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The Existence of Set-Indexed Lévy Processes

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Summary. This paper considers the problem of the existence of set-indexed Lévy processes having regular sample paths defined over as large a class, $\mathscr{A}$, as possible of subsets of the unit cube in $\mathbb{R}^d$. Regular sample paths means here the natural generalization of right continuity and left limits, to concepts of outer continuity and inner limits. A general integral condition involving the Lévy measure and the entropy $\exp(H(\delta))$ of the class $\mathscr{A}$ is obtained that is sufficient for the existence of such regular processes. In the particular case where the process is stable of index α, $\alpha \in (1, 2)$, the condition becomes

$$\int_0^1 (H(x)/x)^{1-1/\alpha}\, dx < \infty.$$

1. Introduction

Processes with independent increments have been extensively studied, with the earliest work including that by de Finetti (1929), Lévy (1937, 1948) and Ito (1942) among others. The processes were viewed as functions of a real parameter, say $0 \leqq t < \infty$, and the focus was upon their existence as right continuous processes with left limits, their characterizations, and other aspects of their sample function behavior. The two special cases of Brownian Motion and a Poisson process, which were of course studied much earlier, form the basic building blocks for these processes. In particular, for infinitely divisible (ID) processes (those with stationary as well as independent increments) the main structure of the non-Gaussian part is that of a series of independent compound Poisson processes (e.g. Ito (1942), Beněs (1958), Ferguson and Klass (1972), Kallenberg (1974)).

A compound Poisson process has a specific structure that allows it to be viewed as a random measure. This follows since the process is equivalent to a sequence of pairs $\{(T_n, Y_n): n \geqq 1\}$ where the T_n's are the ordered discontinuity

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points of the process and the Y_n's are the iid jump heights that occur at these points. Thus almost surely, such a process may be viewed as a signed measure defined for *all* Borel sets, $\mathscr{B}^d$. In contrast to this, the sample paths of a Brownian Motion are not of bounded variation. Consequently, when viewed as a continuous set-function, its domain cannot be enlarged to interesting classes much larger than the class of intervals. Thus we see in these two main special cases the two extremes that are possible when considering to what extent ID processes can be considered as random functions of sets rather than points. [Fristedt (1974) and Taylor (1973) may be referenced for details on ID processes.]

When one extends the idea of ID processes to higher dimensions the questions of existence and maximal domains become much more challenging. The concept of independent increments is of course straightforward; $Z(A)$ and $Z(B)$ are independent whenever A and B are disjoint members of the domain of Z. The first domain one would consider in higher dimensions would be the class of 'lower orthants' $(\mathbf{0}, \mathbf{t}]$ for $\mathbf{0}$, $\mathbf{t} \in I^d = (0,1]^d$. (Throughout this paper we restrict attention to the unit cube of R^d. Extensions to the entire upper orthant R^d_+ or to R^d are straightforward.) The compound Poisson case does not change in higher dimensions since the sample paths may again be identified as the assignment of masses at separated locations thereby determining a signed measure defined on $\mathscr{B}^d$, the class of all Borel sets in the domain I^d. This extension to orthants in higher dimensions of non-Gaussian processes with independent increments has been fully developed in Adler et al. (1983).

The case of a multidimensional extension of Brownian Motion was first considered in 1956 when Chentsov showed the existence of a continuous 2-dimensional Gaussian process $\{Z(\mathbf{t})\colon \mathbf{t} \in I^2\}$ that had zero means and covariance structure given by $EZ(\mathbf{t})Z(\mathbf{s}) = (t_1 \wedge s_1)(t_2 \wedge s_2)$. This is the same as a Gaussian ID process indexed by the lower orthants when one equates $Z(\mathbf{t})$ with $Z((\mathbf{0}, \mathbf{t}])$. This process was called a Brownian Sheet in Pyke (1973), where further references are given.

Although in one dimension it is not possible to view a Brownian Motion as a continuous set function over interesting classes much larger than the class of intervals, it is possible in higher dimensions to extend the domain for a Brownian Sheet to much larger families of sets than the finite union of d-dimensional intervals $(\mathbf{s}, \mathbf{t}]$. In Dudley (1973) it is shown that there exists a continuous Brownian process $\{Z(A)\colon A \in \mathscr{A}\}$ provided that the class $\mathscr{A}$ is not so large as to cause the divergence of the integral $\int_{0+} \{H(u)/u\}^{\frac{1}{2}}\, du$ where H, the log-entropy of $\mathscr{A}$, is defined below. This result implies for example that such a Brownian Process exists when $\mathscr{A}$ is the class of closed convex subsets in I^2 but not when $\mathscr{A}$ is the class of closed convex subsets in I^d for $d > 3$. Subsequently Dudley (1979) showed that $d = 3$ is also a case for which a continuous version does not exist; actually it is shown that not even a bounded version exists.

The purpose of this paper is to provide a criterion, in terms of both the entropy of $\mathscr{A}$ and the Lévy measure of the process, under which the existence of a suitably regular version of general ID set-indexed processes can be established. Under this condition it will be seen that the domains $\mathscr{A}$ will vary in size from the relatively smaller families for the Brownian process to the largest possible case of $\mathscr{A}=\mathscr{B}^d$ for a compound Poisson process. In the latter case, the regularity imposed is that of being signed measures, almost surely. By regular sample paths we mean the natural generalization of 'right continuity with left limits' to 'outer continuity with inner limits.' Since this sample regularity of ID processes in one dimension is due to Lévy they are often referred to as *Lévy processes*, and we do the same; see Definition 2.3. Set-indexed stable processes will be special cases of Lévy processes. In this case our integral criterion for existence simplifies to where only the stable index α $(0<\alpha\leqq 2)$ is relevant. It is of interest to see that the convex sets in I^d form a suitable index family $\mathscr{A}$ for a stable process of index α provided only that $\alpha<(d+1)/(d-1)$.

We believe that our criteria are close to being optimal; in particular we conjecture that when $\alpha>(d+1)/(d-1)$, the closed convex sets in I^d do not form a suitable index family. We do not, however, have any results in this direction.

Sect. 2 contains the notation and preliminaries necessary to state our main results. Sect. 3 contains the statement and proof of these results; most ID processes are covered by Theorem 3.1 whereas Theorem 3.2 covers the near-normal case and Theorem 3.3 the Cauchy case.

