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Regular Birth and Death Times

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§ 1. Introduction

The purpose of some recent work ([6, 11, 12]) on Markov processes has been to obtain intrinsic characterizations of classes of random times having what Jacobsen [7] calls operational properties. For example, a regular birth time R is defined by the operational properties of conditional independence of post-R and pre-R information, given X_R, and the homogeneous Markov character of the post-R process. Regular death times are defined in an analogous operational manner, the pre-R process now being a temporally homogeneous Markov process. We defer the precise definitions to Sects. 4 and 5. Jacobsen and Pitman [6] gave intrinsic characterizations of regular birth times and regular death times in the case of discrete time parameter and countable state space. Pittenger [11] extended their characterization of regular birth times to general (right) Markov processes, and Sharpe [12] did the same for regular death times.

Another result of [6] was a characterization of random times which are both regular birth and regular death times, in the special case where all states of the Markov chain communicate. Their result states that such a random time in this special case must be either coterminal or terminal, and they gave an example showing this is not the case if the communication condition is relaxed. The purpose of this note then is to provide a general characterization of times which are both regular birth and regular death. Our result is the same as that of [6] if the process has no proper absorbing subsets. The statement of the main theorem is given in (6.21) and, shorn of technicalities, states that if R is both regular birth and regular death, there exists a coterminal time L_0 and a terminal time T_0 such that $L_0 \leqq R \leqq T_0$, $[[R]] \subset [[L_0]] \cup [[T_0]]$, and there exists an absorbing set A_0 such that if $L_0 < T_0$ and T_0 equals the first hit of A_0, then $R = L_0$; otherwise $R = T_0$. Moreover, it turns out that these conditions on R are sufficient for R to be both regular birth and regular death.

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There are two other results which are either new or improvements over earlier versions and to which we wish to draw attention. One is (5.12) in which we give an alternative definition for regular birth times in a manner which emphasizes the Markov properties involved as opposed to the existence of entrance laws. The second result is in the Appendix and is an improved version of the absolute continuity result on Markovian measures, a result which is key to the approach used in [11], but which is somewhat buried in [11] and can be expressed in a more general way. Both results may be useful in other investigations.

§ 2. Definitions and Preliminary Results

The underlying Markov process $X=(\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ is assumed to be a right process with state space E. Passing to the Ray topology if necessary, it may be assumed that the semigroup (P_t) of X maps the class $b\mathfrak{E}$ of bounded Borel functions on E into itself. We use [13] as a reference for technical facts about right processes, but emphasize that X is right continuous, strong Markov, and that every point x of E is assumed normal for X - i.e., $P^x\{X_0=x\}=1$.

As in [13], we use $\mathfrak{P}$ and $\mathfrak{O}$ to denote the σ-fields of predictable and optional processes respectively: $\mathfrak{P}$ (resp., $\mathfrak{O}$) is the σ-field on $\mathbb{R}^+\times\Omega$ generated by processes which are evanescent relative to every P^μ and by processes which are adapted to $(\mathfrak{F}_t)$ and are left continuous with right limits (resp., right continuous with left limits).

Given a random time R, $\mathfrak{F}(R)$ will be used to denote the σ-field defined [10, 5] as either the σ-field on Ω generated by the random variables $Z_R 1_{\{R<\infty\}}+F 1_{\{R=\infty\}}$, with $Z\in\mathfrak{O}$ and $F\in\mathfrak{F}_\infty$, or equivalently, the σ-field generated by $\mathfrak{F}(S)|_{\{S\leqq R\}}$ as S varies over optional times (i.e., stopping times) and $\mathfrak{F}(S)$ has its usual meaning in that case. It is immediate from either prescription that $\{R<S\}\in\mathfrak{F}(R)$ for every optional S. Similarly $\mathfrak{F}(R-)$ is defined either as the σ-field on Ω generated by $\mathfrak{F}(S)|_{\{S<R\}}$ as S varies over optional times, or equivalently, as the σ-field generated by $Z_R 1_{\{R<\infty\}}+F 1_{\{R=\infty\}}$ with $Z\in\mathfrak{P}$ and $F\in\mathfrak{F}_\infty$.

The σ-field $\mathfrak{H}^d$ (resp., $\mathfrak{H}^g$) of homogeneous processes is defined on $\mathbb{R}^+\times\Omega$ (resp., $\mathbb{R}^{++}\times\Omega$ where $\mathbb{R}^{++}\equiv]0,\infty[)$ as that generated by evanescent processes and those measurable processes Z such that $Z_t=Z_0\circ\theta_t$ for all $t\geqq 0$ (resp., $Z_{s+t}=Z_s\circ\theta_t$ for all $s>0$ and $t\geqq 0$) and which are right continuous with left limits (resp., left continuous with right limits). See [5] or [13] for further discussion. Given a random time R, the relatively homogeneous σ-fields $\mathfrak{H}^d([[0,R[[)$, $\mathfrak{H}^g([[0,R[[)$ are defined in a similar way, the condition on Z being restricted to shifts by $t<R$.

The left germ field $\mathfrak{F}[R-]$ at R is defined to be the least σ-field on Ω whose trace on $\{0<R<\infty\}$ contains $Z_R 1_{\{0<R<\infty\}}$ with $Z\in\mathfrak{P}\cap\mathfrak{H}^g$. By results from [5], $\mathfrak{F}[R-]$ is the σ-field on Ω generated by $f(X_R)_-\equiv\lim_{t\uparrow\uparrow R} f(X_t)$ with f an α-excessive function. In a similar vein, $\mathfrak{F}[R]$ is generated by Z_R with $Z\in\mathfrak{O}\cap\mathfrak{H}^d$ and is equal to that generated by $f(X_R)1_{\{R<\infty\}}$ with f α-excessive.

Following [5], a random time R is called a *Markov time* (resp., *left Markov time*) if for every $F \in b\mathfrak{F}^0$ there exists $G \in b\mathfrak{F}[R]$ (resp., $b\mathfrak{F}[R-]$) such that for every initial law μ, (2.1) (resp., (2.2)) holds:

(2.1) $$E^{\mu}\{F \circ \theta_R 1_{\{0<R<\infty\}} | \mathfrak{F}(R)\} = G 1_{\{0<R<\infty\}};$$

(2.2) $$E^{\mu}\{F \circ \theta_R 1_{\{0<R<\infty\}} | \mathfrak{F}(R-)\} = G 1_{\{0<R<\infty\}}.$$

Informally, the requirement is conditional independence of the past (resp., strict past) and future given the present (resp., infinitesimal past).

