Z. Wahrscheinlichkeitstheorie verw. Gebiete
54, 161 – 219 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Semimartingales and Markov Processes

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[1] The work of this author was supported by the National Science Foundation Grant No. ENG77-02529 and by the Air Force Office of Scientific Research Grant No. AFOSR74-2733, both to Northwestern University

[2] The work of this author was partially supported by the Air Force Office of Scientific Research through their contract No. F49620-79-C-0080 with Northwestern University

[3] The work of this author was supported by the National Science Foundation Grant No. MCS79-02746 to Purdue University

[4] The work of this author was supported by the National Science Foundation Grant No. MCS76-80623 to University of California, San Diego

0044-3719/80/0054/0161/$11.80

1. Introduction

There are three main problems that are solved here. First, given a Markov process X with some state space E, we characterize all functions f such that $f(X)$ is a semimartingale; basically, f is such a function if and only if it is locally the difference of two excessive functions.

Second, when the state space of the Markov process X is $\mathbb{R}^m$, we give necessary and sufficient conditions for X to be a semimartingale. In particular, a quasi-left-continuous strong Markov process X is a semimartingale if and only if there is a random time change that transforms X into a process whose extended generator is of the form

$$
\begin{aligned}
Gf(x) = & \sum_{i \leq m} b^i(x) D_i f(x) + \tfrac{1}{2} \sum_{i,j \leq m} c^{ij}(x) D_{ij} f(x) \\
& + \int K(x, dy) [f(x+y) - f(x) - 1_{[0,1]}(|y|) \sum_{i \leq m} y^i D_i f(x)]
\end{aligned} \tag{1.1}
$$

for $f \in \mathbb{C}^2(\mathbb{R}^m)$.

Third, given an additive process Y that is a semimartingale with respect to every probability P^x (corresponding to the initial position x of X), we show that the decomposition of Y into a martingale and a process with finite variation and various other processes such as the continuous local martingale part of Y and the quadratic variation of Y can be so constructed that they are all additive and are the same under every P^x. Our original motivation for this and related matters came from the first two problems mentioned above, whose proofs require these. However, these results are more basic and have a larger domain of applicability, for they settle a good part of stochastic calculus on Markov processes. This explains why we choose to give a systematic treatment with full proofs and in as great a generality as possible.

2. Summary of Main Results

Our aim in this section is to discuss the main results of the paper in an informal style, and to describe its organization.

Throughout this paper we follow the notational conventions of Blumenthal and Getoor [3]. The following are some particulars and extensions. As usual we write $\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{R}^m$, etc. for $[0, \infty)$, $(-\infty, +\infty)$, m-dimensional Euclidean space, etc. For any topological space E, $\mathscr{E}$ denotes its Borel σ-field. For any measurable space $(E, \mathscr{E})$, $\mathscr{E}^*$ denotes the universal completion of $\mathscr{E}$. If $(E, \mathscr{E})$ and $(F, \mathscr{F})$ are measurable spaces and if $f\colon E \to F$ is measurable with respect to $\mathscr{E}$ and $\mathscr{F}$, we write $f \in \mathscr{E}/\mathscr{F}$; when $F = \mathbb{R}^m$ we write $f \in \mathscr{E}$ instead of $f \in \mathscr{E}/\mathscr{R}^m$. Moreover, we let $p\mathscr{E}$, $b\mathscr{E}$, $pb\mathscr{E}$ denote the sets of all positive (≥ 0), bounded, positive and bounded $\mathscr{E}$-measurable functions respectively.

For the purposes of this expository section, let $\mathbf{X} = (\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ be a right continuous strong Markov process with some topological state space E, and with $\mathscr{F}_t$ being the usual completion of the σ-field generated by X_s, $s \leq t$. Weaker or different assumptions will be discussed below.

2a) Additive Semimartingales and Random Measures

Let Y be a process which is a semimartingale over $(\Omega, \mathscr{F}, \mathscr{F}_t, P^x)$ for every x. A priori, its decomposition as a sum of a local martingale and a process with finite variation, its quadratic variation process, its continuous local martingale part, and stochastic integrals with respect to it are all dependent on the measure P^x being used. However, a slight extension of the recent work of Stricker and Yor [46] shows that such decompositions and processes can be defined in such a way as to be the same for all P^x; (this property has in fact nothing to do with the Markov property of $\mathbf{X}$; see (3.12)ff.). Somewhat more surprisingly, Y is then a semimartingale over $(\Omega, \mathscr{F}, \mathscr{F}_t, P^\mu)$ for all initial laws μ, and the above mentioned decompositions and processes are also fitted to the measures P^μ; see (3.13). One of our major results concerns the case where Y is additive (that is, $Y_0 = 0$ a.s. and $Y_{t+s} = Y_t + Y_s \circ \theta_t$ a.s. with the exceptional set possibly depending on s and t): then, the above mentioned decompositions and processes are also additive; see Theorem (3.18). All these results are stated in § 3a, b and proved in § 3c, d.

As a corollary, we obtain that, if Y is an increasing additive process that is P^x-locally integrable for every x, there exists an additive process which is a version of the dual predictable projection of Y for every P^x (and even P^μ). This result extends over to the case of random measures Γ which satisfy a suitable condition of σ-integrability with respect to every P^x. In addition, if Γ is an integer valued additive random measure, its additivity property is inherited by the stochastic integrals with respect to it; see § 6a, b.

Suppose further that the integer valued additive random measure Γ is quasi-left-continuous. Then, by a standard argument based upon Motoo's theorem, (a slightly generalized version of which is stated and proved in § 3f) the dual predictable projection $\tilde{\Gamma}$ of Γ admits the factorization $\tilde{\Gamma}(\omega; dt, dy) = dF_t(\omega)\, K(X_t(\omega), dy)$ for some increasing continuous additive process F and some positive kernel K. The proof, given in § 6c, is essentially the same as in Benveniste and Jacod [1]. This theorem is the key step in proving many results such as the existence of Lévy systems for $\mathbf{X}$, last exit decompositions for $\mathbf{X}$, and entrance-exit decompositions for regenerative systems. Similar results would hold for the dual optional projection as well, hence allowing one to prove the Markov property at certain times other than stopping times.

Finally, the previous results are applied in § 6d to obtain the additivity of, and a nice factorization for, the local characteristics of an additive semimartingale.

The results of Sects. 3 and 6 are in some sense mere extensions of those in the fundamental paper by Kunita and Watanabe [27]: most of these were either known at least in some special cases (see the various references in the text itself), or strongly suspected to be true by all specialists. Even then, we have chosen to present this material systematically and with full proofs. Our choice seems necessary not only because our results are formally new, but also because we want to achieve both the best possible measurability properties and the weakest possible conditions.

Concerning these conditions, it is worth pointing out two features. First, for most of the results presented in Sects. 3 and 6, the ordinary Markov property is sufficient. Second, we are forced to work with a filtration $(\mathscr{M}_t)$ which is larger than $(\mathscr{F}_t)$, and in fact our semimartingales Y are on the space $(\Omega, \mathscr{M}, \mathscr{M}_t, P^x)$. Moreover, in order to obtain the additivity of the various decompositions and processes related to an additive Y, we need an "extended" version of the Markov property, namely that the future $\theta_t^{-1}\mathscr{M}$ (and not only $\theta_t^{-1}\mathscr{F}$) be conditionally independent of the past $\mathscr{M}_t$ given X_t. (Thus, when this property is in force and Y is additive, the pair (X, Y) is a Markov additive process in the sense of [5].) This property yields some surprising results; for example, any increasing $(\mathscr{M}_t)$-predictable additive process is $(\mathscr{F}_t)$-predictable; see Theorem (3.26) and its proof in § 3e.

2b) Semimartingale Functions

For any deterministic function f, $Y_t = f(X_t) - f(X_0)$ defines an additive process. If Y is further a semimartingale, then our previous results will apply. In the case of continuous strong Markov processes $\mathbf{X}$ on $\mathbb{R}$, the classical result of Feller is that there always is such a good function f, in fact a strictly increasing and continuous one, and this fact is the key step in characterizing such $\mathbf{X}$. So, the natural question is, given the Markov process $\mathbf{X}$, for what functions f is $f(X)$ a semimartingale for every P^x?

In Sect. 4 we answer this question as follows when $\mathbf{X}$ is a right process: $f(X)$ is a semimartingale for every P^x if and only if there exist finely open sets E_n and 1-excessive functions g_n and h_n of the process $\mathbf{X}$ killed at the time T_n of exit from E_n such that $\bigcup_n E_n = E$, $\sup T_n = +\infty$ a.s., and $f = g_n - h_n$ on E_n.

In Sect. 5 we answer the same question for more specific processes, namely, for linear Brownian motion, linear Brownian motion reflecting at 0 or absorbed at 0, and more generally, diffusions on $\mathbb{R}$. For instance, $f(X)$ is a semimartingale, when $\mathbf{X}$ is the linear Brownian motion, if and only if f is locally the difference of two convex functions. This result implies that Meyer's Theorem (according to which a convex function of a semimartingale is a semimartingale) cannot be substantially extended. If $\mathbf{X}$ is a regular conservative diffusion on $\mathbb{R}$, then X is itself a semimartingale if and only if its inverse scale function is locally the difference of two convex functions. The proofs of Sect. 5 do not rely upon Sect. 4; otherwise, the result just mentioned for the Brownian motion case can be obtained at once from the main characterization for f given in Sect. 4 and the well-known fact that every excessive function of a Brownian motion on an interval is concave.

2c) Markov Processes That are Semimartingales

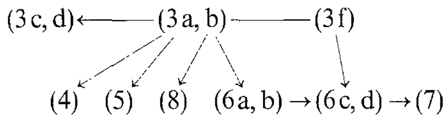
Section 7 contains our third major result: a characterization of strong Markov processes on $\mathbb{R}^m$ that are semimartingales. Our results are particularly pleasant for Hunt processes: an $\mathbb{R}^m$ valued Hunt process is a semimartingale if and only if there is a random time change that transforms it into a process whose extended generator has the form (1.1) for $f \in \mathbb{C}^2(\mathbb{R}^m)$.

Thus, the processes whose extended generators have the form (1.1) are of central importance among semimartingale Markov processes, and deserve a name of their own. We call them *Ito processes*. We believe this choice of the name is particularly appropriate, not only because of Ito's pioneering contributions to Markov processes and stochastic calculus, but also because the processes under discussion are exactly those that were first introduced by Ito [19] as solutions of certain stochastic integral equations; see [6] for this.

2d) Stochastic Differential Equations and Markov Processes

In the final section, Sect. 8, we consider the stochastic equation $Y=y+H+\int F(Y)\,dZ$ with a given additive process H and additive semimartingale Z. Under some conditions making F homogeneous and insuring a solution Y, we show that the pair (X, Y) is a Markov process. When, in addition, X is a right process, so is (X, Y). This extends a result of [37].

The following is a logic chart for reading the paper.



3. Semimartingales Defined on a Markov Process

3a) Basic Setup

Although we will follow [3] closely, we find it useful to recall here the basic ingredients of a Markov process and the particular assumptions and conventions being made in this paper.

Throughout, E is a topological space whose Borel σ-field $\mathscr{E}$ is separable; we write $\mathscr{E}^*$ for the σ-field of universally measurable subsets of E.

Let Ω be a space on which there are defined

(i) a semi-group $(\theta_t)_{t\geq 0}$ of operators;

(ii) a right continuous process $X=(X_t)_{t\geq 0}$ taking values in $(E,\mathscr{E})$ and such that $X_{t+s}=X_s\circ\theta_t$; we let $\mathscr{F}_t^0=\sigma(X_s; s\leq t)$ and $\mathscr{F}^0=\bigvee_t \mathscr{F}_t^0$; note that each $\mathscr{F}_t^0$ is separable and that $\mathscr{F}_{t+s}^0=\mathscr{F}_t^0\vee\theta_t^{-1}(\mathscr{F}_s^0)$;

(iii) an increasing family $(\mathscr{M}_t^0)_{t\geq 0}$ of separable σ-fields on Ω such that $\mathscr{F}_t^0\subset\mathscr{M}_t^0$; we let $\mathscr{M}^0=\bigvee_t \mathscr{M}_t^0$; we assume that $\theta_t\in\mathscr{M}^0/\mathscr{M}^0$;

(iv) a probability kernel $P^x(d\omega)$ from $(E,\mathscr{E}^*)$ into $(\Omega,\mathscr{M}^0)$.

