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Some Martingales Associated with Queueing and Storage Processes*

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1. Introduction

Given a Markov process $x(t,w)$ with semigroup $T(t)f(x)=E_x f(x(t,w))$ and infinitesimal generator A there is a well known recipe due to Dynkin (1965) for constructing martingales associated with $x(t)$[1]:

(1.1) **Lemma.** *For $f\in\mathscr{D}(\tilde{A})$, $\tilde{A}$ the weak infinitesimal operator, the process $f(x(t)) - \int_0^t \tilde{A}f(x(s))\,ds$ is a martingale.*

A useful extension of Lemma (1.1) is contained in

(1.2) **Theorem.** *Suppose $U(t,x)\in\mathscr{D}(\tilde{A})$, $U_t(t,x)\in\mathscr{D}(\tilde{A})$ for each $t\geqq 0$, $U(t,x)$ and $\tilde{A}U(t,x)$ are absolutely continuous in t for each x. Then the process $U(t,x(t)) - \int_0^t [U_s(s,x(s))+\tilde{A}U(s,x(s))]\,ds$ is a martingale.*

Proof. This is a routine generalization of Theorem 1 on p. 269 of Rosenkrantz (1975) and the proof is therefore omitted. A similar result under slightly different hypotheses has been obtained by Kurtz (1981), Proposition 3.5.

It is worth pointing out that some restrictions on the function f are essential if the process defined in Lemma (1.1) is to be a martingale. For example, consider the reflecting Brownian motion process $x(t)=|w(t)|$, where $w(t)$ is the standard Wiener process and $Af(x)=(1/2)f''(x)$. Now if $f(x)=x$ then $Af(x)=0$ and (1.1), if true, would imply $f(x(t))=x(t)=|w(t)|$ is a martingale – a result which is clearly absurd. The difficulty here is due to the fact that f will not be in $\mathscr{D}(A)$ unless $f'(0)=0$. Since $f(x)=x$ has derivative $f'(0)=1\neq 0$ we see at once that $f(x)=x\notin\mathscr{D}(A)$.

Now the problem of constructing a nice class of functions f for which $f(x(t))-\int_0^t Af(x(s))\,ds$ is a martingale is equivalent to solving the martingale

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[1] To simplify the notation we suppress the w and write $x(t)$ for $x(t,w)$ etc.

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problem for the operator A and this is not an easy matter in general, see, e.g. Stroock-Varadhan (1979). Some idea of the subtleties involved can be gained by considering the following class of stochastic processes studied by Cinlar-Pinsky (1971). In their paper $x(t)$ denotes the content of a dam at time t with release rate $r(x)$ and random cumulative input $A(t)=A(t,w)$ assumed to be a process with stationary and nonnegative independent increments. In addition it is assumed the jump rate b of $A(t)$ is finite and $H(y)$ denotes the distribution of the size of the jump. Under the additional hypotheses that $r(0)=0$, $r(x)$ is strictly increasing and Lipschitz continuous they showed that $x(t)$ is a strong Markov process with state space $R^+=[0,\infty)$ whose corresponding semigroup $T(t)f(x)=E_x f(x(t))$ satisfies the integral equation

$$(1.3)\qquad T(t)f(x)=e^{-bt}f(q(x,t))+\int_0^t\int_0^\infty be^{-bs}\,T(t-s)f(y+q(x,s))\,dH(y)\,ds.$$

Here $q(x,t)$ is the unique solution to the ordinary differential equation

$$(1.4)\qquad z'(t)=a-r(z(t)),\qquad z(0)=x.$$

The following facts concerning $q(x,t)$ are easily established by explicitly solving (1.4).

(1.5) (i) For fixed $x, t\to q(x,t)$ is monotone continuous.
(ii) For fixed t, $x\to q(x,t)$ is nondecreasing and continuous.
(iii) $q(q(x,s),t)=q(x,s+t)$.
(iv) $q_t(x,t)=(a-r(x))\,q_x(x,t)$.

Making use of the integral representation (1.3) and the properties of $q(x,t)$ listed at (1.5) Cinlar-Pinsky remark without proof that the function $v(t,x)=T(t)f(x)$ satisfies the integro-differential equation:

(1.6) (i) $v_t=A\,v(t,x)$, $v(0,x)=f(x)$, $x\in R^+$
(ii) $Af(x)=(a-r(x))f'(x)+b\int_0^\infty (f(x+y)-f(x))\,dH(y)$.

One of the main purposes of this paper is to show that the class of Markov processes constructed by Çinlar-Pinsky solves the martingale problem corresponding to the operator A defined in (1.6(ii)) above. Specifically, let $C_c(R^+)$ denote the space of bounded, continuous functions on R^+ with compact support; let $C_c^1(R^+)=\{f: f, f'\in C_c(R^+)\}$. We then show, in Corollary 2.3, that $C_c^1(R^+)\subset\mathscr{D}(\tilde{A})$. It is important to note that this result does not follow directly from the integral equation (1.3); it is however a consequence of Theorem 2.2 which asserts that $T(t)\,C(\bar{R}^+)\subset C(\bar{R}^+)$ and $C(\bar{R}^+)=\{f: f$ bounded, continuous on $\bar{R}^+=R^+\cup\{\infty\}$, i.e. $\lim_{x\to\infty} f(x)=f(\infty)$ exists$\}$. For future reference we define $C_0(R^+)=\{f: f\in C(\bar{R}^+), f(\infty)=0\}$.

Before proceeding further it is worth noting that our methods can be extended to cover the very important special case of the virtual waiting time process $\eta(t)$ for the $M/GI/1$ queue, for the definition of which we refer the reader to

Takács (1962) p. 49. Indeed the virtual waiting time process corresponds to the case where $a=0$, $r(x)=1$ for $x>0$ and $r(0)=0$. The fact that $r(x)$ is discontinuous is relatively unimportant because it can be shown that the integral representation (1.3) remains valid in this case too, provided we set $q(x,t)=[x-t]^+$, $x^+=\text{Max}(x,0)$. Indeed, it is only necessary to repeat the argument of Lemma 2.18 of Çinlar-Pinsky to establish (1.3) for the virtual waiting time process.

In part 3 we apply Lemma 1.1 to the special case where $x(t)=\eta(t)$ (the virtual waiting time process), $f(x)=x^n$, $n\geqq 1$ to derive a new recursive formula for the moments $m_n(t,x)=E_x(\eta(t)^n)$ - see (3.4). A formula for $m_1(t,x)$ was first derived by Takács (1962) p. 55; thus the recursive formula (3.4) may be regarded as a significant generalization of his result. It is to be observed that x^n is unbounded hence $x^n \notin C(\bar{R}_+)$ and to justify the use of Lemma 1.1 we found it necessary to resurrect the notion of a "semigroup of type Γ", an idea due to Doob (1955) in the context of classical one dimensional diffusion processes - this is carried out in part 2.

