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On Invariance Principles with Limit Processes Satisfying Strong Laws*

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1. Introduction

This paper investigates convergence in distribution $U_n \Rightarrow U$ of stochastic processes in D when the limit process U satisfies, e.g., the strong law of large numbers or the law of the iterated logarithm. Usually the linear space $D = D[0, \infty)$ of real-valued functions defined on $[0, \infty)$ that are right-continuous and have left-hand limits, is equipped with one of the Skorohod topologies $S \in \{J_1, J_2, M_1, M_2\}$ [13]. Let $D[0, T]$ be the space of restrictions of functions $x \in D$ to $[0, T]$. For convenience we write x for both an element of D and its restriction to $[0, T]$. By this notation $U_n \Rightarrow U$ in (D, S) is equivalent to $U_n \Rightarrow U$ in $(D[0, T], S)$ for all $T \in T_U = \{t > 0\colon P[U(t-) = U(t)] = 1\}$ (Pollard [10], Theorem 6). Hence, from an invariance principle $U_n \Rightarrow U$ in (D, S) we immediately obtain only limit theorems $f(U_n) \Rightarrow f(U)$ for the distributions of functionals $f\colon D \to \mathbb{R}$ which do not depend essentially on the behaviour of the processes outside a sufficiently large bounded interval $[0, T]$.

To obtain limit theorems for functionals which also relate to the behaviour of the tails of the processes, Müller ([9], § 1) introduced a metric

$$e(x, y) := \sup_{t \geq 0} \frac{|x(t) - y(t)|}{\pi(t)}$$

on the subset of continuous functions in $\Lambda(\pi, 0)$, where

$$\Lambda(\pi, a) = \left\{ x \in D\colon \overline{\lim_{t \to \infty}} \frac{|x(t)|}{\pi(t)} \leqq a \right\}$$

(positive continuous $\pi \in D$, $a \geqq 0$). The topology generated by e is finer than that of uniform convergence on compact intervals. A corresponding refinement of the J_1 topology on $\Lambda(\pi, 0)$ is due to Whitt [15, 16]. These topologies are well adapted for the study of processes obeying a strong law of large numbers. To treat processes satisfying the law of the iterated logarithm, one wishes to

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extend the Müller-Whitt topologies to the case $a=1$ and $\pi(t) \sim (2t \log\log t)^{1/2}$ as $t \to \infty$. But ([9], § 3) for $a>0$ the metric e generates a nonseparable topology on the subset of continuous functions in $\Lambda(\pi, a)$. Borovkov and Sakhanenko [4] pointed out that in this case difficulties arise. If one accepts the continuum hypothesis, then for a wide class of stochastic processes (including the Wiener process) it is impossible to construct the corresponding distributions on the Borel σ-field.

Borovkov [3] and Sakhanenko [12] avoided this difficulty by making no use of metrics on D. They introduced the notion of $(\rho, \Lambda(\pi, a))$-continuity of functionals (see Sect. 2). A different possibility is to study weak convergence of probability measures on D relative to incompatible topology and σ-field (Miller and Sentilles [8]). The purpose of this paper is to introduce metrics $\hat{\rho}$ on D (Sect. 3) such that the usual theory of convergence in distribution in metric spaces applies. These metrics $\hat{\rho}$ generate separable topologies $\hat{S}$ which are refinements of the various Skorohod topologies and which include extensions of the Müller-Whitt ones to the case $a>0$. Further the class of $(\rho, \Lambda(\pi, a))$-continuous functionals coincides with the class of functionals that are continuous on $\Lambda(\pi, a)$ with respect to $\hat{\rho}$. A corresponding metrization problem was solved by Sakhanenko ([12], § 3) for the case of discrete time, i.e. for the space of sequences instead of D. In Sect. 4 we study necessary and sufficient conditions for the validity of invariance principles $U_n \Rightarrow U$ in $(D, \hat{S})$ with limit processes U in $\Lambda(\pi, a)$. If the processes U_n are in $\Lambda(\pi, a)$ too, we may, as it is done in Sect. 5, restrict our attention to the Polish space $(\Lambda(\pi, a), \hat{S})$. Finally in Sect. 6 we prove a limit theorem for last entrance times. This theorem illustrates the possible applications of the invariance principles treated here.