As usual, for each probability measure μ on $(E,\mathscr{E})$ we write $P^\mu=\int\mu(dx)\,P^x$. We let $\mathscr{M}^\mu$ be the P^μ-completion of $\mathscr{M}^0$; $\mathscr{M}_t^\mu$ is the σ-field generated by $\mathscr{M}_t^0$ and the P^μ-null sets of $\mathscr{M}^\mu$; and $\mathscr{M}=\bigcap_\mu \mathscr{M}^\mu$, $\mathscr{M}_t=\bigcap_\mu \mathscr{M}_t^\mu$. We define $\mathscr{F}$ and $\mathscr{F}_t$ similarly. Throughout we assume that the following holds:

(3.1) **Hypothesis.** *The collection* $\mathbf{X}=(\Omega,\mathscr{M},\mathscr{M}_{t+},\theta_t,X_t,P^x)$ *is a Markov process, that is, for every* $Z\in b\mathscr{F}$, $t\geq 0$, *and all* μ,

$$E^\mu[Z\circ\theta_t\,|\,\mathscr{M}_{t+}]=E^{X_t}[Z].$$

In many places we will need to assume the following stronger form of Markov property.

(3.2) *Definition.* The filtration $(\mathcal{M}_t)$ is a *Markov filtration* if

(i) $\mathcal{M}_{t+s}^0 = \mathcal{M}_t^0 \vee \theta_t^{-1}(\mathcal{M}_s^0)$ for all $s, t \geqq 0$;
(ii) $E^\mu[Z \circ \theta_t | \mathcal{M}_{t+}] = E^{X_t}[Z]$ for all μ, t and all $Z \in b\mathcal{M}$.

Of course, under (3.1), $(\mathcal{F}_t)$ is always a Markov filtration. It is shown in [3], (8.12) of Chap. I, that $\mathcal{F}_t = \mathcal{F}_{t+}$ under (3.1); and a similar proof shows that $\mathcal{M}_t = \mathcal{M}_{t+}$ when $(\mathcal{M}_t)$ is a Markov filtration. To illustrate the difference between (3.1) and (3.2) we give the following examples with the reflecting Brownian motion.

(3.3) *Examples.* Let $(\Omega, \mathcal{B}, \mathcal{B}_t, \theta_t, B_t, P^x)$ be a Brownian motion on $\mathbb{R}$.

(i) Let $X_t = |B_t|$. Then, the process $\mathbf{X} = (\Omega, \mathcal{B}, \mathcal{B}_t, \theta_t, X_t, P^x)$ satisfies (3.1) with $\mathcal{M}_t = \mathcal{B}_t$. But $(\mathcal{B}_t)$ is not a Markov filtration for $\mathbf{X}$, since the future of B after time t depends not only on $X_t = |B_t|$ but also on the sign of B_t.

(ii) Set $A_t = \int_0^t 1_{\mathbb{R}_+}(B_s)\, ds$ and $\tau_t = \inf\{s: A_s > t\}$. Let $X'_t = B_{\tau_t}$. Then, $\mathbf{X}' = (\Omega, \mathcal{B}, \mathcal{B}_{\tau_t}, \theta_{\tau_t}, X'_t, P^x)$ satisfies again (3.1) with $\mathcal{M}_t = \mathcal{B}_{\tau_t}$; furthermore, now $(\mathcal{M}_t)$ is a Markov filtration for X'.

Note that the processes X and X' have the same distributions under each P^x: they are reflecting Brownian motions on $\mathbb{R}_+$.

The following is the extended strong Markov property in accordance with (3.2):

(3.4) *Definition.* The filtration $(\mathcal{M}_t)$ is a *strong Markov filtration* if for every finite $(\mathcal{M}_{t+}^0)$-stopping time T we have

(i) $\mathcal{M}_{(T+s)+}^0 = \mathcal{M}_{T+}^0 \vee \theta_T^{-1}(\mathcal{M}_{s+}^0)$ for all $s \geqq 0$,
(ii) $Z \circ \theta_T \in \mathcal{M}$ and $E^\mu[Z \circ \theta_T | \mathcal{M}_{T+}] = E^{X_T}[Z]$ for all μ and all $E \in b\mathcal{M}$.

When $\mathcal{M}_t^0 = \mathcal{F}_t^0$, (3.4, i) is satisfied automatically, and (3.4, ii) reduces to the usual strong Markov property. Note that (3.4, ii) implies $E^\mu[Z \circ \theta_T | \mathcal{M}_T] = E^{X_T}[Z]$ on $\{T < \infty\}$ for every (possibly nonfinite) stopping time T of $(\mathcal{M}_t)$.

Going back to the general assumptions, we note that the lifetime of $\mathbf{X}$ is infinite. In addition, we will assume the following to hold:

(3.5) **Hypothesis.** Either $\mathbf{X}$ is normal, (that is, $P^x\{X_0 = x\} = 1$ for every $x \in E$,) or else θ_0 is the identity mapping on Ω and the property (3.2, ii) holds for $t = 0$.

In order to unify the treatment of various measurability properties, we introduce the following convention:

(3.6) **Convention.** Throughout, $\mathcal{E}_0$, $\mathcal{H}_t$, $\mathcal{H}$ will satisfy one of the following three cases:

(i) $\mathcal{E}_0 = \mathcal{E}^*$, $\mathcal{H}_t = \mathcal{M}_{t+}$, $\mathcal{H} = \mathcal{M}$.
(ii) $\mathcal{E}_0 = \mathcal{E}^e$, $\mathcal{H}_t = \mathcal{F}_{t+}^e$, $\mathcal{H} = \bigvee_t \mathcal{F}_t^e$, where $\mathcal{E}^e$ is the σ-field on E generated by the α-excessive functions ($\alpha > 0$) and $\mathcal{F}_t^e = \sigma(f(X_s): s \leqq t, f \in \mathcal{E}^e)$; in this case it is assumed that $\mathcal{M}_t^0 = \mathcal{F}_t^0$ and that $\mathbf{X}$ is a "right" process (see [13, 41]), and we have $\mathcal{E} \subset \mathcal{E}_0 \subset \mathcal{E}^*$ and $\mathcal{F}_{t+}^0 \subset \mathcal{H}_t \subset \mathcal{F}_t$.

(iii) $\mathscr{E}_0=\mathscr{E}, \mathscr{H}_t=\mathscr{M}_{t+}^0, \mathscr{H}=\mathscr{M}^0$; in this case it is assumed that $P^x(d\omega)$ is a transition kernel from $(E,\mathscr{E})$ into $(\Omega,\mathscr{M}^0)$.

In fact, as far as measurability properties are concerned, we need only the following properties, which always hold under the above convention.

(3.7) $P^x(d\omega)$ is a probability kernel from $(E,\mathscr{E}_0)$ into $(\Omega,\mathscr{H})$;

(3.8) $X_t\in\mathscr{H}_t/\mathscr{E}_0$;

(3.9) $\mathscr{H}_t=\mathscr{H}_{t+}$, and $\mathscr{H}_t$ is contained in the $(\mathscr{M},P^x)$-completion of the separable σ-field $\mathscr{M}_{t+s}^0$ for all $s>0$.

Finally, we recall that the $(\mathscr{H}_t)$-optional (resp. $(\mathscr{H}_t)$-predictable) σ-field is the σ-field on $\Omega\times\mathbb{R}_+$ generated by all $(\mathscr{H}_t)$-adapted processes that are right-continuous and admit left-hand limits (resp. that are left-continuous), without any reference to a specific probability measure on $(\Omega,\mathscr{H})$.

3b) Semimartingales

Let P be a probability measure on $(\Omega,\mathscr{H})$. A *semimartingale* on $(\Omega,\mathscr{H},\mathscr{H}_t,P)$ is a P-a.s. right continuous $(\mathscr{H}_t)$-optional process Y which is the sum $Y=M+A$ of a local martingale M and a $(\mathscr{H}_t)$-optional process A with a.s. finite variation over finite intervals. (Note that we are always working with a $(\mathscr{H}_t)$-optional version of Y.)

When the total variation $\int_0^t |dA_s|$ of A is locally integrable, we call Y a special semimartingale, in which case there exists a unique (up to a P-null set) decomposition $Y=M+A$ with A predictable and $M_0=0$; this decomposition is called the *canonical decomposition* of the special semimartingale Y.

Let Y be a semimartingale. As usual, we denote by Y^c the "continuous local martingale" part of Y and by $[Y,Y]$ its "quadratic variation process". If Y is a $(\mathscr{H}_t)$-optional process with P-locally integrable variation, we denote by $\tilde{Y}$ its dual predictable projection. We denote by $L(Y,P)$ the linear space of all $(\mathscr{H}_t)$-predictable processes which are integrable with respect to Y; see [21]. If $H\in L(Y,P)$, we denote by $H\cdot Y$ the "stochastic integral process" of H with respect to Y; (this may happen to be an ordinary Stieltjes integral). We do not need a precise description of $L(Y,P)$, but we will need the following facts:

(3.10) Every bounded $(\mathscr{H}_t)$-predictable process is in $L(Y,P)$.

(3.11) Let $K\in L(Y,P)$, and let (H^n) be a sequence of $(\mathscr{H}_t)$-predictable processes converging pointwise to a process H and such that $|H^n|\leq K$. Then, H^n and H are in $L(Y,P)$, and $P\text{-}\lim_n H^n\cdot Y_t=H\cdot Y_t$ for every $t\geq 0$; (P-lim means "limit in measure"; this is the Lebesgue dominated convergence theorem for stochastic integrals).

For all facts about semimartingales and stochastic integrals, we refer to [21] and [34].

Returning to the Markov process $\mathbf{X}$, we introduce the following notations for semimartingales on it:

$\mathscr{S} \quad = \{Y\colon Y$ is a semimartingale on $(\Omega, \mathscr{H}, \mathscr{H}_t, P^x)$ for every $x \in E\}$
$\mathscr{S}_p \quad = \{Y \in \mathscr{S}\colon Y$ is P^x-special for every $x \in E\}$
$\mathscr{L} \quad = \{Y \in \mathscr{S}\colon Y_0 = 0$ a.s., Y is a P^x-local martingale for every $x \in E\}$
$\mathscr{V}^+ = \{Y \in \mathscr{S}\colon Y \geq 0$ a.s., Y is increasing a.s.$\}$
$\mathscr{V} \quad = \mathscr{V}^+ - \mathscr{V}^+ = \{Y \in \mathscr{S}\colon Y$ a.s. has finite variation over finite intervals$\}$.
$\mathscr{A}_{\mathrm{loc}} = \{Y \in \mathscr{V}\colon$ for every $x \in E$, Y admits a P^x-locally integrable variation$\}$.
$\mathscr{P} \cap \mathscr{V} = \{Y \in \mathscr{V}\colon$ for every $x \in E$, Y is P^x-indistinguishable
from a $(\mathscr{H}_t)$-predictable process$\}$.

A priori, for $Y \in \mathscr{S}$, the various decompositions such as $Y = M + A$, such terms as Y^c and $[Y, Y]$, and stochastic integrals $H \cdot Y$ all depend on the measure P^x being used. The fact that these terms can all be defined in such a way as to be the same for all P^x is one of our basic results.

(3.12) **Theorem.** *Let* $Y \in \mathscr{S}$.

(i) *There exist* $M \in \mathscr{L}$ *and* $A \in \mathscr{V}$ *such that* $Y = M + A$.

(ii) *If* $Y \in \mathscr{S}_p$, *there exist* $M \in \mathscr{L}$ *and* $A \in \mathscr{P} \cap \mathscr{V}$ *such that* $Y = M + A$; *(this is the canonical decomposition of* Y*).*

(iii) *If* $Y \in \mathscr{A}_{\mathrm{loc}}$, *there exists* $\tilde{Y} \in \mathscr{P} \cap \mathscr{V}$ *which is a version of the* P^x*-dual predictable projection of* Y *for every* $x \in E$.

(iv) *There exists an a.s. continuous* $Y^c \in \mathscr{L}$ *which is a version of the* P^x*-continuous local martingale part of* Y *for every* $x \in E$.

(v) *There exists* $[Y, Y] \in \mathscr{V}^+$ *which is a version of the* P^x*-quadratic variation of* Y *for every* $x \in E$.

(vi) *For every* $H \in \bigcap_{x \in E} L(Y, P^x)$ *there exists* $H \cdot Y \in \mathscr{S}$ *which is a version of the* P^x*-stochastic integral process of* H *with respect to* Y *for every* $x \in E$.

The preceding theorem is very closely related to the results of Stricker and Yor [46]; it will be proved in § 3d together with the following corollary, which gives an affirmative answer to a question of Meyer [36], p. 777. (In fact, the Markov property (3.1) is not used for proving (3.12) and (3.13): all that will be used is (3.7) and, in case $\mathbf{X}$ is not normal, property (3.2, ii) at time $t = 0$.)

(3.13) **Corollary.** *Let* μ *be a probability measure on* $(E, \mathscr{E})$.

(i) *If* Y *is in* $\mathscr{S}$ *(resp.* $\mathscr{L}$, *resp.* $\mathscr{S}_p$, *resp.* $\mathscr{A}_{\mathrm{loc}}$ *and* $E^\mu[|Y_0|] < \infty$*), then* Y *is a semimartingale (resp. a local martingale, resp. a special semimartingale, resp. a process with locally integrable variation) over the space* $(\Omega, \mathscr{H}, \mathscr{H}_t, P^\mu)$.

