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Killing Times for Markov Processes

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1. Introduction

It is a well known fact in Markov process theory that if a Markov process is killed at a terminal time - typically the hitting time of some subset of the state space - then the killed process is also Markovian. In addition, the process X killed at a terminal time T is conditionally independent of the future $(X_{T+t}; t \geqq 0)$ given X_T. Of more recent origin [5] is the fact that if L is a co-optional time then X killed at L is Markovian, and though it is not in general true that X killed at L is conditionally independent of $(X_{L+t}; t \geqq 0)$ given X_L, there is a conditional independence property [3] in which, roughly speaking, past and future are conditionally independent on $\{0<L<\infty\}$ given the left germ field at L. The first results aiming at a characterization of those random times at which killing preserves the Markov property with conditional independence of the future were obtained by Jacobson and Pitman [4] for Markov chains with countable stable spaces. Their result states that such a time is in essence a co-optional time before some terminal time. The main result of this paper is an extension of that result to right processes, and though the technicalities are considerably more burdensome, one obtains essentially the same result, the left germ field at L seeming to be the correct analogue in continuous time of $\sigma(X_{L-1})$ for chains.

It should be mentioned that Jacobson and Pitman also characterized for chains birth times with conditional independence of the past, and Pittenger [6] has recently extended their characterization to right processes. It would be interesting to put the result of this paper together with Pittenger's to obtain a characterization of those random times at which killing and birthing produce conditionally independent Markov processes, but such a result is lacking except for chains in which all states communicate [4].

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2. Preliminaries

It is assumed throughout that $X=(\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ is a right process with state space E and transition semigroup (P_t). The work [7] will serve as a reference for technical results needed here, and [2] contains some of the material on additive functionals, shifts and homogeneity. As some of the objects discussed below are not quite standard, we shall give careful definitions. A stopping time T is a *terminal time* if for every stopping time S over $(\mathfrak{F}_t)$, $T=S+T\circ\theta_S$ a.s. on $\{S<T\}$. An $\mathfrak{F}$-measurable random variable with values in $[0,\infty]$ will be called a random time. Given a terminal time T and a random time R, we say that R is *co-optional* for (X,T) if

(2.1) (i) *$R\leqq T$ a.s.;*
(ii) *for every stopping time S, $R\circ\theta_S=(R-S)^+$ a.s. on $\{S<T\}$.*

In case $T=\zeta$, the lifetime of X, the definition agrees with the usual meaning of a co-optional time, though (2.1)(ii) may be replaced then by the weaker condition with S replaced by an arbitrary constant time. The fact that the weaker condition suffices comes from a perfection result ([7], (25.6), for example) which is not available in the case of an arbitrary terminal time. Examples of random times R co-optional for (X,T) include (i) $R=T$; (ii) $R=L\,1_{\{L\leqq T\}}$, where L is co-optional for X. The latter example points out that R depends in general not only on the process X killed at T but possibly on information in the future from T.

By a *multiplicative functional* (MF) for X we mean a process (M_t) satisfying

(2.2) (i) *M_t is positive and optional over $(\mathfrak{F}_t)$;*
(ii) *for every stopping time S and every $t\geqq 0$, $M_{t+S}=M_S M_t\circ\theta_S$ almost surely;*
(iii) *$M_t=0$ for all $t\geqq\zeta$.*

We are not insisting here that $t\to M_t$ be right continuous or decreasing, but we are assuming the strong multiplicative property. If M is a right continuous MF then $S_M=\inf\{t: M_t=0\}$ is a terminal time with $M_{S_M}=0$ on $\{S_M<\infty\}$ and one sees from (2.2)(ii) that a.s. $M_t=0$ for all $t\geqq S_M$.

