f. The indicator function of a set $B = \{\cdot\}$ is denoted by 1_B or $1\{\cdot\}$, and we shall prefer to write $B\mu$ in place of $1_B \mu$. If $\mu B > 0$, $\mu(A \mid B)$ will denote the ratio $\mu(A \cap B)/\mu B$. If μ and ν are purely atomic, we define $\mu \cdot \nu = \sum \mu\{s\}\, \nu\{s\}\, \delta_s$, where δ_s is the Dirac measure at s, i.e. $\delta_s B = 1_B(s)$. The letter λ is reserved for Lebesgue measure on Euclidean spaces and for uniform measures on the spaces Φ_k^d.

All random elements are defined on some common measurable space $\Omega = \{\omega\}$ with probability measure P and expectation = integration E. In particular, *random measures* and *point processes* on S are random elements in $\mathfrak{M}(S)$ and $\mathfrak{N}(S)$ respectively. A point process ξ is said to be *simple* if all its atoms have size 1. A *homogeneous thinning* (*p-thinning*) of ξ is obtained by deleting the atoms independently with a fixed probability $1-p$. For the definition and basic properties of conditional intensities, we refer to [5].

2. Degeneracies of Random Flats

In this section we shall consider the degeneracies of a fixed measure $\mu \in \mathfrak{M}(M_k^d)$ or $\mu \in \mathfrak{M}(\Phi_k^d)$. To simplify notations, let us identify with a flat $u \in M_m^d$ the set of all flats of arbitrary dimensionality which lie in u or contain u. We shall say that μ has a *degeneracy of order m*, if $\mu u > 0$ for some $u \in M_m^d$. The degeneracy is called *outer* if $m \geqq k$ and *inner* if $m \leqq k$.

Note that outer and inner degeneracies are *dual* in the following sense. Let $\mu\in\mathfrak{M}(\Phi_k^d)$, and let $\tilde{\mu}\in\mathfrak{M}(\Phi_{d-k}^d)$ be the dual measure induced by the mapping $x\to\tilde{x}$. Then $\mu u>0$ iff $\tilde{\mu}\tilde{u}>0$, so each outer (inner) degeneracy of μ corresponds uniquely to an inner (outer) degeneracy of $\tilde{\mu}$.

We shall further use the fact that a measure $\mu\in\mathfrak{M}(\Phi_k^d)$ may be decomposed uniquely according to its outer (or inner) degeneracies. To this aim, let m be the minimum order of outer degeneracy, and let $u_1, u_2\in\Phi_m^d$ be distinct degeneracy flats. Then $u_1\cap u_2$ cannot be an outer degeneracy flat because of the minimality of m, so the sets $A_j=\{x\in\Phi_k^d\colon x\subset u_j\}$ are disjoint a.e. μ. Thus there can be at most countably many degeneracy flats $u_j\in\Phi_m^d$ with corresponding μ-components $\mu_j=A_j\mu$. Since the degeneracies of $\mu-\Sigma\mu_j$ have clearly order $>m$, the argument may be continued recursively, leading ultimately to the *outer degeneracy decomposition* $\mu=\sum_{m=k}^{d}\sum_j\mu_{mj}$. Here each μ_{mj} is clearly confined to some $u_{mj}\in\Phi_m^d$ and has no outer degeneracies of lower order. Note that the μ_{mj} and u_{mj} are unique apart from their order. We shall refer to the latter as the *minimal outer degeneracy flats* of μ. A similar argument (or duality) leads to the *inner degenercy decomposition* and to the family of *maximal inner degeneracy flats.*

In several proofs of the subsequent sections, we shall proceed by a successive reduction of dimensionality, where we turn in each step from the original flats to their intersections with a suitable fixed flat $u\in M_{d-1}^d$. A basic role will then be played by the following lemma. Here the phrase "almost every u" refers to the homogeneous measures on M_{d-1}^d and Φ_{d-1}^d.

Lemma 2.1. *Let $1\leqq k, m\leqq d$, and let $\mu\in\mathfrak{M}(M_k^d)$ be such that $\mu v=0$ for all $v\in M_m^d$. Then almost every $u\in M_{d-1}^d$ is such that $\sigma_u x\in M_{k-1}^{(u)}$, $x\in M_k^d$ a.e. μ, and moreover $\mu\sigma_u^{-1}v=0$ for all $v\in M_{m-1}^{(u)}$. For $2\leqq k, m\leqq d$, this remains true with each M replaced by Φ.*

Proof. Notice first that the M-version of the lemma for some triple (d, k, m) follows from the Φ-version for $(d+1,\ k+1,\ m+1)$. To see this, imbed R^d into R^{d+1} as a flat $w\in M_d^{d+1}\setminus\Phi_d^{d+1}$, and make the corresponding imbedding of M^d into M^{d+1}. Then the operator $\mathscr{L}$ in R^{d+1} defines a $1-1$ correspondence between M^d and $\Phi^{d+1}\setminus\Phi^{(\pi w)}$ (the inverse of $\mathscr{L}$ being σ_w), and it is easily verified that both the hypothesis and the conclusion of the lemma are simultaneously fulfilled for a measure $\mu\in\mathfrak{M}(M_k^d)$ and its image $\mu\mathscr{L}^{-1}\in\mathfrak{M}(\Phi_{k+1}^{d+1})$. It is enough to consider the case $\mu\in\mathfrak{M}(\Phi_k^d)$.

In the case $m\leqq k$, let $v_0, v_1, \ldots$ be the maximal degeneracy flats of μ, and let $\mu_0, \mu_1, \ldots$ be the corresponding components of μ. By assumption, $\dim v_j<m$ for all j, and we may assume that $\dim v_j\geqq 1$ iff $j\geqq 1$. Since for any $u\in\Phi_{d-1}^d$ and $v\in\Phi^d$, $\dim\rho_u v=\dim v$ iff $\tilde{u}\notin v$, we get $\dim\rho_u\tilde{v}_j=\dim\tilde{v}_j>d-m$ for almost every u and for $j\geqq 1$. Now $\tilde{\mu}_j$ is non-degenerate on $\tilde{v}_j$ by definition, so it follows that $\tilde{\mu}_j\rho_u^{-1}$ is non-degenerate on $\rho_u\tilde{v}_j$ for any such u and j. Letting $v\in\Phi_{m-1}^{(u)}$ be arbitrary and noting that $\dim(\tilde{v})_u=d-m$, we thus obtain

$$\mu_j\sigma_u^{-1}v=(\mu_j\sigma_u^{-1})_u^{\sim}(\tilde{v})_u=\tilde{\mu}_j\rho_u^{-1}(\tilde{v})_u=0,\qquad j\geqq 1,$$

and this remains true for $j=0$ since $\mu_0\sigma_u^{-1}$ is non-degenerate on u.

In the case $k \leqq m < d$ (the case $m = d$ is trivial), a probabilistic argument is convenient, so assume without loss that μ is a probability measure, and let ξ be a random flat with distribution μ. Further suppose that ζ is a random line in ξ at unit distance from the origin but otherwise uniformly distributed. Writing γ for the point where ζ touches the unit sphere, it is seen from the hypothesis that

$$\mathsf{P}\{\zeta \subset v\} \leqq \mathsf{P}\{\gamma \in v\} = \mathsf{E}\mathsf{P}[\gamma \in v \mid \xi] = \mathsf{P}\{\xi \subset v\} = 0, \qquad v \in \Phi_m^d. \tag{1}$$

We further introduce an independent and uniformly distributed random flat η in Φ_{d-1}^d, and put $\alpha = \zeta \cap \eta$. Then α is a.s. a point. To see this, let η be generated by $d-1$ independent and uniformly distributed random lines $\kappa_1, \ldots, \kappa_{d-1}$ in Φ_1^d, and note that $\pi\zeta, \kappa_1, \ldots, \kappa_{d-1}$ are a.s. linearly independent. It follows in particular that $\xi \not\subset \eta$ a.s., and hence that $\dim(\xi \cap \eta) = k-1$ a.s.

Now suppose that $(\xi_1, \zeta_1), (\xi_2, \zeta_2), \ldots$ are independent duplicates of (ξ, ζ), independent of η, and put $\alpha_j = \zeta_j \cap \eta$. We shall prove by induction in $n = 1, \ldots, m$ that $\dim \mathscr{L}(\alpha_1, \ldots, \alpha_n) = n$ a.s. For $n = m$, it will then follow by Fubini's theorem that, for almost every $u \in \Phi_{d-1}^d$,

$$\dim \mathscr{L}(\sigma_u \xi_1, \ldots, \sigma_u \xi_m) \geqq \dim \mathscr{L}(\sigma_u \zeta_1, \ldots, \sigma_u \zeta_m) = m \qquad \text{a.s.},$$

and this will clearly imply that $\sigma_u \xi \not\subset v$ a.s. for every $v \in \Phi_{m-1}^{(u)}$, as asserted.