While the results of this paper were being prepared for typing, we received from R.J. Adler a copy of the paper Adler and Feigin (1984). In this paper the authors also formulate the question of the existence of set-indexed Lévy processes. They independently obtain an integral criterion for existence that relates entropy and the Lévy measure. Their result is not as strong as that given here; they make a conjecture for the stable case that is much closer to the results we obtain. Two interesting examples are included that indicate clearly the necessity of entropy conditions on $\mathscr{A}$.

2. Notation and Preliminary Propositions

We use the Lévy representation of an ID characteristic function, namely,

$$\ln\phi(u)=iu\mu-\sigma^2u^2/2+\int\left(e^{iux}-1-\frac{iux}{1+x^2}\right)\nu(dx) \tag{2.1}$$

where ν, called the Lévy measure, is defined on the Borel subsets of $R_0=(-\infty,0)\cup(0,\infty)$ and satisfies

$$\int(x^2\wedge 1)\,\nu(dx)<\infty.$$

A process $Z=\{Z(B): B\in\mathscr{B}^d\}$ is said to be an *ID process with Lévy measure* ν if it has independent "increments" in the sense that $Z(B_1),\ldots,Z(B_k)$ are independent whenever $B_1,\ldots,B_k$ are disjoint, and the marginal distributions are given by

$$\ln E\{e^{iuZ(B)}\}=|B|\,\ln\phi(u),\qquad u\in R^1,\ B\in\mathscr{B}^d,$$

where $|B|$ denotes Lebesgue measure of B. All finite dimensional distributions are uniquely determined by these properties in a consistent way, showing existence of such processes by Kolmogorov's consistency theorem. Since we use $|B|$ rather than $g(|B|)$ with g finitely additive, these ID processes have no fixed "points" of discontinuities.

We assume throughout that all underlying probability spaces are complete.

As mentioned in Sect. 1, the basic structure of the non-Gaussian part of an ID process is that of a convergent series of compound Poisson processes. The convergence of the series entails the limit of compound Poisson processes as smaller and more frequent point masses are permitted. The known structure of a compound Poisson process is as follows: Let $\{Y_n : n \geqq 1\}$ be a sequence of iid random masses with distribution F. Let W be a Poisson-(λ) r.v. representing the number of masses in I^d, and let $\{U_n : n \geqq 1\}$ be iid uniform r.v.'s on I^d representing the locations of the masses. Define, for any $B \in \mathscr{B}^d$,

$$Z(B) = \sum_{n=1}^{W} Y_n 1_B(U_n),$$

interpreting $Z(B)=0$ if $W=0$. Then it is known that

$$\log E\{e^{iuZ(B)}\} = |B| \lambda E(e^{iuY_1} - 1) = |B| \int (e^{iux} - 1)\, \nu(dx)$$

where $\nu = \lambda F$, restricted to R_0, is a bounded Lévy measure. Conversely, given a bounded measure ν one can choose $F = \nu/\nu(R_0)$ and repeat the above construction. Thus for any Lévy measure ν, its restriction ν_ε to $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ determines an ID process in the form of a compound process which is defined on all of $\mathscr{B}^d$. Moreover, Lévy's 1-dimensional theory states that if ν satisfies

$$\int (|x| \wedge 1)\, \nu(dx) < \infty \tag{2.2}$$

then $Z(I^d)$ exists as the limit of compound Poisson processes in which the sum over I^d of all the positive (resp. negative) masses is finite. Thus any such process has a representation which is the difference of two purely atomic measures. This proves

Theorem 2.1. *If* (2.2) *holds, an ID process* $\{Z(A): A \in \mathscr{B}^d\}$ *exists that is almost surely a signed measure.*

In the case of positive Z, a representation of ID processes is included in the study by Kingman (1967) of random measures with independent increments.

When (2.2) does not hold it is necessary to restrict $\mathscr{A}$ to be smaller than $\mathscr{B}^d$. Recall that in the 1-dimensional case the sample paths are no longer of bounded variation even though the non-deterministic part of the sample paths move by jumps only. The reason is that the sum of all of the positive jumps does not converge, although when one adds all jumps in an interval, there is sufficient cancellation to make the centered series converge. In our situation this translates into being able to use only families $\mathscr{A}$ of sufficiently smooth sets, like the intervals on the line, in which a similar cancellation can take place.

The index families $\mathscr{A}$ considered for the case in which (2.2) is not satisfied are familes of subsets of I^d that satisfy the following basic property:

(TBI) *Totally bounded with inclusion:* For every $\delta>0$, there exists a finite set $\mathscr{A}_\delta \subseteq \mathscr{A}$ such that for any $A \in \mathscr{A}$, there exists A_δ, $A_\delta^+ \in \mathscr{A}_\delta$ such that $A_\delta \subseteq A \subseteq A_\delta^+$ and $d_L(A_\delta, A_\delta^+) = |A_\delta^+ \setminus A_\delta| \leqq \delta$.

This concept was introduced by Dudley (1978). Note that $\mathscr{A}_\delta$ is a δ-net with respect to d_L for $\mathscr{A}$. As an example of a family that can be shown to be (TBI) we mention the class of closed convex sets in I^d.

A second assumption about $\mathscr{A}$ is also needed which imposes a restriction, in terms of the given Lévy measure ν, upon the size of $\mathscr{A}$ through its log-entropy H where $H(\delta)$ is defined to be the logarithm of the cardinality of the smallest δ-net $\mathscr{A}_\delta$. When $\mathscr{A}$ is the family of closed convex sets in I^d for example, it is known (cf. Dudley (1974)) that $H(\delta) \leqq K\delta^{-r}$ for $r=(d-1)/2$. Another example is the class $\mathscr{A}(d,q,c)$, introduced by Dudley (cf. Dudley (1974)), of closed sets in I^d whose boundaries are represented parametrically as continuous mappings of the $d-1$ dimensional unit sphere, which have bounded (by c) derivatives of orders up to q. For this class $r=(d-1)/q$.