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2. Preliminaries, $(\rho, \Lambda(\pi, a))$-continuity

Let $\mathfrak{D}$ be the usual σ-field on D generated by the evaluation maps from D to the real line $\mathbb{R}$. Let $(\Omega, \mathfrak{A}, P)$ be a probability space. Let random variables always be defined on Ω. By a stochastic process we mean a family $U=\{U(t); t \geqq 0\}$ of real-valued random variables.

Throughout this paper $\pi \in D$ is a positive continuous function and $a \geqq 0$ is a constant. We put

$$\Lambda_{T,\alpha} = \left\{ x \in D \colon \sup_{t \geqq T} \frac{|x(t)|}{\pi(t)} < \alpha \right\} \in \mathfrak{D},$$

$$\Lambda(\pi, a) = \bigcap_{\alpha > a} \bigcup_{T>0} \Lambda_{T,\alpha} = \left\{ x \in D \colon \overline{\lim_{t \to \infty}} \frac{|x(t)|}{\pi(t)} \leqq a \right\} \in \mathfrak{D}.$$

We assume a metric ρ on D generating one of the Skorohod topologies $S=J_1$ (see [2, 7, 17]), $S=J_2$, $S=M_1$ (see [11, 14, 18]) or $S=M_2$. Let ρ_T be a metric generating the corresponding topology on $D[0, T]$.

Example. For $S=M_2$ we may define $\rho_T(x,y):=H(\Gamma(x,T),\Gamma(y,T))$, where H is the Hausdorff metric in the plane and

$$\Gamma(x,T):=\{(t,s)\in\mathbb{R}^2\colon 0\leqq t\leqq T,\ x(t-)\leqq s\leqq x(t)\}$$

is the completed graph of $x\in D[0,T]$ with $x(0-):=x(0)$ (cf. Pomarede [11], p. 83, Theorem 4.2). Further we may set

$$\rho(x,y):=\int_0^\infty e^{-s}\min(1,\rho_s(x,y))\,ds.$$

Let $f\colon E\to\mathbb{R}$ be a functional with $E\subset D$. Then f is said to be $(\rho, \Lambda(\pi,a))$-*continuous* at $x\in E\cap\Lambda(\pi,a)$, if the following relation holds.

(B) *For each $\varepsilon>0$ there exist a continuity point $N=N(x,\varepsilon)>0$ of x, $\beta=\beta(x,\varepsilon)>0$ and $\delta=\delta(x,\varepsilon)>0$ such that $x\in\Lambda_{N,a+\beta}$ and the following condition is satisfied: if $y\in E\cap\Lambda_{N,a+\beta}$ and $\rho_N(x,y)<\delta$, then $|f(x)-f(y)|<\varepsilon$.*

Definition (B) is used by Borovkov ([3], p. 68). We slightly modified the definition of $\Lambda_{T,\alpha}$ to obtain equivalence of (B) and the corresponding definition of Sakhanenko ([12], p. 74).

3. Metrization of D

We define

$$f_s(x)=\min\left(1,\max\left(0,\sup_{t\geqq s}\frac{|x(t)|}{\pi(t)}-a\right)\right)\qquad(s\geqq 0,\ x\in D),$$

$$d(x,y)=\int_0^\infty e^{-s}|f_s(x)-f_s(y)|\,ds\qquad(x,y\in D),$$

$$\hat\rho(x,y)=\max(\rho(x,y),d(x,y))\qquad(x,y\in D).$$

The function $\hat\rho$ is a metric on D. We denote by $\hat S_{\pi,a}$ the topology generated by $\hat\rho$ on D or a subset of D. If no confusion may arise, we write $\hat S$ instead of $\hat S_{\pi,a}$. With the lemma below it is easy to show that the metric space $(D,\hat\rho)$ is separable (cf. [2], Sect. 14). But $(D,\hat\rho)$ is not complete (A divergent Cauchy sequence is defined by $x_n(t)=(a+1)\pi(t)$ for $n-1\leqq t<n$ and $x_n(t)=0$ elsewhere).

Lemma 3.1. *Let $x_n, x\in D$. Then $x_n\to x$ in $(D,\hat S)$ if and only if $x_n\to x$ in (D,S) and $f_T(x_n)\to f_T(x)$ for all continuity points $T>0$ of x.*

Convergence to an element $x\in\Lambda(\pi,a)$ is characterized as follows.