(ii) *If* $Y \in \mathscr{S}$ *then the processes* Y^c *and* $[Y, Y]$ *defined in* (3.12, iv, v) *are versions, respectively, of the continuous local martingale part and the quadratic variation of* Y *with respect to the measure* P^μ. *If* $Y \in \mathscr{A}_{\mathrm{loc}}$ *and* $E^\mu[|Y_0|] < \infty$, *then* $\tilde{Y}$ *defined in* (3.12, iii) *is a version of the dual predictable projection of* Y *with respect to* P^μ.

(iii) *If* $Y \in \mathscr{S}$ *and* $H \in \bigcap_{x \in E} L(Y, P^x)$, *then* $H \in L(Y, P^\mu)$ *and the process* $H \cdot Y$ *defined in* (3.12, vi) *is a version of the* P^μ*-stochastic integral process of* H *with respect to* Y.

Next we examine the homogeneity properties which can be deduced from the Markov property. First, in accordance with [40] and [41], we introduce the "big shifts" Θ_s by setting, for every process Y,

$$(3.14) \qquad (\Theta_s Y)_t = Y_{t-s} \circ \theta_s 1_{[s,\infty)}(t), \qquad s,t \geqq 0.$$

For example, if $Y = I_{[\![T,\infty[\![}$ for some stopping time T, then $\Theta_s Y = I_{[\![s+T\circ\theta_s,\infty[\![}$ if $Y_t = f(X_t)$ for all t, then $(\Theta_s Y)_t = Y_t 1_{[s,\infty)}(t)$. In view of Theorem (3.12), a natural question to ask is: if $s \geqq 0$ and $Y \in \mathscr{S}$, do we have $\Theta_s Y \in \mathscr{S}$? If the answer is yes, do we have $(\Theta_s Y)^c = \Theta_s(Y^c)$ or $[\Theta_s Y, \Theta_s Y] = \Theta_s([Y,Y])$, etc.? The answers are not always positive, but are so under fairly broad conditions:

(3.15) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a Markov filtration. Let $Y \in \mathscr{S}$ and $s \geqq 0$.*

(i) *We have $\Theta_s Y \in \mathscr{S}$. If $Y \in \mathscr{S}_p$ (resp. $\mathscr{L}$, $\mathscr{V}$, $\mathscr{A}_{\mathrm{loc}}$, $\mathscr{P} \cap \mathscr{V}$), then $\Theta_s Y \in \mathscr{S}_p$ (resp. $\mathscr{L}$, $\mathscr{V}$, $\mathscr{A}_{\mathrm{loc}}$, $\mathscr{P} \cap \mathscr{V}$).*

(ii) *If $Y \in \mathscr{S}_p$ admits the canonical decomposition $Y = M + A$, then $\Theta_s Y = \Theta_s M + \Theta_s A$ is the canonical decomposition of $\Theta_s Y$.*

(iii) *If $Y \in \mathscr{A}_{\mathrm{loc}}$ then $\widetilde{\Theta_s Y} = \Theta_s \tilde{Y}$.*

(iv) *We have $(\Theta_s Y)^c = \Theta_s(Y^c)$.*

(v) *We have $[\Theta_s Y, \Theta_s Y] = \Theta_s([Y,Y])$.*

(vi) *Let $H \in \bigcap_{x\in E} L(Y, P^x)$. Then, $\Theta_s H \in \bigcap_{x\in E} L(\Theta_s Y, P^x)$ and $(\Theta_s H)\cdot(\Theta_s Y) = \Theta_s(H \cdot Y)$.*

Moreover, if $(\mathscr{M}_t)$ is a strong Markov filtration, all these statements hold when s is replaced by any finite $(\mathscr{H}_t)$-stopping time S.

The proof of (3.15) will be given in § 3d. We now consider the questions of additivity for semimartingales. We say that a process Y is *additive* (resp. *strongly additive*) if

(3.16) (i) $Y_0 = 0$ a.s.

(ii) for every $s,t \geqq 0$ we have $Y_{s+t} = Y_s + Y_t \circ \theta_s$ a.s. (resp. for all $t \geqq 0$ and all $(\mathscr{M}_t)$-stopping times S we have $Y_{S+t} = Y_S + Y_t \circ \theta_S$ a.s.).

We denote by $\mathscr{S}_{\mathrm{ad}}$ (resp. $\mathscr{S}_{p,\mathrm{ad}}$, $\mathscr{L}_{\mathrm{ad}}$, $\mathscr{V}_{\mathrm{ad}}$, $\mathscr{A}_{\mathrm{loc,ad}}$) the set of all additive processes that are in $\mathscr{S}$ (resp. $\mathscr{S}_p$, $\mathscr{L}$, $\mathscr{V}$, $\mathscr{A}_{\mathrm{loc}}$). We say that a process H is *homogeneous* (resp. strongly *homogeneous*) if,

(3.17) for every $s \geqq 0$ (resp. every finite $(\mathscr{M}_t)$-stopping time S), the processes H and $\Theta_s H$ (resp. H and $\Theta_S H$) are indistinguishable on (s,∞) (resp. on (S,∞)).

The following theorem is a simple corollary of Theorems (3.12) and (3.15). This is the main result of the present section.

(3.18) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a Markov filtration, and let $Y \in \mathscr{S}_{\mathrm{ad}}$.*

(i) *There exist $M \in \mathscr{L}_{\mathrm{ad}}$ and $A \in \mathscr{V}_{\mathrm{ad}}$ such that $Y = M + A$.*

(ii) *If $Y \in \mathscr{S}_{p,\mathrm{ad}}$ there exist $M \in \mathscr{L}_{\mathrm{ad}}$ and $A \in \mathscr{P} \cap \mathscr{V}_{\mathrm{ad}}$ such that $Y = M + A$.*

(iii) *If $Y \in \mathscr{A}_{\mathrm{loc,ad}}$, there exists $\tilde{Y} \in \mathscr{P} \cap \mathscr{V}_{\mathrm{ad}}$ which is a version of the P^x-dual predictable projection of Y for every $x \in E$.*

(iv) *There exists an a.s. continuous $Y^c \in \mathscr{L}_{\mathrm{ad}}$ which is a version of the P^x-continuous local martingale part of Y for every $x \in E$.*

(v) *There exists $[Y,Y] \in \mathscr{V}_{\mathrm{ad}}^+$ which is a version of the P^x-quadratic variation of Y for every $x \in E$.*

(vi) *Let $H \in \bigcap_{x\in E} L(Y, P^x)$ be homogeneous. Then, there exists a $H \cdot Y \in \mathscr{S}_{\mathrm{ad}}$ which is* *a version of the P^x-stochastic integral process of H with respect to Y for every $x \in E$.*

Moreover, if $(\mathcal{M}_t)$ is a strong Markov filtration and if Y is strongly additive, then we may find strongly additive versions of M and A in (i), (ii), *of $\tilde{Y}$ in* (iii), *of Y^c in* (iv), *of $[Y, Y]$ in* (v), *and of $H \cdot Y$ in* (vi) *when H is strongly homogeneous.*

Proof. First we note that a process Z is additive if and only if

(3.19) $Z_0 = 0$ a.s. and for all $s, t \geqq 0$ we have $(\Theta_s Z)_t = Z_t - Z_{t \wedge s}$ a.s.

Let $Y \in \mathcal{S}_{p,\text{ad}}$; then, by Theorem (3.12), Y admits a canonical decomposition $Y = M + A$. Then Theorem (3.15) implies that $\Theta_s Y = \Theta_s M + \Theta_s A$ is the canonical decomposition of $\Theta_s Y$. Since Y satisfies (3.19), this canonical decomposition is also $\Theta_s Y = M' + A'$, where $M'_t = M_t - M_{t \wedge s}$ and $A'_t = A_t - A_{t \wedge s}$. It follows that M and A satisfy (3.19) as well, and (ii) is proved. Statements (iii), (iv), (v), and (vi) are proved similarly, using (3.12), (3.15), (3.19), and the definition (3.17) and the fact that $Y'_0 = 0$ for (vi).

Next we prove (i). Let $Y \in \mathcal{S}_{\text{ad}}$, and let ΔY denote the jump process of Y with the convention that $\Delta Y_t = 0$ whenever Y_{t-} does not exist (this can happen only on an evanescent set). Set

$$Y_t^e = \sum_{0 < s \leqq t} \Delta Y_s I_{\{|\Delta Y_s| > 1\}}. \tag{3.20}$$

We obviously have $Y^e \in \mathcal{V}_{\text{ad}}$, and thus $Y' = Y - Y^e$ belongs to $\mathcal{S}_{\text{ad}}$. Since $|\Delta Y'| \leqq 1$, we have $Y' \in \mathcal{S}_{p,\text{ad}}$; and (ii) implies the existence of $M \in \mathcal{L}_{\text{ad}}$ and $B \in \mathcal{P} \cap \mathcal{V}_{\text{ad}}$ such that $Y' = M + B$. Putting $A = B + Y^e$, we obtain the decomposition $Y = M + A$ satisfying (i).

There remains to prove the statements about strong additivity. Using the last statement of Theorem (3.15) and replacing s by a finite $(\mathcal{H}_t)$-stopping time S in all places above, we obtain that the various processes Z for which we want to prove the strong additivity satisfy $Z_{S+t} = Z_S + Z_t \circ \theta_S$ a.s. for every finite $(\mathcal{H}_t)$-stopping time S. If S is a finite $(\mathcal{M}_t)$-stopping time (recall here that $\mathcal{M}_t = \mathcal{M}_{t+}$), for every $x \in E$ there exists a stopping time S^x of $(\mathcal{H}_t)$ (and even of $(\mathcal{M}_{t+}^0)$) such that $S = S^x$ P^x-a.s. From what precedes, we obtain that $Z_{(S^x \wedge n)+t} = Z_{S^x \wedge n} + Z_t \circ \theta_{S^x \wedge n}$ P^x-a.s. for every $n \geqq 1$, and it follows that $Z_{S+t} = Z_S + Z_t \circ \theta_S$ a.s. □

The statements (3.18, iii) for $Y \in \mathcal{A}_{\text{loc,ad}}$ and (3.18, iv, v, vi) for $Y \in \mathcal{L}_{\text{ad}}$ were proved a long time ago by Kunita and Watanabe [26, 27, 51] and Meyer [32] under the assumption that $\mathbf{X}$ is a standard process and that Hypothesis (L) holds (plus some other minor assumptions on Y). Hypothesis (L) simplifies matters considerably, since it allows one to work with only one measure instead of the family $(P^x)_{x \in E}$. Without Hypothesis (L), when Y has a bounded 1-potential, (3.18, iii) was proved in [14] and [2], while (3.18, iv, v) for $Y \in \mathcal{L}_{\text{ad}}$ was proved by Meyer [36].

We have established a careful distinction in (3.16) between additivity and strong additivity. It is well known (see [3] for instance) that any right continuous additive functional (i.e. $(\mathcal{F}_t)$-adapted) of a strong Markov process is strongly additive. Walsh [48] has proved that a finite valued additive functional is indistinguishable from a "perfect" additive process Y, that is, a process Y

which satisfies $Y_0=0$ a.s. and $Y_{t+s}=Y_t+Y_s\circ\theta_t$ outside a null set that does not depend on s, t. Of course, a perfect additive process is strongly additive. The following clarifies the relationships between additivity, strong additivity, and perfectness for $(\mathscr{H}_t)$-adapted processes:

(3.21) **Proposition.** *Suppose $(\mathscr{M}_t)$ is a Markov filtration. Suppose that for every $Z\in b\mathscr{M}^0$ the mapping $s\to Z\circ\theta_s(\omega)$ on $\mathbb{R}_+$ is Borel measurable for all $\omega\in\Omega$; (this assumption is automatically satisfied when $\mathscr{M}^0=\mathscr{F}^0$). Let Y be a $(\mathscr{H}_t)$-adapted right continuous real valued additive process. Then,*

(i) *Y is indistinguishable from a perfect additive process Y';*

(ii) *if $(\mathscr{M}_t)$ is a strong Markov filtration, Y is strongly additive.*

Remark. The process Y' above is $(\mathscr{M}_t)$-adapted, but it is possible that it is not adapted to $(\mathscr{H}_t)$.

Proof. (i) The result is obtained by applying the proof of [48] to $M=e^Y$. In [48] it is assumed that $\mathscr{M}_t\subset\mathscr{F}$ and $M_t\in(0,1]$ for all $t\geqq 0$. The former assumption is used only through the facts that $s\to Z\circ\theta_s$ is Borel for all $Z\in b\mathscr{F}^0$ and that the Markov property (3.1) applies; (see [48], p. 235;) here we replace these facts by the Borel measurability of $s\to Z\circ\theta_s$ for all $Z\in b\mathscr{M}^0$ and by the Markov property (3.2, ii). The latter assumption that $M_t\in(0,1]$ has been weakened to the assumption $M_t\in(0,\infty)$ by Meyer [33], which is fulfilled here by $M=e^Y$; (in [33] the strong Markov property is assumed but not used for this result when $M>0$).