By a *raw additive functional* (RAF) of (X,M), where M is a MF, is meant an increasing positive process A satisfying

(2.3) (i) *for every $t\geqq 0$, A_t is $\mathfrak{F}$-measurable;*
(ii) *$t\to A_t$ is a.s. right continuous and finite valued;*
(iii) *for every stopping time S and every $t\geqq 0$, $A_{t+S}=A_S+M_S A_t\circ\theta_S$ a.s.*

If A satisfies (2.3) and A is adapted to $(\mathfrak{F}_t)$ we say that A is an *AF* of (X,M). It is easy to see that if A is a RAF of (X,M) then $A_t=A_{S_M}$ for all $t\geqq S_M$, almost surely. It is permitted however that $A(S_M)>A(S_M-)$ on $\{S_M<\infty\}$. If A is a (raw) *AF* of (X,M) where $M=1_{[\![0,T[\![}$ for some terminal time T, then A is also called a (raw) *AF* of (X,T). A simple calculation shows that if T is a terminal time and R is co-optional for (X,T) then $A_t=1_{[R,\infty[}(t)\,1_{\{0<R<\infty\}}$ is a RAF of (X,T). The converse result will be important in the characterization of killing times.

(2.4) **Proposition.** *Let M be a MF and let A be a* RAF *of (X, M). Then*

$$R = \sup\{t\colon \Delta A_t > 0\} \qquad (\textit{with } \sup\emptyset = 0)$$

defines a co-optional time for (X, S_M).

Proof. Let T be an arbitrary stopping time. Then since $R \leqq S_M$,

$$R \circ \theta_T = \sup\{0 < u \leqq S_M \circ \theta_T\colon \Delta A_u \circ \theta_T > 0\}.$$

But $M_T \Delta A_u \circ \theta_T = \Delta A_{u+T}$ for all $u \geqq 0$, a.s. on $\{T < \infty\}$, and $M_T > 0$ a.s. on $\{T < S_M\}$. Consequently, a.s. on $\{T < S_M\}$,

$$\begin{aligned} R \circ \theta_T &= \sup\{0 < u \leqq S_M - T\colon \Delta A_{u+T} > 0\} \\ &= (R - T)^+. \end{aligned}$$

Let $\mathfrak{M}$ and $\mathfrak{P}$ denote respectively the measurable and predictable σ-fields on $\mathbb{R}^+ \times \Omega$ (see [7], §20) and let $\mathfrak{H}^d, \mathfrak{H}^g$ be the σ-fields defined by Azéma [1] (see also [7]; §24) generated respectively by measurable processes which are right continuous with left limits (resp., left continuous with right limits) and satisfy, except on a null set

(2.5) $$Z_s \circ \theta_t = Z_{s+t} \quad \textit{for all } s \geqq 0,\ t \geqq 0 \ (\textit{resp., } s > 0, t \geqq 0).$$

Given a right continuous measurable increasing process A with $A_0 = 0$ and A locally integrable relative to every P^x, $\tilde{A}$ will denote the dual predictable projection (or *compensator*) of A. More precisely, we need the version of $\tilde{A}$ which works simultaneously for all P^x. See ([7], §31). For $t \geqq 0$, Θ_t denotes the shift operator on measurable processes defined by

(2.6) $$(\Theta_t Z)(s, \omega) = Z_{s-t}(\theta_t \omega)\, 1_{[t, \infty[}(s).$$

If $Z \in b\mathfrak{M}_+$, and A is an increasing process as described above, $Z * A$ is the increasing process defined by $(Z * A)_t = \int_{]0,t]} Z_s \, dA_s$. It is easily checked that $\Theta_T(Z * A) = \Theta_T Z * \Theta_T A$. In addition, one has the commutation property

(2.7) $$1_{]\!]T, \infty[\![} * (\Theta_T A)^{\sim} = 1_{]\!]T, \infty[\![} * \Theta_T(\tilde{A})$$

for every stopping time T. See [7], §31.