One additional consequence of the martingale methods of part 3 is a new bound on the distribution of the supremum of the virtual waiting time process. For example if

$$\mu_m=\int_0^\infty y^m\,dH(y)<\infty, \qquad m\geqq 1 \text{ and } g(x)=1+x^m$$

then

$$P_x(\sup_{0\leqq s\leqq t}\eta(s)>y)\leqq e^{\lambda_0 t}\,g(x)/g(y) \tag{1.7}$$

where λ_0 is a constant depending only on m and μ_j; $1\leqq j\leqq m$. The constant λ_0 is defined in the proof of Lemma (3.9). In particular if $\mu_m<\infty$ then (1.7) implies

$$P_x(\sup_{0\leqq s\leqq t}\eta(s)>y)=O(y^{-m}) \qquad \text{as } y\to\infty,\ t \text{ held fixed.} \tag{1.8}$$

It is interesting to compare this estimate with the result in Cohen (1969) pp. 607-608, that $\sup\eta(s)$, taken over a busy cycle, is in L_2 iff $\mu_2<\infty$. Cohen assumes that the traffic intensity $\rho<1$. Our estimate is valid for all m and even for $\rho\geqq 1$. In particular applying Chebyshev's inequality to Cohen's estimate in the case $m=2$ yields an estimate on the tail no better than estimate (1.8).

In part 4 we turn our attention to the integro-differential equation (1.6) itself and study the conditions that must be imposed on f if the function $v(t,x)=T(t)f(x)$ is to satisfy (1.6) in the classical sense. Indeed to give a rigorous proof of the Çinlar-Pinsky remark it is first necessary to prove that $T(t)$ preserves the differentiability of the initial datum f; but this is equivalent to establishing a regularity theorem for the solutions to the integro-differential equation (1.6). One method for solving the Kolmogorov backward equation (1.6) is to study $T(t)$ as an equation of evolution in a suitable Banach Space and then characterize the domain $\mathscr{D}(A)$ of $T(t)$ - this is carried out in part 4.

It is noteworthy that $\mathscr{D}(A)$ depends on a and $r(x)$. For example, when

(1.9 i) $\quad a=0$ we have $\mathscr{D}(A)=\{f: f, rf'\in C(\bar{R}^+), \lim_{x\to 0} r(x)f'(x)=0\}$, and in the case of the virtual waiting time process we have

(1.9ii) $$\mathscr{D}(A)=\{f: f, f' \in C(\bar{R}^+), f'(0)=0\}.$$

In general a complete analysis of the integro-differential equation (1.6) requires that we distinguish 4 cases which we present here for future reference.

(1.10) *Case 1.* $x(t)=\eta(t)$ (the virtual waiting time process).

In this case $q(x,t)=[x-t]^+$ and

$$\tilde{A}f(x)=-f'(x)+b\int_0^\infty [f(x+y)-f(x)]\,dH(y), \qquad x>0,$$

$$\tilde{A}f(0)=b\int_0^\infty [f(y)-f(0)]\,dH(y).$$

Note. In case 2–4 it is assumed $r(0)=0$, $r(x)$ is strictly increasing and Lipschitz continuous.

(1.11) *Case 2.* $a=0$.

$$\tilde{A}f(x)=r(x)f'(x)+b\int_0^\infty [f(x+y)-f(x)]\,dH(y), \qquad x>0,$$

$$\tilde{A}f(0)=b\int_0^\infty [f(y)-f(0)]\,dH(y).$$

Remark. Under the additional condition that $\int_0^x (1/r(y))\,dy<\infty$, $x>0$ Harrison and Resnick (1976) characterize $\mathscr{D}(\tilde{A})$ in cases 1 and 2. However in order to solve (1.6) in the classical sense it is necessary to study $\mathscr{D}(A)$ – see Theorem 4.5.

(1.12) *Case 3.* $a>0$, $\sup_{0\leqq x<\infty} r(x)\leqq a$.

$$\tilde{A}f(x)=(a-r(x))f'(x)+b\int_0^\infty [f(x+y)-f(x)]\,dH(y), \qquad 0\leqq x<\infty.$$

(1.13) *Case 4.* $a>0$, $\sup_{0\leqq x<\infty} r(x)>a$. Let x^* be the unique solution to the equation $r(x^*)=a$. Then

$$\tilde{A}f(x)=(a-r(x)f'(x)+b\int_0^\infty [f(x+y)-f(x)]\,dH(y), \qquad x\neq x^*,$$

$$\tilde{A}f(x^*)=b\int_0^\infty [f(x^*+y)-f(x^*)]\,dH(y).$$

In part 4 we explicitly characterize domains $\mathscr{D}(A)$ in each of the 4 cases above – see (4.1) through (4.4). This leads to the following strong regularity theorem for the solutions to (1.6) – see Theorem (4.5).

Theorem. *The function $v(t,x)=T(t)f(x)$, $f\in\mathscr{D}(A)$, as defined in* (4.1)–(4.4), *is the unique solution to the integro differential equation* (1.6) *satisfying the conditions*

(i) *$v(t,x)$ is strongly differentiable in t with strongly continuous derivative $v_t(t,x)$;*

(ii) *$\|v(t)\|$ is bounded;*

(iii) $\lim_{t\to 0}\|v(t,x)-f(x)\|=0$.

In Case 1 we can actually prove a stronger result namely that $T(t)(C_0^1(R^+))\subset C_0^1(R^+)$.

A novelty of our approach is the use we make of a perturbation theorem of Hille-Phillips to characterize $\mathscr{D}(A)$. More precisely we decompose the operator A in (1.6ii) into the sum

$$
\begin{aligned}
&Af(x)=Bf(x)+Cf(x) \quad \text{where}\\
(1.14)\qquad &Bf(x)=(a-r(x))f'(x)-bf(x)\\
&Cf(x)=b\int_0^\infty f(x+y)\,dH(y).
\end{aligned}
$$

A trivial calculation shows that B is the infinitesimal generator of the semigroup $U(t)f(x)=e^{-bt}f(q(x,t))$ and thus $T(t)=e^{tA}=e^{t(B+C)}$ is a perturbation (as defined by Hille-Phillips) of the semigroup $U(t)=e^{tB}$; note that C is a bounded linear operator and hence $\mathscr{D}(A)=\mathscr{D}(B)$. In addition our approach yields a sharp regularity theorem for the solutions to the integro-differential equation (1.6) which we believe to be new even in the case of the so-called Takács integro-differential equation for the virtual waiting time process.