(ii) Apply the strong additivity of Y' and the almost sure equalities $Y_S=Y'_S$, $Y_{S+t}=Y'_{S+t}$, and $Y_t\circ\theta_S=Y'_t\circ\theta_S$ for all stopping times S (the last equality uses the strong Markov property (3.4, ii)).

(3.22) *Remark.* Let us recall that, if $Y\in\mathscr{P}\cap\mathscr{V}$, then for every $x\in E$ there exists a $(\mathscr{H}_t)$-predictable process which is P^x-indistinguishable from Y. When in addition Y is additive, $\mathscr{M}_t^0=\mathscr{F}_t^0$, and $\mathbf{X}$ is a right process, we can find a $(\mathscr{H}_t)$-predictable process which does not depend on x and which is P^x-indistinguishable from Y for every $x\in E$; see [41].

It may be of interest to consider simultaneously the filtration $(\mathscr{H}_t)$ and a smaller filtration related to $(\mathscr{F}_t^0)$. We introduce the following convention complementing (3.6).

(3.23) **Convention.** *Throughout, $\mathscr{H}'_t$ will satisfy the following:*

(i) *$\mathscr{H}'_t=\mathscr{F}_t$ when* (3.6, i) *holds;*

(ii) *$\mathscr{H}'_t=\mathscr{F}^e_{t+}$ when* (3.6, ii) *holds, (then $\mathscr{H}'_t=\mathscr{H}_t$);*

(iii) *$\mathscr{H}'_t=\mathscr{F}^0_{t+}$ when* (3.6, iii) *holds.*

For purposes of avoiding confusion, we will indicate the filtration being used in discussing such classes as $\mathscr{S}$, $\mathscr{V}$,... by writing $\mathscr{S}(\mathscr{H}_t)$, $\mathscr{V}(\mathscr{H}_t)$,..., $\mathscr{S}(\mathscr{H}'_t)$, $\mathscr{V}(\mathscr{H}'_t)$, ...

Note that $(\mathscr{H}'_t)$ satisfies (3.7), (3.8), (3.9) with $\mathscr{H}$ and $\mathscr{M}_t^0$ there replaced by $\mathscr{F}^0$ and $\mathscr{F}_t^0$. The Markov property in (3.1) implies immediately that every bounded martingale on $(\Omega, \mathscr{H}, \mathscr{H}'_t, P^x)$ is also a martingale on $(\Omega, \mathscr{H}, \mathscr{H}_t, P^x)$. From [21] § IX-2-c, we deduce the following:

(3.24) (i) $\mathscr{S}(\mathscr{H}'_t)\subset\mathscr{S}(\mathscr{H}_t)$; in fact, due to a result of Stricker [45], we have $\mathscr{S}(\mathscr{H}'_t)=\{Y\in\mathscr{S}(\mathscr{H}_t)$: Y is $(\mathscr{H}'_t)$-adapted$\}$,

(ii) $\mathscr{L}(\mathscr{H}_t') \subset \mathscr{L}(\mathscr{H}_t)$, $\mathscr{S}_p(\mathscr{H}_t') \subset \mathscr{S}_p(\mathscr{H}_t)$;

(iii) if $Y \in \mathscr{S}(\mathscr{H}_t')$ then the various decompositions and processes appearing in (3.12) are relative to the filtration $(\mathscr{H}_t')$ as well as to the filtration $(\mathscr{H}_t)$, (provided that in (3.12, vi) H be $(\mathscr{H}_t')$-predictable).

(3.25) *Remark.* Even when the filtration $(\mathscr{M}_t)$ is not Markov, Theorems (3.15) and (3.18) are still valid for a $(\mathscr{H}_t')$-adapted process Y. This can be proved as follows. The Markov process $(\Omega, \mathscr{F}, \mathscr{F}_t, \theta_t, X_t, P^x)$ satisfies (3.2) and Y is $(\mathscr{H}_t')$-adapted. Thus, (3.15) and (3.18) apply relative to $(\mathscr{H}_t')$, and because of (3.24) above, they apply relative to $(\mathscr{H}_t)$ as well.

When the filtration $(\mathscr{M}_t)$ is Markov, the next theorem shows that $(\mathscr{M}_t)$ cannot be "too much bigger" than $(\mathscr{F}_t)$ at least as far as its ability to hold additive functionals is concerned. Within the theorem, by saying that $(\Omega, \mathscr{M}^0)$ is a "nice measurable space", we mean that for every probability measure P on $(\Omega, \mathscr{M}^0)$ there exists a regular version of the conditional probability $P(\cdot|\mathscr{F}^0)$. For instance, $(\Omega, \mathscr{M}^0)$ is nice if it is a U-space in the sense of Getoor [12].

(3.26) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a Markov filtration.*

(i) *If $Y \in (\mathscr{P} \cap \mathscr{V}_{\mathrm{ad}})(\mathscr{H}_t)$ is such that $E^x\left[\int_0^t |dY_s|\right] < \infty$ for all $x \in E$, $t \geqq 0$, then there exists $Y' \in (\mathscr{P} \cap \mathscr{V})(\mathscr{H}_t')$ which is P^x-indistinguishable from Y for every x, and hence Y is $(\mathscr{F}_t)$-adapted.*

(ii) *The same conclusion holds for every continuous $Y \in \mathscr{V}_{\mathrm{ad}}(\mathscr{H}_t)$.*

(iii) *Suppose $(\Omega, \mathscr{M}^0)$ is a nice measurable space. Then, the same conclusion holds for every $Y \in (\mathscr{P} \cap \mathscr{V}_{\mathrm{ad}})(\mathscr{H}_t)$.*

(iv) *Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. Then, the same conclusion holds for every strongly additive $Y \in (\mathscr{P} \cap \mathscr{V}_{\mathrm{ad}})(\mathscr{H}_t)$.*

Proof will be given in §3.e except for (iii), for which we refer to [23]. The key point for obtaining (i) was pointed out to us by Maisonneuve, while (ii) is a trivial consequence of the results in [5]. In fact, when $(\mathscr{M}_t)$ is a Markov filtration and Y is additive, the pair (X, Y) is a Markov additive process in the sense of [5]. The latter provides examples of processes $Y \in \mathscr{S}_{\mathrm{ad}}(\mathscr{H}_t)$ which are not $(\mathscr{F}_t)$-adapted: either because they are not $(\mathscr{H}_t)$-predictable although in $\mathscr{V}_{\mathrm{ad}}$, like (τ_t) in Example (3.3, ii), or because they are not in $\mathscr{V}_{\mathrm{ad}}$ although $(\mathscr{H}_t)$-predictable, like continuous elements of $\mathscr{L}_{\mathrm{ad}}(\mathscr{H}_t)$.

In the same line of thought, the following may be deduced from [5]: When $(\mathscr{M}_t)$ is a Markov filtration, every right continuous $(\mathscr{H}_t)$-adapted additive process that is *not* a semimartingale is the sum of a process in $\mathscr{S}_{\mathrm{ad}}(\mathscr{H}_t)$ and a right continuous $(\mathscr{H}_t')$-adapted additive process that is not a semimartingale.

3c) Some Measurability Properties

We start with the following

(3.27) **Lemma.** *Let $(Y^x)_{x \in E}$ be a family of processes such that*

(i) *for every $x \in E$, Y^x is right-continuous and left-hand limited P^x-a.s.*

(ii) *for every $t \geqq 0$, there exists $Z_t \in \mathscr{H}_t$ such that $Z_t = Y_t^x$ P^x-a.s. for every $x \in E$.*

Then, there exists a right-continuous process Y adapted to $(\mathscr{H}_t)$ such that Y and Y^x are P^x-indistinguishable for every $x \in E$.

Remark. It follows that Y is a.s. left-hand limited. However, unlike right-continuity, we cannot in general obtain a process whose every path is left-hand limited.

Proof. Let A_t be the set of all ω such that, for some right-continuous and left-hand limited function f, $Z_r(\omega)=f(r)$ for every $r \in \mathbb{Q} \cap [0,t]$. For every t, $A_t \in \mathscr{H}_t$; [7] IV-T-18; and A_t decreases when t increases. Thus,

$$Y_t(\omega)=\begin{cases} \lim\limits_{r\downarrow t, r>t, r\in\mathbb{Q}} Z_r(\omega) & \text{if } \omega \in \bigcup\limits_{s>t} A_s, \\ 0 & \text{otherwise,} \end{cases}$$

defines a right-continuous $(\mathscr{H}_t)$-adapted process. From (i) we obtain $P^x[A_t]=1$, and $Y_t^x = Y_t$ P^x-a.s. The P^x-indistinguishability of Y and Y^x follows from the right-continuity.

The next several results provide criteria ensuring that conditions (3.27, i) and (3.27, ii) are met. The first one is basically due to Doléans-Dade [8]; the proof here follows [36] and [46].

(3.28) **Lemma.** *Let (V^n) be a sequence of $\mathscr{H}_t$-measurable variables such that P^x-$\lim_n V^n$ exists for every $x \in E$. Then, there exists $V \in \mathscr{H}_t$* such that *P^x-$\lim_n V^n = V$ for every $x \in E$.*

Proof. Put $n_0(x)=0$ and

$$n_k(x)=\inf\{m>n_{k-1}(x) \colon \sup_{p,q \geqq m} P^x[|V^p - V^q| > 2^{-k}] \leqq 2^{-k}\}.$$

By (3.7), each n_k is $\mathscr{E}_0$-measurable, and thus $(x,\omega) \to Z_k^x(\omega) = V^{n_k(x)}(\omega)$ is $\mathscr{E}_0 \otimes \mathscr{H}_t$-measurable as well as $(x,\omega) \to Z^x(\omega) = \liminf_k Z_k^x(\omega)$. Since

$$P^x[|Z_k^x - Z_{k+1}^x| > 2^{-k}] \leqq 2^{-k},$$

the Borel-Cantelli lemma implies that $Z_k^x \to Z^x$ P^x-a.s. Since P^x-$\lim_n V^n$ exists by hypothesis, we have $n_k(x) \to \infty$ for every $x \in E$, from which we deduce that P^x-$\lim_n V^n = Z^x$.

Now we set $V(\omega)=Z^{X_0(\omega)}(\omega)$. V is obviously $\mathscr{H}_t$-measurable. If $\mathbf{X}$ is normal, the proof is finished. Suppose $\mathbf{X}$ is not normal, but that (3.5, ii) holds for $t=0$. Saying that P^x-$\lim_n V_n = Z^x$ amounts to saying that $E^x[1 \wedge |V^n - Z^x|] \to 0$ as $n \to \infty$. Applying the Markov property (3.2) at $t=0$, and using the fact that θ_0 is the identity mapping on Ω, we obtain

$$E^x[1 \wedge |V^n - V|] = \int P^x(d\omega)\, E^{X_0(\omega)}[1 \wedge |V^n - Z^{X_0(\omega)}|],$$

which goes to 0 by the bounded convergence theorem. This completes the proof.

The following is an immediate consequence of (3.27) and (3.28).

(3.29) **Lemma.** *Let* $(Y^x)_{x\in E}$ *be a family of processes such that*

(i) *for every* $x\in E$, Y^x *is* P^x-a.s. *right-continuous and left-hand-limited;*

(ii) *there exists a sequence* (Z^n) *of* $(\mathscr{H}_t)$*-adapted processes with* $P^x\text{-}\lim_n Z_t^n = Y_t^x$ *for all* $t\geqq 0$, $x\in E$.

Then, there exists a right-continuous $(\mathscr{H}_t)$*-adapted process* Y *which is* P^x*-indistinguishable from* Y^x *for every* $x\in E$.

(3.30) **Lemma.** *Let* $(V^x)_{x\in E}$ *be a family of variables such that*

(i) *for every* $x\in E$, $V^x\in\mathscr{M}_{t^+}$ *and* $E^x[|V^x|]<\infty$*;*

(ii) *for every* $A\in\mathscr{M}^0$, *there exists* $Z_A\in\mathscr{H}$ *such that* $E^x[V^x 1_A]=E^x[Z_A]$ *for every* $x\in E$.

Then, there exists $V\in\mathscr{H}_t$ *such that* $V=V^x$ P^x-a.s. *for every* $x\in E$.