(2.8) **Proposition.** *Let M be a MF, and let A be a right continuous increasing measurable process with $A_0 = 0$ and $E^x A_\infty < \infty$ for all $x \in E$. Suppose that for every $Z \in b\mathfrak{H}^d_+$, $(Z * A)^{\sim}$ is an AF of (X, M). Then A is a RAF of (X, M).*

Proof. For any stopping time T, $(Z * A)^{\sim}$ being an AF of (X, M) implies

$$M_T 1_{]\!]T, \infty[\![} * \Theta_T (Z * A)^{\sim} = 1_{]\!]T, \infty[\![} * (Z * A)^{\sim}.$$

Because of (2.7) this implies that

$$M_T 1_{]\!]T, \infty[\![} * (\Theta_T Z * \Theta_T A)^{\sim} = M_T 1_{]\!]T, \infty[\![} * (Z * A)^{\sim}.$$

But, since $Z\in\mathfrak{H}^d$, $\Theta_T Z = Z 1_{[\![T,\infty[\![}$. Therefore

$$M_T 1_{]\!]T,\infty[\![} * (Z * \Theta_T A)^{\sim} = 1_{]\!]T,\infty[\![} * (Z * A)^{\sim}.$$

Integration relative to an arbitrary $Y\in b\mathfrak{P}_+$ now gives, using the fact that $M_T 1_{]\!]T,\infty[\![}\in\mathfrak{P}$,

$$E^x \int_{]T,\infty[} M_T Y_u Z_u \Theta_T A(du) = E^x \int_{]T,\infty[} Y_u Z_u\, dA_u.$$

However, products YZ with $Y\in b\mathfrak{P}_+$ and $Z\in b\mathfrak{H}^d_+$ generate $\mathfrak{M}$ ([7], §24) so by monotone classes,

$$E^x \int_{]T,\infty[} W_u M_T \Theta_T A(du) = E^x \int_{]T,\infty[} W_u\, dA_u$$

for every $W\in b\mathfrak{M}$. This last equality implies that the random measures $M_T\Theta_T A(du)$ and dA_u agree on $]\!]T,\infty[\![$, which is the same as saying that A is a RAF of (X, M).

Let R be a random time. Following [3], we say that R is a left Markov time if the strict past and future from R are conditionally independent on $\{0<R<\infty\}$ given the left germ field at the present. The operational meaning which will be used in the next section is that for every $Z\in b\mathfrak{H}^d_+$ there exists $Z'\in b(\mathfrak{P}\cap\mathfrak{H}^g)_+$ such that $Z*(\varepsilon_R 1_{\{0<R<\infty\}})$ and $Z'*(\varepsilon_R 1_{\{0<R<\infty\}})$ have the same dual predictable projection. Here $\varepsilon_R 1_{\{0<R<\infty\}}$ is shorthand for the random measure generated by the increasing process $A_t = 1_{[R,\infty[}(t)\, 1_{\{0<R<\infty\}}$.

A function f on E is *nearly optional* if the process $t\to f(X_t)$ is (P^μ indistinguishable from) a process which is optional over $(\Omega, \mathfrak{F}^\mu_t, P^\mu)$ for every initial law μ, and a subset Λ of E is called nearly optional if its indicator 1_Λ is a nearly optional function. A terminal time T is a *right terminal time* if $\operatorname{reg}(T)\equiv\{x\in E\colon P^x\{T=0\}=1\}$ is a nearly optional subset of E. See [7], VII, where it is shown that if T is a right terminal time, then T is a.s. equal to $\overline{T}\wedge S$, where $\overline{T}$ is the perfect exact regularization of T and $S=\inf\{t\geqq 0\colon X_t\in\operatorname{reg}(T)\}$, and that a.s., $X_t\notin\operatorname{reg}(T)$ for all $t\in]0,\overline{T}[$. In particular, right terminal times are perfectable, and it can be shown that a right process killed at a right terminal time remains a right process. If T is a right terminal time and if R is co-optional for (X, T), then since T may be assumed perfect, one may modify the proof of the perfection theorem for ordinary co-optional times ([7], (25.6), for example) to show that one may replace R in (2.1) by an equivalent random time R' satisfying $R'\circ\theta_t=(R'-t)^+$ for all $t<T$ - that is, R' is a perfect co-optional time for (X, T).