Lemma 3.2. *Let x_n, $x\in D$. Then $x\in\Lambda(\pi,a)$ and $x_n\to x$ in $(D,\hat S)$ if and only if $x_n\to x$ in (D,S) and the following condition is satisfied:*

$$\lim_{T\to\infty}\overline{\lim_n}\sup_{t\geqq T}\frac{|x_n(t)|}{\pi(t)}\leqq a. \tag{1}$$

Proof of Lemma 3.2. Necessity. We only have to show that (1) is necessary. This follows by Lemma 3.1 from the equivalence of

$$\sup_{t \geqq T} \frac{|x(t)|}{\pi(t)} < a + \varepsilon \quad \text{and} \quad f_T(x) < \varepsilon \qquad (0 < \varepsilon < 1).$$

Sufficiency. Let $T > 0$ be a continuity point of x. Given $\varepsilon > 0$, by (1) there exists a continuity point $T_1 > T$ of x such that

$$\overline{\lim_n} \sup_{t \geqq T_1} \frac{|x_n(t)|}{\pi(t)} < a + \varepsilon.$$

From $x_n \to x$ in (D, S) we obtain the relation

$$\sup_{t \geqq T_1} \frac{|x(t)|}{\pi(t)} \leqq a + \varepsilon. \tag{2}$$

Hence for n sufficiently large, $|f_T(x_n) - f_T(x)| \leqq \varepsilon$. By Lemma 3.1 it follows that $x_n \to x$ in $(D, \hat{S})$, whereas (2) ensures $x \in \Lambda(\pi, a)$.

Lemma 3.3. *Let $f \colon E \to \mathbb{R}$ be a functional with $E \subset D$, and let $x \in E \cap \Lambda(\pi, a)$. Then f is $(\rho, \Lambda(\pi, a))$-continuous at x if and only if f is continuous at x with respect to $\hat{S}_{\pi, a}$.*

Proof. 1) Assume that f is continuous at x with respect to $\hat{S}$. To prove that f is $(\rho, \Lambda(\pi, a))$-continuous at x we assume the contrary, say, Definition (B) is not satisfied. Now we choose an increasing sequence (N_n) of continuity points of x with $x \in \Lambda_{N_n, a+\frac{1}{n}}$ for all n and $N_n \to \infty$. Then for some $\varepsilon > 0$ and for each sequence (δ_n) of positive numbers there exists a sequence (y_n) such that for all n there holds

$$y_n \in E \cap \Lambda_{N_n, a+\frac{1}{n}}, \tag{3}$$

$$\rho_{N_n}(x, y_n) < \delta_n, \tag{4}$$

$$|f(x) - f(y_n)| \geqq \varepsilon. \tag{5}$$

We can choose a sequence (δ_n) such that from (4) it follows that $\rho(x, y_n) \to 0$ as $n \to \infty$. Further the sequence (δ_n) can be choosen such that for $m \leqq n$ there holds

$$\left| \sup_{N_m \leqq t \leqq N_n} \frac{|y_n(t)|}{\pi(t)} - \sup_{N_m \leqq t \leqq N_n} \frac{|x(t)|}{\pi(t)} \right| < \frac{1}{n}.$$

Using (3) it follows that

$$\sup_{t \geqq N_m} \frac{|y_n(t)|}{\pi(t)} \leqq a + \frac{1}{m} + \frac{1}{n}.$$

From Lemma 3.2 we obtain $y_n \to x$ in $(D, \hat{S})$. Hence $f(y_n) \to f(x)$, in contradiction to (5).

2) The other direction of the lemma follows straightforward from Definition (B) and Lemma 3.1.

4. Convergence in Distribution in $(D, \hat{S})$

Before characterizing convergence in distribution of processes in $(D, \hat{S}_{\pi,a})$ we observe that the Borel σ-field associated with $\hat{S}_{\pi,a}$ is the usual σ-field $\mathfrak{D}$ on D. (Using the definition of $\hat{\rho}$, we easily obtain the $\mathfrak{D}$-measurability of the open spheres in the separable space $(D, \hat{\rho})$.)