The decomposition (1.14) of the infinitesimal generator A into a sum $A=B+C$ where C is bounded and B generates a semigroup whose domain $\mathscr{D}(B)=\mathscr{D}(A)$ is easily computed is applicable to a much larger class of Markov processes than is considered here. It is particularly useful in proving heavy traffic limit theorems via the Trotter-Kato theorem. For example in the recent thesis of D. Burman (1979) we are confronted with a sequence of integro-partial differential operators of the form:

$$
\begin{aligned}
&A_nf(x,y)=B_nf(x,y)+C_nf(x,y), \qquad (x,y)\in R\times R^+\\
(1.15)\qquad &B_nf(x,y)=\sqrt{n}\,f_x(x,y)+nf_y(x,y)\\
&C_nf(x,y)=nr(y)\,[f(x+n^{-1/2},0)-f(x,0)].
\end{aligned}
$$

It is not too difficult to see that B_n is the infinitesimal generator of the semigroup $U(t)f(x,y)=f(x-n^{1/2}t,y+nt)$ and with a little more work one can compute $\mathscr{D}(B_n)$. Now the Trotter-Kato-Kurtz approach to the heavy traffic approximation requires the construction of a sequence of functions $f_n\in\mathscr{D}(A_n)$ with the property that $\lim_{n\to\infty}A_nf_n=Af$ in some suitable sense. If the perturbation theorem of Hille-Phillips is valid (and it is under suitable conditions) we can compute $\mathscr{D}(A_n)$ by observing that $\mathscr{D}(A_n)=\mathscr{D}(B_n)$, the computation of the latter being often a much simpler task. These and other aspects of the functional

analytic approach to queueing theory are currently the subject of a joint study with D. Burman of Bell Laboratories. It is also a pleasure to acknowledge some useful conversations ith W. Whitt, also of Bell Laboratories.

2. The Martingale Problem and Its Solution

We now show that the Markov processes $x(t)$, whose corresponding semigroups $T(t)$ satisfy the integral equation (1.3), solve the martingale problem; more precisely we shall prove the following result:

(2.1) **Theorem.** *Let $f \in C_c^1(R^+)$. Then the process $f(x(t)) - \int_0^t \tilde{A} f(x(s))\, ds$ is a martingale.*

Proof. By a Theorem of Dynkin (1965) p. 133 it suffices to show that $C_c^1(R^+) \subset \mathscr{D}(\tilde{A})$ and this in turn is a consequence of

(2.2) **Theorem.** *Let $f \in C(\bar{R}^+)$. Then there exists a unique solution $T(t)f(x)$ to the integral equation* (1.3) *satisfying the conditions*

(i) $\|T(t)f\| \leqq \|f\|$ (ii) $T(t)f \in C(\bar{R}^+)$ and (iii) $T(t)f(x)$: $R^+ \times R^+ \to R$ *is continuous in* (t, x).

(2.3) **Corollary.** $C_c^1(R^+) \subset \mathscr{D}(\tilde{A})$.

We shall begin with the proof of Corollary (2.3). It suffices to show that $\lim_{t \to 0} (T(t)f(x) - f(x))/t = \tilde{A} f(x)$ in the sense of bounded pointwise convergence – see Dynkin (1965) p. 55.

Step 1. Set $g(s,t,x) = \int_0^\infty b e^{-bs} T(t-s) f(y + q(x,s))\, dH(y)$, and hold x fixed. By Theorem (2.2) $T(t-s)f(y+q(x,s))$ is jointly continuous in (s,t,x) and this implies the continuity of $g(s,t,x)$. In particular, by the fundamental theorem of the calculus, we have

$$\lim_{t \to 0} (1/t) \int_0^t g(s,t,x)\, ds = g(0,0,x) = b \int_0^\infty f(x+y)\, dH(y) \tag{2.4}$$

in the sense of bounded pointwise convergence; it is easily checked that

$$\left| (1/t) \int_0^t g(s,t,x)\, ds \right| \leqq \|f\| (1/t) \int_0^t b e^{-bs}\, ds.$$

Step 2. In terms of the function $g(s,t,x)$ the integral equation (1.3) can be rewritten as

$$T(t) f(x) = e^{-bt} f(q(x,t)) + \int_0^t g(s,t,x)\, ds. \tag{2.5}$$

Thus

$$(T(t)f(x) - f(x))/t = (e^{-bt} f(q(x,t)) - f(x))/t + (1/t) \int_0^t g(s,t,x)\, ds.$$

Now an easy calculation using (1.4) and (1.5) shows that

$$\lim_{t\to 0}(e^{-bt}f(q(x,t))-f(x))/t=\frac{d}{dt}(e^{-bt}f(q(x,t)))$$

evaluated at $t=0$ and this equals $-bf(x)+(a-r(x))f'(x)$. Moreover the hypothesis $f\in C_c^1(R^+)$ implies that the above limit exists in the sense of bounded pointwise convergence. Summing up then we've shown that $f\in C_c^1(R^+)$ implies:

$$\lim_{t\to 0}(T(t)f(x)-f(x))/t$$
$$=(a-r(x))f'(x)+b\int_0^\infty [f(x+y)-f(x)]\,dH(y)=\tilde{A}f(x)$$

in the sense of bounded pointwise convergence. For a complete description of $\tilde{A}f$ the reader is referred back to (1.10)-(1.13). Hence $C_c^1(R^+)\subset\mathscr{D}(\tilde{A})$. □

Remark. When r is bounded, as in Cases 1 and 3, one can replace $C_c^1(R^+)$ in Theorem 2.1 by $C_0^1(R^+)$. If r is unbounded then Theorem (2.1) remains valid for those functions f, $rf'\in C(\bar{R}^+)$ – we omit the details of the proof.

We turn now to Theorem (2.2) the proof of which has been adapted from Kato (1976) 2nd edition, pp. 497–498.

(2.6) *Definition.* (i) $U(t)f(x)=e^{-bt}f(q(x,t))$,

(ii) $Cf(x)=b\int_0^\infty f(x+y)\,dH(y), \quad f\in C(\bar{R}^+)$.

(2.7) **Lemma.** *$U(t)f(x)$: $C(\bar{R}^+)\to C(\bar{R}^+)$ is a strongly continuous contraction semigroup and C is a bounded linear operator on $C(\bar{R}^+)$ with $\|C\|\leqq b$.*

Proof. From property 1.5 (iii) and the compactness of $\bar{R}^+$ it is easy to check that $U(t)$ is a semigroup and that $\lim_{t\to 0} U(t)f(x)=f(x)$ in the sense of bounded pointwise convergence since $\lim_{t\to 0} q(x,t)=x$. Moreover $\lim_{x\to\infty}$ q(x, t) exists as a finite or infinite limit and $f\in C(\bar{R}^+)$ implies $\lim_{x\to\infty} U(t)f(x)=\lim_{x\to\infty} e^{-bt}f(q(x,t))$ exists also, so $U(t)f\in C(\bar{R}^+)$. Thus $U(t)f$ is weakly right continuous in t and hence by a well known result, e.g. Dynkin (1965) p. 35, Theorem 1.5, $U(t)f$ is strongly continuous. It is trivial to see that $\|Cf\|\leqq b\|f\|$ since H is a probability distribution. In addition $\lim_{x\to\infty} f(x+y)=f(\infty)$ in the sense of bounded pointwise convergence and hence $\lim_{x\to\infty} Cf(x)$ exists also. □

(2.8) **Lemma.** *The integral equation* (1.3) *can be written as:*

(2.9) $$T(t)f(x)=U(t)f(x)+\int_0^t U(t-\sigma)\,CT(\sigma)f(x)\,d\sigma.$$

Moreover the solution $T(t)f(x)$ is unique and can be expressed as

(2.10) $$T(t)f(x)=\sum_{n=0}^\infty U_n(t)f(x)$$

where

$$U_0(t)=U(t), \qquad U_n(t)f(x)=\int_0^t U(t-\sigma)\,CU_{n-1}(\sigma)f(x)\,d\sigma, \qquad n=1,2,\ldots$$

Remark. $U_n(t)f(x)$ is for each n jointly continuous in (t,x).