The induction hypothesis is automatically true for $n = 1$ since $\alpha_1 \neq 0$. It remains to prove it for arbitrary fixed $n \in \{2, \ldots, m\}$, given that it is true for $n-1$. Under this assumption, $\alpha_1, \ldots, \alpha_n$ are a.s. points satisfying

$$\dim \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}) = \dim \mathscr{L}(\alpha_1, \ldots, \alpha_{n-2}, \alpha_n) = n-1. \tag{2}$$

By Fubini's theorem, almost all $\eta, \zeta_1, \ldots, \zeta_{n-1}$ are such that (2) holds a.s. for ζ_n. Fix $\eta, \zeta_1, \ldots, \zeta_{n-1}$ accordingly, and let β denote the orthogonal complement of $\mathscr{L}(\alpha_1, \ldots, \alpha_{n-1})$ in η. Since $\zeta_{n-1} \not\subset \eta$, almost every $\alpha'_{n-1} \in \zeta_{n-1} \setminus \{\alpha_{n-1}\}$ is such that $\eta' = \mathscr{L}(\alpha_1, \ldots, \alpha_{n-2}, \alpha'_{n-1}, \beta)$ has dimension $d-1$ and intersects $\zeta_1, \ldots, \zeta_{n-1}$ at unique points $\alpha_1, \ldots, \alpha_{n-2}, \alpha'_{n-1}$ and ζ_n at an a.s. unique point α'_n. By the choice of α'_{n-1}, we get from (2)

$$\dim \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_{n-1}) = n \leqq m, \tag{3}$$

so by (1) we have a.s.

$$\zeta_n \not\subset \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_{n-1}). \tag{4}$$

Fix ζ_n accordingly, and such that moreover ζ_n intersects η and η' uniquely at α_n and α'_n respectively.

Assume that $\zeta_1, \ldots, \zeta_n$ have linearly dependent intersections with both η and η'. Then by (3)

$$\alpha_n \in \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}) \quad \text{and} \quad \alpha'_n \in \mathscr{L}(\alpha_1, \ldots, \alpha_{n-2}, \alpha'_{n-1}). \tag{5}$$

If $\alpha_n = \alpha'_n$, we get by (5) and (3)

$$\alpha_n \in \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}) \cap \mathscr{L}(\alpha_1, \ldots, \alpha_{n-2}, \alpha'_{n-1}) = \mathscr{L}(\alpha_1, \ldots, \alpha_{n-2}),$$

which is excluded by (2). On the other hand, $\alpha_n \neq \alpha'_n$ implies by (5) that

$$\zeta_n \subset \mathscr{L}(\alpha_n, \alpha'_n) \subset \mathscr{L}(\alpha_1, \ldots, \alpha_{n-1}, \alpha'_{n-1}),$$

contrary to (4). This contradiction implies by Fubini's theorem that, for almost every choice of $\zeta_1, \ldots, \zeta_n$, there is at least one flat $u \in \Phi^d_{d-1}$ such that the intersections $\sigma_u \zeta_1, \ldots, \sigma_u \zeta_n$ are unique and linearly independent.

Fix $\zeta_1, \ldots, \zeta_n$ with this property, and choose $a_i \in R^d$ and $b_i \in R^d \setminus \{0\}$ such that $\zeta_i \equiv \{a_i + t b_i; t \in R\}$. Let further $r \in R^d \setminus \{0\}$ be arbitrary. If $r b_i \neq 0$ for all i, then the ζ_i will intersect the flat $u = \{x \in R^d; rx = 0\}$ uniquely at the points $a_i - b_i(r a_i)/(r b_i)$, so there are unique linearly independent intersections whenever the $(d \times n)$-matrix

$$(r b_i)(a_i(r b_i) - b_i(r a_i)), \qquad i = 1, \ldots, n, \tag{6}$$

has rank n, i.e. when at least one of the $\binom{d}{n}$ $(n \times n)$-determinants of (6) is non-zero. Now this has just been shown to occur for at least one r, and since a polynomial in the components of r is a.e. non-zero unless it vanishes identically, the intersections are indeed unique and linearly independent for almost every u. By Fubini's theorem, this completes the induction step, and hence concludes the proof. □

2. Degeneracies of Flat Processes

In this section we consider the degeneracies in the space of directions of stationary random measures η on M^d_k, and in the particular case of simple point processes ξ, we discuss the relationship between the degeneracies of ξ and those of the conditional intensity of a homogeneous thinning of ξ. Our key result is the following $0-\infty$ law for the case when η a.s. gives mass zero to any set of parallel flats.

Theorem 3.1. *Let η be a stationary first order random measure on M^d_k such that a.s. $\eta \pi^{-1} v = 0$, $v \in \Phi^d_k$. Then a.s. $\eta \pi^{-1} u = 0$ or ∞ for all $u \in \Phi^d_m$, $m \leq d$.*

Proof. We shall only consider the case $m \geq k$, the argument for $m \leq k$ being similar. Let $C_1, C_2, \ldots \in \mathscr{B}(M^d_k)$ be a disjoint partition of M^d_k, and define the random measure ζ_1 on Φ^d_k by

$$\zeta_1 = \sum_n 2^{-n} \frac{(C_n \eta) \pi^{-1}}{1 + \eta C_n}.$$

Next choose a measurable mapping g_1: $\Phi^d_k \times \Phi^k_1 \to \Phi^d_1$ such that $g_1(v, \cdot)$ is an isomorphism of Φ^k_1 onto $\Phi^{(v)}_1$ for each $v \in \Phi^d_k$. Letting λ be an invariant measure on Φ^k_1, it follows that $\zeta_2 = (\zeta_1 \times \lambda) g_1^{-1}$ is a random measure on Φ^d_1, and hence that $(\zeta_2)^m$ is a random measure on $(\Phi^d_1)^m$.

For fixed $\omega \in \Omega$, let m be the smallest integer $\geq k$ such that $\eta \pi^{-1} u \neq 0$ for some $u \in \Phi^d_m$. Then $x_1, \ldots, x_m \in \Phi^d_1$ are clearly linearly independent a.e. $(\zeta_2)^m$, and in that case they span a flat $u \in \Phi^d_m$ which is a measurable function g_2 of

$(x_1, \ldots, x_m)$. Thus $\zeta_3 = (\zeta_2)^m g_2^{-1}$ is a random measure on Φ_m^d, and it is easily seen that $\zeta_3\{u\} > 0$ iff $\eta\pi^{-1}u \neq 0$. By Lemma 2.3 in [2], it follows in particular that the integer m above as well as the corresponding sequence of degeneracy flats $u \in \Phi_m^d$ are measurable. We may therefore redefine m to be the smallest integer $\geqq k$ such that, with positive probability, $\eta\pi^{-1}u \neq 0$ for some $u \in \Phi_m^d$.

For $x \in M_k^d$ and $u \in \Phi_m^d$, put $g_3(x, u) = x$ if $\pi x \subset u$ and let g_3 remain undefined otherwise. Note that both g_3 itself and the induced mapping $g_3^{-1}: \mathfrak{M}(M_k^d \times \Phi_m^d) \to \mathfrak{M}(M_k^d)$ are measurable. Applying g_3^{-1} to the pairs $(\eta, \delta_{\vartheta_n})$, where $\{\vartheta_n\}$ is a measurable enumeration of the atom positions of ζ_3 (cf. Lemma 2.3 in [2]), we obtain a sequence $\{\eta_n\}$ of random measures on M_k^d, representing the restrictions of η to the π-inverses of the degeneracy flats in Φ_m^d. Note that the η_n are a.s. mutually singular by the choice of m, and hence that $\sum \eta_n \leqq \eta$. From $\{\eta_n\}$ we further define the random measures η'_n on M_k^d by $\eta'_n = \eta_n$ if $\eta_n M_k^d < \infty$, and $\eta'_n = 0$ otherwise.

By Lemma 2.1 and Fubini's theorem, we may fix a flat $v \in M_{d-k+1}^d$ such that a.s., $\sigma_v u \equiv u \cap v \in M_1^{(v)}$ for $u \in M_k^d$ a.e. η, and such that moreover $\eta\sigma_v^{-1}\pi^{-1}\{x\} = 0$ for all $x \in \Phi_1^{(v)}$. The mapping σ_v being measurable, we may define random measures ζ'_n on $M_1^{(v)}$ by $\zeta'_n = \eta'_n \sigma_v^{-1}$. Then $\zeta_n'^2$ will automatically be random measures on $(M_1^{(v)})^2$. For each pair $x_1, x_2 \in M_1^{(v)}$ with $\pi x_1 \neq \pi x_2$, there is clearly a unique point $g_4(x_1, x_2)$ at equal minimum distance from x_1 and x_2, and since g_4 is measurable, we may define random measures χ_n on v by $\chi_n = \zeta_n'^2 g_4^{-1}$. To every bounded measure μ on v we may next associate measurably a point $g_5(\mu) \in v$, such that translations of μ yield the corresponding translations of $g_5(\mu)$, and then define the random elements γ_n in v by $\gamma_n = g_5(\chi_n)$. (There are many ways to define g_5, one being based on the medians in $d-k+1$ directions.)

For each $x \in M_1^{(v)}$ and $y \in v$, let $g_6(x, y)$ be the point on x which is closest to y, and note that g_6 is measurable. Then so is $g_7(x, y) = (\pi x, g_6(x, y))$. Moreover, $\zeta'_n \times \delta_{\gamma_n}$ is for each n a random measure on $M_1^{(v)} \times v$, so we may define a random measure ζ on $\Phi_1^{(v)} \times v$ by $\zeta = \sum_n (\zeta'_n \times \delta_{\gamma_n}) g_7^{-1}$. Let us further put $\zeta' = \sum \zeta'_n$, and note that $\zeta' = \zeta g_8^{-1}$ where $g_8(x, y) = x + y$, $x \in \Phi_1^{(v)}$, $y \in v$. Note also that ζ is v-stationary, since our construction doesn't depend on any particular choice of coordinate system, and that ζ' is of first order, since the σ_v-inverses of bounded sets are bounded and moreover $\sum \eta'_n \leqq \eta$.