Proof. Let $s>t$. We define two finite transition kernels from $(E,\mathscr{E}_0)$ into $(\Omega,\mathscr{M}_s^0)$ by setting

$$\hat{P}_s^x[A]=P^x[A], \qquad Q_s^x[A]=E^x[Z_A], \qquad A\in\mathscr{M}_s^0.$$

Since $Q_s^x[A]=E^x[V^x 1_A]$, we have $Q_s^x \ll P_s^x$. Since $\mathscr{M}_s^0$ is separable, Doob's theorem on Radon-Nikodym derivatives (see [30], p. 154) implies the existence of a $\mathscr{E}_0\otimes\mathscr{H}_s$-measurable function: $(x,\omega)\to Z_s^x(\omega)$ such that $Q_s^x[A]=E^x[Z_s^x 1_A]$. Since $V^x\in\mathscr{H}_t$ and since $\mathscr{H}_t$ is contained in the P^x-completion of $\mathscr{M}_s^0$, the fact that $E^x[V^x 1_A]=E^x[Z_s^x 1_A]$ for all $A\in\mathscr{M}_s^0$ implies that $V^x=Z_s^x$ P^x-a.s. Set $Z^x=\liminf_{s\downarrow t, s\in\mathbb{Q}, s>t} Z_s^x$: we have $V^x=Z^x$ P^x-a.s. again, and $(x,\omega)\to Z^x(\omega)$ is measurable with respect to $\bigcap_{s>t}(\mathscr{E}_0\otimes\mathscr{H}_s)$. Finally, set $V(\omega)=Z^{X_0(\omega)}(\omega)$. Since $\mathscr{H}_t=\mathscr{H}_{t^+}$, the variable V is obviously $\mathscr{H}_t$-measurable. If $\mathbf{X}$ is normal the proof is finished. If $\mathbf{X}$ is not normal, θ_0 is the identity mapping on Ω and, applying (3.2, ii) for $t=0$ twice, we have

$$\begin{aligned}E^x[V1_A]&=E^x[Z^{X_0}\circ\theta_0 1_A\circ\theta_0]\\&=\int P^x(d\omega)\,E^{X_0(\omega)}[Z^{X_0(\omega)}1_A]\\&=\int P^x(d\omega)\,E^{X_0(\omega)}[Z_A]\\&=E^x[Z_A\circ\theta_0]=E^x[Z_A]=E^x[V^x 1_A],\end{aligned}$$

which again yields the desired conclusion that $V^x=V$ P^x-a.s.

The following is a result on the interchangeability of limits and time shifts.

(3.31) **Lemma.** *Suppose the filtration* $(\mathscr{M}_t)$ *is Markov. Let* (V^n) *be a sequence of variables such that* $P^x\text{-}\lim_n V^n=V$ *for every* $x\in E$. *Then* $P^x\text{-}\lim_n V^n\circ\theta_s=V\circ\theta_s$ *for all* $s\geqq 0$, $x\in E$.

Proof. This is immediate from the following consequence of the Markov property (3.2) (see the proof of (3.28)):

$$E^x[1\wedge|V^n\circ\theta_s-V\circ\theta_s|]=E^x[E^{X_s}[1\wedge|V^n-V|]].$$

Note that, when $(\mathscr{M}_t)$ is not Markovian, the preceding result is not true in general unless the V^n above are $\mathscr{F}$-measurable. This is a permanent feature for all homogeneity properties.

(3.32) **Lemma.** *Let* $V \in b\mathscr{H}$.

(i) *There exists a right-continuous* $(\mathscr{H}_t)$*-adapted process* ${}^{\pi}V$ *which is a version of the martingale* $E^x[V|\mathscr{H}_t]$ *for every* $x \in E$. *Moreover, if* $V \in b\mathscr{F}$, *then* ${}^{\pi}V$ *is* $(\mathscr{F}_t)$*-adapted.*

(ii) *Suppose the filtration* $(\mathscr{M}_t)$ *is Markov. Then, for all* $t \geqq s \geqq 0$, *we have* $(\Theta_s {}^{\pi}V)_t = {}^{\pi}(V \circ \theta_s)_t$ a.s.

Proof. We follow [2] closely. Let Y^x be a right-continuous version of the martingale $E^x[V|\mathscr{M}_t]$, so that the family $(Y^x)_{x \in E}$ satisfies (3.27, i). For every $A \in \mathscr{M}_t$, $Z_A = V 1_A$ is $\mathscr{H}$-measurable and $E^x[Y_t^x 1_A] = E^x[Z_A]$, hence by Lemma (3.30) we obtain that the family $(Y^x)_{x \in E}$ satisfies (3.27, ii) as well. Then, there exists a right continuous $(\mathscr{H}_t)$-adapted process ${}^{\pi}V$ such that ${}^{\pi}V_t = Y_t^x$ P^x-a.s. for all $t \geqq 0$, $x \in E$.

Let $t \geqq 0$, $U \in b\mathscr{F}_t$, $W \in b\mathscr{F}$, and suppose $V = UW \circ \theta_t$. Then, $E^x[V|\mathscr{H}_t] = UE^{X_t}[W] = E^x[V|\mathscr{F}_t]$. Since $\mathscr{F}_t$ is generated by the random variables V of this form, a monotone class argument shows that, for every $V \in b\mathscr{F}$, $E^x[V|\mathscr{H}_t] = E^x[V|\mathscr{F}_t]$. This proves the last assertion in (i) (this property is related to (3.24)).

(ii) Since (3.2, ii) holds, we have for all $t \geqq s \geqq 0$, $U \in b\mathscr{M}_s^0$, $W \in b\mathscr{M}_{t-s}^0$

$$\begin{aligned}
E^x[UW \circ \theta_s (\Theta_s {}^{\pi}V)_t] &= E^x[UE^{X_s}[W {}^{\pi}V_{t-s}]] \\
&= E^x[UE^{X_s}[WV]] \\
&= E^x[UW \circ \theta_s V \circ \theta_s] \\
&= E^x[UW \circ \theta_s {}^{\pi}(V \circ \theta_s)_t],
\end{aligned}$$

since $UW \circ \theta_s \in \mathscr{H}_t$. Since by (3.2, i) the random variables of the form $UW \circ \theta_s$ generate $\mathscr{M}_t^0$, while $(\Theta_s {}^{\pi}V)_t \in \mathscr{H}_t$, and $\mathscr{H}_t \in \mathscr{M}_t$, we obtain the desired conclusion.

(3.33) *Remark.* When the filtration $(\mathscr{M}_t)$ is strong Markov, we can replace s in Lemmas (3.31) and (3.32, ii) by any finite $(\mathscr{M}_t)$-stopping time S; the proof is exactly the same.

(3.34) *Remark.* Let H be a bounded $\mathscr{H}$-measurable process. It follows from (3.32) that there exists a process ${}^0H$ (resp. ${}^pH$) which is a version of the $(\mathscr{H}_t)$-optional (resp. predictable) projection of H for every measure P^x (or even P^μ): it is sufficient to prove this for H of the form $H = V 1_{]\!]u,v]\!]}$ with $V \in b\mathscr{H}$, $0 \leqq u < v$, and in this case, we have ${}^0H = {}^{\pi}V 1_{]\!]u,v]\!]}$ and ${}^pH = ({}^{\pi}V)_- 1_{]\!]u,v]\!]}$. Moreover, (3.32, ii) and (3.33) imply that, when the filtration $(\mathscr{M}_t)$ is Markov (resp. strong Markov), we have ${}^0(\Theta_s H) = \Theta_s({}^0H)$ and ${}^p(\Theta_s H) = \Theta_s({}^pH)$ for every $s \geqq 0$ (resp. ${}^0(\Theta_S H) = \Theta_S({}^0H)$ and ${}^p(\Theta_S H) = \Theta_S({}^pH)$ for every finite $(\mathscr{H}_t)$-stopping time S). See [41] for many more facts about this question.

3d) Proofs of (3.12), (3.13) *and* (3.15)

Our proof of (3.12) will follow closely Stricker and Yor [46], from which our Theorem (3.12) might be deduced up to some minor details mainly concerned with measurability properties. However, for the sake of completeness, and also because we will need many of the intermediate steps of this proof to prove theorem (3.15), we present the proof in full.

We begin by a series of lemmas. In the first one, P is any probability measure on $(\Omega, \mathscr{H})$, and $\mathscr{A}^+_{loc}(P)$ denotes the class of all nonnegative increasing right-continuous processes Y which are P-locally integrable (we do not require here that Y be $(\mathscr{H}_t)$-adapted, but there exists a sequence (T_n) of finite $(\mathscr{H}_t)$-stopping times increasing to $+\infty$, such that $E[Y_{T_n}] < \infty$). Recall that $\tilde{Y}$ denotes the $(\mathscr{H}_t)$-dual predictable projection of such a Y; $\tilde{Y}$ is uniquely determined up to a P-null set.

(3.35) **Lemma.** *Let (Y^n) be a sequence of elements of $\mathscr{A}^+_{loc}(P)$ such that $Y^{n+1} - Y^n \in \mathscr{A}^+_{loc}(P)$. Then, $\tilde{Y}^{n+1} - \tilde{Y}^n \geqq 0$ a.s. Moreover, if $Y = \sup_n Y^n$ and $Y' = \sup_n \tilde{Y}^n$, then $Y \in \mathscr{A}^+_{loc}(P)$ if and only if $Y' \in \mathscr{A}^+_{loc}(P)$, in which case $\tilde{Y} = Y'$ a.s.*

Proof. The first statement is obvious. Since the dual predictable projection $\tilde{Z}$ of $Z \in \mathscr{A}^+_{loc}(P)$ is characterized by its predictability and the property that $E[\tilde{Z}_T] = E[Z_T]$ for every finite $(\mathscr{H}_t)$-stopping time T, the final statement readily follows from the monotone convergence theorem.

We turn back to our Markov process $\mathbf{X}$.

(3.36) **Lemma.** *Let $Y \in \mathscr{S}$. If H is a bounded $(\mathscr{H}_t)$-predictable process, there exists $H \cdot Y \in \mathscr{S}$ which is a version of the P^x-stochastic integral process of H with respect to Y for every $x \in E$.*

Proof. Put

(3.37) $$H_t = U^0 1_{\{0\}}(t) + \sum_{i=1}^{n} U^i 1_{(s_i, t_i]}(t),$$

where $U^0 \in b\mathscr{H}_0$, $U^i \in b\mathscr{H}_{s_i}$. An $(\mathscr{H}_t)$-optional version of the stochastic integral process $H \cdot Y$ is

(3.38) $$H \cdot Y_t = U^0 Y_0 + \sum_{i=1}^{n} U^i (Y_{t \wedge t_i} - Y_{s \wedge t_i}),$$

regardless of the measure P^x.

Let $\mathscr{K}$ be the linear space of all bounded $(\mathscr{H}_t)$-predictable processes for which the conclusion of our lemma holds. Let (H^n) be a sequence of elements of $\mathscr{K}$ which converges uniformly (resp. increasingly) to a bounded process H. Let Z^x denote a version of the P^x-stochastic integral process of H with respect to Y. Property (3.11) implies that $P^x\text{-}\lim_n H^n \cdot Y_t = Z^x_t$ for all $t \geqq 0$, $x \in E$. Hence, (3.29) implies the existence of a $(\mathscr{H}_t)$-optional process $H \cdot Y$ which is P^x-indistinguishable from Z^x for every $x \in E$. Thus $H \in \mathscr{K}$. Since $\mathscr{K}$ contains all processes (3.37), a monotone class argument shows that $\mathscr{K}$ is exactly the set of all bounded $(\mathscr{H}_t)$-predictable processes.

(3.39) **Lemma.** *If $Y \in \mathscr{S}$, there exists $[Y, Y] \in \mathscr{V}^+$ which is a version of the P^x-quadratic variation of Y for every $x \in E$.*

Proof. Let A^x be a version of the P^x-quadratic variation of Y. The family $(A^x)_{x\in E}$ satisfies (3.27, i). If $I(n,t)=\{0=t_0<t_1<\ldots<t_n=t\}$ is a subdivision of $[0, t]$, whose mesh goes to 0 when $n\to\infty$, and if

$$V_{I(n,t)}=Y_0^2+\sum_{i=1}^{n}(Y_{t_i}-Y_{t_{i-1}})^2, \tag{3.40}$$

then $P^x\text{-}\lim_n V_{I(n,t)}=A_t^x$. The result now follows from (3.29).

(3.41) **Lemma.** *Let $Y\in\mathscr{S}$ be such that $Y_0=0$ and that the jump process ΔY is bounded by a constant c. Then, there exists $M\in\mathscr{L}$ and $A\in\mathscr{P}\cap\mathscr{V}$ such that $Y=M+A$. Moreover, up to an evanescent set, we have $|\Delta M|\leqq 2c$, and $|\Delta A|\leqq c$.*

Proof. Because of (3.27), we may, and will, replace Y (resp. $[Y,Y]$) by an indistinguishable process which is still denoted by the same symbol Y (resp. $[Y,Y]$), and which is $(\mathscr{H}_t)$-adapted and everywhere right-continuous. Hence,

$$T_n=\inf\{t\colon |Y_t|\geqq n, \text{or } [Y,Y]_t\geqq n\}$$

is a $(\mathscr{H}_t)$-stopping time, even in case $\mathscr{H}_t=\mathscr{M}_{t+}^0$.