Proof. Set $v(t,x)=T(t)f(x)$ and note that $v(t-s,q(x,s)+y)=T(t-s)f(q(x,s)+y)$. Now $CT(\sigma)f(x)=Cv(\sigma,x)=\int_0^\infty bv(\sigma,x+y)\,dH(y)$ and therefore

$$\begin{aligned}\int_0^t U(t-\sigma)\,CT(\sigma)f(x)\,d\sigma &= \int_0^t be^{-b(t-\sigma)}\int_0^\infty v(\sigma,q(x,t-\sigma)+y)\,dH(y)\,d\sigma\\ &= \int_0^t\int_0^\infty be^{-bs}v(t-s,q(x,s)+y)\,dH(y)\,ds\\ &= \int_0^t\int_0^\infty be^{-bs}T(t-s)f(q(x,s)+y)\,dH(y)\,ds.\end{aligned}$$

Since $U(t)f(x)=e^{-bt}f(q(x,t))$ this completes the proof of (2.9).

We next prove that the infinite series (2.10) converges uniformly for $x\in\bar{R}^+$ and t belonging to compact subintervals of R^+. Since $U(t)$ is a contraction semigroup we have $\|U_n(t)f\|\leqq b\int_0^t e^{-b(t-\sigma)}\|U_{n-1}(\sigma)f\|\,d\sigma$ and hence by induction

$$\|U_n(t)f\|\leqq(e^{-bt}(bt)^n/n!)\,\|f\|, \qquad n=0,1,2,\ldots \tag{2.11}$$

From which it follows at once that $w(t,x)=\sum_{n=0}^\infty U_n(t)f(x)$ converges uniformly for $x\in\bar{R}^+$ and t belonging to compact subsets of R^+. We now show that $w(t,x)$ is a solution to the integral equation (2.9).

$$\begin{aligned}\sum_{n=0}^\infty U_n(t)f(x) &= U(t)f(x)+\sum_{n=1}^\infty\int_0^t U(t-\sigma)\,CU_{n-1}(\sigma)f(x)\,d\sigma\\ &= U(t)f(x)+\int_0^t U(t-\sigma)\,C\left(\sum_{n=0}^\infty U_n(\sigma)f(x)\right)d\sigma\end{aligned}$$

or

$$w(t,x)=U(t)f(x)+\int_0^t U(t-\sigma)\,Cw(\sigma,x)\,d\sigma.$$

The proof of the Lemma will be completed by showing that $h(t,x)=w(t,x)-T(t)f(x)\equiv 0$. It is clear that $\|w(t,x)\|\leqq\|f\|$ and that $\|T(t)f\|\leqq\|f\|$ and thus $\sup_{0\leqq\sigma\leqq t}\|h(\sigma,x)\|\leqq 2\|f\|=M$. In addition $h(t,x)$ satisfies the integral equation:

$$h(t,x)=\int_0^t U(t-\sigma)\,Ch(\sigma,x)\,d\sigma. \tag{2.12}$$

Thus

$$\|h(t,x)\| \leqq \int_0^t b\,\|h(\sigma,x)\|\,d\sigma \leqq 2bt\,\|f\|.$$

In particular $\|h(\sigma,x)\| \leqq 2b\sigma\,\|f\|$ on $[0,t]$. By mathematical induction one sees that

(2.13) $$\|h(\sigma,x)\| \leqq ((2bt)^n/n!)\,\|f\|, \quad 0 \leqq \sigma \leqq t, \quad n=1,2,\ldots$$

from which it follows at once that $h(\sigma,x)\equiv 0$. □

The martingales produced by Theorem (3.1) require that $f\in C_0^1(R^+)$ but for many purposes this is unnecessarily restrictive. For example polynomial functions of the form $f(x)=x^m$ are excluded and yet it is known that if $\mu_m = \int_0^\infty y^m\,dH(y) < \infty$ and $x(t)=\eta(t)$ (the virtual waiting time process for the $M/GI/1$ queue) then $T(t)f(x)=E_x(\eta(t)^m)<\infty$. Thus $T(t)f(x)$ is defined for functions $f(x)$ satisfying a growth condition of the form $\lim_{x\to\infty} |f(x)|/(1+x^m)<\infty$. The possibility of extending the action of $T(t)$ to a much larger class of unbounded continuous functions was first noted by Doob (1955) who was interested in constructing martingales and supermartingales associated with Markov processes. For example it is not too difficult to show that if $g\in\mathscr{D}(\tilde{A})$ and $\tilde{A}g(x)\leqq\lambda g(x)$ then $e^{-\lambda t}g(x(t))$ is a supermartingale - see Theorem 3.5. Unfortunately in many cases g is unbounded and hence is not in $\mathscr{D}(\tilde{A})$ as defined, say, in Corollary (2.3).

To circumvent these difficulties Doob introduced the notion of a semigroup of type Γ. For us this means the introduction of a growth function $\phi(x)$ which is (i) strictly positive on R^+, (ii) continuous and (iii) $\lim_{x\to\infty}\phi(x)=+\infty$. We then define a Banach Space $C(\phi,\bar{R}^+)$ via the recipe

(2.14) $$C(\phi,\bar{R}^+)=\{f\colon f(x)/\phi(x)\in C(\bar{R}^+)\}$$
$$\|f\|_\phi = \sup_{0\leqq x<\infty} |f(x)/\phi(x)|.$$

Our choice of ϕ depends on the existence of μ_m.

(2.15) *Definition.* If $\mu_m<\infty$ then $\phi_m(x)=1+x^m$, $m\geqq 1$.

Remark. From now on we restrict our attention to the case where $x(t)=\eta(t)$ = virtual waiting time process.

(2.16) **Lemma.** *$f\in C(\phi_m,\bar{R}^+)$ implies $U(t)f(x)=e^{-bt}f([x-t]^+)\in C(\phi_m,\bar{R}^+)$ and moreover $\|U(t)f\|_{\phi_m}\leqq e^{-bt}\,\|f\|_{\phi_m}$.*

The proof is simple and straightforward and therefore omitted.