§ 3. Terminal and Coterminal Times

In this section we collect the various definitions of the random times with which we will be working. For a more complete exposition the reader should consult [13] or [5].

Assume that X is a right process as described in § 2. By a *right terminal time* is meant an optional time T such that

(3.1) *for every optional time S,* $T = S + T \circ \theta_S$ *a.s. on* $\{S < T\}$

and

(3.2) $\mathrm{reg}(T) \equiv \{x \in E : P^x(T=0)=1\}$ *is a nearly optional subset of E.*

(Recall that a function f on E is *nearly optional* if for every initial law μ, $f(X)$ is (P^{μ} indistinguishable from) a process which is optional over $(\Omega, \mathfrak{F}_t^{\mu}, P^{\mu})$, and a subset Λ of E is *nearly optional* if 1_{Λ} is nearly optional.)

Much of our discussion requires the perfection of the various random times, and the definition of a right terminal time leads to the following result (see [13] or [15]):

(3.3) if T is a right terminal time then there exists a perfect terminal time $\bar{T}$ such that $\bar{T} = T$ a.s.

A random time R is *co-optional* if for every stopping time S

(3.4) $$R \circ \theta_S = (R-S)^+ \text{ a.s.}$$

If T_0 is a right terminal time, then following [12] we define R as being *co-optional for* (X, T_0) if (3.4) is modified to

(3.5) $$R \circ \theta_S = (R-S)^+ \text{ a.s. on } \{S < T_0\}$$

for every stopping time S.

This variant of co-optionality turns out to be a key concept in the characterization of regular death times as discussed in § 4 below. Moreover, since T_0 may be assumed perfect, the usual proof that co-optional times may be perfected ([13, 15]) goes over to show that co-optional times for T_0 may also be perfected. Thus, with T_0 a perfect right terminal time and after modification of

R on a null set, one may assume that $R \circ \theta_t = (R-t)^+$ for every $t < T_0$, except on a null set independent of t.

The last type of random time we need is the coterminal time, and since this term has been used in several different ways, we will provide some background to the definition given in (3.10) below. Moreover, since killing operators are used, it is convenient to assume that X is realized as the coordinate process on the space Ω^E of all Ray-right continuous maps ω of $\mathbb{R}^+$ into $E \cup \{\Delta\}$ admitting Δ as a trap – i.e., $\omega(t) = \Delta$ implies $\omega(s) = \Delta$ for all $s \geqq t$. For later reference we observe that if $F \subset E$ and is nearly optional with respect to X, then the debut of F^c is a stopping time ([13]) and thus $\Omega^F \subset \Omega^E$ belongs to $\mathfrak{F}$.

The killing operator k_t is defined by

$$k_t \omega(s) = \begin{cases} \omega(s) & s < t \\ \Delta & s \geqq t \end{cases},$$

and ([13]) the coordinate realization of a Ray process supports such operators. These operators were used in the definition of coterminal times L given in [9] as random times satisfying

(3.6) (i) *L is a perfect, co-optional time with $L \leqq \zeta$;*
(ii) *$L \circ k_t = L$ on $\{L < t\}$.*

As was pointed out in [10], one needs the following additional hypothesis to make things work.

(3.7) *For every $t \geqq 0$, $\omega \to L(k_t \omega)$ is $\mathfrak{F}_t$ measurable.*

The *exact regularization L' of L* is defined by

$$L'(\omega) = \sup_t L(k_t \omega), \tag{3.8}$$

so that $L' \leqq L$ and L' is coterminal. One calls L *exact* if $L = L'$.

One defect with these definitions is that because killing operators k_t do not respect null sets, it is possible to construct coterminal times L_1, L_2 such that $L_1 = L_2$ a.s. but $L'_1 < L'_2$ a.s. To avoid such difficulties, we shall use a different formulation of the coterminal condition and refer to random times satisfying (3.6) and (3.7) as being *strict co-terminal* times. The first definition comes from [4], although the word "exact" was not used there.

(3.9) *Definition. A random time L is an exact coterminal time if there exists an optional random set $M \subset \mathbb{R}^{++} \times \Omega$ which is perfectly homogeneous on R^{++} and such that $L(\omega) = \sup\{t\colon (t, \omega) \in M\}$.*

It was shown in [4] that if L is a strict exact coterminal time, then L is an exact coterminal time in the above sense. Using [8, p. 185] it is not difficult to show that if L is exact coterminal, there exists a strict exact coterminal time $\bar{L}$ such that $L = \bar{L}$ a.s.

Throughout this paper we will use the following formulation:

(3.10) *Definition. A random time L is coterminal if there exists an exact coterminal time $\bar{L}$ such that*

(i) $\bar{L}=L$ *if* $L<\infty$

(ii) *the set* $\Lambda=\{L=\infty\}$ *is perfectly invariant, (i.e., outside some null set,* $\Lambda=\Lambda\circ\theta_t$ *for all* $t\geqq 0$).

It is a relatively routine matter (see [13], for example) to show the following facts:

(3.11) *If* L *is coterminal then there exists a strict coterminal time* L^0 *such that* $L=L^0$ *a.s.*

(3.12) *If* L *is coterminal, then one may choose the exact coterminal time* $\bar{L}$ *in* (3.10) *so that* $P^x(\bar{L}<L)<1$ *for all* $x\in E$. *If* $\bar{L}$ *is so chosen, it is essentially unique and is called the exact regularization of* L.

§ 4. Regular Death Times

Assume X is a right process as defined in § 2. Then we have from [12]

(4.1) *Definition. A random time* R *is a* regular death time *if*

(i) R *is a left Markov time;*

(ii) $c_R(x)=P^x(R>0)$ *is a nearly optional function on* E*;*

(iii) *the process* X *killed at* R *is (temporally homogeneous) strong Markov on the state space* $C=\{x\colon c_R(x)>0\}$ *with respect to the family of probability laws* $P^x[\cdot]=P^x[\cdot|R>0]$.

As one would expect, a right terminal time T is a regular death time. This follows from

(4.2) the Markov property and the left Markov property hold at T,

a result established in [5], and

(4.3) the process (X,T) obtained by killing X at T is a right process on the state space $E-\operatorname{reg}(T)$,

which was shown in [13], for example.