Theorem 4.1. *Let U_n, U be stochastic processes with almost all paths in $E \in \mathfrak{D}$. Then propositions* (a) *and* (b) *are equivalent.*

(a1) $U_n \Rightarrow U$ *in* (D, S),

(a2) $\lim_{T\to\infty} \overline{\lim_n} P[U_n \notin \Lambda_{T,a+\beta}] = 0$ *for each* $\beta > 0$.

(b1) $U \in \Lambda(\pi, a)$ *P-a.s.*,

(b2) $U_n \Rightarrow U$ *in* $(E, \hat{S}_{\pi,a})$.

Remark 4.2. 1) In view of Lemma 3.3 proposition (b2) in Theorem 4.1 may be replaced by

(b3) *$f(U_n) \Rightarrow f(U)$ for all $\mathfrak{D}$-measurable functionals f on E, which are $(\rho, \Lambda(\pi, a))$-continuous P_U-a.s.*

Theorem 1 of Sakhanenko [12] corresponds to the statement: (a) and (b1) imply (b3).

2) Theorem 4.1 remains true for k-tuples of stochastic processes, if we replace sets and spaces by corresponding k-fold products of sets and spaces throughout (cf. Corollary 5.2).

Proof of Theorem 4.1. 1) We assume that (b) holds. From the continuity of the canonical embedding from $(D, \hat{S})$ to (D, S) we obtain (a1). For $0 < \beta < 1$ and $T \in T_u$ we have

$$\overline{\lim_n} P[U_n \notin \Lambda_{T,a+\beta}] = \overline{\lim_n} P[f_T(U_n) \geqq \beta] \leqq P\left[f_T(U) \geqq \frac{\beta}{2}\right]$$

$$= P[U \notin \Lambda_{T,a+\frac{\beta}{2}}] \to 0 \quad \text{as} \quad T \to \infty.$$

Thus (a2) holds.

2) We assume that (a) holds. Given $\delta > 0$, by (a2) there exists a sequence of points $T_j \in T_U$ $(j = 0, 1, 2, \ldots)$ such that $T_j \uparrow \infty$ as $j \to \infty$ and such that for each j,

$$\overline{\lim_n} P\left[h_j(U_n) \geqq a + \frac{1}{j}\right] < \delta\, 2^{-j},$$

where

$$h_j(x) = \sup_{T_{j-1} \leqq t \leqq T_j} \frac{|x(t)|}{\pi(t)} \qquad (x \in D).$$

By (a1) we have $h_j(U_n) \Rightarrow h_j(U)$ as $n \to \infty$. Hence

$$P[U \notin \Lambda(\pi, a)] \leqq \sum_{m=1}^{\infty} P[U \notin \Lambda_{T_m, a+\frac{2}{m}}]$$

$$\leqq \sum_{m=1}^{\infty} P\left(\bigcup_{j=m+1}^{\infty} \left[h_j(U) \geqq a + \frac{2}{j}\right]\right)$$

$$\leqq \sum_{m=1}^{\infty} \sum_{j=m+1}^{\infty} \overline{\lim_n}\, P\left[h_j(U_n) \geqq a + \frac{1}{j}\right] < \delta.$$

Since $\delta > 0$ is arbitrary, this yields (b1).

We denote by $\bar{A}$ respectively $\bar{A}^{\wedge}$ the closure of a set $A \subset E$ in (D, S) respectively $(E, \hat{S})$. Let F be a closed subset of $(E, \hat{S})$ and let $\eta > 0$ be arbitrary. By (a2) we can choose a sequence (t_m) in $(0, \infty)$ such that

$$\overline{\lim_n}\, P[U_n \notin \Lambda_{t_m, a+\frac{1}{m}}] < \eta 2^{-m}$$

for each m. If we put

$$B_j = \bigcap_{m=1}^{j} \Lambda_{t_m, a+\frac{1}{m}},$$

then we obtain for each j the relation $\overline{\lim_n}\, P[U_n \notin B_j] < \eta$. By (a1) it follows that

$$\overline{\lim_n}\, P[U_n \in F] \leqq \overline{\lim_n}\, P[U_n \in \overline{F \cap B_j}] + \eta$$

$$\leqq P[U \in \overline{F \cap B_j}] + \eta = P[U \in E \cap \overline{F \cap B_j}] + \eta$$

for all j. As $j \to \infty$, $F \cap B_j$ is nonincreasing. Hence

$$\overline{\lim_n}\, P[U_n \in F] \leqq P\left[U \in E \cap \bigcap_{j=1}^{\infty} \overline{F \cap B_j}\right] + \eta \leqq P[U \in F] + \eta.$$

(The last inequality follows from Lemma 3.2 and $\bar{F}^{\wedge} = F$.) Since $\eta > 0$ is arbitrary, the portmanteau theorem yields (b2).