Before stating the second assumption the following notation is introduced. Define

$$\begin{aligned} M(x) &= \nu((-\infty,-x)\cup(x,\infty)), \quad x>0,\\ Q(x) &= \int_{|u|\leqq x} u^2\,\nu(du), \quad x>0,\\ M^{-1}(y) &= \inf\{x\colon M(y)<x\}, \quad y>0, \quad \text{and}\\ G(y) &= yM^{-1}(y), \quad y>0. \end{aligned} \tag{2.3}$$

With this notation, an example of the third type of assumption on $\mathscr{A}$ is that needed in Theorem 3.1, namely

$$\int_0^1 G(H(u)/u)\,du < \infty. \tag{2.4}$$

Further assumptions, aimed primarily at giving some technical simplicity without too much essential loss of generality are the following:

(A1) $H(x) = x^{-c_0} L(x)$ for some constant $c_0>1$, where L is a slowly varying function near 0 such that $x^{(c_0+1)/2} H(x)$ increases as $x \searrow 0$. (In particular, H is regularly varying of order greater than 1.)

(A2) H is regularly varying near 0 and $xH(x)$ is monotone.

Assumption (A1) is used in Theorem 3.1 whereas (A2) is required in Theorem 3.2.

We also impose some general conditions on the Lévy measure ν that do not involve $\mathscr{A}$. They are

(B1) $\limsup_{x\to 0} x^2 |\ln x|^9 M(x) < \infty$.

(B2) For some $\tau>0$, $x^\tau M(x)$ increases as $x \searrow 0$.

The stable processes form an important subclass of ID processes. We denote the Lévy measure of a stable distribution of index α, $0<\alpha<2$, by ν_α. In view of Theorem 2.1, we are interested only in $1 \leqq \alpha < 2$. Here $\nu_\alpha(dx)$ is proportional to $|x|^{-\alpha-1}\,dx$ and so $M(x)$ is proportional to $|x|^{-\alpha}$. In this case

$$M^{-1}(y) = \text{const.} \times y^{-1/\alpha} \quad \text{and} \quad G(y) = \text{const.} \times y^{1-1/\alpha}.$$

Consequently, (2.4) is satisfied provided (A1) holds with $c_0 < 1/(\alpha-1)$.

Some remarks on the preceding assumptions are in order. First of all, as the proof of Theorem 3.1 shows, $\limsup_{x\to 0} x^2 M(x) = 0$. Therefore (B1) is an extremely mild condition. The exponent of $\ln x$ in the condition may be any number >4, provided the definition of η_n and γ_d in Theorem 3.1 are suitably modified. For the stable processes, $M(x)$ is a multiple of $x^{-\alpha}$, and so (B1) is trivially satisfied.

(B2) is also satisfied for the stable processes. The only use of this condition in the proof of Theorem 3.1 is to show that k_n must grow with n at some minimum rate, which in turn is used only in showing that the integrability condition (3.1) implies summability of η_n.

For most Lévy measures ν, one could circumvent (B2) as follows. If

$$\int_{-1}^{1} |x|^{2-2\tau}\,\nu(dx) < \infty$$

for some $\tau > 0$, let

$$\nu_1(dx) = \tau x^{-\tau-1} M(x)\,dx + x^{-\tau}\nu(dx) \quad \text{and} \quad \nu_2(dx) = \nu_1(dx) - \nu(dx).$$

It is not hard to check that $\int_{-1}^{1} x^2 \nu_i(dx) < \infty$, $i = 1, 2$. Define M_1 and M_2 analogously to M, and then define $G_i(y) = y M_i^{-1}(y)$, $i = 1, 2$. Replacing G in the integrability condition (3.1) by G_1 and G_2, construct Lévy processes Y_1, Y_2 (see Definition 2.3), and finally, let $Y = Y_1 - Y_2$. Y will be the desired Lévy process corresponding to ν. (B2) is easily seen to be satisfied for M_1, M_2, since $x^\tau M_1 = M$. Alternately, see the comment following the proof of Theorem 3.1.

If a Lévy process can be defined over a class $\mathscr{A}$, it certainly can be defined over any subclass. Thus, there is no loss of generality in taking $\mathscr{A}$ as large as possible. In particular, requiring $H(x)$ to be regularly varying entails very little loss of generality. H regularly varying is used to show that (3.1) implies summability of η_n.

The assumption that $H(x) = x^{-c_0} L(x)$, $c_0 > 1$ is not a restrictive condition in most cases. For example, for a stable process, $G(y) = y^{1-1/\alpha}$, and we can therefore take $H(x)$ as large as x^{-r} as long as $r < (\alpha-1)^{-1}$. In this case, we take $1 < c_0 \leqq (\alpha-1)^{-1}$. In general (A1) is a restrictive condition only when the Lévy measure concentrates most of its mass very close to the origin, that is, when $x^{2-\tau} M(x) \to \infty$ for all $\tau > 0$. This case is treated in Theorem 3.2.

Before proceding to the main results it is necessary to define the type of sample path regularity that we wish our processes to possess in the situations where (2.2) is not satisfied. As we have mentioned above, it will not be possible in these cases for the sample paths to be signed measures over all Borel sets; rather, $\mathscr{A}$ must be restricted to a smaller family of sets. We nevertheless want the sample paths to have a measure-like continuity which we describe as having inner limits and being outer continuous. The precise definition is given in

Definition 2.1. A set function $\psi: \mathscr{A} \to R^1$ is said to have *inner limits and outer continuity at* $A \in \mathscr{A}$ if

i) $\psi(A_n) \to \psi(A)$ for any (outer) sequence $\{A_n\} \subset \mathscr{A}$ such that $A_n \supset A$ for all n and $\lim_{n\to\infty} A_n = A$, and

ii) $\lim_{n\to\infty} \psi(A_n)$ exists for any (inner) sequence $\{A_n\} \subset \mathscr{A}$ such that $A_n \subset A^0$ for all n and $\lim_{n\to\infty} A_n = A^0$.

Recall that a sequence of sets, $\{A_n\}$, is said to have a limit if

$$\bigcap_n \bigcup_{m>n} A_m = \bigcup_n \bigcap_{m>n} A_m.$$

Also, notice that no monotoneity is imposed on these sequences; if monotoneity were required this concept would not even be adequate for the orthant case of real functions over R^2 having left limits and right continuity.

Definition 2.2. The space of all functions $\psi: \mathscr{A} \to R^1$ that have inner limits and outer continuity at each $A \in \mathscr{A}$ is denoted by $\mathscr{D}(\mathscr{A})$.