(2.17) **Lemma.** *Suppose $\mu_m<\infty$ and $f\in C(\phi_m,\bar{R}^+)$. Then*

(i) *$Cf(x)\in C(\phi_m,\bar{R}^+)$ and*

(ii) *$\|Cf\|_{\phi_m}\leqq M'b\,\|f\|_{\phi_m}$ where*

$$M' = \sup_{0 \leqq x < \infty} \left(1 + \sum_{j=0}^{m} \binom{m}{j} \mu_{m-j} x^j\right) / (1 + x^m).$$

Proof. By hypothesis $|f(x)| \leqq \|f\|_{\phi_m}(1+x^m)$, thus

$$|Cf(x)| \leqq b \|f\|_{\phi_m} \int_0^\infty (1 + (x+y)^m)\, dH(y)$$

$$= b \|f\|_{\phi_m} \left(1 + \sum_{j=0}^{m} \binom{m}{j} \mu_{m-j} x^j\right).$$

Thus $|Cf(x)/(1+x^m)| \leqq M' b \|f\|_{\phi_m}$. We leave it to the reader to check that $f \in C(\phi_m, \bar{R}^+)$ implies $Cf \in C(\phi_m, \bar{R}^+)$.

Combining Lemmas (2.16) and (2.17) allows us to solve the integral equation (1.3) in the much larger space $C(\phi_m, \bar{R}^+)$. More precisely we have the following result.

(2.18) **Theorem.** *Suppose* $\mu_m < \infty$ *and* $f \in C(\phi_m, \bar{R}^+)$. *Then the integral equation* (1.3) *possesses a unique solution denoted by* $T(t)f$ *with the following properties:* (i) $T(t)f \in C(\phi_m, \bar{R}^+)$ *and* (ii) $\|T(t)f\|_{\phi_m} \leqq e^{\gamma t} \|f\|_{\phi_m}$ *where* $\gamma = (M'-1)b$, M' *defined as in Lemma* (2.17).

Remark. The estimate (ii) implies that $T(t)$ is a semigroup of type Γ in the sense of Doob (1955) - see p. 168.

Proof. The proof of Theorem (2.18) proceeds exactly as in Lemma (2.8) with the sup norm replaced by $\| \;\|_{\phi_m}$. In particular making use of Lemmas (2.16) and (2.17) it is easy to show that the operators $U_n(t)$ defined in (2.10) satisfy the analogue of the estimate (2.11) i.e.

$$\|U_n(t)f\|_{\phi_m} \leqq (e^{-bt}(M'bt)^n/n!) \|f\|_{\phi_m}, \qquad n = 0, 1, 2, \ldots \tag{2.19}$$

Hence the series $\sum_{n=0}^{\infty} U_n(t) f(x)$ converges uniformly in the $\| \;\|_{\phi_m}$ norm to a solution $T(t)f(x)$ satisfying (2.18) (i) and (ii). □

(2.20) *Definition.* $C^1(\phi_m, \bar{R}^+) = \{f : f$ and $f' \in C(\phi_m, \bar{R}^+)\}$.

Example. If f is a polynomial of degree m then $f \in C^1(\phi_m, \bar{R}^+)$.

(2.21) **Theorem.** *Assume* $\mu_m < \infty$ *and* $f \in C^1(\phi_m, \bar{R}^+)$. *Then*

$$f(\eta(t)) - \int_0^t \tilde{A} f(\eta(s))\, ds \quad \textit{is a martingale.}$$

Proof. Let $j(x)$ denote a C^∞ function with the property that $j(x) = 1$ on $x \leqq 0$; $0 \leqq j(x) \leqq 1$ on $0 \leqq x \leqq 1$ and $j(x) = 0$ for $x \geqq 1$; set $f_n(x) = f(x) j(x-n)$. It is clear that (i) $|f_n(x)| \leqq |f(x)|$, (ii) $(f_n'(x)| \leqq |f'(x)| + c|f(x)|$ where $c = \|j'\|$ and that $f_n \in C_0^1(\bar{R}^+) \subset \mathscr{D}(\tilde{A})$. Thus $f_n(\eta(t)) - \int_0^t \tilde{A} f_n(\eta(s))\, ds$ is a martingale for each n. We would like to pass to the limit and conclude that $f(\eta(t))-\int_0^t \tilde{A}f_n(\eta(s))\,ds$ is a martingale too. Now $E_x(|f_n(\eta(t))|)\leqq E_x(|f(\eta(t))|)=T(t)|f|(x)\leqq e^{\gamma t}\|f\|_{\phi_m}$ and so as a consequence of the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} E(f_n(\eta(t))|F(\sigma))=E(f(\eta(t))|F(\sigma)), \qquad 0\leqq\sigma\leqq t.$$

Similarly it is easily seen that

$$\begin{aligned}|\tilde{A}f_n(x)| &\leqq |f_n{}'(x)|+b\int_0^\infty |f_n(x+y)-f_n(x)|\,dH(y)\\ &\leqq |f'(x)|+c|f(x)|+b\int_0^\infty |f(x+y)|\,dH(y)+b\int_0^\infty |f(x)|\,dH(y)\\ &\leqq |f'(x)|+(c+b)|f(x)|+C|f|(x).\end{aligned}$$

Thus

$$\begin{aligned}\|\tilde{A}f_n\|_{\phi_m} &\leqq \|f'\|_{\phi_m}+(b+c)\|f\|_{\phi_m}+M'b\|f\|_{\phi_m}\\ &=\|f'\|_{\phi_m}+(b+c+M'b)\|f\|_{\phi_m}.\end{aligned}$$

By Theorem (2.18) $E_x(|\tilde{A}f_n(\eta(s))|)\leqq e^{\gamma t}(\|f'\|_{\phi_m}+(b+c+M'b\|f\|_{\phi_m}),\ s\leqq t$, and thus we can apply the Lebesgue dominated convergence theorem again to conclude:

$$\begin{aligned}\lim_{n\to\infty} E_x\left(\int_0^t \tilde{A}f_n(\eta(s))\,ds|F(\sigma)\right) &=\int_0^t \lim_{n\to\infty} E_x(\tilde{A}f_n(\eta(s))|F(\sigma))\,ds\\ &=\int_0^t E_x(\tilde{A}f(\eta(s))|F(\sigma)\,ds\\ &=E_x\left(\int_0^t \tilde{A}f(\eta(s))\,ds|F(\sigma)\right).\end{aligned}$$

Puttung these results together yields the result

$$\begin{aligned}E(f(\eta(t))-\int_0^t \tilde{A}f(\eta(s))\,ds|F(\sigma)) &=\lim_{n\to\infty} E(f_n(\eta(t))-\int_0^t \tilde{A}f(\eta(s))\,ds|F(\sigma))\\ &=\lim_n (f_n(\eta(\sigma))-\int_0^\sigma \tilde{A}f_n(\eta(s))\,ds)\\ &=f(\eta(\sigma))-\int_0^\sigma \tilde{A}f(\eta(s))\,ds. \quad \square\end{aligned}$$

3. Some Martingales Associated with the Virtual Waiting Time Process

Let ρ denote the traffic intensity of the $M/GI/1$ queue, i.e. $\rho=\mu_1 b$. In his book Takács' (1962) p. 50 derived in a fairly straight-forward way the following formula for $E_x(\eta(t))=m_1(t,x)$.