Let $w \in \Phi_{d-k}^{(v)}$ be arbitrary, and divide v into congruent slices S_j, $j \in Z$, parallel to w. Let κ_j be the restriction of ζ to $\Phi_1^{(v)} \times S_j$, and note that each κ_j is w-stationary, and further that the κ_j have the same distribution for all j apart from a translation. These properties will clearly be carried over to the random measures $\kappa_j g_8^{-1}$, and so it follows by Lemma 2.2 in [3] that the latter have the same intensity measure. Hence $\mathsf{E}\zeta' B = \sum \mathsf{E}\kappa_j g_8^{-1} B$ is either 0 or ∞ for every $B \in \mathscr{B}(M_1^{(v)})$, and since the latter possibility is excluded, we have in fact $\zeta' = 0$ a.s. By the definition of ζ', this means that $\eta'_n = 0$ a.s. for all n, i.e. that $\eta_n M_k^d = 0$ or ∞ a.s. This proves the assertion for degeneracy flats of dimension $\leqq m$. Since $\eta - \sum \eta_n$ is stationary, and since a.s. $(\eta - \sum \eta_n)\pi^{-1}u = 0$ for all $u \in \Phi_m^d$, we may proceed recursively to complete the proof. (In fact, we have proved the slightly stronger assertion that the degeneracy components of η are a.s. infinite.) □

Our next aim is to analyse the degeneracies of point processes ξ on M_k^d. Note that Theorem 3.1 doesn't apply directly, since ξ is degenerate in the sense of that theorem at all flats containing or contained in the directions at the atoms of ξ. Instead, we shall apply the theorem to the conditional intensity ζ of ξ (or of a thinning of ξ). This requires ξ to be *regular*, in the sense that the hypothesis of the theorem is fulfilled for ζ, i.e. that a.s. $\zeta\pi^{-1}v=0$ for all $v\in\Phi_k^d$. In order to check this condition, it is enough to look at the conditional behavior of ξ at the directions of its atoms. The precise statement is given by the following lemma by taking $S=M_k^d$, $S'=\Phi_k^d$, and $f=\pi$.

Lemma 3.2. *Let S and S' be lcsc, let $f: S\to S'$ be measurable, and let $B\in\mathscr{B}(S)$ be fixed. Suppose that ξ and $\xi'=(B\xi)f^{-1}$ are simple point processes on S and S', and let ζ and ζ' be the conditional intensities of ξ and $(B^c\xi,\xi')$ respectively. Then $(B\zeta)f^{-1}$ is a.s. diffuse iff $\xi'\perp\zeta'$ a.s.*

As in [5], we define for any simple point process ξ on some lcsc S the condition (Σ): $\mathsf{P}[\xi B=0\,|\,B^c\xi]>0$ a.s. on $\{\xi B=1\}$, $B\in\mathscr{B}(S)$.

Proof. Assume without loss that $B=S$. Since (Σ) is violated simultaneously for ξ and ξ', and since neither condition is true in that case, we may assume that (Σ) is fulfilled. Let us first consider the case when f is the identity mapping on S. If $\mathsf{P}\{\zeta\notin\mathfrak{M}_d\}>0$, there exists by Theorem 2.1 in [5] some fixed set $I\in\mathscr{B}$ such that

$$\mathsf{P}\{\mathsf{E}[I\xi;\,\xi I=1\,|\,I^c\xi]\notin\mathfrak{M}_d\}>0.$$

By Lemma 2.3 in [2], there is then some $I^c\xi$-measurable random element σ in I such that

$$\mathsf{P}\{\mathsf{P}[\xi\{\sigma\}=\xi I=1\,|\,I^c\xi]>0\}>0.$$

By another application of Theorem 2.1 in [5], we then obtain

$$\begin{aligned}\mathsf{P}\{\xi\cdot\zeta\neq 0\}&\geqq\mathsf{P}\{(\xi\cdot\zeta)\{\sigma\}>0\}\geqq\mathsf{P}\{\xi\{\sigma\}=\xi I=1,\,\mathsf{P}[\xi\{\sigma\}=\xi I=1\,|\,I^c\xi]>0\}\\&=\mathsf{E}[\mathsf{P}[\xi\{\sigma\}=\xi I=1\,|\,I^c\xi];\,\mathsf{P}[\xi\{\sigma\}=\xi I=1\,|\,I^c\xi]>0]>0,\end{aligned}$$

as desired. For general f, we may apply this result to both ξ and ξ' and use Theorem 3.3 in [5] to see that $\xi'\perp\zeta'$ a.s. iff $\zeta'\in\mathfrak{M}_d$, i.e. iff $\eta f^{-1}=\eta'\in\mathfrak{M}_d$ a.s., and finally iff $\zeta f^{-1}\in\mathfrak{M}_d$ a.s., where η and η' are related to ξ and ξ' as in §3 of [5]. □

Since a point process on M_k^d is always degenerate in the sense of Theorem 3.1, we shall modify the definition in this case and allow a small number of flats with directions in (or through) a common linear subspace of R^d. More precisely, when $1\leqq m\leqq d-1$, we allow at most $|m-k|+1$ flats with directions in/through a subspace of dimension m. (For $k=d-1$ and $m=1$, this coincides with Krickeberg's definition of degeneracy in [9].) Note that, in case of more than $|m-k|+1$ flats with this property, one of the directions must lie in the proper subspace spanned by the others, or must contain the intersection of the others, respectively. With this modified definition of degeneracy, we get for regular point processes ξ a direct counterpart to Theorem 3.1. Moreover, ξ and the conditional intensity of a homogeneous thinning [2] of ξ are simultaneously a.s. degenerate.

Theorem 3.3. *Fix* $p\in(0,1)$. *Let* ξ *be a simple stationary first order point process on* M_k^d, *and let* η *be the conditional intensity of a p-thinning* ξ' *of* ξ. *Suppose that* $\eta\pi^{-1}v=0$, $v\in\Phi_k^d$, *a.s. Then a.s., for every* $u\in\Phi_m^d$, $m\leqq d$, $\xi\pi^{-1}u<\infty$ *implies* $\xi\pi^{-1}u\leqq|m-k|+1$ *and* $\eta\pi^{-1}u=0$. *Conversely, for fixed* $m\leqq d$, $\eta\pi^{-1}u=0$ *for all* $u\in\Phi_m^d$ *a.s. implies* $\xi\pi^{-1}u<\infty$ *for all* $u\in\Phi_m^d$ *a.s.*

Proof. Assume throughout that $m\geqq k$, the case $m\leqq k$ being similar. Our proof will proceed in four steps. For the needs in steps 2 and 3, introduce a countable DC-ring [2] $\mathscr{U}\subset\mathscr{B}(M_k^d)$, and note that all a.s. relations in [5] may be assumed to hold simultaneously for sets in $\mathscr{U}$.

1. We shall first prove that, with probability one, $\eta\pi^{-1}u=\infty$ for a minimal flat $u\in\Phi_m^d$, $k\leqq m\leqq d$, implies that $\xi\pi^{-1}u=0$ or ∞. For this purpose, divide η as above into outer degeneracy components $\eta_1,\eta_2,\ldots$, and for each $n\in N$, put $\eta_n'=\eta_n$ if $0<\xi\pi^{-1}u<\infty$ for the corresponding degeneracy flat $u\in\Phi_m^d$, and otherwise $\eta_n'=0$. Let ξ_n be the corresponding restriction of ξ. Next choose a fixed flat $v\in M_{d-k+1}^d$ such that a.s. $\sigma_v u\equiv u\cap v\in M_1^{(v)}$ for $u\in M_k^d$ a.e. $\xi+\eta$, and moreover $\eta\sigma_v^{-1}\pi^{-1}\{x\}=0$ for all $x\in\Phi_1^{(v)}$, (cf. Lemma 2.1). Put $\xi_n'=\xi_n\sigma_v^{-1}$ and $\zeta_n'=\eta_n'\sigma_v^{-1}$.

For any $\mu=\sum_{j=1}^{r}\delta_{y_j}\in\mathfrak{N}(M_1^{(v)})$ with $r\in N$ and any $x\in M_1^{(v)}$ with $\pi x\neq\pi y_1,\ldots,\pi y_r$, let $z_1,\ldots,z_r$ be the points on x which are closest to $y_1,\ldots,y_r$, and let $g_1(x,\mu)$ be the mean value of $z_1,\ldots,z_r$. Note that g_1 is jointly measurable. Thus $\eta_n\times\delta_{\xi_n}$ is a random measure on the subset of $M_1^{(v)}\times\mathfrak{N}(M_1^{(v)})$ where g_1 is defined, and we may further define a random measure ζ on $\Phi_1^{(v)}\times v$ by $\zeta=\sum_n(\eta_n\times\delta_{\xi_n})g_2^{-1}$ where $g_2(x,\mu)\equiv(\pi x, g_1(x,\mu))$. As in the proof of Theorem 3.1, $\zeta'\equiv\sum\zeta_n'=\zeta g_3^{-1}$ where $g_3(x,y)=x+y$, $x\in\Phi_1^{(v)}$, $y\in v$, and moreover ζ is v-stationary whereas ζ' is of first order. Arguing as before, we may conclude that $\zeta'=0$ a.s. Hence $\eta\pi^{-1}u=\infty$ with u minimal implies that $\xi\pi^{-1}u=0$ or ∞, which in turn trivially implies $\xi'\pi^{-1}u=0$ or ∞.