5. The Subspace $\Lambda(\pi, a)$

Let $x_n, x \in D$. Then from Lemma 3.2 it follows that $x_n, x \in \Lambda(\pi, a)$ and $x_n \to x$ in $(\Lambda(\pi, a), \hat{S})$ if and only if the following conditions are satisfied:

$$x_n \to x \quad in\ (D, S), \qquad \overline{\lim_{t \to \infty}} \sup_n \frac{|x_n(t)|}{\pi(t)} \leqq a.$$

The subspace $\Lambda(\pi, a)$ with the metric $\hat{\rho}$ is not complete (see counterexample in Sect. 3). But by introducing a modified metric $\hat{\rho}_1$ on $\Lambda(\pi, a)$, which was suggested by Ward Whitt for this purpose, we obtain

Lemma 5.1. *The space* $(\Lambda(\pi, a), \hat{S})$ *is Polish.*

Proof. Let $x \in \Lambda(\pi, a)$. Then M_x defined by

$$M_x(s) = \begin{cases} 0 & \text{for } s<0 \\ 1-f_s(x) & \text{for } s \geqq 0 \end{cases}$$

is a distribution function of a probability measure on $\mathbb{R}$. We set

$$\hat{\rho}_1(x, y) = \max(\rho(x, y), L(M_x, M_y)) \qquad (x, y \in \Lambda(\pi, a)),$$

where L is the Lévy metric ([6], p. 33). In view of Lemma 3.1 and [6], p. 33, Theorem 1, the metrics $\hat{\rho}$ and $\hat{\rho}_1$ are equivalent on $\Lambda(\pi, a)$. From Lemma 3.2 it follows easily that $(\Lambda(\pi, a), \hat{\rho}_1)$ is complete, if (D, ρ) is complete.

From Theorem 4.1 with Remark 4.2.2 we obtain

Corollary 5.2. *Let* U_n^i, U^i *be stochastic processes with paths in* D. *If we put* $U_n = (U_n^1, \ldots, U_n^k)$ *and* $U = (U^1, \ldots, U^k)$, *then propositions* (α) *and* (β) *are equivalent.*

(α1) $U_n \Rightarrow U$ *in* $(D, S)^k$,

(α2) $\lim\limits_{T\to\infty} \sup\limits_n P[U_n^i \notin \Lambda_{T, a+\beta}] = 0$ *for* $i = 1, \ldots, k$ *and each* $\beta > 0$.

(β1) $U_n \in \Lambda(\pi, a)^k$ *P-a.s. for all* n, $U \in \Lambda(\pi, a)^k$ *P-a.s.*,

(β2) $U_n \Rightarrow U$ *in* $(\Lambda(\pi, a), \hat{S})^k$.

Remark 5.3. 1) We consider the special case $a = 0$, $S = J_1$. As usual we denote by Λ the class of all strictly increasing, continuous mappings of $[0, \infty)$ onto itself. Let $x_n, x \in \Lambda(\pi, 0)$. Then $x_n \to x$ in $(\Lambda(\pi, 0), \hat{J}_1)$ if and only if there exists a sequence (λ_n) in Λ such that

$$\sup_{t\geqq 0} |\lambda_n t - t| \to 0 \quad \text{and} \quad \sup_{t \geqq 0} \left| \frac{x_n(t)}{\pi(t)} - \frac{x(\lambda_n t)}{\pi(\lambda_n t)} \right| \to 0.$$

If $\pi(t) = 1 + t^\gamma$ for some $\gamma > 0$, then $(\Lambda(\pi, 0), \hat{J}_1)$ is the metrizable space used by Whitt ([15], Sect. 3). An error in the definition in [15] was noted in [16].