(3.1) **Theorem** (Takács' (1962) p. 55).

$$m_1(t,x)=x+(\rho-1)t+\int_0^t P_x(\eta(s)=0)\,ds.$$

It is instructive to derive this formula via Theorem (2.21) with $f(x)=x$. A routine calculation yields

$$\tilde{A}f(x)=(\rho-1)I_{(0,\infty)}(x)+\rho I_{(0)}(x)=\rho-1+I_{(0)} \tag{3.2}$$

Thus

$$f(x)=E_x\left\{f(\eta(t))-\int_0^t \tilde{A}f(\eta(s))ds\right\}$$
$$=E_x\left(\eta(t)-\int_0^t [(\rho-1)+I_{(0)}(\eta(s))]\,ds\right)=x$$

or

$$E_x(\eta(t))=x+(\rho-1)t-\int_0^t p_x(\eta(s)=0)\,ds,$$

since

$$E_x(I_{(0)}(\eta(s)))=p_x(\eta(s)=0). \quad \square$$

The process by which we just computed $m_1(t,x)$ is easily extended to derive a new recursive formula for $m_n(t,x)=E_x(\eta(t)^n)$, $n=0,1,2,\ldots,m_0(t,x)\equiv 1$. Set $f(x)=x^n$, assume $\mu_n<\infty$, $n>1$ and apply Theorem (2.21) to conclude:

$$E_x\left(\eta(t)^n-\int_0^t\left\{-n\,\eta(s)^{n-1}+\int_0^\infty b\sum_{j=1}^n \binom{n}{j}\eta(s)^{n-j}y^j\,dH(y)\right\}ds\right)=x^n. \tag{3.3}$$

Of course $\mu_j<\infty$, $1\leqq j\leqq n$ and thus (3.3) can be rewritten as

$$m_n(t,x)=x^n+\int_0^t\left\{n(\rho-1)m_{n-1}(s,x)+b\sum_{j=2}^n \mu_j\binom{n}{j}m_{n-j}(s,x)\right\}ds. \tag{3.4}$$

(3.5) **Theorem.** *Assume $\mu_m<\infty$, $g\in C^1(\phi_m,\bar{R}^+)$ and that g satisfies the integro-differential inequality $\tilde{A}g(x)\leqq\lambda g(x)$ for some constant λ. Then $e^{-\lambda t}g(\eta(t))$ is a supermartingale.*

Proof. The proof is very similar to that of Theorem (2.21) so we shall only sketch the details. Set $g_n(x)=g(x)j(x-n)$, where $j(x)$ is the same function defined at the beginning of the proof of Theorem (2.21). Then $g_n\in\mathscr{D}(\tilde{A})$ and $U^{(n)}(t,x)=e^{-\lambda t}g_n(x)$ satisfies the conditions of Theorem (1.2) from which it follows that

$$U^{(n)}(t,\eta(t))-\int_0^t\{U_s^{(n)}(s,\eta(s))+\tilde{A}U^{(n)}(s,\eta(s))\}\,ds \tag{3.6}$$

is a martingale for each n. It is easy to check that the pointwise limit of (3.6) as $n\to\infty$ *is*

$$U(t,\eta(t))-\int_0^t\{U_s(s,\eta(s))+\tilde{A}U(s,\eta(s))\}\,ds \tag{3.7}$$

with $U(t,x)=e^{-\lambda t}g(x)$. Indeed, using estimates similar to those obtained in the course of proving Theorem (2.21), it can be shown that the Lebesgue dominated convergence theorem is applicable and hence (3.7) is a martingale, too. An easy calculation shows that

$$U_t(t,x)+\tilde{A}U(t,x)\leqq 0 \qquad \text{all } (t,x) \tag{3.8}$$

and this clearly implies that $U(t,\eta(t))=e^{-\lambda t}g(\eta(t))$ is a supermartingale. □

(3.9) **Lemma.** *Assume $\mu_m<\infty$, $g(x)=1+x^m$. Then there exists a constant λ_0 depending only on the moments μ_j, $1\leqq j\leqq m$, of the service time distribution $H(y)$, b and m such that $\tilde{A}g(x)\leqq\lambda_0 g(x)$. Hence $e^{-\lambda_0 t}g(\eta(t))$ is a nonnegative supermartingale.*

Proof.

Case 1. $m\geqq 2$. An easy calculation shows that $\tilde{A}g(x)=\sum_{j=0}^{m-1}a_j x^j$ is a polynomial of degree $m-1$ where the coefficients a_j depend only on b, m, j and μ_j, $0\leqq j\leqq m$. If we now set $\lambda_0=\sup_{0\leqq x<\infty}\left(\sum_{j=0}^{m-1}a_j x^j/(1+x^m)\right)$ then it is easy to see that $\tilde{A}g(x)\leqq\lambda_0 g(x)$.

Case 2. When $g(x)=1+x$ then $\tilde{A}g(x)$ is discontinuous at $x=0$, i.e. $\tilde{A}g(x)=-1+\rho$ for $x>0$ and $\tilde{A}g(0)=\rho$. In any event it is still true that $\tilde{A}g(x)\leqq\rho g(x)$. □

From Lemma (3.9) we deduce the following estimate on the supremum:

(3.10) **Theorem.** *Assume $\mu_m<\infty$. Then*

$$P_x(\sup_{0\leqq s\leqq t}\eta(s)>y)\leqq e^{\lambda_0 t}(1+x^m)/(1+y^m).$$

Hence

$$P_x(\sup_{0\leqq s\leqq t}\eta(s)>y)=0(y^{-m}) \qquad \textit{as } y\to\infty.$$

Notation. $\tau_y=\inf\{t>0\colon \eta(t)>y\}$.

Proof. By Doob's optional stopping theorem applied to the supermartingale $e^{-\lambda t}g(\eta(t))$ we get

$$g(x)\geqq E_x(e^{\lambda_0(\tau_y\wedge t)}g(\eta(\tau_y\wedge t)))\geqq E_x(e^{-\lambda_0\tau_y}g(\eta(\tau_y);\ \tau_y\leqq t)).$$

But on the set where $\tau_y\leqq t$ we have $e^{-\lambda_0\tau_y}\geqq e^{-\lambda_0 t}$ and $g(x)$ is monotone increasing implies $g(\eta(\tau_y))\geqq g(y)$, thus

$$E_x(e^{-\lambda_0\tau_y}g(\eta(\tau_y));\ \tau_y\leqq t)\geqq e^{-\lambda_0 t}g(y)P_x(\tau_y\leqq t).$$

From which it follows at once that

$$P_x(\sup_{0\leqq s\leqq t}\eta(s)>y)=P_x(\tau_y\leqq t)\leqq(e^{\lambda_0 t}g(x))/g(y). \quad □$$

An immediate consequence of Theorem (3.10) is the integrability of the $\sup_{0\leqq s\leqq t}\eta(s)=\eta^*(t)$. More precisely if $\mu_m>\infty$, $m>1$, $0<\varepsilon<m$, then

$\eta^*(t)\in L^{m-\varepsilon}(R^+)$. T. Kurtz, in a private communication, has observed that this result can be obtained much more simply by noting that $\eta^*(t)\leqq\xi(t)+x$ where $\xi(t)=\sum_{i=1}^{N_b(t)} x_i$, where $N_b(t)$ is a Poisson process with parameter b and the X_i are i.i.d. with distribution H. Thus $P_X(\eta^*(t)>y)<P(\xi(t)>y-x)\leqq E(\xi(t)^m)/(y-x)^m$, where $E(\xi(t))=O(t^m)$ if $\mu_m<\infty$. Note that this eliminates the unpleasant exponential factor $e^{\lambda t}$.