2. Next we show that, if there exists with positive probability some flat $u\in\Phi_m^d$, $m\geqq k$, with $\eta\pi^{-1}u=\infty$ but $\xi'\pi^{-1}u=0$, there is with positive probability some flat $u\in\Phi_m^d$ with the same m such that $r\equiv|m-k|+1<\xi'\pi^{-1}u<\infty$. To see this, fix $\omega\in\Omega$ and a u with $\eta\pi^{-1}u=\infty$ and $\xi'\pi^{-1}u=0$, and choose $B\in\mathscr{U}$ with $\xi'B=0$ and $\eta(B\cap\pi^{-1}u)\geqq q^{-r-1}$, where $q=1-p$. Using Theorem 3.1 in [5] and the fact that $B\xi'$ remains a p-thinning of $B\xi$ even after conditioning on $B^c\xi'$, we obtain

$$\mathsf{P}[\xi B\leqq r\mid B^c\xi']\leqq q^{-r}\,\mathsf{E}[q^{\xi B}\mid B^c\xi']=q^{-r}\,\mathsf{P}[\xi'B=0\mid B^c\xi']\leqq q^{-r}(\eta B)^{-1}\leqq q,^{+2}.$$

and moreover

$$\mathsf{P}[\xi'B>r\mid\xi B>r, B^c\xi']\geqq p^{r+1},$$

so

$$\mathsf{P}[\xi'B>r\mid B^c\xi']=\mathsf{P}[\xi B>r\mid B^c\xi']\,\mathsf{P}[\xi'B>r\mid\xi B>r, B^c\xi']\geqq(1-q)\,p^{r+1}=p^r \quad (1)$$

For given $B^c\xi'$, we may consider u as fixed, and we may choose a sequence $B_1, B_2,\ldots\in\mathscr{U}$ such that $B\supset B_n\downarrow(B\cap\pi^{-1}u)$. Using the consistency relation (6) in § 2 of [5] and noting that (1) remains valid with B replaced by B_n, we get

$$\begin{aligned}\mathsf{P}[\xi'B_n>r\mid B^c\xi']&\geqq\mathsf{P}[\xi'B=\xi'B_n>r\mid B^c\xi']\\&=\mathsf{P}[\xi'B_n>r\mid B_n^c\xi']\,P[\xi'(B\setminus B_n)=0\mid B^c\xi']\geqq p^{r+2}\,\mathsf{P}[\xi'B=0\mid B^c\xi'].\end{aligned}$$

Letting $n\to\infty$ and using the thinning nature of ξ', we hence obtain

$$\mathsf{P}[r<\xi'\pi^{-1}u<\infty\,|\,B^c\xi']=\mathsf{P}[\xi'(B\cap\pi^{-1}u)>r\,|\,B^c\xi']\geqq p^{r+2}\,\mathsf{P}[\xi'B=0\,|\,B^c\xi']>0.$$

But $\mathscr{U}$ being countable, there is then some fixed $B\in\mathscr{U}$ with

$$\mathsf{P}\{\mathsf{P}\Big[\bigcup_{u\in\Phi_m^d}\{r<\xi'\pi^{-1}u<\infty\}\,|\,B^c\xi'\Big]>0\}>0,$$

so

$$\mathsf{P}\bigcup_{u\in\Phi_m^d}\{r<\xi'\pi^{-1}u<\infty\}>0, \tag{2}$$

as asserted.

3. Our next aim is to show that, if (2) holds for some $m\geqq k$, there exists with positive probability some $u\in\Phi_m^d$ with $\eta\pi^{-1}u=\infty$ and $\xi'\pi^{-1}u=0$. To see this, fix ω and u such that $r<\xi'\pi^{-1}u<\infty$, and let $x_1,\ldots,x_{r'}$ be the atom positions of ξ' in $\pi^{-1}u$. Assume without loss that $u=\mathscr{L}(\pi x_1,\ldots,\pi x_{r'})$. Since $r'>r$, already $r'-1$ of the flats x_j, say $x_2,\ldots,x_{r'}$, have this property, which means that $\pi x_1\subset\mathscr{L}(\pi x_2,\ldots,\pi x_{r'})$. Since ξ' is simple, there exists some set $B\in\mathscr{U}$ such that $x_1\in B$ and $\xi'B=1$. The class $\mathscr{U}$ being countable, there must exist some fixed set $B\in\mathscr{U}$ such that, if $u_1,u_2,\ldots$ denote the random flats in Φ_m^d which are spanned by at most finitely many flats πx with $x\notin B$ and $\xi'\{x\}=1$, we have

$$\mathsf{P}\bigcup_n\{\xi'B=\xi'(B\cap\pi^{-1}u_n)=1\}>0.$$

(The measurability of this set may be established as in the proof of Theorem 3.1.) Since there are at most countably many flats u_n, we get the corresponding relation for one of them, say for u, i.e.

$$\mathsf{P}\{\xi'B=\xi'(B\cap\pi^{-1}u)=1\}>0. \tag{3}$$

(If $\{u_n\}=\emptyset$, take u to be a fixed flat with $\xi'\pi^{-1}u<\infty$.)

By the definition of conditional probabilities, the event in (3) implies a.s. that $\mathsf{P}[\xi'B=\xi'(B\cap\pi^{-1}u)=1\,|\,B^c\xi']>0$, and hence by (Σ) that $\mathsf{P}[\xi'B=0\,|\,B^c\xi']>0$. Thus by (3)

$$\mathsf{P}\{\mathsf{P}[\xi'B=0\,|\,B^c\xi']>0,\,\mathsf{P}[\xi'B=\xi'(B\cap\pi^{-1}u)=1\,|\,B^c\xi']>0\}>0,$$

so by Theorem 3.1 above and Theorem 3.1 in [5],

$$\begin{aligned}\mathsf{P}\{\eta\pi^{-1}u=\infty\}&\geqq\mathsf{P}\{\eta(B\cap\pi^{-1}u)>0\}\\&\geqq\mathsf{P}\{\xi'B=0,\,\mathsf{P}[\xi'B=0\,|\,B^c\xi']>0,\,\mathsf{P}[\xi'B=\xi'(B\cap\pi^{-1}u)=1\,|\,B^c\xi']>0\}\\&=\mathsf{E}[\mathsf{P}[\xi'B=0\,|\,B^c\xi'];\,\mathsf{P}[\xi'B=\xi'(B\cap\pi^{-1}u)=1\,|\,B^c\xi']>0]>0.\end{aligned}$$

Since $\xi'\pi^{-1}u<\infty$ holds by construction, this yields the desired result.

4. The degeneracy flat u for η constructed above cannot be minimal, since the relations $0<\xi'\pi^{-1}u<\infty$ would then contradict the first part of the proof, so there must exist with positive probability some flat $u\in\Phi_{m'}^d$ with $k\leqq m'<m$ such that $\eta\pi^{-1}u=\infty$ and $\xi'\pi^{-1}u=0$. We may thus apply the second part of the proof to show that (2) remains true with m replaced by some smaller number.

Continuing recursively, we ultimately get a contradiction to the hypothesis on η. This disproves (2) and shows at the same time that $\eta\pi^{-1}u=\infty$ and $\xi'\pi^{-1}u=0$ are incompatible. Combining this with the first part of the proof, it is seen that $\eta\pi^{-1}u=\infty$ a.s. implies $\xi\pi^{-1}u \geqq \xi'\pi^{-1}u=\infty$. Since (2) is equivalent to the corresponding relation for ξ, the proof of the first assertion is hence complete. To establish the second assertion, we may proceed as in the third part of the proof, except that $\xi'\pi^{-1}u$ is now allowed to be infinite. □

Just as in the diffuse case, we get unique degeneracy decompositions of ξ:

Corollary 3.4. *Let ξ be such as in Theorem 3.2. Then every ξ-atom belongs to unique minimal outer and maximal inner degeneracy flats.*

Proof. Suppose that ξ has with positive probability some atom position x such that πx belongs to two different minimal outer degeneracy flats. Then the same thing is true for ξ'. Arguing as in the third part of the preceding proof, it follows that, with positive probability, ξ' has two different degeneracy flats such that their intersection u satisfies $\eta\pi^{-1}u=\infty$ and $\xi'\pi^{-1}u<\infty$. But this contradicts Theorem 3.3. The proof for inner degeneracies is similar. □

4. The Absolutely Continuous Case

From here on, we shall mainly be concerned with the basic problem of proving a.s. invariance or Cox structure of a stationary diffuse random measure η or simple point process ξ respectively on M_k^d. By Theorem 5.1 in [5], every result of this type for η will immediately yield a corresponding result for ξ. For the sake of brevity, we shall usually omit the latter.

The present section is devoted to the case when the projections of η onto Φ_k^d are a.s. absolutely continuous with respect to some fixed measure μ. A basic role is then played by the following lemma, which shows that diffuseness alone of μ implies a certain weak invariance property. Note that, when M_1^{d+1} is identified with R^{2d}, the first component $q\in R^d$ is the point of intersection with a fixed flat $u\in M_d^{d+1}$, while the second component $p\in R^d$ is the slope, measured as the rate at which the projection on u changes relative to a change in (the signed) distance from u for a point moving along the line, (cf. [6]).