Definition 2.3. $\{Z(A): A \in \mathscr{A}\}$ is said to be a Lévy process indexed by $\mathscr{A}$ and having Lévy measure v if it is an ID process with Lévy measure v and if the sample paths are almost surely $\mathscr{D}(\mathscr{A})$.

With this notation, Theorem 2.1 can be restated by saying that when (2.2) is satisfied there exists a Lévy process with paths in $\mathscr{D}(\mathscr{B}^d)$. Let $\|\cdot\|_{\mathscr{A}}$ denote the sup-norm defined by

$$\|\psi\|_{\mathscr{A}} = \sup_{A \in \mathscr{A}} |\psi(A)|. \tag{2.5}$$

Note that if ψ is a finite signed measure with Jordan decomposition $\psi = \psi^+ - \psi^-$ then $\|\psi\|_{\mathscr{B}^d} = \psi^+(I^d) \vee \psi^-(I^d)$ in contrast to the variation norm $\psi^+(I^d) + \psi^-(I^d)$.

The reader should also note that in this paper we do not need $\|Z\|_{\mathscr{A}}$ to be measurable. This is because only almost sure results are proved; specifically our results state that except on a null set the sample paths have a certain structure. In a forthcoming paper (Bass and Pyke, 1984) we study the Central Limit problem for arrays of independent r.v.'s in the domain of attraction of ID distributions. There we provide a suitable topology on $\mathscr{D}(\mathscr{A})$ which permits us to establish the measurability of the partial-sum processes considered therein; this is needed for the existence of image laws and the study of their weak convergence.

Our method of proof for Theorems 3.1, 3.2, and 3.3 is to establish the uniform convergence with respect to $\|\cdot\|_{\mathscr{A}}$ of a sequence of Lévy processes with bounded Lévy measures. For these proofs and for general application the following elementary properties are needed.

Lemma 2.1. (i) *If* Y_1 *and* Y_2 *are two independent Lévy processes indexed by* $\mathscr{A}$ *and defined on the same probability space, then* Y_1+cY_2 *is also a Lévy process indexed by* $\mathscr{A}$, *for any constant* c.

(ii) *If* Y *is the degenerate process defined by* $Y(A)=c|A|$ *for some constant* c, Y *is a continuous Lévy process defined over all Borel sets.*

(iii) *If* v *is a finite measure on* $\mathbb{R}^1\setminus\{0\}$, *there exists a Lévy process defined over all Borel sets with Lévy measure* v.

(iv) *If* Y_n *is a sequence of Lévy processes, all defined over the same class* $\mathscr{A}$, *and* $\|Y_n-Y_m\|_{\mathscr{A}}\to 0$ a.s. *as* $n, m\to\infty$, *then there is a Lévy process* Y *defined on* $\mathscr{A}$ *such that* $\|Y_n-Y\|_{\mathscr{A}}\to 0$ a.s.

Proof. (i) and (ii) are clear. (iii) was proved above. Alternatively, let $X(\mathbf{t})$ be a process with stationary, independent increments indexed by points in I^d, and let $Y(A)=\sum_{\mathbf{t}\in A}\Delta X_{\mathbf{t}}$, the sum of the jumps of $X_{\mathbf{t}}$ for $\mathbf{t}\in A$.

To show (iv), first observe that $\|\cdot\|_{\mathscr{A}}$ defines a complete norm for set-indexed functions. Then note that the uniform limit of elements of $\mathscr{D}(\mathscr{A})$ is again in $\mathscr{D}(\mathscr{A})$. □

In view of the properties, we may restrict ourselves without any loss of generality to Lévy measures with support in $[-1, 1]$ and for which the Gaussian component is zero as well as the mean.

Two probability inequalities are central to the proofs. One is a bound on the tail of a Poisson r.v., while the second is a necessary analogue of Bernstein's inequality for unbounded partial sums that can be obtained for ID random variables.

Lemma 2.2 *If* W *is a Poisson* (λ) *random variable then*

$$\begin{aligned} P(W>s) &\leqq \exp\{-s(\ln(s/\lambda)-1+\lambda/s)\} && \text{if } s>\lambda \\ &\leqq \exp(-s) && \text{if } s>e^2\lambda. \end{aligned} \tag{2.5}$$

Proof. [This result is obtained in Pyke (1983) although it is probably older. The proof is standard.] Observe

$$\begin{aligned} P(W>s) &\leqq P(e^{cW}\geqq e^{cs})\leqq e^{-cs}\,E e^{cW} \\ &=\exp\{-cs+\lambda(e^c-1)\} \\ &=\exp\{-s(\ln(s/\lambda)-1+\lambda/s)\} \quad \text{at} \quad c=\ln(s/\lambda). \end{aligned}$$

The result is immediate. □

Lemma 2.3. *Suppose* X *is* ID *with*

$$E e^{iuX}=\exp\left\{\int_{-a}^{a}(e^{iux}-1-iux)\,v(dx)\right\}$$

and define $\theta=Q(a)=\int_{-a}^{a}x^2\,v(dx)<\infty$. *Then for any* $\lambda>0$

$$P(X\geqq\lambda)\leqq\exp\{-\lambda^2/2(\theta+a\lambda/3)\} \tag{2.6}$$

and

$$P(|X|\geqq\lambda)\leqq 2\exp\{-\lambda^2/2(\theta+a\lambda/3)\}. \tag{2.7}$$

Proof. First of all,

$$\begin{aligned}|e^y-1-y|&\leqq\tfrac{1}{2}y^2(1+|y|/3+y^2/4\cdot 3+\dots)\\ &\leqq\tfrac{1}{2}y^2(1-|y|/3)^{-1}\end{aligned}$$

for $|y|<3$. The moment generating function for X is, for $u>0$

$$\begin{aligned}Ee^{uX}&=\exp\left\{\int_{-a}^{a}(e^{ux}-1-ux)\,\nu(dx)\right\}\\ &\leqq\exp\left\{(1-ua/3)^{-1}\int_{-a}^{a}\frac{u^2x^2}{2}\,\nu(dx)\right\}\\ &\leqq\exp\{(1-ua/3)^{-1}\,\theta u^2/2\}\end{aligned}$$

provided $ua<3$. Set $g(u)=(1-ua/3)^{-1}$. Then by Čebyšev's inequality,

$$\begin{aligned}P(X\geqq\lambda)&=P(e^{uX}\geqq e^{u\lambda})\leqq e^{-u\lambda}E(e^{uX})\\ &\leqq\exp(-u\lambda+g(u)\,u^2\,\theta/2)\end{aligned}$$

if $ua<3$. Let $u_0=\lambda(\theta+a\lambda/3)^{-1}$ so that $u_0=\lambda/g(u_0)\,\theta$. Then

$$P(X\geqq\lambda)\leqq\exp(-\lambda^2/2g(u_0)\,\theta)=\exp(-\lambda^2/2(\theta+a\lambda/3))$$

since in this case $u_0\,a=a\lambda(\theta+a\lambda/3)^{-1}<3$ as required, provided only that $\theta>0$. However, if $\theta=0$, $X=0$ a.s. and the proof of (2.6) is complete. Since θ is unchanged when X is replaced by $-X$, the 2-sided inequality (2.7) is immediate. □