4. Solution to the Kolmogorov Backward Differential Equation (1.6)

It was noted in the introduction that $\mathscr{D}(A)$ depends on a and r. Specifically there are 4 cases to be considered and these are listed below:

Case 1. $x(t)=\eta(t)$ (virtual waiting time) process, then

(4.1) $\mathscr{D}(A)=\{f: f\in C(\bar{R}^+), f'\in C(\bar{R}^+), f'(0)=0\}$, and $q(x,t)=[x-t]^+$.

Note. In case 2–4 below it is assumed that $r(x)$ is monotone, strictly increasing, Lipschitz continuous with $r(0)=0$.

Case 2. $a=0$, then

(4.2) $\mathscr{D}(A)=\{f: f\in C(\bar{R}^+), rf'\in C(\bar{R}^+), \lim_{x\downarrow 0} r(x)f'(x)=0, f'(x)$ exists for $x>0.\}$

Note. $\lim_{x\downarrow 0} r(x)f'(x)=0$ does not imply that $f'(0)$ exists.

Case 3. $a>0$, $\sup_{0\leqq x<\infty} r(x)\leqq a$, then

(4.3) $$\mathscr{D}(A)=\{f: f, (r(x)-a)f'(x)\in C(\bar{R}^+), \lim_{x\to\infty} (r(x)-a)f'(x)=0\}.$$

Case 4. $a>0$, $\sup_{0\leqq x<\infty} r(x)>a$, then

(4.4) $$\mathscr{D}(A)=\{f: f\in C(\bar{R}_+), (r(x)-a)f'(x)\in C(\bar{R}^+), \lim_{x\to x^*} (r(x)-a)f'(x)=0\}$$

where $r(x^*)=a$, $0<x^*<\infty$.

Note. As in case (4.2) $f'(x)$ need not exist for $x=x^*$.

The correct version of remark (2.26) of Cinlar-Pinsky can now be given:

(4.5) **Theorem.** *The function $v(t,x)=T(t)f(x)$, $f\in\mathscr{D}(A)$ (as defines in (4.1)–(4.4) is the unique solution to the integro-differential equation (1.6) satisfying the conditions:*

(i) *$v(t,x)$ is strongly differentiable with strongly continuous derivative $v_t(t,x)$.*

(ii) *$\|v(t)\|$ is bounded.*

(iii) $\lim_{t\downarrow 0}\|v(t,x)-f(x)\|=0$.

Remark. This is a consequence of Theorem (4.6) below that $T(t)$ is a strongly continuous semigroup acting on $C(\bar{R}^+)$ with domain $\mathscr{D}(A)$ - see Dynkin p. 28. The proof of Theorem (4.6) follows from the observation that $T(t)$ is a perturbation of the semigroup $U(t)=e^{tB}$ with $T(t)=e^{t(B+C)}=e^{tA}$, where $U(t)$, C were defined at (2.6) and the definition of B depends on $q(x,t)$, see e.g. (4.7), (4.14).

We can now state the main theorem of this section.

(4.6) **Theorem.** *The semigroup* $T(t)$ (*for which* $T(t)f(x)$ *is the unique solution to the integral equation* (1.2)) *is a strongly continuous semigroup* $T(t)$: $C(\bar{R}^+)\to C(\bar{R}^+)$ *with infinitesimal generator* A *as in* (1.6ii) *and* $\mathscr{D}(A)$ *as in* (4.1)–(4.4).

Proof. We shall give the proofs for Cases 1 and 2 only since Cases 3 and 4 are quite similar to Case 2.

Case 1 (the virtual waiting time process): $q(x,t)=[x-t]^+$.

We've already seen that $U(t)f(x)=e^{-bt}f([x-t]^+)$ is a strongly continuous semigroup and it is easy to see that for $f\in\mathscr{D}(A)$ given by (4.1), its infinitesimal generator is

$$\begin{aligned}(4.7)\qquad Bf(x)&=\lim_{t\downarrow 0}(U(t)f(x)-f(x))/t=-f'(x)-bf(x),\ x>0\\ &=-bf(0),\ x=0.\end{aligned}$$

Clearly

$$f\in\mathscr{D}(B)\quad\text{iff } f(x)=\int_0^\infty e^{-\lambda t}U(t)g(x)=\int_0^\infty e^{-\lambda t}g([x-t]^+)\,dt$$

for some $\lambda>0$ and some $g\in C(\bar{R}^+)$ i.e. f belongs to the range of the resolvent. An easy calculation shows that

$$(4.8)\qquad f(x)=g(0)\,e^{-(\lambda+b)x}/(\lambda+b)+e^{-(\lambda+b)x}\int_0^x e^{(\lambda+b)y}g(y)\,dy.$$

The representation (4.8) implies that $f\in C(\bar{R}^+)$, f' exists and that

$$(\lambda-B)f(x)=(\lambda+b)f(x)+f'(x)=g(x).$$

Now $f'(x)=[-(\lambda+b)f(x)+g(x)]\in C(\bar{R}^+)$ so if $f\in\mathscr{D}(B)$ then $f,f'\in C(\bar{R}^+)$ and since $f(0)=(\lambda+b)^{-1}g(0)$ we see that $f'(0)=0$. We have thus shown that $\mathscr{D}(B)\subset\{f\colon f,f'\in C(\bar{R}^+),\ f'(0)=0\}$; a direct calculation as in (4.7) shows that $\{f\colon f,f'\in C(\bar{R}^+),\ f'(0)=0\}\subset\mathscr{D}(B)$ and hence

$$(4.9)\qquad \mathscr{D}(B)=\{f\colon f,f'\in C(\bar{R}^+),\ f'(0)=0\}.$$

As is customary we write $f=(\lambda-B)^{-1}g$.

Turing now to the proof of Theorem (4.6) we begin by writing the integro-differential operator A as *a* sum.

(4.10) $Af=Bf+Cf$, C as in definition (2.6) and invoking the following perturbation theorem of Hille-Phillips - see Kato (1976) pp. 497–498.