In similar vein, a MF $(M_t)_{t\geqq 0}$ will be called a *right* MF if $t\to M_t$ is a.s. right continuous and if $E_M\equiv\{x\in E\colon P^x\{M_0=1\}=1\}$ is nearly optional relative to X. It is easy to see that if M is a right MF, then $S_M\equiv\inf\{t\geqq 0\colon M_t=0\}$ is a right terminal time.

3. Characterization of Killing Times

For ease of manipulations we shall suppose that X is realized as the coordinate process on the space Ω of all right continuous maps of $\mathbb{R}_+$ into E_Δ

$=E\cup\{\Delta\}$ admitting Δ as a trap. Let ζ denote the lifetime variable on Ω and let ω_Δ denote the path constantly equal to Δ. Take (k_t) to be the killing operators on Ω and let $\mathfrak{F}_t^{0*}=\sigma\{f(X_s): 0\leq s\leq t, f\in b\,\mathfrak{E}^*\}$.

Given a random time R, let $c_R(x)=P^x\{R>0\}$, a universally measurable function on E, and let $C_R=\{x\in E: c_R(x)>0\}$. By the process X killed at R is meant the process Y defined by

$$(3.1)\qquad \begin{aligned} Y_t&=X_t \quad \text{if } t<R\\ &=\Delta \quad \text{if } t\geq R \end{aligned}$$

under the probability measures $P^x\{\cdot|R>0\}$, $x\in C_R$. We shall say that R is a *killing time* if

(3.2) *c_R is a nearly optional function relative to X;*

(3.3) *The process Y with the measures $P^x\{\cdot|R>0\}$ is temporally homogeneous strong Markov on the state space C_R.*

It is implicit in (3.3) that a.s. $X_t\in C_R$ for all $t<R$. The condition (3.2) is a technical condition which will be needed for comparison of Y and X. It also follows from (3.2) that $S=\inf\{t: X_t\notin C_R\}$ is a stopping time over $(\mathfrak{F}_t)$. Obviously (3.2) is satisfied if c_R is Borel on E. We suppose now that R is a killing time and, to simplify notation, write c for c_R and C for C_R.

The killed process Y may be handled more expeditiously by means of its distribution on the path space Ω. For $x\in E_\Delta-C$ let $\hat{P}^x=\varepsilon_{\omega_\Delta}$ and for $x\in C$ let $\hat{P}^x$ be the distribution of Y under $P^x\{\cdot|R>0\}$. That is, for $0\leq t_1<\ldots<t_n$, $f_1,\ldots,f_n\in b\mathfrak{E}^*$ and $x\in C$.

$$(3.4)\qquad \hat{E}^x\{f_1(X_{t_1})\ldots f_n(X_{t_n})\}=E^x\{f_1(X_{t_1})\ldots f_n(X_{t_n})\,1_{\{t_n<R\}}|R>0\}.$$

We are of course using the convention $f(\Delta)=0$ for $f\in b\mathfrak{E}^*$. The correspondence between $\hat{P}^x$ and P^x is more compactly expressed in terms of killing operators. A monotone class argument based on (3.4) shows that for $H\in b\mathfrak{F}^{0*}$ and $x\in E$,