Since $|\Delta Y|\leqq c$, we have $Y\in\mathscr{S}_p$. Let $Y=M^x+A^x$ be a version of the P^x-canonical decomposition of Y, and let $[M^x,M^x]$ and $[A^x,A^x]$ be the P^x-quadratic variations of M^x and A^x. By a well-known property of canonical decompositions (see [45], for instance), we have $E^x[[A^x,A^x]_{T_n}]\leqq E^x[[Y,Y]_{T_n}]$. Since $M^x=Y-A^x$, we have $[M^x,M^x]\leqq 2([Y,Y]+[A^x,A^x])$, which is a general property of quadratic variations. Since $|\Delta Y|\leqq c$, we have $[Y,Y]_{T_n}\leqq n+c^2$. Therefore,

$$E^x[[M^x,M^x]_{T_n}]\leqq 4E^x[[Y,Y]_{T_n}]\leqq 4(n+c^2).$$

Hence, $(M^x_{T_n\wedge t})_{t\geqq 0}$ is a P^x-square integrable martingale. Since $|Y_{t\wedge T_n}|\leqq n+c$, it follows that

$$\sup_{t\leqq T_n}(A_t^x)^2\leqq(n+c+\sup_{t\leqq T_n}|M_t^x|)^2,$$

which is P^x-integrable.

Let $V\in b\mathscr{H}$. We consider the martingale ${}^{\pi}V$ introduced in (3.32), and we put $B=[{}^{\pi}V,{}^{\pi}V]$. Since ${}^{\pi}V$ is bounded, we have $E^x[B_\infty]<\infty$. Since A^x is predictable, the change of variables formula yields

$$A^x_{t\wedge T_n}\,{}^{\pi}V_{t\wedge T_n}=(A^x\cdot{}^{\pi}V)_{t\wedge T_n}+({}^{\pi}V_-\cdot A^x)_{t\wedge T_n}. \tag{3.42}$$

The P^x-quadratic variation of the P^x-local martingale $A^x\cdot{}^{\pi}V$ is $C=(A^x)^2\cdot B$. We have

$$\begin{aligned}E^x[C_{T_n}^{1/2}]&\leqq E^x[(\sup_{t\leqq T_n}|A_t^x|)B_{T_n}^{1/2}]\\&\leqq\{E^x[\sup_{t\leqq T_n}|A_t^x|^2]\,E^x[B_{T_n}]\}^{1/2},\end{aligned}$$

which is finite, from what precedes. Thus $((A^x \cdot {}^\pi V)_{T_n \wedge t})_{t \geqq 0}$ is a P^x-uniformly integrable martingale. See [34]. Taking the expectation on both sides of (3.42) yields

(3.43) $$E^x[A^x_{t \wedge T_n} {}^\pi V_{t \wedge T_n}] = E^x[({}^\pi V_- \cdot A^x)_{t \wedge T_n}].$$

The process ${}^\pi V_-$ is bounded, and $(M^x_{T_n \wedge t})_{t \geqq 0}$ is a P^x-square integrable martingale. Thus $(({}^\pi V_- \cdot M^x)_{T_n \wedge t})_{t \geqq 0}$ is again a P^x-square integrable martingale, and we have $E^x[({}^\pi V_- \cdot M^x)_{T_n \wedge t}] = 0$. The definition of ${}^\pi V$ implies

$$E^x[A^x_{t \wedge T_n} {}^\pi V_{t \wedge T_n}] = E^x[A^x_{t \wedge T_n} V].$$

Using $Y = M^x + A^x$ and (3.43), we obtain at last

(3.44) $$E^x[A^x_{t \wedge T_n} V] = E^x[({}^\pi V_- \cdot Y)_{t \wedge T_n}].$$

Lemmas (3.32) and (3.36) imply that $({}^\pi V_- \cdot Y)_{t \wedge T_n} \in \mathscr{H}_t$. Thus, we deduce from (3.30) and (3.27) the existence of a right-continuous $(\mathscr{H}_t)$-adapted process $A(n)$ such that $A(n)_t = A^x_{t \wedge T_n}$ P^x-a.s. for all $t \geqq 0$, $x \in E$. Since $\lim_n T_n = \infty$ a.s., the process

$$A_t = \sum_n A(n)_t \, 1_{[T_n, T_{n+1})}(t)$$

is right-continuous, $(\mathscr{H}_t)$-adapted, P^x-indistinguishable from A^x for every $x \in E$. Thus, $A \in \mathscr{P} \cap \mathscr{V}$; and $M = Y - A$ is P^x-indistinguishable from M^x for every $x \in E$, and hence $M \in \mathscr{L}$.

Since $Y = M + A$ is the P^x-canonical decomposition of Y, and since $|\Delta Y| \leqq c$, it is well known (see [21] (2.15) for instance) that $|\Delta M| \leqq 2c$ and $|\Delta A| \leqq c$, up to a P^x-evanescent set for every $x \in E$.

(3.45) **Lemma.** *Suppose the filtration $(\mathscr{M}_t)$ is Markov. Let $Y \in \mathscr{L}$ be such that the jump process ΔY is bounded by a constant c. Then, $\Theta_s Y \in \mathscr{L}$ for every $s \geqq 0$.*

Proof. Put $T_n = \inf\{t: |Y_t| > n\}$ and $T_n' = \inf\{t: |(\Theta_s Y)_t| > n\}$. We have

$$T_n' = \inf\{t > s: |Y_{t-s} \circ \theta_s| > n\} = s + T_n \circ \theta_s,$$

which implies $[\Theta_s(Y_{T_n \wedge \cdot})]_t = (\Theta_s Y)_{T_n' \wedge t}$. Since $(Y_{t \wedge T_n})_{t \geqq 0}$ is a martingale bounded by $n + c$, with the notation of (3.32) we have $Y_{T_n \wedge t} = {}^\pi(Y_{T_n})_t$. Thus, (3.32, ii) implies $(\Theta_s Y)_{T_n' \wedge t} = {}^\pi(Y_{T_n} \circ \theta_s)_t$ a.s. if $t \geqq s$, while $(\Theta_s Y)_t = 0$ if $t \leqq s$ because $Y_0 = 0$. Therefore $((\Theta_s Y)_{T_n' \wedge t})_{t \geqq 0}$ is a martingale, and, since $\lim_n T_n' = \infty$ a.s., we obtain the desired conclusion that $\Theta_s Y \in \mathscr{L}$.

The following proves (3.12, iii), (3.15, iii), and part of (3.15, i).

(3.46) **Lemma.** *Let $Y \in \mathscr{A}_{\text{loc}}$. There exists $\tilde{Y} \in \mathscr{P} \cap \mathscr{V}$ which is a version of the P^x-dual predictable projection of Y for every $x \in E$. Moreover, if the filtration $(\mathscr{M}_t)$ is Markov, we have $\Theta_s Y \in \mathscr{A}_{\text{loc}}$ and $\widetilde{\Theta_s Y} = \Theta_s \tilde{Y}$ for every $s \geqq 0$.*

Proof. It is clearly sufficient to prove the result when Y is positive and increasing. Set $Y^n = Y \wedge n$. We have $Y^n \in \mathscr{S}_p$ and $|\Delta Y^n| \leqq n$, thus (3.41) implies the existence of $M^n \in \mathscr{L}$ and $\tilde{Y}^n \in \mathscr{P} \cap \mathscr{V}$ such that $Y^n = M^n + \tilde{Y}^n$, which means in particular that $\tilde{Y}^n$ is the P^x-dual predictable projection of Y^n for every $x \in E$. If we set $\tilde{Y} = \sup_n \tilde{Y}^n$, (3.35) implies that $\tilde{Y}$ is the dual predictable projection of Y for every P^x.

Suppose $(\mathscr{M}_t)$ is a Markov filtration. By (3.29) we have $|\Delta M^n| \leqq 2n$; thus (3.45) implies that $\Theta_s M^n \in \mathscr{L}$. Hence $\Theta_s \tilde{Y}^n$ is the dual predictable projection of $\Theta_s Y^n$. Since $\tilde{Y} \in \mathscr{P} \cap \mathscr{V}$, it is obvious that $\Theta_s \tilde{Y} \in \mathscr{P} \cap \mathscr{V}$ and, since $(\Theta_s \tilde{Y})_0 = 0$, it follows that $\Theta_s \tilde{Y} \in \mathscr{A}_{\text{loc}}$ (see [34], or [21], (1.37)). Obviously $\Theta_s Y = \sup_n \Theta_s Y^n$ and $\Theta_s \tilde{Y} = \sup_n \Theta_s \tilde{Y}^n$, thus applying once more (3.35) completes the proof.

(3.47) **Lemma.** *If $Y \in \mathscr{L}$, then there exists an a.s. continuous $Y^c \in \mathscr{L}$ which is a version of the P^x-continuous local martingale part of Y for every $x \in E$. Moreover, if $(\mathscr{M}_t)$ is a Markov filtration we have $\Theta_s Y \in \mathscr{L}$ and $(\Theta_s Y)^c = \Theta_s(Y^c)$.*

Proof. We follow [36]. As in (3.41) we may assume that Y is everywhere right-continuous. Therefore the following process

(3.48) $$A(n)_t = \sum_{0<s\leqq t} \Delta Y_s 1_{\{|\Delta Y_s| > 1/n\}}$$

is $(\mathscr{H}_t)$-adapted and right-continuous (recall the convention: $\Delta Y_s = 0$ when Y_{s-} does not exist). It is well known that $A(n) \in \mathscr{A}_{\text{loc}}$, and we set $N(n) = A(n) - \tilde{A}(n)$, where $\tilde{A}(n)$ is the dual predictable projection of $A(n)$. Let N^x be a version of the P^x-continuous local martingale part of Y. We know that $P^x\text{-}\lim_n N(n)_t = Y_t - N_t^x$ for all $t \geqq 0$, $x \in E$. Then, the existence of a $Y^c \in \mathscr{L}$ which is P^x-indistinguishable from N^x for every $x \in E$ follows from (3.29).

Suppose $(\mathscr{M}_t)$ is a Markov filtration. Since $|\Delta(Y - N(1))| \leqq 2$, (3.45) implies $\Theta_s(Y - N(1)) \in \mathscr{L}$, while (3.46) implies $\Theta_s N(1) \in \mathscr{L}$. Therefore $\Theta_s Y \in \mathscr{L}$, and from what precedes we can consider $(\Theta_s Y)^c$. Moreover, the process associated to $\Theta_s Y$ by (3.48) is $\Theta_s A(n)$, and from (3.45) we have $\widetilde{\Theta_s A(n)} = \Theta_s \widetilde{A(n)}$; thus we have $P^x\text{-}\lim_n (\Theta_s N(n))_t = (\Theta_s Y)_t - (\Theta_s Y)_t^c$. Since $P^x\text{-}\lim_n N(n)_t = Y_t - Y_t^c$, (3.31) implies that $(\Theta_s Y)_t^c = \Theta_s(Y^c)_t$ a.s. for all $t \geqq s$. Since $(\Theta_s Y)_t^c = \Theta_s(Y^c)_t = 0$ for $t < s$, we have finished the proof.

Proof of (3.12). We may assume that Y is everywhere right continuous, so the process Y^e defined by (3.20) is in $\mathscr{V}$, and $Y' = Y - Y_0 - Y^e$ belongs to $\mathscr{S}_p$ (since $|\Delta Y'| < 1$). (3.41) implies the existence of $N \in \mathscr{L}$ and $B \in \mathscr{P} \cap \mathscr{V}$ such that $Y' = N + B$.

We obtain (i) by setting $M = N$ and $A = Y_0 + Y^e + B$. Statements (iii) and (iv) have been proved in (3.46) and (3.39). Since the continuous local martingale parts of Y and N coincide, (v) follows from (3.47). When $Y \in \mathscr{S}_p$, we have $Y^e \in \mathscr{A}_{\text{loc}}$ and we obtain (ii) by setting $M = N + Y^e - \tilde{Y}^e$ and $A = Y_0 + \tilde{Y}^e + B$.

Let $H \in \bigcap_{x \in E} L(Y, P^x)$. We put $H^n = H\, 1_{\{|H| \leqq n\}}$. We have constructed the stochastic integral processes $H^n \cdot Y \in \mathscr{S}$ in (3.26). Let Z^x be a version of the P^x-stochastic integral process of H with respect to Y. By (3.11) we have $P^x\text{-}\lim_n H^n \cdot Y_t = Z_t^x$ for all $t \geqq 0$, $x \in E$. Hence, (vi) immediately follows from (3.29).

Proof of (3.13). We divide the proof into several steps.