The general characterization of regular death times was given in [12]:

(4.4) **Theorem.** *A random time* R *is a regular death time if and only if there exists a right terminal time* $T_0\geqq R$ *such that* R *is co-optional for* (X,T_0).

We will make use of the existence of such a T_0 in § 6. In addition to that we need some observations which were not explicit in [12]. It was shown there that if R is a regular death time and $c(x)=P^x(R>0)$, then there exists a right continuous supermartingale (M_t) which is also a multiplicative functional and such that for every stopping time S

$$P^x[R>S|\mathfrak{F}(S)]=c(X_0)\,M_S. \tag{4.5}$$

In addition if S_0 denotes $\inf\{t\colon X_t\in E-C\}$, then $M_t=0$ for all $t\geqq S_0$. The terminal time T_0 of (4.4) was defined as $\inf\{t\colon M_t=0\}$. Consequently, if S is optional and $S\geqq R$ a.s., (4.5) implies that $c(X_0)\,M_S=0$. This proves

(4.6) **Proposition.** *The terminal time T_0 is the smallest optional time dominating R.*

Moreover, the form of (M_t) may be computed explicitly.

(4.7) **Proposition.** *The multiplicative functional (M_t) is indistinguishable from $(c(X_t)/c(X_0))\, 1_{[0,T_0[}(t)$. In particular, $t \to c(X_t)$ is a.s. right continuous on $[0, T_0[$.*

Proof. Because of the section theorem, it sufficies to show that for every stopping time S

$$E^x[M_S 1_{\{S<\infty\}}] = E^x[(c(X_S)/c(X_0))\, 1_{\{S<T_0\}}].$$

As we observed following (3.7), $T_0 \leqq S_0$ a.s., and so it is enough to show

$$c(x)\, E^x[M_S \cdot 1_{\{S<\infty\}}] = E^x[c(X_S)\, 1_{\{S<T_0\}}].$$

However, (4.5) shows that

$$\begin{aligned} c(x)\, E^x[M_S \cdot 1_{\{S<\infty\}}] &= P^x[R>S] \\ &= P^x[R>S, T_0>S]. \end{aligned}$$

Since R is co-optional for (X, T_0), $R>S$ and $T_0>S$ if and only if $R\circ\theta_S>0$ and $T_0>S$. Therefore, by the strong Markov property

$$\begin{aligned} c(x)\, E^x[M_S 1_{\{S<\infty\}}] &= E^x[P^{X(S)}(R<\infty); S<T_0] \\ &= E^x[c(X_S); S<T_0], \end{aligned}$$

completing the proof.

§ 5. Regular Birth Times

In [11] the definition of a regular birth time R and a characterization of such times was given. This characterization involves finding a coterminal time L preceding R and then showing R is optional for the post-L process. In fact a detailed description of $R-L$ is given in [11], but for our purposes we need only the existence of such an L and certain measurability properties of R with respect to σ-algebras related to L. In addition we provide an alternate formulation of the definition of regular birth times, a formulation which emphasizes the various Markov properties and permits the derivation of the entrance laws used in (5.3) below.

As in § 3, we now assume the process is realized as a coordinate process. Then, given a coterminal time L, we set

$$\breve{X}_t = \begin{cases} X_{L+t} & \text{if } L<\infty \\ \Delta & \text{if } L=\infty \end{cases}$$

and call $\breve{X}$ the post-L process. Let $\breve{\mathfrak{F}}_t = \mathfrak{F}((L+t)+)$ and $\breve{E} = \{x \in E\colon P^x(L=0)>0\}$. (Recall that if S is a random time $\mathfrak{F}(S+) \equiv \bigcap [\mathfrak{F}(S+r);\ r>0]$. See [10]). If one defines kernels $\breve{P}_t$ on $\breve{E} \cup \{\Delta\}$ by

(5.1) $$\breve{P}_t f(x) = E^x[f(X_t) | L=0],$$

where $f \in b\, \breve{\mathfrak{E}}^*$, the trace of $\mathfrak{E}^*$ on $\breve{E}$, then it is easy to check that $(\breve{P}_t)$ is a right semigroup on $\breve{E} \cup \{\Delta\}$. In [9] it is shown that for any initial law μ such that $P^\mu(L<\infty)>0$, the process $(\breve{X}_t)_{t>0}$ is Markov relative to $(\breve{\mathfrak{F}}_t, P^\mu(\cdot \mid L<\infty))$ and has transition semigroup $(\breve{P}_t)$. Actually they prove this assuming L is exact (and strict) but their proof is easily modified to give the above statement.

Given a right semigroup (Q_t) defined on a subset F of E with $F \in \mathfrak{E}^*$ one calls a family (η_t) of measures on $(E, \mathfrak{E}^*)$ an entrance law if $\eta_t Q_s = \eta_{t+s}$ for every $t>0$, $s \geqq 0$. Evidently $\eta_t(E-F)=0$ for all $t>0$. If $\sup \eta_t(F)<\infty$ one may construct a measure Q^η on the space $\Omega(F)$ of right continuous maps of $]0,\infty[$ into $F \cup \{\Delta\}$ making the coordinate process $(X_t)_{t>0}$ Markovian with transition semigroup (Q_t) and entrance law η. We may consider Q^η as defined on $\Omega(E)$, passing to its image under the injection map.

Following [11] a random time will be called a *regular birth time* if:

(5.2) *R has the Markov property* (2.1);

(5.3) *there exists a right semigroup (Q_t) on a subset F of E and a family $\{\eta^x, x \in E\}$ of probability entrance laws relative to (Q_t) such that*

(i) *for every $t>0$ and $f \in b\mathfrak{E}$, $x \to \eta_t^x(f)\, 1_F(x)$ is nearly optional relative to X;*

(ii) *for every initial law μ and every $H \in b\mathfrak{F}^0$*

$$E^\mu[H \circ \theta_R 1_{\{R<\infty\}} \mid \mathfrak{F}(R)] = Q^{\eta^{X(R)}}(H)\, 1_{\{R<\infty\}}. \tag{5.4}$$

If the conditions of (5.2) and (5.3) are met, then the post-R process $(\breve{X}_t)_{t>0}$ defined above is, under $P^\mu(\cdot \mid R<\infty)$, $\breve{\mathfrak{F}}_t = \mathfrak{F}((R+t)+)$, $t>0$, and $\breve{\mathfrak{F}}_0 = \mathfrak{F}(R)$, a Markov process with entrance law $\eta^{X(R)}$ and semigroup (Q_t) on F. Every coterminal time is a regular birth time. The exact case was proved in [10], and the general case is easily deduced from the exact case.