$$(3.5)\qquad \hat{E}^x\{H\}\,c(x)=E^x\{H\circ k_R;\, R>0\}.$$

That is, for every $x\in C$, the trace of $\hat{P}^x_{|\mathfrak{F}_t^{0*}}$ on $\{t<\zeta\}$ is absolutely continuous relative to the trace of $P^x_{|\mathfrak{F}_t^{0*}}$ on $\{t<\zeta\}$. A well known representation theorem ([7], VII for example) shows that there exists then a right continuous supermartingale MF $(M_t)_{t\geq 0}$ generating $\hat{P}^x$ from P^x. Spelling this out, $(M_t)_{t\geq 0}$ is right continuous and satisfies the conditions (2.2), $E^x M_t\leq 1$ for all $x\in E$ and all $t\geq 0$, and for every stopping time T over $(\mathfrak{F}_{t+}^{0*})$,

$$(3.6)\qquad \hat{E}^x\{H 1_{\{T<\zeta\}}\}=E^x\{HM_T\} \quad \text{if } H\in b\mathfrak{F}_{T+}^{0*}.$$

Because of the section theorem, (3.6) uniquely determines $(M_t)_{t\geq 0}$, up to evanescence. Taking $H=1_{\{X_0=x\}}$ with $x\in C$ one sees that $E_M=C$, and as C is nearly optional by hypothesis, M is a right MF, using the terminology established at the end of §2.

(3.7) **Proposition.** *For every $t>0$ and $x\in E$,*

$$P^x\{R>t|\mathfrak{F}_t\}=c(x)\,M_t.$$

Proof. It suffices to show that if $H \in b\mathfrak{F}_t^{0*}$ then

$$E^x\{H 1_{\{t<R\}}\} = c(x) E^x\{H M_t\}.$$

Because $1_{\{t<R\}} = 1_{\{t<\zeta \circ k_R\}}$, $H 1_{t<R\}} = H \circ k_R 1_{\{t<\zeta \circ k_R\}}$ and so, by (3.5)

$$E^x\{H 1_{\{t<R\}}\} = c(x) \hat{E}^x\{H 1_{\{t<\zeta\}}\}.$$

Then (3.6) applies to evaluate the right side as $c(x) E^x\{H M_t\}$.

If $x \in E_\Delta - C$, (3.6) shows that $P^x\{M_0 = 0\} = 1$. If $x \in C$, (3.7) yields

(3.8) $$c(x) M_t = W_t + Z_t$$

where W_t is a right continuous version, independent of x, of the uniformly integrable martingale $P^x\{R = \infty | \mathfrak{F}_t\}$ with final value $1_{\{R=\infty\}}$ and Z_t is the potential of the random measure $\varepsilon_R 1_{\{0<R<\infty\}}$ generated by the increasing process $A_t = 1_{[R,\infty[}(t) 1_{\{0<R<\infty\}}$. It follows then that for every stopping time T,

(3.9) $$P^x\{R > T | \mathfrak{F}_T\} = W_T + Z_T = c(x) M_T \quad \textit{where } M_\infty = \lim_{t\to\infty} M_t.$$

Now let S denote the debut of $E_\Delta - C$. Then S is a stopping time because of hypothesis (3.2). Let $T_0 = \inf\{t: M_t = 0\}$, so that T_0 is a terminal time and $M_t = 0$ for all $t \geqq T_0$, almost surely. Because M is a right MF, T_0 is in fact a right terminal time.

(3.10) **Proposition.** *Almost surely,* $R \leqq T_0 \leqq S$.

Proof. If $x \in E_\Delta - C$, $P^x\{M_0 = 0\} = 1$ as we noted above and so $P^x\{T_0 = 0\} = 1$. Therefore $P^x\{T_0 \leqq S\} = 1$ if $x \in E_\Delta - C$. If $x \in C$, (3.9) gives

$$E^x M_S = \frac{1}{c(x)} P^x\{R > S\},$$

but $P^x\{R > S\} = 0$ because of the hypothesis (3.3). Therefore $P^x\{S \geqq T_0\} = 1$ if $x \in C$, proving that a.s., $T_0 \leqq S$. For $x \in E_\Delta - C$, $P^x\{R = 0\} = 1$ so $P^x\{R \leqq T_0\} = 1$. If $x \in C$, (3.9) gives

$$0 = E^x M_{T_0} = \frac{1}{c(x)} P^x\{R > T_0\}$$

and we conclude that $P^x\{R \leqq T_0\} = 1$.