(4.11) **Theorem.** *Suppose B generates a strongly continuous semigroup on the Banach Space X with* $\|U(t)\| \leqq M e^{\beta t}$. *Let C denote a bounded linear operator with norm* $\|C\|$. *Then* $T(t) = e^{t(B+C)}$ *is a strongly continuous semigroup on X such that* $\|T(t)\| \leqq M e^{(\beta + M\|C\|)t}$ *and* $\mathscr{D}(A) = \mathscr{D}(B+C) = \mathscr{D}(B)$. *Moreover T(t) satisfies the integral equation*

$$T(t)f = U(t)f + \int_0^t U(t-s)\, CT(s) f\, ds. \tag{4.12}$$

Remark. Note that in all our cases $\beta = -b$, $M = 1$, $\|C\| = b$ so $\|T(t)\| \leqq 1$, as is to be expected. Since $\mathscr{D}(A) = \mathscr{D}(B)$ as defined in (4.9), the proof of Theorem (4.6) Case 1 is finished. □

Case 2. In this case $q(x,t) = \phi^{-1}(\phi(x) - t)$ where $\phi(x) = \int_1^x r(z)^{-1}\, dz$, $\phi'(x) = r(x)^{-1}$.

Exactly the same reasoning used in the proof of Lemma (2.7) shows that

$$U(t) f(x) = e^{-bt} f(q(x,t)) \colon C(\bar{R}^+) \to C(\bar{R}^+) \tag{4.13}$$

is a strongly continuous semigroup with $\|U(t)\| \leqq e^{-bt}$ and for $f \in \mathscr{D}(A)$ given by (4.2) its infinitesimal generator B is

$$Bf(x) = \lim_{t \to 0} (U(t) f(x) - f(x))/t = -r(x) f'(x) - bf(x), \qquad x \geqq 0. \tag{4.14}$$

Now $f \in \mathscr{D}(B)$ if and only if there exists $\lambda > 0$, $g \in C(\bar{R}^+)$ such that $f(x) = \int_0^\infty e^{-\lambda t} g(q(x,t))\, dt$ from which it follows at once that

$$\|f\| \leqq (\lambda + b)^{-1} \|g\|, \quad f \in C(\bar{R}^+) \quad \text{and} \quad f(0) = (\lambda + b)^{-1} g(0),$$

since $q(x,t) \equiv 0$. Fix $x > 0$ and make the change of variable $y = q(0,t) = \phi^{-1}(\phi(x) - t)$ so $t = \phi(x) - \phi(y)$ and $0 \leqq t < \infty$ implies $0 < y \leqq x$ with $dt = -\phi'(y)\, dy = -r(y)^{-1}\, dy$. Thus

$$f(x) = e^{-(\lambda + b)\phi(x)} \int_0^x e^{(\lambda + b)\phi(y)} (g(y)/r(y))\, dy. \tag{4.15}$$

From (4.15) one sees at once that $f'(x)$ exists for $x > 0$ and that

$$(\lambda - B) f(x) = (\lambda + b) f(x) + r(x) f'(x) = g(x), \qquad x > 0. \tag{4.16}$$

Since $r(x) f'(x) = g(x) - (\lambda + b) f(x) \in C(\bar{R}^+)$ it follows that $\lim_{x \to 0} r(x) f'(x) = g(0) - (\lambda + b) f(0) = 0$. Thus

$$\mathscr{D}(B) \subset \{f \colon f, rf' \in C(\bar{R}^+), \lim_{x \to 0} r(x) f'(x) = 0\}.$$

Once again it is very easy to show that if $f, rf' \in C(\bar{R}^+)$, $\lim_{x \to 0} r(x) f'(x) = 0$ and $(\lambda - B) f(x) = g(x)$ then $f(x) = \int_0^\infty e^{-\lambda t} g(q(x,t)\, dt = (\lambda - B)^{-1} g(x)$ and hence in Case 2

(4.17) $$\mathscr{D}(B)=\{f: f, rf' \in C(\bar{R}^+), \lim_{x\to 0} r(x)f'(x)=0\}.$$

At this point we just apply the perturbation theorem (4.11) to conclude in Case 2 that $T(t)=e^{tA}=e^{t(B+C)}$ with $\mathscr{D}(A)=\mathscr{D}(B)$. □

The proofs in Cases 3 and 4 are quite similar and are therefore omitted.

Finally we show that the semigroup $T(t)$ associated with the virtual waiting time process $\eta(t)$ is strongly continuous on $C_0^1(\bar{R}_+)$ i.e., if f and f' are in $C_0(\bar{R}_+)$ then the same is true for $T(t)f(x)$. To see this introduce the norm $\|f\|_1=\|f\|+\|f'\|$ for $f\in C_0^1(\bar{R}_+)$. It is clear that $C_0^1(\bar{R}_+)$ is a Banach space with respect to the norm $\|\ \|_1$. It is also easily seen that the operator $Cf(x) = b\int_0^\infty f(x+y)\,dH(y)$ is a bounded linear operator on $C_0^1(\bar{R}_+)$ since differentiation under the integral sign is easily justified to conclude $(Cf(x))'=Cf'(x)$ and thus

(4.18) $$\|Cf\|_1 \leqq b\|f\|_1.$$

The only thing left to show is that to every $g\in C_0^1(\bar{R}_+)$ there exists a unique $f\in\mathscr{D}_1(B)=\{f: f\in C_0^2(\bar{R}_+), f'(0)=0\}$ satisfying the equation

(4.19) $$(\lambda+b)f(x)+f'(x)=g(x) \quad \text{and the apriori inequality}$$

(4.20) $$(\lambda+b)\|f\|_1 \leqq \|g\|_1.$$

That there exists a function $f\in\mathscr{D}_1(B)$ satisfying (4.19) follows at once from the representation (4.8). Note however that $f''(x)$ exists because $f'(x)=g(x)-(\lambda+b)f(x)$ and we are assuming that $g'(x)$ exists too. Thus $f'(x)$ satisfies the differential equation

(4.21) $$(\lambda+b)f'(x)+f''(x)=g'(x).$$

Since $f'(0)=0$ familiar reasoning with the maximum principle yields the estimate $(\lambda+b)\|f'\|\leqq\|g'\|$. We already know that $(\lambda+b)\|f\|\leqq\|g\|$ and adding these two inequalities yields the estimate:

(4.22) $$(\lambda+b)(\|f\|+\|f'\|)\leqq\|g\|+\|g'\| \quad \text{or equivalently } (\lambda+b)\|f\|_1\leqq\|g\|_1.$$

The Hille-Yosida theorem and the Hille-Phillips perturbation theorem now tell us that A is the infinitesimal generator of a semigroup $T(t) = e^{tA}: C_0^1(\bar{R}_+)\to C_0^1(\bar{R}_+)$, such that $\|T(t)f\|_1\leqq\|f\|_1$. The analogue of Theorem (4.5) with respect to the norm $\|\ \|_1$ is left to the reader.

It is to be observed that the above method for showing that $T(t)$ preserves the differentiability of f is applicable to a much larger class of Markov processes than is considered here and we refer the interested reader to Brezis, Rosenkrantz and Singer (1971), C.C.Y. Dora (1976). S.Ethier (1978) where similar computations are performed.

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