This definition of a regular birth time differs slightly from that given in [11]. First of all, the semigroup (Q_t) of (5.3) was not explicitly assumed to be a right semigroup of [10]. However, it was assumed there that (Q_t) had a right continuous strong Markov realization and mapped Borel functions into nearly Borel functions. These conditions imply, however, that (Q_t) is a right semigroup. (See [13, (7.6)] for example.) Second, the change in (5.3)(i) from nearly Borel to nearly optional necessitates a change in the proof of the fundamental absolute continuity lemma of [11]. This change is straight-forward, and a complete proof is presented in the appendix to this paper.

Suppose now that R is a regular birth time for X. The arguments in [11] produce a coterminal time $L \leqq R$ such that

(5.5) *for every $t>0$, $\{R<t\}$ belongs to the trace of $\mathfrak{F}_t$ on $\{L<t\}$.*

Then (5.5) leads to the following result.

(5.6) **Proposition.** *For every $t>0$, $\{R<L+t\} \in \mathfrak{F}(L+t)$.*

Proof.

$$\{R<L+t\} = \lim_n \bigcup_k \{R<t+k2^{-n}\} \cap \{k2^{-n} \leqq L < (k+1)2^{-n}\}.$$

Using (5.5), there exists $\Lambda(k,n,t)\in\mathfrak{F}(t+k2^{-n})$ so that

$$\{R<t+k2^{-n}\}=\Lambda(k,n,t)\cap\{L<t+k2^{-n}\}.$$

Therefore

$$\{R<L+t\}=\lim_n \bigcup_k \Lambda(k,n,t)\cap\{L<t+k2^{-n}\}\cap\{k2^{-n}\leqq L<(k+1)2^{-n}\}.$$

If n is so large that $t>2^{-n}$, then each term on the right will be in $\mathfrak{F}(L+t)$:

$$\begin{aligned}&\Lambda(k,n,t)\cap\{L<t+k2^{-n}\}\cap\{k2^{-n}\leqq L<(k+1)2^{-n}\}\\&=\Lambda(k,n,t)\cap\{k2^{-n}\leqq L<(k+1)2^{-n}\}\\&=\Lambda(k,n,t)\cap\{t+k2^{-n}\leqq t+L<t+(k+1)2^{-n}\}\\&\in\mathfrak{F}(L+t),\end{aligned}$$

completing the proof.

As noted above a detailed description of $R-L$ was given in [11]. For our purposes (5.5) and (5.6) together with their consequences below will suffice for the proof of our main theorem.

(5.7) **Proposition.** *There exists an* $(\mathfrak{F}_t)$ *optional time* ρ *with* $\rho=R$ *on* $\{L=0\}$.

Proof. Because of (5.5), there exists $\Lambda_t\in\mathfrak{F}_t$ such that $\{R<t\}=\Lambda_t\cap\{L<t\}$. Thus $\{R<t\}\cap\{L=0\}=\Lambda_t\cap\{L=0\}$. Then define

$$\rho(\omega)=\inf\{r>0\colon r \text{ rational}, \omega\in\Lambda_r\}$$

so that for any $t>0$

$$\{\rho<t\}=\bigcup\{\Lambda_r\colon 0<r<t, r \text{ rational}\}.$$

Consequently,

$$\begin{aligned}\{\rho<t\}\cap\{L=0\}&=\bigcup\{\Lambda_r\cap\{L=0\}\colon 0<r<t, r \text{ rational}\}\\&=\bigcup\{\{R<r\}\cap\{L=0\}\colon 0<r<t, r \text{ rational}\}\\&=\{R<t\}\cap\{L=0\}.\end{aligned}$$

It follows immediately that $\rho=R$ on $\{L=0\}$.

(5.8) **Corollary.** *If* $P^x(L=0)>0$, *then* $P^x\{R=0|L=0\}$ *is equal to either* 0 *or* 1.

As a final topic in our summary of properties of regular birth times, we give an alternate definition of regular birth times which emphasizes the various Markov properties and permits the derivation of the entrance laws of (5.3). To do this we need a mild measurability condition on the process:

(5.9) *The measurable space* $(\Omega,\mathfrak{F}^0)$ *is (measurably) isomorphic to a U-space.*

We then prescribe:

(5.10) *for every* $F\in b\mathfrak{F}^0$ *there exists a nearly optional function* f *on* E *such that*

$$E^\mu[F\circ\theta_R 1_{\{R<\infty\}}|\mathfrak{F}(R)]=f(X_R)1_{\{R<\infty\}}$$

for every initial law μ;

(5.11) *there exists a right semigroup* (Q_t) *on a subset* $F \subset E$ *such that for every initial law* μ *carried by* $\{x: P^x(R<\infty)>0\}$ *and every set* $\Lambda \in \mathfrak{F}(R)$ *with* $P^\mu(\Lambda)>0$, *the post-R process* $(\check{X}_t)_{t>0}$ *is Markovian in* F *with semigroup* (Q_t) *relative to* $P^\mu[\cdot \mid \Lambda \cap \{R<\infty\}]$.

The measurability condition (5.9) is satisfied if, for example, $\Omega = \Omega(E)$ and E is Lusinian. See [2, p. 147]. Note that (5.10) is a slightly stronger form of the Markov property (2.1) at R, with universal measurability of f replaced by nearly optionality. However, both (5.10) and (5.11) are satisfied if R is a regular birth time, and we prove the converse next.

(5.12) **Theorem.** *Suppose* (5.9) *holds. Then R is a regular birth time if and only if* (5.10) *and* (5.11) *hold.*

Proof. All that is needed is to construct a family of entrance laws, assuming (5.10) and (5.11). Using standard methods for the construction of kernels [3], one may define a probability kernel $K(x, d\omega)$ on $E \times \mathfrak{F}^0$ such that for every $F \in b\mathfrak{F}^0$, $K(\cdot, F)$ is nearly optional and such that

$$E^\mu[F \circ \theta_R 1_{\{R<\infty\}} \mid \mathfrak{F}(R)] = K(X(R), F)\, 1_{\{R<\infty\}}.$$

Here the hypotheses (5.9) and (5.10) are invoked.