Lemma 4.1. *Let $\mu\in\mathfrak{M}_d(R^d)$, and let η be a stationary random measure on M_1^{d+1} such that $\eta(B\times\cdot)\ll\mu$ a.s. for all $B\in\mathscr{B}(R^d)$. Then there exists a measurable mapping $\varphi: R^d\to\Phi_1^d$ such that $\eta(\cdot\times dp)/\mu(dp)$ a.s. has a version η_p which is $\varphi(p)$-invariant for all $p\in R^d$.*

Proof. Let η_p be a stationary and measurable measure valued version of $\eta(\cdot\times dp)/\mu(dp)$, (cf. the proof of Theorem 3.1 in [3] for the existence). Choose a measure determining sequence $\{f_k\}\subset\mathscr{F}_c(R^d)$, and define $Y_k(q,p)\equiv(\eta_p*\delta_q)f_k$. Let us further introduce a net $\{I_{nj}\}$ (in the sense of [12], p. 208) in (R^d,μ), and for $p\in R^d$ and $n\in N$, write $I_n^{(p)}$ for the unique set I_{nj} containing p. Define $\mu_n^{(p)} = I_n^{(p)}\mu/\mu I_n^{(p)}$ whenever $\mu I_n^{(p)}>0$. Writing $g(x)\equiv 1-e^{-|x|}$, we get by Theorem 5 in [12], p. 220, for every $\omega\in\Omega$ and $k\in N$

$$\begin{aligned}&\int g(Y_k(0,p)-Y_k(0,x))\,\mu_n^{(p)}(dx)\\&\quad\leqq\int|Y_k(0,p)-Y_k(0,x)|\,\mu_n^{(p)}(dx)\to 0,\qquad p\in R^d \text{ a.e. } \mu.\end{aligned}$$

By Fubini's theorem and dominated convergence, we hence obtain for $p \in R^d$ a.e. μ

$$\int \sum_k 2^{-k} \mathsf{E} g(Y_k(0,p) - Y_k(0,x))\, \mu_n^{(p)}(dx) \to 0. \tag{1}$$

Fix an arbitrary $p \in R^d$ satisfying (1). Since (1) may be interpreted as convergence in $L_1(\prod_n \mu_n^{(p)})$, we may conclude that the integrand converges along some fixed subsequence for almost every point in that space. The coordinates of the point being a.s. non-zero, we may hence fix $h_1, h_2, \ldots \in R^d \setminus \{0\}$ such that

$$\sum_k 2^{-k} \mathsf{E} g(Y_k(0,p) - Y_k(0, p+h_n)) \to 0,$$

and therefore

$$Y_k(0, p+h_n) \xrightarrow{\mathsf{P}} Y_k(0,p), \qquad k \in N. \tag{2}$$

Assume without loss that $h_n/|h_n| \to$ some $q \in R^d \setminus \{0\}$, fix a $t>0$, and put $r_n \equiv t/|h_n|$. Write

$$\begin{aligned} &|Y_k(tq,p) - Y_k(0,p)| \\ &\quad \leqq |Y_k(tq,p) - Y_k(r_n h_n, p)| + |Y_k(r_n h_n, p) - Y_k(r_n h_n, p+h_n)| \\ &\qquad + |Y_k(r_n h_n, p+h_n) - Y_k(0,p)| = \vartheta_1 + \vartheta_2 + \vartheta_3. \end{aligned}$$

Here $\vartheta_1 \to 0$ since $r_n h_n \to tq$ while $Y_k(\cdot, p)$ is continuous. Using (2) and the space and time stationarity of Y_k, it is further seen that

$$\vartheta_2 \stackrel{d}{=} |Y_k(0,p) - Y_k(0,p+h_n)| \xrightarrow{\mathsf{P}} 0,$$
$$\vartheta_3 \stackrel{d}{=} |Y_k(-r_n p, p+h_n) - Y_k(-r_n p, p)| \stackrel{d}{=} |Y_k(0,p+h_n) - Y_k(0,p)| \xrightarrow{\mathsf{P}} 0.$$

Hence $Y_k(tq,p) = Y_k(0,p)$ a.s., and since the f_k are measure determining while t was arbitrary, it follows that η_p is a.s. $\mathscr{L}(q)$-invariant.

Next we define

$$f(q,p) = \int_0^1 \sum_k 2^{-k} \mathsf{E} g(Y_k(tq,p) - Y_k(0,p))\, dt, \qquad q, p \in R^d,$$

and note that η_p is a.s. $\mathscr{L}(q)$-invariant iff $f(q,p)=0$. For fixed $p \in R^d$, the set of all such q is clearly a linear subspace A_p of R^d, and we have just shown that $\dim A_p \geqq 1$ a.e. μ. Since $f(q,p)$ is clearly continuous in q for fixed p and measurable in p for fixed q, the set $\{p \in R^d : A_p \cap F = \emptyset\}$ is measurable for every closed set $F \subset R^d$. In fact, choosing compact sets C_n with union F and letting $\{q_{nj}\}$ be dense in C_n for each n, it is seen that

$$\begin{aligned} \{p: A_p \cap F = \emptyset\} &= \{p: f(q,p) > 0, q \in F\} = \bigcap_n \{p: f(q,p)>0, q \in C_n\} \\ &= \bigcap_n \{p: \inf_j f(q_{nj}, p) > 0\} = \bigcap_n \bigcup_k \bigcap_j \left\{p: f(q_{nj}, p) > \frac{1}{k}\right\}. \end{aligned}$$

Writing S for the unit sphere in R^d, we may hence define a measurable mapping $k: R^d \to N$ by

$$k(p) = \min\{k: \dim A_p \cap (R^k \times \{0\}^{d-k}) = 1\} = \min\{k: A_p \cap S \cap (R_k \times \{0\}^{d-k}) \neq \emptyset\}, \qquad p \in R^d.$$

We finally put

$$\varphi(p) = A_p \cap (R^{k(p)} \times \{0\}^{d-k(p)}), \qquad p \in R^d,$$

and note that φ is measurable since, whenever F is a closed subset of some fixed M^d_{d-1}-flat, $\varphi(p)$ intersects F iff p belongs to

$$\{p: A_p \cap (R^{k(p)} \times \{0\}^{d-k(p)}) \cap F \neq \emptyset\} = \bigcup_k \{p: k(p) = k, A_p \cap (R^k \times \{0\}^{d-k}) \cap F \neq \emptyset\}.$$

Moreover, η_p is clearly a.s. $\varphi(p)$-invariant for $p \in R^d$ a.e. μ, and since the $(\Omega \times R^d)$-set where η_p is $\varphi(p)$-invariant is measurable (cf. Exercise 10.10 in [2]), Fubini's theorem ensures the existence of a (possibly different) version of η_p with the desired property. □

In the case of an absolutely continuous conditional intensity (possibly after thinning), regularity of a point process ξ on M^d_k is enough to ensure Cox structure. Note, however, that the directing random measure of ξ will not be invariant in general.

Theorem 4.2. *Let $\mu \in \mathfrak{M}_d(\Phi^d_k)$, and let ξ be a stationary simple point process on M^d_k whose conditional intensity η satisfies $(B\eta)\pi^{-1} \ll \mu$ a.s. for every $B \in \mathscr{B}(M^d_k)$. Then ξ is a Cox process directed by η.*

Proof. By Lemma 2.1 and an obvious approximation argument based on Exercise 4.5 in [2], it is enough to consider the case $k=1$, so we may assume that ξ is a point process on M^{d+1}_1. Let φ and η_p be such as in Lemma 4.1, and fix $u \in \Phi^d_{d-1}$ such that $\varphi(p) \not\subset u$, $p \in R^d$ a.e. μ. (Cf. Lemma 2.1. Only $p \in R^d$ with $\varphi(p) \not\subset u$ will be considered below.) Choose a unit vector e in R^d with $e \perp u$, and write $a_p \equiv (e+u) \cap \varphi(p)$. Define

$$h(q,p) = (f_p(q), p) = (q + (e\,q)(e - a_p), p), \qquad q, p \in R^d, \tag{3}$$

and note that h is measurable and has the unique measurable inverse

$$h^{-1}(q,p) = (q - (e\,q)(e - a_p), p), \qquad q, p \in R^d. \tag{4}$$

From (3) and the disintegration $\eta = \int (\eta_p \times \delta_p)\,\mu(dp)$ we get

$$\eta h^{-1} = \int (\eta_p \times \delta_p) h^{-1} \mu(dp) = \int (\eta_p f_p^{-1} \times \delta_p)\,\mu(dp). \tag{5}$$

Writing S_x for a shift in R^d by the vector x, we further obtain for any $t \in R$ and $q, p \in R^d$

$$f_p \circ S_{ta_p}(q) = q + t\,a_p + [e(q + t\,a_p)](e - a_p) = q + (e\,q)(e - a_p) + t\,e = S_{te} \circ f_p(q), \tag{6}$$

so by the $\varphi(p)$-invariance of η_p,

$$\eta_p f_p^{-1} S_{te}^{-1} = \eta_p S_{ta_p}^{-1} f_p^{-1} = \eta_p f_p^{-1}, \qquad t \in R,\ p \in R^d \text{ a.e. } \mu,$$

outside a fixed P-null set. Hence by (5), ηh^{-1} is a.s. $\mathscr{L}(e)$-invariant. Now ηh^{-1} is the conditional intensity of ξh^{-1} (cf. [5], p. 216), so by Theorem 5.1 in [5], ξh^{-1} is a Cox process directed by ηh^{-1}. Applying the inverse mapping (4) to both ξh^{-1} and ηh^{-1}, it follows easily (e.g. by using Laplace transforms [2]) that ξ is a Cox process directed by η. □

Diffuseness of the projections of η is not enough to guarantee a.s. invariance in all directions. In fact, the counterexamples of Papangelou [11] show that we must also exclude the possibility of outer degeneracies. The following theorem implies that the resulting assertion is true at least in the absolutely continuous case.