(3.11) **Proposition.** *For every stopping time* T, *a.s.*,

$$\{R = \infty, T < T_0\} = \{R \circ \theta_T = \infty, T < T_0\}.$$

Proof. Letting $t \to \infty$ in (3.7) gives

$$1_{\{R=\infty\}} = c(X_0) M_\infty.$$

By the multiplicative property of M,

$$\begin{aligned} M_T 1_{\{R\circ\theta_T = \infty\}} &= c(X_T) M_T M_\infty \circ \theta_T \\ &= c(X_T) M_\infty \end{aligned}$$

a.s. on $\{T<\infty\}$. On $\{T<T_0\}$, $T\leqq S$ and (3.10) gives $c(X_T)>0$ and $M_T>0$, so that $1_{\{R\circ\theta_T=\infty\}}>0$ if and only if $M_\infty>0$. Because $T<T_0$ implies $c(X_0)>0$, it follows that $R\circ\theta_T=\infty$ if and only if $R=\infty$, a.s. on $\{T<T_0\}$.

Define now $\bar{M}_t=c(X_0)M_t/c(X_t)$. Because of (3.10), $\bar{M}$ is well defined (setting $0/0=0$) and $T_0=\inf\{t:\bar{M}_t=0\}$. Though $\bar{M}_t$ need not be right continuous, it is obviously a MF as described in (2.2).

(3.12) **Proposition.** *Let $\tilde{A}$ denote the dual predictable projection of $A_t=1_{[R,\infty[}(t)\,1_{\{0<R<\infty\}}$. Then $\tilde{A}$ is an AF of $(X,\bar{M})$.*

Proof. As in (3.8) let Z be the potential of A so that Z is also the potential of $\tilde{A}$. Setting $h(x)=E^x M_\infty$ one sees that for every stopping time T

$$\begin{aligned}E^x\{M_\infty|\mathfrak{F}_T\}\,1_{\{T<\infty\}}&=E^x\{M_T M_\infty\circ\theta_T|\mathfrak{F}_T\}\,1_{\{T<\infty\}}\\&=M_T\,h(X_T)\,1_{\{T<\infty\}}.\end{aligned}$$

Therefore, from (3.8) and the fact that $1_{\{R=\infty\}}=c(X_0)M_\infty$

$$\begin{aligned}W_T\,1_{\{T<\infty\}}&=P^x\{R=\infty|\mathfrak{F}_T\}\,1_{\{T<\infty\}}\\&=c(X_0)M_T\,h(X_T)\,1_{\{T<\infty\}}.\end{aligned}\tag{3.13}$$

(If we knew that h were an optional function, this would imply that W_t and $c(X_0)\,M_t\,h(X_t)$ were indistinguishable.) From (3.8), almost surely

$$\begin{aligned}Z_T=Z_T\,1_{\{T<\infty\}}&=[c(X_0)\,M_T-W_T]\,1_{\{T<\infty\}}\\&=c(X_0)\,M_T[1-h(X_T)]\,1_{\{T<\infty\}}.\end{aligned}$$

It follows that for each $t\geqq 0$, almost surely

$$Z_t\circ\theta_T\,1_{\{T<\infty\}}=c(X_T)\,M_t\circ\theta_T[1-h(X_{t+T})]\,1_{\{T<\infty\}}$$

so

$$Z_t\circ\theta_T\,1_{\{T<T_0\}}=\frac{c(X_T)\,c(X_0)}{c(X_0)\,M_T}\,M_{t+T}[1-h(X_{t+T})]\,1_{\{T<T_0\}}.$$