We now wish to regularize K so that for every $x \in E$, $K(x, \cdot)$ is a measure on $(\Omega, \mathfrak{F}^0)$ making $(X_t)_{t>0}$ Markovian with semigroup (Q_t). To this end, let Q^x denote the law of $(X_t)_{t \geqq 0}$ starting at x with transitions (Q_t). If $F = f_1(X_{t_1}) \cdot \ldots \cdot f_n(X_{t_n})$ with $0 \leqq t_1 < \ldots < t_n$ and the f_i uniformly continuous on E, then for every $t>0$, $\Lambda \in \mathfrak{F}(R)$ and μ as in (5.11):

$$\begin{aligned} &E^\mu[F \circ \theta_{R+t} 1_\Lambda 1_{\{R<\infty\}} \mid \mathfrak{F}(R)] \\ &\quad = E^\mu[Q^{X(R+t)}(F)\, 1_\Lambda 1_{\{R<\infty\}} \mid \mathfrak{F}(R)]. \end{aligned} \tag{5.13}$$

Since (5.13) is trivially valid for μ carried by $\{x: P^x(R<\infty)=0\}$, it is valid for arbitrary μ. It follows that

$$K(X(R), F \circ \theta_t) = K(X(R), Q^{X(t)}(F))$$

a.s. on $\{R<\infty\}$.

Let $\mathfrak{C}$ be a countable dense set of uniformly continuous functions on E and let $\mathfrak{H} = \{F \circ \theta_t: 0<t, t \text{ rational}, F \text{ as above with } f_i \in \mathfrak{C} \text{ and the } t_i \text{ rational}\}$. Define $B = \{x \in E: K(x, F \circ \theta_t) \neq K(x, Q^{X(t)}(F)) \text{ for some } F \circ \theta_t \in \mathfrak{H}\}$. Then $P^\mu[X_R \in B, R<\infty]=0$ for every μ and, by hypothesis, B is nearly optional. Define then

$$\bar{K}(x, \cdot) = K(x, \cdot)\, 1_{B^c}(x) + Q^x(\cdot)\, 1_B(x).$$

Obviously $\bar{K}$ is a probability kernel on $E \times \mathfrak{F}^0$, and since $\mathfrak{H}$ is a multiplicative class generating $\mathfrak{F}^0$, a monotone class argument shows that for every $x \in E$, $F \in b\mathfrak{F}^0$ and $t>0$

$$\bar{K}(x, F \circ \theta_t) = \bar{K}(x, Q^{X(t)}(F)).$$

It then follows that if one defines the measures η_t^x on E by $\eta_t^x(f) = \bar{K}(x, f(X_t))$, then every η^x is a (Q_t) entrance law having the properties described in (5.3).

§ 6. Times of Regular Birth and Death

It is assumed throughout this section that R is both a regular birth time and a regular death time. The main result here is (6.21) which gives a complete characterization of such times.

Since R is a regular birth time, the results of § 5 obtain, and there exists a coterminal time $L \leqq R$ such that (5.5)-(5.8) hold. The regular death time property, as discussed in § 4, gives us a right terminal time $T_0 \geqq R$ such that R is co-optional for (X, T_0).

Define now $\varphi(x) = P^x(L > 0)$. Then it is easy to check that φ is excessive. Further if

(6.1) $$E_0 = \{x : \varphi(x) < 1\},$$

then using elementary properties of coterminal times, it is straight-forward to show that $(X_{L+t})_{t>0}$ stays in E_0 a.s. on $\{L < \infty\}$. (This assertion is part of the result cited after (5.1).) We can further subdivide E_0 as $A_0 \cup A_1$, where

(6.2) $$A_0 = \{x : \varphi(x) = 0\}, \qquad A_1 = \{x : 0 < \varphi(x) < 1\}.$$

Because φ is excessive, A_0 is a finely closed absorbing set for X, and we denote its hitting time by S:

(6.3) $$S = \inf\{t : X_t \in A_0\}.$$

Note that $\varphi(X_S) = 0$.

(6.4) **Lemma.** *Almost surely,* $L \leqq S \leqq T_0$.

Proof. Since

$$\begin{aligned} P^x[L > S] &= P^x[L \circ \theta_S > 0, S < \infty] \\ &= E^x[P^{X_S}(L > 0), S < \infty] \\ &= E^x[\varphi(X_S), S < \infty] \\ &= 0, \end{aligned}$$

it follows that $L \leqq S$ a.s. On the other hand, $L \leqq T_0$ almost surely, so that $L \circ \theta_{T_0} = (L - T_0)^+ = 0$ a.s., and

$$\begin{aligned} 0 &= P^x[L \circ \theta_{T_0} > 0, T_0 < \infty] \\ &= E^x[P^{X(T_0)}(L > 0), T_0 < \infty] \\ &= E^x[\varphi(X(T_0)), T_0 < \infty]. \end{aligned}$$

Hence, $\varphi(X(T_0)) = 0$ a.s. on $T_0 < \infty$, and that forces $S \leqq T_0$ a.s. by the definition of S.

For every $x \in E_0$, $P^x(L = 0) > 0$, so the elementary conditional probabilities $P^x[\cdot | L = 0]$ are defined unambiguously. If we set for $x \in E_0$

(6.5) $$h(x) = P^x[R > 0 | L = 0],$$

(6.6) $$g(x) = P^x[R > 0, L = 0],$$

then $g(x)=h(x)\,(1-\varphi(x))$ on E_0. Moreover, from (5.8) h takes only the values zero and one, and we define two more subsets of E_0 by

(6.7) $$A_i=\{x\in E_0\colon h(x)=1\},\qquad A_r=\{x\in E_0\colon h(x)=0\}$$

so that $E_0=A_i\cup A_r$, and A_i, A_r are respectively the irregular and regular points for R, conditional on $L=0$.

The sets $A_i\cap A_1$ and $A_r\cap A_0$ play a crucial role in our characterization of R. Unfortunately we need better measurability than $A_i, A_r\in\mathfrak{E}^*$ to complete our argument, and the discussion from now to the statement of Lemma (6.15) is primarily concerned with this technical point.