Theorem 4.3. *Let $\mu \in \mathfrak{M}(\Phi_k^d)$ and $y \in \Phi_1^d$. Then every stationary random measure η on M_k^d satisfying $(B\eta)\pi^{-1} \ll \mu$ a.s. for all $B \in \mathscr{B}(M_k^d)$ is a.s. y-invariant, iff y lies in every $v \in \Phi_{d-1}^d$ with $\mu v > 0$.*

Proof. Let y be such as stated. The outer degeneracy decomposition of μ induces a corresponding decomposition of η, and it is clearly enough to prove the y-invariance of each component. We may therefore assume that μ is supported by the set of flats in some $v \in \Phi_m^d$ with $y \subset v$, and that v contains no proper degeneracy subspaces. But in that case it sufficies to prove the invariance of the projections onto v of the restrictions of η to any v-parallel slices. Thus we may assume without loss that $v = R^d$, i.e. that μ has no outer degeneracies of order $< d$, and prove that η is then a.s. invariant in all directions.

According to Lemma 2.1 applied recursively $k-1$ times, we may choose a flat $u \in \Phi_{d-k+1}^d$ with $\sigma_u x \in \Phi_1^{(u)}$ for $x \in \Phi_k^d$ a.e. μ, and such that moreover $\mu \sigma_u^{-1} v = 0$ for every $v \in \Phi_{d-k}^{(u)}$. Since clearly $\sigma_u x \in \Phi_1^{(u)}$ implies $\sigma_u y \in M_1^{(u)}$ for every $y \in M_k^d$ with $\pi y = x$, η is a.s. such that $\sigma_u y \in M_1^{(u)}$ a.e. η. Thus the function $y \to (\sigma_u y, \pi y)$ maps η into a random measure η' on $(M_1^{(u)} \times \Phi_k^d)$, whereas σ_u maps μ into some measure $\mu' \equiv \mu \sigma_u^{-1}$ on $\Phi_1^{(u)}$ with $\mu' v = 0$ for all $v \in \Phi_{d-k}^{(u)}$. Moreover, η' is clearly u-stationary, and we have $\eta'(B \times \cdot \times \Phi_k^d) \ll \mu'$ a.s. for all $B \in \mathscr{B}(M_1^{(u)}/\Phi_1^{(u)})$. If the theorem is true for $k=1$, we may conclude that $(M_1^{(u)} \times A)\eta'$ is a.s. invariant for every $A \in \mathscr{B}(\Phi_k^d)$, and hence that η' itself is a.s. invariant. By Lemma 2.2 in [3], this yields the a.s. invariance of η. We may thus assume from now on that $k=1$.

We now proceed by induction in d. For $d=2$, the statement follows from Lemma 4.1 above and Lemma 2.2 in [3], or from Theorem 3.2 in [3]. Next assume that the statement is true for a certain $d \geqq 2$, and let η and μ fulfill the hypothesis for $d+1$ in place of d. Let us identify M_1^{d+1} with R^{2d}, and note that $\mu R^d < \infty$. We may assume that $\eta \ll \lambda_d \times \mu$, since we may otherwise consider $\eta * v$ in place of η for an arbitrary probability measure v with bounded support on R^d, and then truncate the density at a fixed level, (cf. Lemma 2.5 in [6]). Let φ and η_p be such as in Lemma 4.1. Defining $\mu' \in \mathfrak{M}(M_1^d)$ by $\mu' B \equiv \mu\{p: p + \varphi(p) \in B\}$, we get by the assumption

$$\mu' v \leqq \mu v = 0, \qquad v \in M_{d-1}^d,$$

so by Lemma 2.1 we may choose a flat $u \in M_{d-1}^d$ such that

$$\mu\{p: (p+\varphi(p)) \cap u \in v\} = \mu'\{x: x \cap u \subset v\} \equiv 0, \qquad v \in M_{d-2}^{(u)}. \tag{7}$$

Write $u=\pi u+ce$, where $e\in R^d$ with $|e|=1$ and $e\perp u$ while $c\in R$, put $a_p\equiv(\pi u+e)\cap\varphi(p)$, and define

$$f(q,p)=(f_p(q),g(p))=(q+(eq)(e-a_p),p+(ep-c)(e-a_p)-ce),\qquad q,p\in R^d.$$

Note that ηf^{-1} is a.s. locally finite, since clearly $\lambda f_p^{-1}=\lambda$ for all $p\in R^d$, and therefore

$$\begin{aligned}(\lambda\times\mu)f^{-1}&=\int(\lambda\times\delta_p)f^{-1}\mu(dp)\\&=\int(\lambda f_p^{-1}\times\delta_{g(p)})\mu(dp)=\int(\lambda\times\delta_{g(p)})\mu(dp)=\lambda\times\mu g^{-1}.\end{aligned}\tag{8}$$

Defining $\{T_t\}$ and $\{S_x\}$ by $T_t(q,p)=(q+tp,p)$ and $S_x(q,p)=(q+x,p)$, we further obtain

$$\begin{aligned}T_t\circ f(q,p)&=(q+(eq)(e-a_p)+tp+t(ep-c)(e-a_p)-tce,g(p))\\&=(q+t(p-ce)+[e(q+t(p-ce))](e-a_p),g(p))\\&=f(q+t(p-ce),p)=f\circ S_{-tce}\circ T_t(q,p),\end{aligned}$$

and since the shifts in M_1^{d+1} correspond to arbitrary combinations of the transformations T_t and S_x in R^{2d} (cf. [6]), we get by the stationary of η

$$\eta f^{-1}T_t^{-1}=\eta T_t^{-1}S_{-tce}^{-1}f^{-1}\overset{d}{=}\eta f^{-1},\qquad t\in R.\tag{9}$$

Next we obtain for arbitrary $x\in\pi u$

$$S_x\circ f(q,p)=(q+x+(eq)(e-a_p),g(p))=(q+x[e(q+x)](e-a_p),g(p))=f\circ S_x(q,p),$$

so

$$\eta f^{-1}S_x^{-1}=\eta S_x^{-1}f^{-1}\overset{d}{=}\eta f^{-1},\qquad x\in\pi u.\tag{10}$$

Finally it may be seen as in (6) that

$$f\circ S_{ta_p}(q,p)=S_{te}\circ f(q,p),\qquad t\in R,\ q,p\in R^d,$$

so

$$(\eta_p\times\delta_p)f^{-1}S_{te}^{-1}=(\eta_p\times\delta_p)S_{ta_p}^{-1}f^{-1}=(\eta_p\times\delta_p)f^{-1},\qquad t\in R,\ p\in R^d,\quad\text{a.s.},$$

and hence

$$\eta f^{-1}S_x^{-1}=\eta f^{-1},\qquad x\perp u,\quad\text{a.s.}\tag{11}$$

By (11) there exists a random measure ζ on $u\times R^d$ such that $\eta f^{-1}=\lambda_1\times\zeta$ a.s. Letting r denote projection onto u in the p-component, we get by (8)

$$\zeta r^{-1}\ll(\lambda_{d-1}\times\mu g^{-1})r^{-1}=\lambda_{d-1}\times\mu g^{-1}r^{-1}.$$

Now

$$\begin{aligned}r\circ g(p)&=p+(ep-c)(e-a_p)-ce-(ep-c)e=p-(ep-c)a_p-ce\\&=(p+\varphi(p))\cap u-ce,\end{aligned}$$

so by (7) we obtain $\mu g^{-1}r^{-1}v=0$ for all $v\in M_{d-2}^{(\pi u)}$. Moreover, it is seen from (9) and (10) that ζr^{-1} is stationary when considered as a random measure on M_1^d.

Thus it follows from the induction hypothesis that ζr^{-1} is a.s. invariant, and this implies in turn the a.s. invariance of

$$\eta h^{-1}(\cdot \times R^d) = \eta f^{-1}(\cdot \times R^d) = (\lambda_1 \times \zeta)(\cdot \times R^d),$$

where h is given by (3). Applying this result to $(R^d \times B)\eta$ in place of η, it is seen that $\eta h^{-1}(\cdot \times B) = [(R^d \times B)\eta] h^{-1}(\cdot \times R^d)$ is a.s. invariant for arbitrary $B \in \mathscr{B}(R^d)$, and since ηh^{-1} is determined by countably many such projections, it must be a.s. invariant itself. We may finally apply the inverse mapping h^{-1} in (4) and proceed as in (8) to conclude that η is a.s. invariant.

To prove the assertion in the converse direction, let $\mu \in \mathfrak{M}(\Phi_k^d)$ and $v \in \Phi_{d-1}^d$ be such that $\mu v > 0$. Choose a unit vector $e \in R^d$ with $e \perp v$. Let α be a uniformly distributed random variable on $(0, 2\pi)$, and define the stationary random process Y on M_k^d by

$$Y(x) = \sin^2(\alpha + (e x) e) \cdot 1\{\pi x \subset v\}, \qquad x \in M_k^d,$$

where $e x$ denotes the projection of x onto the e-axis, (which is clearly unique when $\pi x \subset v$). Writing λ for Lebesgue measure on $M_k^d/\Phi_k^d \sim R^{d-k}$, we next define $\eta = Y(\lambda \times \mu)$ and note that η is stationary but a.s. not y-invariant. □

We finally remark that the usefulness of the above results depends on the possibility of proving that a given random measure, such as the conditional intensity of a simple point process, is a.s. absolutely continuous. Thus Problem 6 of Papangelou [10], p. 630, gains in importance. Some sufficient conditions for absolute continuity are given in [8].