2) We denote by $C = C[0, \infty)$ the subset of continuous functions in D. Let $S = M_2$. Then S relativized to C coincides with the topology of uniform convergence on compact intervals ([13], p. 264). Hence $(C \cap \Lambda(\pi, a), \hat{S}_{\pi, a})$ is a metrizable space suitable for §1 ($\pi(t) = t \vee 1$, $a = 0$) and §3 ($\pi(t) = (2(t \vee 3) \log\log(t \vee 3))^{1/2}$, $a = 1$) of Müller [9]. Corollary 5.2 remains true, if we replace D by C and $\Lambda(\pi, a)$ by $C \cap \Lambda(\pi, a)$.

6. Last Entrance Times

Let $\psi \in D$ be a continuous function with the properties $\psi(t) > 0$ for $t > 0$ and

$$\varliminf_{t\to\infty} \frac{\psi(t)}{\pi(t)} > a. \tag{6}$$

Then we define a *last entrance time functional* g_ψ: $\Lambda(\pi, a) \to \mathbb{R}$ by

$$g_\psi(x) = \sup\{t > 0 \colon x(t) > \psi(t)\} \qquad (x \in \Lambda(\pi, a)),$$

where $\sup \emptyset = 0$.

Theorem 6.1. *Let U_n, U be stochastic processes with almost all paths in $\Lambda(\pi, a)$. Assume that $U_n \Rightarrow U$ in $(\Lambda(\pi, a), \hat{M}_2)$ and*

$$\text{if} \quad 0 < p < q, \quad \text{then} \quad P\left[\sup_{p \leqq t \leqq q} \frac{U(t)}{\psi(t)} = 1\right] = 0. \tag{7}$$

Then it follows that $g_\psi(U_n) \Rightarrow g_\psi(U)$.

Remark 6.2. If U is a stable process with exponent α $(0 < \alpha \leqq 2)$, then (7) follows in a similar way as in the proof of Theorem 8.1 of Dudley [5] from the fact that the distribution of $\sup_{p \leqq t \leqq q} (U(t) - \psi(t))$ has a density.

Example. Let (T_n) be a sequence of independent identically distributed real random variables and assume that $0 < V(T_1) < \infty$. Put

$$Y_n(t) = (n\, V(T_1))^{-1/2} \sum_{i=1}^{[nt]} (T_i - E\, T_1) \qquad ([nt] \text{ integer part of } nt).$$

Let W be the standard Wiener process and

$$\pi(t) := (2 \max(t, 3) \log\log \max(t, 3))^{1/2}.$$

From Theorem 2 of Müller [9] and Corollary 5.2 we obtain

$$Y_n \Rightarrow W \quad \text{in } (\Lambda(\pi, 1), \hat{J}_1).$$

Hence by Theorem 6.1 it follows that

$$g_\psi(Y_n) \Rightarrow g_\psi(W),$$

where $\psi(t) = c\,\pi(t)$ with $c > 1$ or $\psi(t) = c\, t^\gamma$ with $c > 0$, $\gamma > \frac{1}{2}$. Corresponding results are valid if the distribution of T_1 belongs to the domain of attraction of a stable distribution with exponent $\alpha < 2$ (see [1], p. 292f).

Proof of Theorem 6.1. It suffices to show

$$P\left[\sup_{t \geqq s} \frac{U_n(t)}{\psi(t)} \leqq 1\right] \to P\left[\sup_{t \geqq s} \frac{U(t)}{\psi(t)} \leqq 1\right] \tag{8}$$

for all $s \in T_U$. Now let $s \in T_U$. By (6) there exist $\beta > 0$ and $T \geqq s$ such that

$$\sup_{t \geqq T} \frac{\pi(t)}{\psi(t)} < \frac{1-\beta}{a+\beta}.$$

We put $f(x) := \max\left(1-\beta, \sup_{t \geqq s} \frac{x(t)}{\psi(t)}\right)$ $(x \in \Lambda(\pi, a))$. Assume that $x_n \to x$ in $(\Lambda(\pi, a), \hat{M}_2)$. If x is continuous at s, we easily see that $f(x_n) \to f(x)$. Hence f is

P_U-a.s. continuous on $(\Lambda(\pi, a), \hat{M}_2)$. By (6) and (7) it follows that 1 is a continuity point of the distribution function of $f(U)$. Therefore $P[f(U_n) \leqq 1] \to P[f(U) \leqq 1]$ as $n \to \infty$, which proves (8).

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