Remark. If one wishes to obtain an analogue of Bennett's inequality (Bennett (1962), Eq. (8b)) for an ID r.v. X, which would be of interest when providing bounds for $P(X\geqq\lambda)$ for *large* values of λ, it is necessary to begin with

$$|e^y-1-y|\leqq 1/2\,y^2\,e^{|y|}$$

which holds for *all* y. From this it follows that

$$Ee^{uX}\leqq\exp\{(\theta/2)\,u^2\,e^{ua}\},\qquad u>0.$$

Choose $r_\lambda>0$, and set

$$u_0=a^{-1}\{\ln\lambda-\ln\ln\lambda+\ln r_\lambda\}.$$

Then

$$\begin{aligned}P(X\geqq\lambda)&\leqq\exp\{-u_0\,\lambda+e^{u_0a}\,u_0^2\,\theta/2\}\\ &=\exp(-(\lambda/a)(\ln\lambda)\{1-d_\lambda+\theta r_\lambda(1+d_\lambda)^2/2a\}),\end{aligned}$$

where $d_\lambda=\ln r_\lambda/\ln\lambda-\ln\ln\lambda/\ln\lambda$. Provided $r_\lambda=o(1)$ and $\ln r_\lambda/\ln\lambda=o(1)$ as $\lambda\to\infty$, we obtain the one-sided bound

$$P(X\geqq\lambda)\leqq\exp\{-(\lambda/a)(\ln\lambda)(1+o(1))\}. \tag{2.8}$$

By minor modifications to this argument, (2.8) still holds if X has non-zero mean and non-zero Gaussian component.

The bound in (2.8) under our assumptions is asymptotically best since Sato (1973) has shown that $-\ln P(X \geqq \lambda) \sim a^{-1} \lambda \ln \lambda$ as $\lambda \to \infty$ in the sense that the ratio converges to 1. Theorem 4.9 of Rossberg, Jesiak, and Siegel (1981) states a related result of Kruglov.

3. Main Results

Our main theorem is

Theorem 3.1. *Suppose $\mathscr{A}$ is a class of sets satisfying* (TBI) *and* (A1). *Suppose v is a Lévy measure satisfying* (B1), (B2). *Suppose*

$$\int_0^1 G(H(x)/x)\,dx < \infty. \tag{3.1}$$

Then there exists a Lévy process indexed by $\mathscr{A}$.

Proof. Let $0<\beta<1$, $\delta_j = \beta^j$, $j=0,1,2,\ldots,n$, the value of β to be chosen later. Let a_n tend to 0, $a_0=1$. Let v_n be the restriction of v to $[-a_{n-1}, -a_n) \cup (a_n, a_{n-1}]$, and let Z_n be a mean 0 Lévy process whose Lévy measure is v_n. Let Y_n be the Lévy process defined by $Y_n(A) = \sum_{j=1}^{n} Z_n(A)$. Let $\eta_n \to 0$ such that

$$\sum_{n=1}^{n} \eta_n < \infty. \tag{3.2}$$

We will show that a_n and η_n may be selected so that

$$\sum_{n=1}^{\infty} P^*(\|Z_n\|_{\mathscr{A}} > 6\eta_n) < \infty, \tag{3.3}$$

where P^* is the outer measured induced by P. It then would follow that $P^*(\|Z_n\|_{\mathscr{A}} > 6\eta_n \text{ i.o.}) = 0$ by the appropriate modification of the Borel-Cantelli lemma. Given ε, take N large so that $\sum_{n=N}^{\infty} 6\eta_n < \varepsilon$, and then, depending on ω, choose $N_\omega \geqq N$ so that if $n \geqq N_\omega$, $\|Z_n(\omega)\|_{\mathscr{A}} \leqq 6\eta_n$. If $n, m \geqq N_\omega$,

$$\|Y_n(\omega) - Y_m(\omega)\|_{\mathscr{A}} < \varepsilon, \quad \text{or} \quad \|Y_n - Y_m\|_{\mathscr{A}} \to 0 \quad \text{a.s.}$$

Applying Lemma 2.1 would then complete the proof of the theorem.

Let us then consider $P^*(\|Z_n\|_{\mathscr{A}} > 6\eta_n)$. For any $A \in \mathscr{A}$, we may write

$$Z_n(A) = \sum_{j=1}^{k_n} [Z_n(A_j) - Z_n(A_{j-1})] + [Z_n(A) - Z_n(A_{k_n})], \tag{3.4}$$

where $A_0 = \emptyset$, $A_j \in \mathscr{A}_{\delta_j}$, $A_{k_n}^+ \in \mathscr{A}_{k_n}$, $|A_j \Delta A_{j-1}| \leqq \delta_{j-1}$, $A_{k_n} \subseteq A \subseteq A_{k_n}^+$, and $|A_{k_n}^+ \setminus A_{k_n}| \leqq \delta_{k_n}$, k_n a number to be chosen later. Since $A_j \in \mathscr{A}_{\delta_j}$ and $A_{j-1} \in \mathscr{A}_{\delta_{j-1}}$, the cardinality of the set of $A_j \Delta A_{j-1}$'s is less than or equal to

$$\exp(H(\delta_j)) \exp(H(\delta_{j-1})) \leqq \exp(2\,H(\delta_j)),$$

while the cardinality of the set of $A_{k_n}^+ \setminus A_{k_n}$'s is less than or equal to $\exp(2\,H(\delta_{k_n}))$.