From § 3 and § 4 we may assume T_0 is a perfect right terminal time. Let M denote the closed random set in $\mathfrak{O}\cap\mathfrak{H}^g$ generated by T_0, so that the ω-section of M is the closure in $]0,\infty[$ of $\{t+T_0(\theta_t\omega)\colon t>0\}$. Then if $\bar{T}=\inf\{t\colon t\in M\}$ there is a nearly optional set $F\subset E$ such that

$$T_0=\bar{T}\wedge(\inf\{t\geqq 0\colon X_t\in F\}).$$

It may be assumed that F is polar for $(X,\bar{T})$. Then set

$$W_t=1_{[0,R[}(t)\,1_{[L,T_0[}(t).$$

It is easy to check that $W\in\mathfrak{H}^d([[0,T_0[[)$. (See § 2.) Using the projection results from [13], it follows that the optional projection of W is equal to $g(X_t)\,1_{[0,T_0[}(t)$, where g was defined in (6.6). Since W is a.s. right continuous, $t\to g(X_t)\,1_{[0,T_0[}(t)$ is necessarily a.s. right continuous. Hence, using $h=g(1-\varphi)$ and the fact that $\varphi(X_t)<1$ for all $t>L$, we have

(6.8) **Lemma.** *$t\to h(X_t)$ is a.s. right continuous on $]L,T_0[$.*

We now make use of the homogeneous extension method described in [14] (see also [13]) to check that the process $g(X_t)\,1_{M^c}(t)$, which is a homogeneous extension of $g(X_t)\,1_{[0,T_0[}(t)$, belongs to $\mathfrak{O}\cap\mathfrak{H}^g$. Then, using (6.7), we conclude immediately that

(6.9) **Lemma.** *Both $\{X\in A_i\}\cap M^c$ and $\{X\in A_r\}\cap M^c$ belong to $\mathfrak{O}\cap\mathfrak{H}^g$.*

Next define times T_1, L_1 by

(6.10) $$T_1(\omega)=\inf\{t\geqq 0\colon X_t(\omega)\in A_0\cap A_r\},$$

(6.11) $$L_1(\omega)=\sup\{t\geqq 0\colon X_t(\omega)\in A_1\cap A_i\}.$$

We are unable to show directly that T_1 is a stopping time or that L_1 is a coterminal time and rely instead on the following weaker assertions.

(6.12) **Lemma.** *$T_0\wedge T_1$ is a stopping time.*

(6.13) **Lemma.** *The time $L_0=L\vee L_1$ is a coterminal time.*

Proof of (6.12). Using the above description of T_0, it is easy to see that $T_0\wedge T_1$ is the debut of the random set

$$\begin{aligned}&M\cup\{X\in A_0\cap A_r\}\cup(\{0\}\times\{X_0\in F\})\\&=M\cup(\{X\in A_r\}\cap M^c\cap\{X\in A_0\})\cup(\{0\}\times\{X_0\in F\}).\end{aligned}$$

The latter set is optional because A_0 is an optional set and $\{X\in A_r\}\cap M^c\in\mathfrak{O}$ was obtained in (6.9).

Proof of (6.13). From the results mentioned in § 3 we may assume that L has the form

$$L(\omega)=(\sup\{t:(t,\omega)\in N\})\vee(\infty 1_\Gamma(\omega))$$

where $N\in\mathfrak{O}\cap\mathfrak{H}^g$ and Γ is perfectly invariant. Then, using (6.9), $\tilde{N}\equiv N\cup(\{X\in A_1\cap A_i\}\cap M^c)$ belongs to $\mathfrak{O}\cap\mathfrak{H}^g$. We claim that

$$L_0(\omega)=(\sup\{t:(t,\omega)\in\tilde{N}\})\vee(\infty 1_\Gamma(\omega)), \tag{6.14}$$

and once this is done the proof will be complete. First, as was shown in (6.4), $X_u\in A_0$ for all $u\geqq T_0$. Therefore, denoting by $\tilde{L}$ the right side of (6.14),

$$\begin{aligned} t\geqq\tilde{L}(\omega) &\Leftrightarrow \omega\notin\Gamma \quad\text{and}\quad \forall s>t,\ (s,\omega)\notin N\cup(\{X\in A_1\cap A_i\}\cap M^c)\\ &\Leftrightarrow \omega\notin\Gamma,\ t\geqq\sup N \quad\text{and}\quad \forall s>t,\ (s,\omega)\notin\{X\in A_1\cap A_i\}\cap M^c\\ &\Leftrightarrow L(\omega)\leqq t \quad\text{and}\quad \forall s\in]t,T_0[,\ (s,\omega)\notin\{X\in A_1\cap A_i\}\\ &\Leftrightarrow L(\omega)\leqq t \quad\text{and}\quad L_1(\omega)\leqq t, \end{aligned}$$

completing the proof of (6.14) and thus of (6.13).

We now have the machinery in place to characterize R, and the first step in this direction is the following lemma.

(6.15) **Lemma.** *Almost surely, $X_t(\omega)\in A_i$ for all $t\in]L(\omega),R(\omega)[$ and $X_t(\omega)\in A_r$ for all $t\in]R(\omega),T_0(\omega)[$.*

Proof. For every $t\geqq 0$ one has

$$\{R>L+t\}=\bigcap_{s>t}\{R\geqq L+s\}\in\mathfrak{F}((L+t)+)$$

from (5.6). Thus, for every $t\geqq 0$,

$$P^x[R>L+t\,|\,\mathfrak{F}((L+t)+)]=1_{[0,R[}(L+t). \tag{6.16}$$

Since R is co-optional for (X,T_0),

$$\{R>L+t\}=\theta_{L+t}^{-1}\{R>0\}\quad\text{on}\quad\{T>L+t\}.$$

Observe that $\{T_0>L+t\}\in\mathfrak{F}(L+t)$, and since $T_0\geqq R$ we deduce from (6.16) that

$$P^x[1_{\{R>0\}}\circ\theta_{L+t}1_{\{T_0>L+t\}}|\mathfrak{F}((L+t)+)]=1_{[0,R[}(L+t). \tag{6.17}$$

Now recall the definitions from § 5 of $(\breve{X}_t,\breve{\mathfrak{F}}_t,\breve{P})_{t>0}$ as $\breve{X}_t=X(L+t)$, $\breve{\mathfrak{F}}_t=\mathfrak{F}((L+t)+)$, $\breve{P}^x(\cdot)=P^x(\cdot|L=0)$ and the fact that this is a Markov process on E_0. Thus for $x\in E_0$ and $t>0$,

$$\begin{aligned} E^x\{1_{\{R>0\}}\circ\theta_{L+t}1_{\{T_0>L+t\}}|\breve{\mathfrak{F}}_t\} &=\breve{P}^{X_t}(R>0)\,1_{\{L+t<T_0\}}\\ &=1_{A_i}(\breve{X}_t)\,1_{\{L+t<T_0\}}. \end{aligned}$$

Substituting this in (6.17) we obtain a.s.

$$1_{[0,R[}(L+t)=1_{A_i}(X_{L+t})\cdot 1_{[0,T_0[}(L+t). \tag{6.18}$$

In view of (6.8) each side of (6.18) is right continuous in t, and so the two sides of (6.18) are indistinguishable. The conclusion of (6.15) is immediate from this.