5. Processes of $(d-2)$-flats in R^d

According to Theorem 3.2 in [3], a stationary first order random measure η on M_k^d is a.s. invariant, provided that

$$\mathsf{P}\{\mathscr{L}(\pi x, \pi y) = R^d, (x, y) \in (M^d)^2 \text{ a.e. } \eta^2\} = 1. \tag{1}$$

Conversely, (1) clearly implies that $k \geqq d/2$ and that the projections of η onto Φ_k^d have a.s. no outer degeneracies. When $k = d-1$, the latter condition is also sufficient, but for general k it may be hard to see whether (1) is fullfilled. In this section we shall consider the case when $k = d-2 \geqq 2$, and show that (1) is then equivalent to the absence of degeneracies of order $\geqq d-3$.

Theorem 5.1. *Let $d \geqq 4$, and let η be a stationary first order random measure on M_{d-2}^d such that a.s. $\eta \pi^{-1} v = 0$ for all $v \in \Phi_{d-1}^d \cup \Phi_{d-3}^d$. Then η is a.s. invariant.*

Proof. By Lemma 2.1 and Fubini's theorem, almost every flat $u \in M_4^d$ is such that the hypothesis is fulfilled (with $d=4$) for the random measure $\eta' = \eta \sigma_u^{-1}$ on $M_2^{(u)}$. If the theorem is true for $d=4$, we may conclude that η' is a.s. u-invariant, and it will follow from Lemma 2.2 in [3] that η is a.s. invariant. We may thus assume that $d=4$.

Write $\Phi_2^4 = \Phi$, and define

$$S = \{(x, y) \in \Phi^2 \colon \mathscr{L}(x, y) \neq R^4\}. \tag{2}$$

Let $A \in \mathscr{B}(M_2^4)$ be arbitrary, and put $\zeta = (A\eta)\pi^{-1}$. For fixed $\omega \in \Omega$, there exists by Lemma 2.1 some $v \in M_3^4 \setminus \Phi_3^4$ with $\sigma_v x \in M_1^{(v)}$, $x \in \Phi$ a.e. ζ. Consider distinct flats $x, y, z \in \Phi$ such that $\sigma_v x, \sigma_v y, \sigma_v z \in M_1^{(v)}$, and note that $(x, y) \in S$ iff $\sigma_v x$ and $\sigma_v y$ lie in a common 2-flat, i.e. iff they intersect or are parallel. Thus, if x, y, z are such that all three pairs belong to S, then the lines $\sigma_v x$, $\sigma_v y$, $\sigma_v z$ must either lie in a common 2-flat, go through a common point, or be parallel. This means that x, y, z must either lie in a common 3-flat or go through a common line. But according to our hypothesis and Fubini's theorem, both possibilities are a.s. excluded almost everywhere with respect to ζ^3. Therefore $(x, y) \in S$ implies that either $(x, z) \in S^c$ or $(y, z) \in S^c$ a.e. ζ^3, a.s. P. This shows in particular that

$$(\zeta B)^2 \leqq 3\zeta^2(B^2 \setminus S) \text{ a.s.,} \qquad B \in \mathscr{B}(\Phi). \tag{3}$$

In fact, if instead $\zeta^2(S|B^2) > 3/2$, the above result would a.s. yield the contradiction

$$\tfrac{2}{3} < \zeta^2(S|B^2) \leqq 2\zeta^2(S^c|B^2) = 2(1 - \zeta^2(S|B^2)) < \tfrac{2}{3}.$$

Assuming without loss that η is ergodic, it may be seen from the proof of Theorem 3.2 in [3] that $\mathsf{E}\zeta^2 = (\mathsf{E}\zeta)^2$ on S^c. Combining this with (3), we get for any $B \in \mathscr{B}(\Phi)$

$$\mathsf{E}(\zeta B)^2 \leqq 3\,\mathsf{E}\zeta^2(B^2 \setminus S) = 3(\mathsf{E}\zeta)^2(B^2 \setminus S) \leqq 3(\mathsf{E}\zeta)^2 B^2 = 3(\mathsf{E}\zeta B)^2.$$

Letting $\{B_j\} \subset \mathscr{B}(\Phi)$ be an arbitrary partition of Φ, it follows that

$$\sum_j \|\zeta B_j\|_2 \leqq \sqrt{3} \sum_j \mathsf{E}\zeta B_j = \sqrt{3}\,\mathsf{E}\zeta\Phi = \sqrt{3}\,\mathsf{E}\eta A < \infty,$$

and since this bound is independent of $\{B_j\}$, we may conclude from Theorem 1 in [8] that $\zeta \ll \mathsf{E}\zeta$ a.e. Thus Theorem 4.3 applies, showing that η is a.s. invariant. □

The last theorem combines with Theorem 3.3 above and Theorem 5.1 in [5] to yield the

Corollary 5.2. *Let $d \geqq 4$, and let ξ be a stationary first order simple and regular point process on M_{d-2}^d such that a.s. $\xi\pi^{-1}v < \infty$ for all $v \in \Phi_{d-1}^d \cup \Phi_{d-3}^d$. Then ξ is a Cox process directed by some a.s. invariant random measure.*

Theorem 5.1 suggests that any measure $\mu \in \mathfrak{M}(\Phi_2^4)$ with neither inner nor outer degeneracies might satisfy $\mathscr{L}(x, y) = R^4$ a.e. μ^2, but the following example shows that this is false. Let ν be the uniform probability measure on the set of lines

$$x = \cos\varphi \pm z\sin\varphi, \quad y = \sin\varphi \mp z\cos\varphi, \qquad \varphi \in [0, 2\pi),$$

in R^3, imbed R^3 into R^4 as a flat $u \in M_3^4 \setminus \Phi_3^4$, and define $\mu \in \mathfrak{M}(\Phi_2^4)$ by $\mu = \nu\mathscr{L}^{-1}$. Then μ is clearly non-degenerate, and still $\mu^2 S = 1/2$, S being defined by (2). In this case, however, μ can be decomposed into two measures which both have the stated property. This reflects the general situation:

Theorem 5.3. *Let $\mu\in\mathfrak{M}(\Phi_2^4)$ satisfy $\mu v=0$, $v\in\Phi_3^4\cup\Phi_1^4$. Then μ is the sum of measures $\mu_1, \mu_2, \ldots$ such that $\mathscr{L}(x,y)=R^4$ a.e. $\sum\mu_n^2$.*

Proof. Write $\Phi=\Phi_2^4$, and put

$$A_1=\{x\in\Phi\colon \mathscr{L}(x,y)=R^4,\ y\in\Phi \text{ a.e. } \mu\}.$$

By Fubini's theorem, we may take $A_1\mu$ as our μ_1, and so we may henceforth assume that μ is supported by A_1^c, i.e. that

$$f(x)\equiv\mu\{y\in\Phi\colon \mathscr{L}(x,y)\neq R^4\}>0, \qquad x\in\Phi \text{ a.e. } \mu.$$

Clearly $\mu\ll v\equiv f\mu$. For any finite measurable partition $\mathscr{I}$ of Φ, let U be a best possible approximation in $L_1(\mu^2)$ of S in (2) by a union of rectangles $I\times J\in\mathscr{I}^2$. By a monotone class argument, the error can be made smaller than any prescribed $\varepsilon>0$, provided that $\mathscr{I}$ is fine enough. Letting B be the projection of U onto one of the component spaces Φ, we then get

$$v B^c=\int_{B^c}\mu(dx)\int 1_S(x,y)\,\mu(dy)=\mu^2(S\cap(B^c\times\Phi))\leqq\mu^2(S\cap U^c)<\varepsilon,$$

and since $\mu\ll v$, we have $\mu B^c\leqq 1/2$ for sufficiently small ε. Repeat this procedure on $(B^c)^2$ with a remainder of μ-measure $\leqq 1/4$, and continue recursively to produce a countable partition of almost all Φ into sets B_j, each being such that there exists a set C_j with $\mu^2(B_j\times C_j)>0$ and $\mu^2(S|B_j\times C_j)\geqq 1/2$.

Fix j, and put $B_j=B$, $C_j=C$. Let $\varepsilon>0$ be arbitrary, and write

$$C_\varepsilon=\{y\in C\colon \mu^2(S|B\times y)>\tfrac{1}{2}-\varepsilon\}, \tag{3}$$

$\mu^2(S|B\times\cdot)$ being defined in the obvious way as a Radon-Nikodym derivative. By Fubini's theorem,

$$\begin{aligned}\tfrac{1}{2}\mu^2(B\times C)\leqq\mu^2(S\cap(B\times C))&=\int_C\mu^2(S|B\times y)\,\mu^2(B\times dy)\\ &\leqq(\tfrac{1}{2}-\varepsilon)\,\mu^2(B\times C_\varepsilon^c)+\mu^2(B\times C_\varepsilon)\\ &=(\tfrac{1}{2}-\varepsilon)\,\mu^2(B\times C)+(\tfrac{1}{2}+\varepsilon)\,\mu^2(B\times C_\varepsilon),\end{aligned}$$

so

$$\mu^2(B\times C_\varepsilon|B\times C)\geqq\frac{2\varepsilon}{1+2\varepsilon}>0.$$

Thus the properties of C carry over to C_ε.