Let γ_j be a sequence (to be chosen later) such that

$$\sum_{j=0}^{\infty} \gamma_j = 1. \tag{3.5}$$

Let $\eta_{nj} = \eta_n \gamma_j$. Let $M_n = M(a_n)$, and let $Q_{n-1} = Q(a_{n-1}) - Q(a_n)$. Since the support of ν_n is contained in $[-a_{n-1}, a_{n-1}] \setminus [-a_n, a_n]$,

$$Q_{n-1} \leqq a_{n-1}^2\, M(a_n) = a_{n-1}^2\, M_n. \tag{3.6}$$

Since

$$|Z_n(A_j) - Z_n(A_{j-1})| \leqq |Z_n(A_j \setminus A_{j-1})| + |Z_n(A_{j-1} \setminus A_j)|,$$

if $|A_j \Delta A_{j-1}| \leqq \delta_{j-1}$, Lemma 2.3 gives us

$$P(|Z_n(A_j) - Z_n(A_{j-1})| > 2\,\eta_{nj}) \leqq 2\,p_{nj}, \tag{3.7}$$

where $p_{nj} = 2\exp(-\eta_{nj}^2/2(Q_{n-1}\,\delta_{j-1} + a_{n-1}\,\eta_{nj}/3))$.

Let $A_{k_n}, A_{k_n}^+ \in \mathscr{A}_{\delta_{k_n}}$ with $|A_{k_n}^+ \setminus A_{k_n}| \leqq \delta_{k_n}$. Then

$$P^*(|Z_n(A) - Z_n(A_{k_n})| > 4\,\eta_n \text{ for some } A \in \mathscr{A},\ A_{k_n} \subseteq A \subseteq A_{k_n}^+) \leqq 2\,q_n, \tag{3.8}$$

where

$$q_n = \max\{P^*(\sup_{A_{k_n} \subset B \subset A_{k_n}^+} |Z_n(B)| > 2\,\eta_n) \colon A_{k_n}, A_{k_n}^+ \in \mathscr{A}_{\delta_{k_n}}, |A_{k_n}^+ \setminus A_{k_n}| \leqq \delta_{k_n}\}.$$

Putting (3.4), (3.7), and (3.8) together

$$\begin{aligned}
&P^*(\|Z_n\|_{\mathscr{A}} > 6\,\eta_n)\\
&\leqq \sum_{j=1}^{k_n} \exp(2\,H(\delta_j)) \max\{P(|Z_n(A_j) - Z_n(A_{j-1})| > 2\,\eta_{nj}) \colon\\
&\qquad\qquad A_j \in A_{\delta_j}, A_{j-1} \in \mathscr{A}_{\delta_{j-1}}, |A_j \Delta A_{j-1}| \leqq \delta_{j-1}\}\\
&\quad + \exp(2\,H(\delta_{k_n})) \max\{P^*(\sup_{\substack{A \in \mathscr{A},\\ A_{k_n} \subseteq A \subseteq A_{k_n}^+}} |Z_n(A) - Z_n(A_{k_n})| > 4\,\eta_n) \colon\\
&\qquad\qquad A_{k_n}, A_{k_n}^+ \in \mathscr{A}_{\delta_{k_n}}, |A_{k_n}^+ \setminus A_{k_n}| \leqq \delta_{k_n}\}\\
&\leqq 2 \sum_{j=1}^{k_n} \exp(2\,H(\delta_j))\, p_{nj} + 2\,q_n \exp(2\,H(\delta_{k_n})).
\end{aligned} \tag{3.9}$$

Suppose that for $j = 1, 2, \ldots, k_n$,

$$2\,H(\delta_j) \leqq \eta_{nj}/8 \max(\delta_{j-1}\,Q_{n-1}/\eta_{nj}, a_{n-1}/3), \tag{3.10}$$

and

$$\eta_{nj}/8 \max(\delta_{j-1}\,Q_{n-1}/\eta_{nj}, a_{n-1}/3) \geqq 2 \ln n + \ln k_n. \tag{3.11}$$

Then,

$$\begin{aligned}\exp(2H(\delta_j))\,p_{nj} &\leqq 2\exp(2H(\delta_j)-\eta_{nj}/2(\delta_{j-1}Q_{n-1}/\eta_{nj}+a_{n-1}/3))\\ &\leqq 2\exp(-\eta_{nj}/8\max(\delta_{j-1}Q_{n-1}/\eta_{nj},\,a_{n-1}/3))\\ &\leqq 2\exp(-2\ln n-\ln k_n).\end{aligned}$$

Hence

$$\sum_{j=1}^{k_n}\exp(2H(\delta_j))\,p_{nj}\leqq 2k_n\exp(-2\ln n-\ln k_n)=2n^{-2},$$

which is summable.

Let us now look at q_n.

$$Z_n(B)=\sum_{\mathbf{t}\in B}\Delta Z_n(\mathbf{t})-c_1|B|,$$

where $c_1=E\sum_{\mathbf{t}\in I^d}\Delta Z_n(\mathbf{t})$. Since the support of v_n is contained in $[-a_{n-1},a_{n-1}]$, $|\Delta Z_n(\mathbf{t})|\leqq a_{n-1}$ for all t, and

$$c_1=\int_{-a_{n-1}}^{a_{n-1}}x\,v_n(dx)\leqq a_{n-1}\,v_n(\mathbb{R})\leqq M_n\,a_{n-1}.$$

Let W_n be a Poisson process on I^d that has a positive jump of size 1 whenever Z_n jumps, that is

$$W_n(B)=\sum_{\mathbf{t}\in B}1_{(\Delta Z_n(\mathbf{t})\neq 0)}. \tag{3.12}$$

Suppose $A\subseteq A^+$ and $|A^+\setminus A|\leqq\delta_{k_n}$. Then

$$\sup_{A\subset B\subset A^+}|Z_n(B)|\leqq a_{n-1}W_n(A^+\setminus A)+a_{n-1}M_n|A^+\setminus A|.$$

Suppose

$$\eta_n/a_{n-1}\geqq c_2\,\delta_{k_n}M_n, \tag{3.13}$$

where $c_2=e^2$. Then

$$\begin{aligned}q_n &\leqq \sup_{\substack{A\subset A^+,\\ |A^+\setminus A|\leqq\delta_{k_n}}} P(W_n(A^+\setminus A)>\eta_n/a_{n-1})\\ &\leqq\exp(-\eta_n/a_{n-1})\end{aligned}$$

by Lemma 2.2, in view of (3.13). If

$$H(\delta_{k_n})\leqq\eta_n/4a_{n-1}, \tag{3.14}$$

and

$$\eta_n/2a_{n-1}\geqq 2\ln n, \tag{3.15}$$

then

$$\begin{aligned}\exp(2H(\delta_{k_n}))\,q_n &\leqq\exp(2H(\delta_{k_n})-\eta_n/a_{n-1})\\ &\leqq\exp(-2\ln n)=n^{-2},\end{aligned}$$

which is summable.