(6.19) **Lemma.** *Almost surely, $X_t \notin A_0 \cap A_r$ for all $t < T_0$.*

Proof. Let $T = T_0 \wedge T_1$. Then from (6.12) T is a stopping time, and (6.15) gives $T \geqq R$. However, as shown in (4.6), T_0 is the minimal stopping time dominating R. Hence $T \geqq T_0$ a.s., and thus $T_1 \geqq T_0$ a.s. as claimed.

Since L is not necessarily the largest coterminal time dominated by R, the situation to the left of R is not completely analogous to that on the right. However, the analogy does obtain if we replace L by the coterminal time L_0 defined in (6.13). From the definition of L_0 and the set A_0 it is obvious that $L_0 \leqq S$. More is true though.

(6.20) **Lemma.** *$L_0 \leqq R$ a.s.*

Proof. Since $S \leqq T_0$ and $L_0 \leqq S$, $L_0 \leqq T_0$. On $\{R < L_0\}$, $R < T_0$ and so by (6.15) $X_t \in A_r$ for all t in $]R, T_0[$. But $X_t \in A_0$ for $t \geqq T_0$, since $S \leqq T_0$, and thus $L_0 \leqq R$ a.s. on $\{R < L_0\}$. Hence $L_0 \leqq R$ a.s.

We are now in a position to state and prove the main result.

(6.21) **Theorem.** *A random time R is both a regular birth time and a regular death time if and only if there exists*

(i) *a finely closed absorbing set $A_0 \subset E$ with hitting time S,*

(ii) *a right terminal time T_0, and*

(iii) *a coterminal time L_0 such that the following are satisfied:*

(6.22) $$L_0 \leqq S \leqq T_0 \quad \textit{and} \quad L_0 \leqq R \leqq T_0,$$

(6.23) *for a.a. ω, the unordered pair $\{R(\omega), S(\omega)\}$ is equal to $\{L_0(\omega), T_0(\omega)\}$.*

Proof. Given a random time R which is both a regular birth time and a regular death time, the arguments of this section give (6.22). Suppose $\omega \in \{R < T_0\}$. Then by (6.15), $X_t \in A_r$ for all $t \in]R, T_0[$. However, it was shown in (6.19) that $X_t \notin A_0 \cap A_r$ for all $t \in]R, T_0[$ and consequently $X_t \in A_1 \cap A_r$ for all $t \in]R, T_0[$. It follows that $S \geqq T_0$ a.s. on $\{R < T_0\}$ and hence $S = T_0$ a.s. on $\{R < T_0\}$.

Now look at $\{L_0 < R\}$. In view of (6.15), $X_t \in A_l$ for all $t \in]L_0, R[$, and by the definition of $L_0, X_t \notin A_1 \cap A_l$ for all $t \in]L_0, R[$. Therefore $X_t \in A_0 \cap A_l$ for all $t \in]L_0, R[$, and hence $S = L_0$ a.s. on $\{L_0 < R\}$. Note in particular that on $\{L_0 < R < T_0\}$ one obtains the absurdity $S = T_0$ and $S = L_0$ a.s.

To obtain the converse we shall prove that under (6.22) and (6.23) R is optional after L_0 [11] and co-optional before T_0 [12]. The conditions (6.22) and (6.23) imply that

$$R = \begin{cases} T_0 & \text{if } L_0 \vee S < T_0 \\ L_0 & \text{otherwise.} \end{cases}$$

Therefore, observing that S is both optional and co-optional, one obtains a.s. on $\{t < T_0\}$

$$R\circ\theta_t=\begin{cases}T_0\circ\theta_t & \text{if } L_0\circ\theta_t\vee S\circ\theta_t<T_0\circ\theta_t\\ L_0\circ\theta_t & \text{otherwise}\end{cases}$$
$$=\begin{cases}T_0-t & \text{if } (L_0-t)^+\vee(S-t)^+<T_0-t\\ (L_0-t)^+ & \text{otherwise}\end{cases}$$
$$=\begin{cases}T_0-t & \text{if } L_0\vee S<T_0\\ (L_0-t)^+ & \text{otherwise}\end{cases}$$
$$=(R-t)^+.$$

Consequently R is co-optional for (X, T_0) so [12] R is a regular death time. Finally, under (6.22) and (6.23), set $S'=\inf\{t>0\colon X_t\notin A_0\}$. Then because A_0 is absorbing we have

$$S'=\begin{cases}0 & \text{on } \{S>0\}\\ \infty & \text{on } \{S=0\}\end{cases}.$$

Define $\rho\colon \Omega\times\Omega\to[0,\infty]$ by

$$\rho(\omega,\omega')=\begin{cases}T_0(\omega')\wedge S'(\omega') & \text{if } L_0(\omega)<T_0(\omega)\\ 0 & \text{if } L_0(\omega)=T_0(\omega)\end{cases}.$$

For fixed $\omega'\in\Omega$, $\omega\to\rho(\omega,\omega')$ is in $\mathfrak{F}(L)$, and for each fixed $\omega\in\Omega$, $\omega'\to\rho(\omega,\omega')$ is obviously a stopping time. In addition for a.a. ω we can write

(6.24) $$R(\omega)=L_0(\omega)+\rho(\omega,\theta_{L_0\omega}).$$

The validity of (6.24) is easy to check by examining separately the cases $\{L_0=T_0\}$, $\{L_0=S<T_0\}$ and $\{L_0<S=T_0\}$, the only cases possible under (6.23). The structure of R as described in (6.24) is what is meant by "optional after L_0" in [11], and R is thus a regular birth time.

From the characterization in (6.21) we obtain the following special case which is the direct analogue of a theorem of Jacobsen and Pitman [6].

