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The Absolute Convergence of Weighted Sums of Dependent Sequences of Random Variables

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Summary. The sum $\sum a_n X_n$ of a weighted series of a sequence $\{X_n\}$ of identically distributed (not necessarily independent) random variables (r.v.s.) is a.s. absolutely convergent if for some α in $0<\alpha\leqq 1$, $\sum |a_n|^\alpha<\infty$ and $E|X_n|^\alpha<\infty$; if $a_n=z^n$ for some $|z|<1$ then it suffices that $E(\log|X_n|)_+<\infty$. Examples show that these sufficient conditions are not necessary. For mutually independent $\{X_n\}$ necessary conditions can be given: the a.s. absolute convergence of $\sum X_n z^n$ (all $|z|<1$) then implies $E(\log|X_n|)_+<\infty$, while if the X_n are non-negative stable r.v.s. of index α, $\sum |a_n X_n|<\infty$ if and only if $\sum |a_n|^\alpha<\infty$.

1. Introduction

This note is concerned with conditions for the a.s. absolute convergence of the weighted sum

$$\sum_{n=1}^{\infty} a_n X_n \tag{1}$$

of identically distributed but not necessarily independent random variables (r.v.s.) $\{X_n\}$ in terms of the weights $\{a_n\}$ and the common distribution of the X_n. This point of view may be contrasted with the emphasis in texts like Breiman (1968) and Chung (1974) which concentrate on the stochastic mode of convergence of the sequence $\{S_n\}$ of partial sums

$$S_n=\sum_{i=1}^{n} a_i X_i \tag{2}$$

or, more generally, of a sequence of r.v.s. $\{Z_n\}$

$$S_n=\sum_{i=1}^{n} Z_i. \tag{2'}$$

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For example, when the X_n are independent and identically distributed (i.i.d.), with zero mean and finite variance, the finiteness of $\sum a_n^2$ ensures the convergence in mean square and the a.s. convergence of $\{S_n\}$. Equally clear, by considering the case where $\Pr\{X_n=1\}=\Pr\{X_n=-1\}=0.5$ and $a_n=1/n$, is the fact that $\sum a_n X_n$ is only a.s. *conditionally* convergent. When our interest in $\sum a_n X_n$ is as a component defining a stochastic model, absolute convergence of the series is important in order that the definition may be unambiguous; this aspect was exemplified in an earlier paper (Daley, 1971; see also Westcott (1974)).

Trivially, the absolute convergence of (1) is equivalent to its convergence when X_n and a_n are non-negative, and in the development below we could have made this specialisation but have preferred not so as to emphasize absolute versus conditional convergence. It may be recalled that a series of real or complex numbers if absolutely convergent if and only if it is conditionally convergent for all possible rearrangements of its terms, but we make no use of this fact.

2. Results

The first observation is surely well known: it comes from a straighforward application of Fubini's theorem and Lebesgue's convergence theorem.

Proposition 1. *$\sum a_n X_n$ is a.s. absolutely convergent if $E|X|<\infty$ and*

$$\sum |a_n| < \infty.$$

The absolute convergence of general power series is the same as considering the convergence of $\sum |X_n| z^n$ for real z in $(0,1)$, and for such series we have the result below which is known for i.i.d. $\{X_n\}$ (see Theorem 14.4.1 of Kawata (1972)).

Proposition 2. *$\sum X_n z^n$ is a.s. absolutely convergent for identically distributed $\{X_n\}$ and all z in $(0,1)$ if $E(\log|X|)_+ < \infty$.*

Proof. Take ζ in $z<\zeta<1$, and define indicator r.v.s. I_n by

$$I_n = \begin{cases} 1 & \text{if } X_n > \zeta^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} |X_n| z^n &\leqq \sum_{n=1}^{\infty} z^n((1-I_n)\zeta^{-n} + I_n|X_n|) \\ &\leqq z/(\zeta - z) + \sum_{n=1}^{\infty} z^n I_n |X_n|. \end{aligned}$$

Appealing to the Borel-Cantelli lemma, there will be a.s. only finitely many non-zero terms in this last sum if

$$\infty > \sum_{n=1}^{\infty} \Pr\{I_n=1\} = \sum_{n=1}^{\infty} \Pr\{|X_n| > \zeta^{-n}\} = \sum_{n=1}^{\infty} (1 - F(\zeta^{-n}))$$

where F is the common distribution function (d.f.) of the $|X_n|$. By an integral comparison test for the convergence of monotonic series, this last sum converges or diverges with

$$\int_0^\infty (1-F(e^{u|\log\zeta|}))\,du = \int_1^\infty (v|\log\zeta|)^{-1}(1-F(v))\,dv$$
$$= E(\log|X|)_+/|\log\zeta|,$$

proving the Proposition.

Recall that if an infinite series $\sum y_n$ of non-negative terms is convergent, then so is $\sum y_n^\beta$ for any $\beta \geqq 1$. Consequently, if $\sum |a_n X_n|^\alpha$ is a.s. convergent for any $0<\alpha<1$, then so is $\sum a_n X_n$. Since the elements of $\{|X_n|^\alpha\}$ are identically distributed whenever those of $\{X_n\}$ are, Proposition 1 can be extended as follows, in part filling the gap between Propositions 1 and 2.