It thus remains to choose $\beta, a_n, \eta_n, \gamma_j, k_n$ to satisfy (3.2), (3.5), (3.10), (3.11), (3.13), (3.14), and (3.15).

Choose β close to 0 so that $(c_0-1)|\ln\beta|>8$. Let $a_n=\beta^n$. Clearly $H(x)/x$ increases as x decreases. Select k_n to be the largest integer such that

$$H(\beta^{k_n})/\beta^{k_n} \leqq M_n. \tag{3.16}$$

For any $t>x>0$,

$$Q(t) \geqq \int_{[-t,t]\setminus[-x,x]} y^2\,\nu(dy) \geqq x^2\,[M(x)-M(t)].$$

Then $\limsup_{x\to 0} x^2 M(x) \leqq Q(t)$ for all t, and since t is arbitrary, $\lim_{x\to 0} x^2 M(x)=0$. Therefore $M_n \leqq a_n^{-2}=\beta^{-2n}$ for n large. This, together with $H \geqq 1$ implies that k_n/n is bounded above by 2.

Let $\eta_n=c_3\max(n^{-2},\ \beta^{k_n} M_n a_{n-1})$, where $c_3=\max(16, 64\beta^{-1})$. (3.13) and (3.15) are then automatically satisfied. We also have (3.14) satisfied since

$$H(\beta^{k_n}) \leqq M_n\,\beta^{k_n} \leqq \eta_n/c_3\,a_{n-1}. \tag{3.17}$$

Let

$$\begin{aligned} \gamma_j &= c_4(k_n+1-j)^{-2}, \quad & j&=1,\dots,k_n,\\ &=0, & j&>k_n \end{aligned} \tag{3.18}$$

where $c_4=\left(\sum_{j=1}^{k_n}(k_n+1-j)^{-2}\right)^{-1}$. Obviously (3.5) holds. Also $\frac{1}{2}\leqq c_4 \leqq 1$. Note that, unlike most entropy arguments, γ_j increases in j. This is necessary to avoid an unwanted $\ln x$ in the integrability condition on H.

Elementary calculus shows that, since $(c_0-1)|\ln\beta|>8$, $\gamma_j^{-2}\,\delta_j^{-(c_0-1)/2}$ increases in j, $j\leqq k_n$. Therefore

$$\delta_j H(\delta_j)\,\gamma_j^{-2} = (\delta_j^{1+(c_0-1)/2} H(\delta_j))(\delta_j^{-(c_0-1)/2}\,\gamma_j^{-2})$$

is increasing in j, $j\leqq k_n$ by (A1).

To verify (3.10), it suffices to show

$$\begin{aligned} &\text{(i)} \quad H(\delta_j) \leqq 3\eta_{nj}/16\,a_{n-1}, \qquad & j&=1,\dots,k_n, \quad \text{and}\\ &\text{(ii)} \quad H(\delta_j) \leqq \eta_{nj}^2/16\,\delta_{j-1}\,Q_{n-1}, & j&=1,\dots,k_n. \end{aligned} \tag{3.19}$$

(i) holds since $\gamma_j^{-1} H(\delta_j)$ is increasing in j, and the inequality holds for $j=k_n$. Recalling (3.6) and that $\delta_j H(\delta_j)\,\gamma_j^{-2}$ increases in j,

$$\begin{aligned} Q_{n-1}\,\delta_{j-1}\,H(\delta_j)\,\gamma_j^{-2} &\leqq c_4^{-2}\,\beta^{-1}\,Q_{n-1}\,\delta_{k_n} H(\delta_{k_n})\\ &\leqq c_4^{-2}\,\beta^{-1}\,a_{n-1}^2\,M_n\,\delta_{k_n} H(\delta_{k_n})\\ &\leqq c_4^{-2}\,\beta^{-1}\,a_{n-1}^2\,M_n^2\,\beta^{2k_n}\\ &\leqq \eta_n^2/256 \end{aligned} \tag{3.20}$$

by the definition of η_n, $j=1,\dots,k_n$. Thus (3.19)(ii) holds.

To verify (3.11), it suffices to show

$$\begin{aligned}&\text{(i)}\quad 3\eta_{nj}/8\,a_{n-1} \geqq 2\ln n+\ln k_n, \quad j=1,\dots,k_n, \quad \text{and}\\ &\text{(ii)}\quad \eta_{nj}^2/8\,\delta_{j-1}\,Q_{n-1} \geqq 2\ln n+\ln k_n, \quad j=1,\dots,k_n.\end{aligned} \tag{3.21}$$

Since k_n/n is bounded, then $\eta_n k_n^{-2}/8a_{n-1} \geqq 2\ln n+\ln k_n$, or (i) holds for $j=1$, and hence for all $j\leqq k_n$. To show (ii), we need only check the case $j=1$. But

$$Q_{n-1}\leqq a_{n-1}^2 M_n \leqq c_5\,\beta^{-2}\,a_n^2\,a_n^{-2}\,|\ln a_n|^{-9}$$

for n large, where by (B1) we can choose c_5 finite and greater than $\limsup M(x)\,x^2\,|\ln x|^9$; (ii) follows.

It now remains to check (3.2), or since n^{-2} is summable, that $\beta^{k_n} M_n\, a_{n-1}$ is summable. We will show this by fixing a positive integer d and then showing $\beta^{k_{nd+i}}\, M_{nd+i}\, a_{nd+i}$ is summable for each $i=0,1,\dots,d-1$.