(a) Every $Y \in \mathscr{L}$ such that $|\Delta Y| \leqq c$ for some $c \in \mathbb{R}_+$ is a P^μ-local martingale: because $E^\mu[Y_{T \wedge T_n}] = \int \mu(dx) E^x[Y_{T \wedge T_n}] = 0$ for all stopping times T, where $T_n = \inf\{t: |Y_t| \geqq n\}$.

(b) Every $Y \in \mathscr{V}$ has P^μ-a.s. finite variation over finite intervals: this is obvious. Since every $Y \in \mathscr{S}$ has a decomposition $Y = M + A$ with $M \in \mathscr{L}$, $|\Delta M| \leqq 2$, $A \in \mathscr{V}$, it follows that every $Y \in \mathscr{S}$ is a P^μ-semimartingale.

(c) Let $Y \in \mathscr{P} \cap \mathscr{V}^+$, $t \geqq 0$, $V \in b\,p\,\mathscr{H}$. Since $E^x[V Y_t] = E^x[({}^\pi V)_- \cdot Y_t]$ for all $x \in E$, we have $E^\mu[V Y_t] = E^\mu[({}^\pi V)_- \cdot Y_t]$ and since ${}^\pi V_t = E^\mu[V | \mathscr{H}_t]$ we deduce that Y is $(\mathscr{M}_t^\mu)$-predictable (this is the identification between predictable and natural processes), so Y is P^μ-indistinguishable from a $(\mathscr{H}_t)$-predictable process.

(d) Let $Y \in \mathscr{A}_{\text{loc}}^+$ with $E^\mu[Y_0] < \infty$. Using (c) and the fact that $\tilde{Y}$ (with the notation of (3.12, iii)), being predictable increasing with $E^\mu[\tilde{Y}_0] < \infty$, is P^μ-locally integrable, see [21] (1.37), we deduce that Y is P^μ-locally integrable and admits $\tilde{Y}$ as a version of its P^μ-dual predictable projection.

(e) Let $Y \in \mathscr{S}$ and let us use the notation of the proof of (3.12). From (a) we deduce that N is a P^μ-local martingale. If $Y \in \mathscr{S}_p$, we have $M = N + Y^e - \tilde{Y}^e$ which is a P^μ-local martingale by (d), so Y is a P^μ-special semimartingale by (c). If $Y \in \mathscr{L}$ we have $Y = M$ which is a P^μ-local martingale. So far we have proved (i) and the assertion in (ii) about $\tilde{Y}$.

(f) To prove the assertion in (ii) about Y^c, it is sufficient to consider the case where $Y \in \mathscr{L}$ and $|\Delta Y| \leqq c$, and by localization we may even assume Y is bounded. With the notations of the proof of (3.47), $E^x[\sup_t (Y_t - Y_t^c - N(n)_t)^2]$ converges to 0 boundedly in x, so $E^\mu[\sup_t (Y_t - Y_t^c - N(n)_t)^2] \to 0$. But Y^c is a continuous P^μ-local martingale by (a), and $N(n)$ is a P^μ-compensated sum of jumps by (b), and it follows that Y^c is the P^μ-continuous local martingale part of Y. Since

$$[Y, Y]_t = [Y^c, Y^c]_t + Y_0^2 + \sum_{0 < s \leqq t} (\Delta Y_s)^2$$

and since $[Y^c, Y^c]$ is the unique continuous increasing process such that $(Y^c)^2 - [Y^c, Y^c]$ is a local martingale, we deduce that $[Y, Y]$ is the P^μ-quadratic variation of Y for every $Y \in \mathscr{S}$.

(g) It remains to prove (iii), and for this we will use freely [21], § II-2-f (what follows will be needed for the proof of (3.15, vi) also). Let $Y \in \mathscr{S}$, $H \in \bigcap_{x \in D} L(Y, P^x)$. Put $D = \{|\Delta Y| > 1\} \cup \{|H \Delta Y| > 1\}$, and

$$Y_t' = Y_0 + \sum_{0 < s \leqq t} \Delta Y_s 1_D(s), \qquad Y'' = Y - Y'.$$

We have $Y' \in \mathscr{V}$, $Y'' \in \mathscr{S}_p$, $|\Delta Y''| \leqq 1$. Moreover, the assumption on H implies that if $Y'' = M + A$ is the P^x-canonical decomposition of Y'' (independent of x and also valid for P^μ from what precedes), then the Stieltjes integral processes $H \cdot Y'$ and $H \cdot A$ exist P^x-a.s. and the P^x-stochastic integral of H with respect to M exists, which amounts to saying that $H^2 \cdot [M, M] \in \mathscr{A}_{\text{loc}}^+(P^x)$, for every $x \in E$. Since the Stieltjes integrals do not depend on the measure, we have $H \cdot Y' \in \mathscr{V}(P^\mu)$, $H \cdot A \in \mathscr{V}(P^\mu)$, and $H^2 \cdot [M, M] \in \mathscr{A}_{\text{loc}}(P^\mu)$ by (d). These facts prove that

$H\in L(Y,P^\mu)$, and it remains to prove that the process $H\cdot M$ constructed in (3.12, vi) is a version of the P^μ-stochastic integral. Since $|\Delta M|\leqq 2$ and $|H\Delta M|\leqq 2$, by localizing we may assume that $[M,M]$ and $H^2\cdot[M,M]$ are P^μ-integrable. Then the result, which is evident for H of the form (3.37), is proved by the same arguments as in (3.36) (for H bounded) and as in the proof of (3.12, vi) (for H unbounded), provided we replace convergence in measure by L^2-convergence as in (f) above. (Note that, unless we use sophisticated arguments such as "medial limits", we cannot conclude immediately from the fact that $P^x\text{-}\lim_n U^n=U$ for every $x\in E$, that $P^\mu\text{-}\lim_n U^n=U$).

Proof of (3.15). We have shown (iii) in (3.46). If $Y\in\mathscr{V}$ (resp. $\mathscr{P}\cap\mathscr{V}$), it is obvious that $\Theta_s Y\in\mathscr{V}$ (resp. $\mathscr{P}\cap\mathscr{V}$). We have proved in (3.45) that $Y\in\mathscr{L}$ implies $\Theta_s Y\in\mathscr{L}$. Using (3.12, i, ii), the remaining assertions in (i), as well as (ii), are obvious. Since $Y^c=M^c$, and similarly $(\Theta_s Y)^c=(\Theta_s M)^c$, whenever $Y=M+A$ with $M\in\mathscr{L}$ and $A\in\mathscr{V}$, (iv) follows from (3.47).

With the notation (3.40), lemma (3.31) implies

$$[Y,Y]_t\circ\theta_s=P^x\text{-}\lim_n V_{I(n,t)}\circ\theta_s$$
$$=P^x\text{-}\lim_n\Big[(\Theta_s Y)_s^2+\sum_{i=1}^n((\Theta_s Y)_{s+t_i}-(\Theta_s Y)_{s+s_i})^2\Big].$$

Since $\Theta_s Y=0$ on $[0,s)$, the above limit equals $[\Theta_s Y,\Theta_s Y]_{s+t}$ a.s., which proves (v).

It remains to prove (vi), and for this we use the notations of part (g) of the proof of (3.13). It is an easy computation to check that

$$(\Theta_s H)\cdot(\Theta_s Y')=\Theta_s(H\cdot Y'),\qquad (\Theta_s H)\cdot(\Theta_s A)=\Theta_s(H\cdot A),$$
$$(\Theta_s H)^2\cdot[\Theta_s M,\Theta_s M]=\Theta_s(H^2\cdot[M,M])$$

(these are Stieltjes integrals, we use (v) for the last one). Because of (i), the first two processes above are in $\mathscr{V}$, and the last one is in $\mathscr{A}_{\text{loc}}$. Thus, $\Theta_s H$ is integrable with respect to $\Theta_s Y'$, to $\Theta_s A$, and to $\Theta_s M$, for every measure P^x, and we have $\Theta_s H\in\bigcap_{x\in E}L(\Theta_s Y,P^x)$.

When H is given by (3.37), a simple computation based upon (3.38) shows that $\Theta_s(H\cdot Y)=(\Theta_s H)\cdot(\Theta_s Y)$. Using (3.31) we obtain that the same property holds, by using the same argument as in (3.36) for H bounded, and then the same argument as in the proof of (3.12, vi) for H unbounded.

Finally, when $(\mathscr{M}_t)$ is a strong Markov filtration, we can replace s by any finite $(\mathscr{H}_t)$-stopping time S in (3.45), (3.46), (3.47) and above, to obtain the final assertion of (3.15) (see remark (3.33)).

3e) *Proof of Theorem* (3.26)

Proof of (3.26, ii). It is sufficient to prove the result when $Y\in(\mathscr{P}\cap\mathscr{V}_{\text{ad}}^{+})(\mathscr{H}_t)$. Since $(\mathscr{M}_t)$ is a Markov filtration, $(\Omega,\mathscr{M},\mathscr{M}_t,\theta_t,(X_t,Y_t),P^x)$ is a Markov additive process, see [5]. Since Y is continuous and increasing, it is shown in [5] that Y is $(\mathscr{F}_t)$-adapted, and hence $(\mathscr{F}_t)$-predictable (although in [5] it is assumed that the transition semi-group of $\mathbf{X}$ is Borel, this fact does not play any role).

It remains to prove that when $\mathscr{H}_t' \neq \mathscr{F}_t$, we can find a $(\mathscr{H}_t')$-adapted process which is indistinguishable from Y. Since $Y_t \in \mathscr{H}_t$, $x \to E^x[Y_t 1_A]$ is $\mathscr{E}_0$-measurable for all $A \in \mathscr{H}' = \bigvee_t \mathscr{H}_t'$. Since $Y_t \in \mathscr{F}_t$, by taking $\mathscr{H}_t'$ in place of $\mathscr{H}_t$ in (3.30) we see that for all $t \geqq 0$ there exists $\tilde{Y}_t \in \mathscr{H}_t$ such that $\tilde{Y}_t = Y_t$ a.s. Then the desired result is obtained by applying (3.27) to the filtration $(\mathscr{H}_t')$.

(3.49) *Remark.* Let us sketch here the proof of (3.26, iii).

(a) The proof of the above referenced result in [5] goes as follows. We denote by $Q^x(\omega, \cdot)$ a version of the conditional distribution of the process Y, conditionally with respect to $\mathscr{F}$, for the measure P^x. It is easy enough to show that for P^x-almost all ω, Y is a process with (non-stationary) independent increments under $Q^x(\omega, \cdot)$. Since Y is continuous increasing, it is deterministic for $Q^x(\omega, \cdot)$, which means that the process Y is $\mathscr{F}$-measurable. Then it is easy to check that $Y_t \in \mathscr{F}_t$.

(b) When $Y \in (\mathscr{P} \cap \mathscr{V}_{\mathrm{ad}}^+)(\mathscr{H}_t)$ is not continuous, we can construct Q^x as before and Y is a process with independent increments for $Q^x(\omega, \cdot)$. However, Y may be $(\mathscr{H}_t)$-predictable without being predictable with respect to the filtration $\mathscr{G}_t = \sigma(Y_s\colon s \leqq t)$, so we cannot conclude that Y is deterministic for $Q^x(\omega, \cdot)$. However when $(\Omega, \mathscr{M}^0)$ is a "nice measurable space", we can consider a regular version $\bar{Q}^x(\omega, \cdot)$ of the conditional probability P^x with respect to $\mathscr{F}$. With some efforts one can prove (see [23]) that $(\Omega, \mathscr{M}^0, \mathscr{M}_t^0, Y_t, \bar{Q}^x(\omega, \cdot))$ is again a process with independent increments and since Y is $(\mathscr{M}_t)$-predictable and increasing we can deduce that Y is deterministic under $\bar{Q}^x(\omega, \cdot)$. It follows like in (a) that $Y_t \in \mathscr{F}_t$ and we prove like for (3.26, ii) that Y is indistinguishable from a $(\mathscr{H}_t')$-adapted process.

(c) It remains to prove that for every $x \in E$, Y is P^x-indistinguishable from a $(\mathscr{H}_t')$-predictable process $\hat{Y}^x$. For this we consider a version $\hat{Y}^{x,n}$ of the $(\mathscr{H}_t', P^x)$-dual predictable projection of $Y \wedge n$. Since $\hat{Y}^{x,n} - Y \wedge n$ is a $(\mathscr{F}_t, P^x)$-martingale, the remark before (3.24) implies that it is also a $(\mathscr{M}_t, P^x)$-martingale, which is $(\mathscr{M}_t)$-predictable and has finite variation. Hence $\hat{Y}^{x,n} = Y \wedge n$ P^x-a.s. and the result follows by taking $\hat{Y}^x = \sup_n \hat{Y}^{x,n}$.