(6.25) **Theorem.** *Suppose that X admits no non-trivial finely closed absorbing sets. Then if R is both a regular birth time and a regular death time, R must be either terminal or coterminal.*

Proof. The hypothesis implies that the time S in (6.21) is either identically zero or identically infinite. If $S\equiv 0$ then $L_0\equiv 0$, since $L_0\leqq S$, and thus $R=T_0$ and is terminal. If $S\equiv\infty$, then $T_0\equiv\infty$ and so (6.21) gives $R=L_0$ and R is thus coterminal.

Appendix

At the heart of the method used to characterize a regular birth time is an absolute continuity theorem for Markovian measures [11; (5.1)]. It was pointed out to the authors by R.K. Getoor that the use of the section theorem in the proof given in [11] permits a weakening of the hypotheses and that the result itself can be expressed in a more general way. Since this result may be useful in other investigations, we present a complete statement and proof here, and we are indebted to Getoor for permitting us to include his improvements.

As in § 5, let $\Omega=\Omega(E)$ be the space of right continuous maps of $\mathbb{R}^+$ into E_Δ admitting Δ as a trap. It is assumed that E is a U-space. The coordinate process is denoted by X_t. Suppose $\{P^x; x\in E\}$ and $\{\bar{P}^x; x\in E\}$ are two families of measures on $(\Omega, \mathfrak{F}^0)$ such that both families make the coordinate process normal and strong Markov. Suppose in addition that *both* $x\to P^x(\Lambda)$ and $x\to \bar{P}^x(\Lambda)$, $\Lambda\in\mathfrak{F}^0$, are nearly optional (P) and nearly optional $(\bar{P})$. That is, both functions are nearly optional using the definition of nearly optional relative to $\{P^x, x\in E\}$ and also the definition relative to $\{\bar{P}^x, x\in E\}$. This measurability condition is satisfied, for example, if both functions are Borel measurable. (See [13, § 5].)

Let

$$S_1=\{x\in E: \bar{P}^x \ll P^x\}, \qquad S_2=\{x\in E, P^x \ll \bar{P}^x\},$$

and $S=S_1\cap S_2$. The following are the principal results of this section.

(A.1) **Theorem.** *If* $x\in S_1$, *then* S_1^c *is* $\bar{P}^x$*-polar.*

(A.2) **Theorem.** *If* $x\in S$, *then* S^c *is* P^x*-polar and* $\bar{P}^x$*-polar.*

Before we turn to the proof of (A.1), we observe that (A.2) follows directly from (A.1). Indeed, assuming (A.1) proven, S_1^c is $\bar{P}^x$-polar if $x\in S$, and since then $P^x\ll\bar{P}^x$, S_1^c is also P^x-polar. Dually, S_2^c is P^x-polar and hence $\bar{P}^x$-polar if $x\in S$. Since a union of two polar sets is polar, (A.2) follows.

Proof of (A.1). The σ-field $\mathfrak{F}^0$ is separable, and we let $\{\Lambda_n\}$ be a countable field in $\mathfrak{F}^0$ generating $\mathfrak{F}^0$. For $\varepsilon>0$, $\delta>0$ and n a positive integer let

$$S(\varepsilon,\delta,n)=\{x\in E: \bar{P}^x(\Lambda_n)\geqq\varepsilon, P^x(\Lambda_n)<\delta\}$$

and note that $S(\varepsilon,\delta,n)$ decreases as δ decreases and decreases as ε increases. Next let

$$S(\varepsilon,\delta)=\bigcup_n S(\varepsilon,\delta,n),$$

so that $S(\varepsilon,\delta)$ has the same monotonicity properties as $S(\varepsilon,\delta,n)$, and then set

$$S(\varepsilon)=\bigcap_m S(\varepsilon,1/m)=\bigcap_{m\geqq M} S(\varepsilon,1/m).$$

(A.3) **Lemma.** $S_1^c=\bigcup S(1/k)=\bigcup_{\varepsilon>0} S(\varepsilon)$.

The proof of (A.3) is a standard exercise in measure theory, and we omit it here.

Now take $\varepsilon>0$ and let T be a stopping time over $(\mathfrak{F}^0_{t+})$. Write

$$A(m,n)=\{\omega\in\Omega: n \text{ is the smallest integer with } X_T(\omega)\in S(\varepsilon,1/m,n)\}.$$

For fixed n the $A(m,n)$ are disjoint and belong to $\mathfrak{F}^0_{T+}$. Moreover, as $m\to\infty$

$$\bigcup_n A(m,n)=\{\omega: X_T(\omega)\in S(\varepsilon,1/m)\}\downarrow\{X_T\in S(\varepsilon)\}.$$

Write

$$A(m)=\bigcup_n [A(m,n)\cap\theta_T^{-1}(\Lambda_n)].$$

Then the strong Markov property relative to $\bar{P}^x$ gives

$$\bar{P}^x[T<\infty, A(m)] = \sum_n \bar{P}^x[\bar{P}^{X_T}(\Lambda_n); A(m,n), T<\infty]$$
$$\geqq \sum_n \bar{P}^x[\varepsilon; A(m,n), T<\infty],$$

by the definition of $S(\varepsilon, 1/m, n)$, and obviously we have

$$\bar{P}^x[T<\infty, A(m)] \geqq \varepsilon \bar{P}^x[T<\infty, \bigcup_n A(m,n)].$$

Thus, letting $m\to\infty$,

(A.4) $$\lim_{m\to\infty} \bar{P}^x[T<\infty, A(m)] \geqq \varepsilon \bar{P}^x[T<\infty, X_T\in S(\varepsilon)].$$

The same calculation with the P^x, but using instead $P^x(\Lambda_n)\leqq 1/m$, if $x\in S(\varepsilon, 1/m, n)$, gives the inequality

(A.5) $$P^x[T<\infty, A(m)] \leqq 1/m P^x[T<\infty, \bigcup_n A(m,n)].$$

Letting $m\to\infty$ in (A.5), we obtain

(A.6) $$P^x[T<\infty, \bigcap_m A(m)] = 0.$$

Take now $x\in S_1$ so that $\bar{P}^x \ll P^x$. From (A.6) we obtain $\bar{P}^x[T<\infty, \bigcap_m A(m)]=0$, and thus from (A.4) $\bar{P}^x[T<\infty, X_T\in S(\varepsilon)]=0$. The section theorem now implies that $S(\varepsilon)$ is $\bar{P}^x$-polar, and because of (A.3) we may conclude that S_1^c is $\bar{P}^x$-polar if x is in S_1.

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