First of all, since $x^\tau M(x)$ increases as x decreases for some $\tau>0$ by (B 2), $a_{j+d}^\tau M_{j+d}\geqq a_j^\tau M_j$, or

$$\begin{aligned}H(\beta^{k_{j+d}+1})/\beta^{k_{j+d}+1} \geqq M_{j+d} &\geqq \beta^{-d\tau} M_j\\ &\geqq \beta^{-d\tau}\, H(\beta^{k_j})/\beta^{k_j}.\end{aligned} \tag{3.22}$$

Choose d large enough so that $d\tau>1+2c_0$. By the regularly varying nature of H, k_{j+d} must be at least as large as k_j+1 for j sufficiently large. In particular, $\beta^{k_j}-\beta^{k_{j+d}}\geqq \beta^{k_j}(1-\beta)$.

If $x<y$, $M^{-1}(x)\geqq M^{-1}(y)$. Then by the definition of G, (3.1), and the monotoneity of $H(x)/x$,

$$\begin{aligned}\infty &> \sum_n \int_{\beta^{k_{(n+1)d+i}}}^{\beta^{k_{nd+i}}} G(H(x)/x)\,dx\\ &\geqq \sum_n (\beta^{k_{nd+i}}-\beta^{k_{(n+1)d+i}})\,[H(\beta^{k_{nd+i}})/\beta^{k_{nd+i}}]\,M^{-1}(H(\beta^{k_{(n+1)d+i}})/\beta^{k_{(n+1)d+i}})\\ &\geqq \sum_n (1-\beta)\,\beta^{k_{nd+i}}\,\beta^{1+2c_0}\,[H(\beta^{k_{nd+i}+1})/\beta^{k_{nd+i}+1}]\,M^{-1}(M_{(n+1)d+i})\\ &\geqq \sum_n (1-\beta)\,\beta^{1+2c_0}\,\beta^{k_{nd+i}}\,M_{nd+i}\,a_{(n+1)d+i}\\ &\geqq \sum_n (1-\beta)\,\beta^{1+2c_0}\,\beta^d\,\beta^{k_{nd+i}}\,M_{nd+i}\,a_{nd+i}.\end{aligned} \tag{3.23}$$

Thus (3.2) follows. $\square$

The focus of Theorem 3.1 is to determine for which families $\mathscr{A}$ there exists a Lévy process indexed by $\mathscr{A}$ for a given Lévy measure ν. One could also take the opposite point of view: given a family $\mathscr{A}$, for which Lévy measures ν does there exist a Lévy process indexed by $\mathscr{A}$? An appropriate condition for the latter approach is

$$\int_0^1 M(x)\,R^{-1}(M(x))\,dx<\infty, \tag{3.24}$$

where $R(x)=H(x)/x$.

To prove this, note that it suffices to prove (3.2), the remainder of the proof being identical. But $\sum_n \beta^{k_n} M_n a_{n-1} < \infty$ when $\sum_n R^{-1}(M(\beta^n)) M(\beta^n) \beta^{n-1} < \infty$ since $\beta^{k_n+1} \leq R^{-1}(M_n)$ by the definition of k_n and the fact that $R(x)$ increases as x decreases. (B2) is not needed for this argument.

To handle the near-normal case, where (3.1) and (A1) may be mutually exclusive, we have

Theorem 3.2. *Suppose $\mathscr{A}$ satisfies* (TBI) *and* (A2). *Suppose ν satisfies* (B1), (B2). *Suppose*

$$\text{for some} \quad \varepsilon > 0, \quad \int |\ln x|^{1+\varepsilon} G(H(x)/x)\, dx < \infty. \tag{3.25}$$

Then there exists a Lévy process indexed by $\mathscr{A}$.

Proof. Define $\gamma_j = c_4 j^{-(1+\varepsilon)}$, where $c_4 = \left(\sum_{j=1}^{\infty} j^{-(1+\varepsilon)}\right)^{-1}$. Let

$$\eta_n = c_3 \max(n^{-2}, k_n^{1+\varepsilon} \beta^{k_n} M_n a_{n-1}),$$

where $\beta = 1/2$, and c_3, δ_n, a_n, M_n, k_n are defined as before. The proof goes through virtually as for Theorem 3.1, except for checking (3.19) (ii). If $\delta_j H(\delta_j)$ increases with j, it suffices to show (3.19) (ii) for $j = k_n$, which follows exactly as in (3.20). If $\delta_j H(\delta_j)$ decreases with j, it suffices to show that $\eta_n^2 \gamma_{k_n}^2 / Q_{n-1} \to \infty$ as $n \to \infty$. But this follows eactly as in the proof of (3.21) (ii). $\square$

We can get a more refined result than Theorem 3.1 for the Cauchy process, either symmetric or asymmetric. There, (A1) imposes a restriction in that classes $\mathscr{A}$ with much larger entropy are allowable. For the Cauchy process, $\alpha = 1$, $M(x)$ is a multiple of x^{-1} and $Q(x)$ is a multiple of x.

Theorem 3.3. *Suppose c_0 is any positive real and $H(x) \leq x^{-c_0|\ln x|}$ for x small. If $\mathscr{A}$ satisfies* (TBI), *there exists a Lévy process defined over $\mathscr{A}$ whose Lévy measure is that of a Cauchy law.*

Proof. We need to show that (3.2), (3.5), (3.10), (3.11), (3.13), (3.14), and (3.15) in the proof of Theorem 3.1 are satisfied by an appropriate choice of k_n, η_n, a_n, and γ_j. Let γ_j be defined as in (3.18). Take β close to 1 so that $c_0|\ln \beta| < \frac{1}{4}$. Let $a_n = \beta^{n(n+1)/2} n^{c_1 n}$, where $c_1 > 1 + \varepsilon$ for some $\varepsilon > 0$. Let $\eta_n = n^{-(1+\varepsilon)}$, and let $k_n = n$. (3.2), (3.5), (3.11), and (3.15) are trivially satisfied.

Straightforward calculation shows that $\delta_n M_n a_{n-1} = o(n^{-(1+\varepsilon)})$, or (3.13) is satisfied. Also $H(\beta^n) \leq (\beta^{-n})^{c_0 n|\ln \beta|} \leq \beta^{-n^2/4}$. But then $a_{n-1} H(\delta_n) = o(\beta^n)$, and (3.14) is satisfied. It is now easy to check that (3.19)(i) and (ii) are satisfied, and hence so is (3.10. $\square$

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