Proposition 3. *$\sum a_n X_n$ is a.s. absolutely convergent for identically distributed $\{X_n\}$ and any sequence $\{a_n\}$ if for some $0<\alpha<1$, $E|X_n|^\alpha<\infty$ and $\sum|a_n|^\alpha<\infty$.*

In the converse direction to these propositions we have the following result for certain sequences $\{a_n\}$ in the case if i.i.d. $\{X_n\}$. The particular case where $a_n = z^n$ leads to the finiteness of $E(\log|X_n|)_+$ being necessary and sufficient for the absolute convergence of $\sum X_n z^n$.

Proposition 4. *If the r.v.s X_n are i.i.d. and $\sum a_n X_n$ is a.s. absolutely convergent where $\{|a_n|\}$ is a monotonic sequence, then*

$$E a^{-1}(|X_n|)<\infty$$

where $a^{-1}(.)$ is the inverse of any monotonic continuous function $a(.)$ for which $a(n)=|1/a_n|$ for all sufficiently large n.

Proof. Without loss of generality, assume $\{a_n\}$ to be non-negative and monotonic. Introduce indicator r.v.s. $I_n'=1$ if $|X_n|>1/a_n$, $=0$ otherwise, and note that these r.v.s inherit from $\{X_n\}$ the property of mutual independence. Then, a.s.,

$$\infty > \sum a_n|X_n| \geqq \sum I_n' a_n |X_n| \geqq I_n',$$

and this a.s. finiteness and independence of $\{I_n'\}$ implies the finiteness of

$$\sum_{n=1}^\infty \Pr\{I_n'=1\} = \sum_{n=1}^\infty (1-F(a(n)))$$

where F is the common d.f. of $|X_n|$. Again by the integral comparison test, and taking $a(.)$ as stated and with a continuous derivative,

$$\infty > \int_0^\infty (1-F(a(u)))\,du$$
$$= \int_{a(0)}^\infty (1-F(v))(a^{-1}(v))'\,dv$$
$$= \int_{a(0)}^\infty a^{-1}(v)\,dF(v),$$

whence the assertion of Proposition 4. It is readily observed that the particular differentiability assumption of $a(.)$ is irrelevant to the conclusion.

Proposition 5. *Let $\{X_n\}$ be i.i.d. non-negative stable r.v.s. of index α for some $0<\alpha<1$, and $a_n \geqq 0$. Then*

$$\sum a_n X_n < \infty \qquad a.s.$$

if and only if

$$\sum a_n^\alpha < \infty.$$

Proof. By Theorem 1 of XIII.6 of Feller (1971), for all $\theta \geqq 0$,

$$-\log E(\exp(-\theta \sum a_n X_n)) = \sum (b\, a_n\, \theta)^\alpha$$

for some scaling constant b. This sum converges to a finite limit, and hence the transform is continuous at $\theta=0$, if and only if $\sum a_n^\alpha < \infty$.

An equivalent formulation of Proposition 5 can be given for monotonic $\{a_n\}$ as follows.

Corollary 5a. *Under the conditions of Proposition 5, with $\{a_n\}$ monotonic, and $a^{-1}(.)$ as in Proposition 4,*

$$\sum a_n X_n < \infty \qquad \text{if and only if} \quad E a^{-1}(X) < \infty.$$

Proof. The necessity of the condition has been proved in Proposition 4. To see the sufficiency, it suffices to observe that $\sum a_n^\alpha < \infty$ if and only if

$$\infty > \int_1^\infty (a(u))^{-\alpha}\, du = \int_{a(1)}^\infty v^{-\alpha}\, d(a^{-1}(v)),$$

and by the same result in Feller (1971), the stable r.v.s. X_n considered here have the property that

$$x^\alpha \Pr\{X_n > x\} \to \text{const.} > 0 \qquad \text{as } x \to \infty.$$

Finally we remark that the proof of Proposition 2 can be mimicked to establish the following result which falls short of being a necessary and sufficient condition (cf. Propositions 4 and 5).

Proposition 6. *$\sum a X_n$ is a.s. absolutely convergent for identically distributed $\{X_n\}$ if there is a monotonic increasing sequence $\{b_n\}$ for which $\sum |a_n b_n| < \infty$ and such that*

$$E b^{-1}(|X_n|) < \infty$$

where $b^{-1}(.)$ is the inverse of a monotonic increasing function $b(.)$ for which $b(n) = b_n$.

3. Concluding Remarks

1. The assumption that $\{a_n\}$ or $\{|a_n|\}$ is monotonic may be replaced by the property of being ultimately monotonic.

2. A non-negative stable r.v. X of index α in $0<\alpha<1$ has $EX^{\alpha}=\infty$, so the sufficient conditions of Proposition 3 are stronger than necessary (cf. Proposition 5) for the convergence of (1).

3. Let Z be any non-negative a.s. finite-valued r.v. with $E(\log Z)_{+}=\infty$, and let $X_n=Z$ (all n). Then the $\{X_n\}$ are identically distributed and $\sum a_n X_n$ is a.s. absolutely convergent if and only if $\sum |a_n|<\infty$, while the condition of Proposition 2 is violated.

4. The assumption of a finite variance for X_n is natural when the series $\sum a_n X_n$ arise in the context of a time series to be subjected to a second order spectral analysis.

5. Recall that the sequence of partial sums of a series of independent r.v.s. is a.s. convergent if and only if it is convergent in distribution, for which a sufficient conditions is its convergence in mean square. When the terms in $\sum Z_n$ are non-negative, the same properties hold, irrespective of independence, because the sequence is a.s. monotonic.

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