As before, we may approximate $S\cap(B\times C_\varepsilon)$ in $L_1(\mu^2)$ by a finite union U' of rectangles $B_i'\times C_j'$, $B_i'\subset B$, $C_j'\subset C$, such that the *relative* error on $B\times C_\varepsilon$ is $<\varepsilon$. Then the relative error on $B\times C_j'$ is also $<\varepsilon$ for at least one j. Writing $U'\cap(B\times C_j')=B''\times C''$, we hence get by (3) for any $\varepsilon<1/12$

$$\mu(B''|B)=\mu^2(U'|B\times C_j')\geqq\mu^2(S|B\times C_j')-\varepsilon\geqq\tfrac{1}{2}-2\varepsilon>\tfrac{1}{3},$$

and moreover

$$\mu^2(S^c|B''\times C'')=\frac{\mu^2(S^c\cap U'|B\times C_j')}{\mu^2(U'|B\times C_j')}<\frac{\varepsilon}{1/3}=3\varepsilon.$$

Continuing recursively, we may construct sequences $B_n'' \subset B$ with $B_n'' \downarrow A$, say, and $C_n'' \subset C$ with $\mu C_n'' > 0$, such that $\mu(B_n''|B) > 1/3$ and $\mu^2(S|B_n'' \times C_n'') \to 1$, and therefore

$$\mu(A|B) \geqq \tfrac{1}{3}, \qquad \mu^2(S|A \times C_n'') \to 1. \tag{4}$$

In this way we have obtained a countable collection of disjoint sets $A \subset \Phi$ as above with total mass $\geqq \mu\Phi/3$, and we may clearly continue the construction recursively to cover almost all of Φ by such sets $A_1, A_2, \ldots$ Put $\mu_j \equiv A_j \mu$.

It remains to prove that $\mu_j^2 S \equiv 0$. To see this, note as in the proof of Theorem 5.1 that $(x, y) \in S$ implies $(x, z) \in S^c$ or $(y, z) \in S^c$ a.e. μ^3, and use (4) to conclude that for any $A = A_j$

$$\mu^2(S|A^2) = \mu^3(S \times C_n''|A^2 \times C_n'') \leqq 2\mu^3(A \times S^c|A^2 \times C_n'') = 2\mu^2(S^c|A \times C_n'') \to 0. \quad \square$$

Note that Theorem 5.1 can not be proved directly from Theorem 5.3, since the η-components provided by the latter result will not be stationary in general.

6. Free Particles in $R \times [0, 1)^d$

As in [6], a line process in R^{d+1} may be interpreted as describing, in a space-time diagram, the motion of a system of free particles (i.e. of non-interacting particles moving with constant velocities) in R^d. In this setting, one of our basic problems becomes that of finding conditions which ensure a space and time stationary particle system to be a Cox process. A complete solution of the corresponding (but much more elementary) problem for free particles on the torus $K^d = [0, 1)^d$ is implicit in §4 of [7]. In the present section, we treat the intermediate case of free particles in $K^d \times R$.

Formally, we introduce the (one particle) phase space $S = (K^d \times R) \times R^{d+1}$ with elements

$$(q, p) = (q', q'', p', p'') = (q_1, \ldots, q_d, q'', p_1, \ldots, p_d, p''),$$

and define the flow $\mathscr{T} = \{T_t\}$ on S by $T_t(q, p) = (q + tp, p)$, where addition in K is modulo 1. Let $\mathscr{S}$ and $\mathscr{S}''$ denote the groups of shifts in q and q'' respectively. Let $\pi(q, p) \equiv p$, and write $\pi_x y = xy$ for the inner product of x and y.

Theorem 6.1. *Let η be a $\mathscr{T}$- and $\mathscr{S}''$-stationary first order random measure on S such that a.s. $\eta \pi^{-1} \pi_x^{-1}\{r\} = 0$, $x \in Z^d \times R \setminus \{0\}$, $r \in R$. Then η is a.s. $\mathscr{S}$-invariant.*

Proof. 1. Assume without loss that η is ergodic. Projecting η onto the (q'', p'')-plane yields a random measure η' on R^2 which is again stationary in q'' as well as under the induced flow. Moreover, the hypothesis on η implies that $\eta' \pi^{-1}$ is a.s. diffuse. By Theorem 3.2 in [3], η' is then a.s. invariant, and η' being ergodic, it may a.s. be written in the form $\lambda \times \mu''$, where λ is Lebesgue measure on R while $\mu'' \in \mathfrak{M}(R)$. Applying this result to $(K^d \times R \times A)\eta$ for arbitrary $A \in \mathscr{B}(R^{d+1})$ shows that η has a.s. the projection $\lambda \times \mu$ on R^{d+2} for some $\mu \in \mathfrak{M}(R^{d+1})$.

2. Let us now assume that $d = 1$. Construct a random measure ζ on R^4 by continuing η periodically outside S, and note that, as far as ζ is concerned, the

periodic continuation of $\mathscr{T}$ is equivalent to the flow of free motion of particles in R^2. Translating ζ randomly in the K-direction according to the uniform distribution on K, we next obtain a random measure ζ' on R^4 which is stationary in both space components. The first part of the proof now implies that $\zeta'(B \times \cdot) \ll \mu$ a.s. for all $B \in \mathscr{B}(R^2)$, and by the hypothesis on η it is further seen that $\mu v = 0$ for all $v \in M_1^2$. We may thus conclude from Theorem 4.3 that ζ' is a.s. invariant, and this yields the asserted invariance of η.

3. Turning to the case $d > 1$, let the vector $z \in Z^d \setminus \{0\}$ be arbitrary, and let n be the greatest common divisor of its components. Then there exists a matrix $M \in Z^{d^2}$ with determinant 1 and with z/n as its first row. Define a mapping of S onto itself by the matrix

$$L = \begin{pmatrix} M & & & 0 \\ & 1 & & \\ & & M & \\ 0 & & & 1 \end{pmatrix},$$

the K-components being reduced modulo 1, and use the same notation for the induced mappings on the spaces of $p \in R^{d+1}$ and $(q'', p) \in R^{d+2}$. It is easily verified that L commutes with the T_t and with all q''-shifts, and hence that the random measure $\eta'' = \eta L^{-1}$ is again $\mathscr{T}$- and $\mathscr{S}''$-stationary. Noting that L^T (the transpose of L) maps $Z^d \times R \setminus \{0\}$ onto itself, we further obtain

$$\eta'' \pi^{-1} \pi_x^{-1}\{r\} = \eta \pi^{-1} \pi_{L^T x}^{-1}\{r\} = 0, \qquad x \in Z^d \times R \setminus \{0\},\ r \in R,$$

so the non-degeneracy condition on η is fulfilled by η'' also. We may thus conclude from the second part of the proof that the projection of $(K^d \times A)\eta''$ onto the first component space K is a.s. invariant for any $A \in \mathscr{B}(R^{d+2})$. For any such A, we hence obtain

$$\int_{K^d \times A} e^{2\pi i z q'} \eta(dq\, dp) = \int_{K^d \times L(A)} e^{2\pi i n q_1} \eta''(dq\, dp) = 0 \quad \text{a.s.}$$

Since this is true for every $z \in Z^d \setminus \{0\}$, $\eta(\cdot \times A)$ must be a.s. invariant, and A being arbitrary, it follows that η is a.s. invariant under any rotations of K^d. According to the first part of the proof, it is then a.s. invariant under $\mathscr{S}''$ also. □

The corresponding (and best possible) result for the phase $S' = K^d \times R^d$ with the induced flow $\mathscr{T}'$ follows easily:

Corollary 6.2. *Let η be a $\mathscr{T}'$-stationary random measure on S' such that a.s. $\eta \pi^{-1} \pi_x^{-1}\{r\} = 0$, $x \in Z^d \setminus \{0\}$, $r \in R$. Then η is a.s. invariant.*

Proof. Apply Theorem 6.1 to $\eta \times \lambda \times \mu / \eta S'$ for arbitrary $\mu \in \mathfrak{M}_d(R)$, or use Theorem 4.1 in [7]. □

For point processes on S we get in analogy with our results on M_k^d the

Corollary 6.3. *Let ξ be a $\mathscr{T}$- and $\mathscr{S}''$-stationary first order simple and regular point process on S such that a.s. $\xi \pi^{-1} \pi_x^{-1}\{r\} < \infty$, $x \in Z^d \times R \setminus \{0\}$, $r \in R$. Then ξ is a Cox process directed by some a.s. $\mathscr{S}$-invariant random measure.*

Proof. Imitate the proofs of Theorems 3.1 and 3.3, or proceed as follows. Since the conditional intensity η of some p-thinning ξ' of ξ is a.s. diffuse by assumption, it satisfies the integral equation in §3 of [5] with ξ' in place of ξ, and invariant ξ'-events being tails events, it follows that η a.s. remains the conditional intensity of ξ', even after conditioning on the σ-field of such events. When discussing pair (ξ', η), we may thus assume that ξ' is ergodic.

In that case η is ergodic also (cf. §3 in [8]), and by the first part of the proof of Theorem 6.1 it is seen that the projection of η onto R^{d+2} a.s. equals $\lambda \times \mu$ for some $\mu \in \mathfrak{M}(R^{d+1})$. If the non-degeneracy assumption in Theorem 6.1 were violated, it would follow by Theorem 4.2 in [5] that

$$\mathsf{E}\,\xi' \pi^{-1} \pi_x^{-1}\{r\} = \mathsf{E}\,\eta\, \pi^{-1} \pi_x^{-1}\{r\} \neq 0$$

for some $x \in Z^d \times R \setminus \{0\}$ and $r \in R$. By Fatou's lemma and the stationarity of ξ', this would imply $\mathsf{P}\{\xi' \pi^{-1} \pi_x^{-1}\{r\} = \infty\} > 0$, contradicting the hypothesis on ξ. Thus Theorem 6.1 applies, showing that η is a.s. $\mathscr{S}$-invariant, and our assertion follows from Theorem 5.1 in [5] and the fact that ξ and ξ' are simultaneously Cox. □

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