That is,

$$\begin{aligned}\bar{M}_T Z_t\circ\theta_T&=c(X_0)\,M_{t+T}[1-h(X_{t+T})]\,1_{\{T<T_0\}}\\&=Z_{t+T}\,1_{\{T<T_0\}}.\end{aligned}$$

By right continuity in t, this implies that $\bar{M}_T Z_t\circ\theta_T$ and $Z_{t+T}\,1_{\{T<T_0\}}$ are indistinguishable. In other words $\bar{M}_T\Theta_T Z$ and $Z\,1_{[\![T,\infty[\![}=Z\,1_{[\![T,\infty[\![}\,1_{\{T<T_0\}}$ are indistinguishable. Now, it is easy to see ([7], (33.12) for example) that the potential Z' of $\Theta_T\tilde{A}$ satisfies

$$Z'\,1_{[\![T,\infty[\![}=(\Theta_T Z)\,1_{[\![T,\infty[\![}$$

so we obtain

$$\bar{M}_T Z'\,1_{[\![T,\infty[\![}=Z\,1_{[\![T,\infty[\![}.$$

But $\bar{M}_T Z'\,1_{[\![T,\infty[\![}$ is the restriction to $[\![T,\infty[\![$ of the potential of the predictable random measure $\bar{M}_T\Theta_T\tilde{A}$ so using ([7], (33.13)) we conclude that $1_{]\!]T,\infty[\![}\bar{M}_T\Theta_T\tilde{A}$

and $1_{]\!]T,\infty[\![} * \tilde{A}$ are distinguishable. This is just another way of saying that $\tilde{A}$ is an AF of $(X, \overline{M})$.

Here then is the principal result.

(3.14) **Theorem.** *Let R be a killing time which is also a left Markov time. Then there exists a right terminal time $T_0 \geqq R$ such that R is co-optional for (X, T_0).*

Proof. If $V \in b\mathfrak{H}^d$ there exists $V' \in \mathfrak{P} \cap \mathfrak{H}^g$ such that the random measure $V * (\varepsilon_R 1_{\{0<R<\infty\}})$ has the same dual predictable projection as $V' * (\varepsilon_R 1_{\{0<R<\infty\}})$. Therefore $V * (\varepsilon_R 1_{\{0<R<\infty\}})$ has dual predictable projection $V' * \tilde{A}$ which is, by (3.12), an AF of $(X, \overline{M})$. In view of (2.8), $\varepsilon_R 1_{\{0<R<\infty\}}$ is a raw AF of $(X, \overline{M})$ so (2.4) shows that $L = \sup\{t: \Delta(\varepsilon_R 1_{\{0<R<\infty\}})_t > 0\} = R 1_{\{0<R<\infty\}}$ is co-optional for (X, T_0), where $T_0 = \inf(t: \overline{M}_t = 0\} = \inf\{t: M_t = 0\}$. Consequently, if T is an arbitrary stopping time,

$$R \circ \theta_T = (R 1_{\{0<R<\infty\}}) \circ \theta_T + (R 1_{\{R=\infty\}}) \circ \theta_T$$
$$= L \circ \theta_T + R \circ \theta_T 1_{\{R \circ \theta_T = \infty\}}.$$

On $\{T < T_0\}$, $L \circ \theta_T = (L-T)^+$ and (3.11) shows that $R = \infty$ if and only of $R \circ \theta_T = \infty$. Thus, a.s. on $\{T < T_0\}$,

$$R \circ \theta_T = (L-T)^+ + R 1_{\{R=\infty\}}$$
$$= (R 1_{\{0<R<\infty\}} - T)^+ + (R 1_{\{R=\infty\}} - T)$$
$$= (R-T)^+,$$

proving that R is co-optional for (X, T_0).

It should be remarked that if R is co-optional for (X, T), where T is a right terminal time, then R has the left Markov property and R is a killing time. The first of these assertions follows from the results of [3] and the proof of the second is a rather trivial modification of the proof ([5]) of the fact that ordinary co-optional times are killing times.

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