In order to prove (3.26, i, iv), we begin with an auxiliary result, which is interesting in itself. We say that a process Y has $(\mathscr{H}_t')$-local integrable variation if it is the difference $Y = Y^1 - Y^2$ of two nonnegative $\mathscr{H}$-measurable increasing right-continuous processes Y^1 and Y^2 such that, for every $x \in E$, there exists a sequence (T_n) of $(\mathscr{H}_t')$-stopping times (possibly depending on x), with $\lim_n T_n = \infty$ a.s. and $E^x[Y_{T_n}^i] < \infty$ for all $n \in \mathbb{N}$, $i = 1, 2$. Note that Y does not need to be adapted, but Y_t is $\mathscr{H}$-measurable for all $t \geqq 0$.

(3.50) **Proposition.** *Let Y be a process with $(\mathscr{H}_t')$-local integrable variation.*

(i) *There exists a process $\hat{Y}$ which is a version of the $(\mathscr{H}_t')$-dual predictable projection of Y for every P^x.*

(ii) *If the filtration* $(\mathcal{M}_t)$ *is Markov (resp. strong Markov) and if* Y *is additive (resp. strongly additive) then* $\hat{Y}$ *is additive (resp. strongly additive).*

An additive process Y such as above is sometimes called a "raw additive functional". (3.50, ii) is well known: see [14] and [2]. Although this result is basically the same as (3.12, iii) and (3.18, iii), it needs a new proof that is reproduced from [2]. Note that the same proof (just replace $({}^{\pi}V)_-$ by ${}^{\pi}V$ below) would give the existence of an (additive) $(\mathcal{H}_t')$-dual optional projection of the (additive) process Y; we would obtain similarly the existence of a $(\mathcal{H}_t)$-dual optional projection. See [41] for a complete treatment of these matters.

Proof. (i) We can suppose Y is positive and increasing. Let $\hat{Y}^x$ be a version of the $(\mathcal{H}_t')$-dual predictable projection of Y for P^x. Let $A \in \mathcal{F}$. (3.32, i) shows that $({}^{\pi}1_A)_t = E^x[1_A | \mathcal{F}_t]$ and, by a well known property of dual predictable projections, we have

$$E^x[\hat{Y}_t^x 1_A] = E^x[(({}^{\pi}1_A)_- \cdot Y)_t]$$

for all $t \geqq 0$, $x \in E$. Since Y and ${}^{\pi}1_A$ are $\mathcal{H}$-measurable processes, (3.7) implies that for every $t \geqq 0$, the family $(Y_t^x)_{x\in E}$ satisfies the conditions of (3.30) with respect to $\mathcal{H}_t'$: take $Z_A = ({}^{\pi}1_A)_- \cdot Y_t$. The existence of $\hat{Y}$ follows from (3.27).

(ii) Let $Y^n = Y \wedge n$, and denote by Y^n and $\Theta_s Y^n$ the $(\mathcal{H}_t')$-dual predictable projections of Y^n and $\Theta_s Y^n$, as constructed in (i). Let $V \in b\mathcal{F}$, $W \in b\mathcal{F}_s$. Since $\Theta_s Y = 0$ on $[0, s)$, and $W^{\pi}(V \circ \theta_s)_t = {}^{\pi}(WV \circ \theta_s)_t$ if $t \geqq s$, we have

$$W[{}^{\pi}(V \circ \theta_s)_- \cdot (\Theta_s Y^n)]_t = [{}^{\pi}(WV \circ \theta_s)_- \cdot (\Theta_s Y^n)]_t. \tag{3.51}$$

If $t \geqq s$, we have

$$\begin{aligned} E^x[WV \circ \theta_s(\Theta_s \hat{Y}^n)_t] &= E^x[W E^{X_s}[V \hat{Y}_{t-s}^n]] \quad \text{(by (3.14))} \\ &= E^x[W E^{X_s}[({}^{\pi}V_- \cdot Y^n)_{t-s}]] \quad \text{(definition of } \hat{Y}^n) \\ &= E^x[W({}^{\pi}V_- \circ \theta_s) \cdot (Y^n \circ \theta_s)_{t-s}] \\ &= E^x[W(\Theta_s({}^{\pi}V)_- \cdot \Theta_s Y^n)_t] \quad \text{(by (3.51))} \\ &= E^x[W({}^{\pi}(V \circ \theta_s)_- \cdot (\Theta_s Y^n))_t] \quad \text{(by (3.32, ii))} \\ &= E^x[({}^{\pi}(WV \circ \theta_s)_- \cdot (\Theta_s Y^n))_t] \quad \text{(by (3.51))} \\ &= E^x[WV \circ \theta_s(\Theta_s Y^n)_t] \quad \text{(by definition of } \Theta_s Y^n). \end{aligned}$$

Since the variables of the form $WV \circ \theta_s$ generate $\mathcal{F}$, we obtain $(\Theta_s \hat{Y}^n)_t = (\widehat{\Theta_s Y^n})_t$ a.s. for all $t \geqq s$. The same equality being trivially fullfilled for $t < s$, we obtain $\Theta_s \hat{Y}^n = \widehat{\Theta_s Y^n}$.

Using (3.35), we prove exactly as in (3.46) that $\Theta_s Y$ has $(\mathcal{H}_t')$-locally integrable variation, and that $\Theta_s \hat{Y}$ is the $(\mathcal{H}_t')$-dual predictable projection of $\Theta_s Y$. Then, using (3.19), we prove as in (3.18) that, since Y is additive, $\hat{Y}$ is also additive.

(3.52) **Proposition.** *If* $(\mathcal{M}_t)$ *is a Markov filtration, we have* (3.26, i).

Proof. It is sufficient to prove the result when $Y \in (\mathcal{P} \cap \mathcal{V}_{ad}^+)(\mathcal{H}_t)$ satisfies $E^x[Y_t] < \infty$ for all $x \in E$, $t \geqq 0$. We can apply (3.50) to such a Y. Since Y and $\hat{Y}$ are additive, and since $\hat{Y}$ is the $(\mathscr{H}_t')$-dual predictable projection of Y, for all $s,t \geqq 0$ we have

$$E^x[\hat{Y}_{t+s} - \hat{Y}_t | \mathscr{H}_t] = E^{X_t}[\hat{Y}_s] = E^{X_t}[Y_s] = E^x[Y_{t+s} - Y_t | \mathscr{H}_t].$$

Since $\hat{Y}$ is P^x-indistinguishable from a $(\mathscr{H}_t)$-predictable process, and since $E^x[Y_t] < \infty$ for all $t \geqq 0$, the above property characterizes $\hat{Y}$ as being the $(\mathscr{H}_t)$-dual predictable projection of Y for P^x. Since Y itself is P^x-indistinguishable from a $(\mathscr{H}_t)$-predictable process, we deduce that $\hat{Y}$ and Y are P^x-indistinguishable, which proves the result.

Although this proposition contains the main idea, (3.26, iv) is far from being an easy corollary of it. The following is a preparatory lemma, more or less well known, and which will be used again later.

(3.53) **Lemma.** *Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. Let $Y \in \mathscr{V}_{ad}^+$ be purely discontinuous and strongly additive, with a.s. finitely many jumps over each finite interval, the size of them being bounded by a constant c. Then, there exists an increasing sequence (D_n) of $\mathscr{E}_0$-measurable sets, such that $\bigcup D_n = E$ and that $\sup_{x \in E} E^x[\int e^{-t} 1_{D_n}(t)\, dY_t] < \infty$.*

Proof. $T = \inf\{t\colon Y_t > 0\}$ is a $(\mathscr{H}_t)$-stopping time, and $D_n = \left\{x\colon E^x[e^{-T}] \leqq 1 - \frac{1}{n}\right\}$ is $\mathscr{E}_0$-measurable. Since $T > 0$ a.s., we have $\bigcup D_n = E$. The process $Z^n = 1_{D_n}(X) \cdot Y$ is in $\mathscr{V}_{ad}^+$, it is purely discontinuous, and its jumps occur at successive times which we label $S_1, S_2, \ldots$ (we have $S_q \to \infty$). We also have $S_1 \geqq T$, and $X_{S_q} \in D_n$ on $\{S_q < \infty\}$, hence $E^{X(S_q)}[e^{-S_1}] \leqq 1 - \frac{1}{n}$ on $\{S_q < \infty\}$. Since Z^n is strongly additive, we have $S_{q+n} = S_q + S_1 \circ \theta_{S_q}$ a.s. Therefore (3.4, ii) implies

$$E^x[e^{-S_{q+1}}] = E^x[e^{-S_q} 1_{[0,\infty)}(S_q) E^{X_{S_q}}[e^{-S_1}]] \leqq \left(1 - \frac{1}{n}\right) E^x[e^{-S_q}],$$

$$E^x[\int e^{-t} 1_{D_n}(X_t)\, dY_t] \leqq \sum_{q \geqq 1} E^x[c\, e^{-S_q}] \leqq c \sum_{q \geqq 1} \left(1 - \frac{1}{n}\right)^{q-1} = c\, n.$$

Proof. of (3.26, iv). It is sufficient to prove the result when $Y \in (\mathscr{P} \cap \mathscr{V}_{ad}^+)(\mathscr{H}_t)$, and because of (3.26, ii), when in addition Y is purely discontinuous. Set

$$Y_t^n = \sum_{0 < s \leqq t} \Delta Y_s 1_{\{1/n \leqq \Delta Y_s \leqq n\}}.$$

Since $Y = \lim_n Y^n$, it is sufficient to prove the result for each Y^n, or, in other words, to prove the result for every $Y \in (\mathscr{P} \cap \mathscr{V}_{ad}^+)(\mathscr{H}_t)$ satisfying the conditions of (3.53). Let Y be such a process, and let (D_n) be the sequence of subsets associated by (3.53). Denote by $\tilde{Z}^n$ the $(\mathscr{H}_t)$-dual predictable projection of $Z^n = 1_{D_n}(X) \cdot Y$. We have $\tilde{Z} \in (\mathscr{P} \cap \mathscr{V}_{ad}^+)(\mathscr{H}_t)$ and (3.53) implies that $E^x[\tilde{Z}_t^n] = E^x[Z_t^n] < \infty$ for all $t \geqq 0$, $x \in E$. Since $Z^{n+1} - Z^n \in \mathscr{V}^+$ and $Y = \sup_n Z^n$ by the monotone convergence theorem, (3.35) implies that $\tilde{Y} = \sup_n \tilde{Z}^n$. Since $\tilde{Y} \in \mathscr{P} \cap \mathscr{V}^+$, we also have $Y = \sup_n \tilde{Z}^n$. Therefore the desired conclusion is obtained by applying (3.26, i) to each $\tilde{Z}^n$.

(3.54) *Remark.* The previous proof for (3.26, iv) hinges upon two non-trivial results, namely (3.26, ii) which relies upon [5], and (3.50). It would be interesting to have a direct proof. We have such a proof in our hands when **X** is a right process and $(\mathscr{M}_t)$ is a strong Markov filtration. For simplicity, we assume that $\mathscr{H}'_t = \mathscr{F}_t$.

(a) Let $Y \in (\mathscr{P} \cap \mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}_t)$ have a bounded 1-potential $u(x) = E^x\left[\int_0^\infty e^{-t}\, dY_t\right]$. Then a classical result (or an application of (3.18, iii) to the process $e^{-t} u(X_t) - u(X_0)$, for the filtration $(\mathscr{F}_t)$) gives the existence of a $\hat{Y} \in (\mathscr{P} \cap \mathscr{V}^+_{\mathrm{ad}})(\mathscr{F}_t)$ such that $u(X_t) = e^t E^x\left[\int_t^\infty e^{-s}\, d\hat{Y}_s \middle| \mathscr{F}_t\right]$, and since $\hat{Y}$ is additive we also have $u(X_t) = e^t E^x\left[\int_t^\infty e^{-s}\, d\hat{Y}_s \middle| \mathscr{M}_t\right]$. A simple computation yields $u(X_t) = e^t E^x\left[\int_t^\infty e^{-s}\, dY_s \middle| \mathscr{M}_t\right]$. Since both processes $\int_0^t e^{-s}\, dY_s$ and $\int_0^t e^{-s}\, d\hat{Y}_s$ are $(\mathscr{M}_t)$-predictable, we deduce that they are a.s. equal (this is basically the same argument as in (3.52), $\hat{Y}$ is exactly the process constructed in (3.50)), and it follows that Y possesses the desired property.

(b) To prove the result for $Y \in (\mathscr{P} \cap \mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}_t)$ the same argument as in the proof of (3.26, iv) shows that we can assume $\Delta Y \leqq c$ for some $c \in \mathbb{R}$. We will see later (4.7) that there exists an increasing sequence (E_n) of finely open sets such that if $T_n = \inf\{t > 0 \colon X_t \notin E_n\}$ the function $E^x\left[\int_{[0, T_n]} e^{-t}\, dY_t\right]$ is bounded and $\lim_n T_n = \infty$ a.s. Then $Y^n_t = Y_{T_n \wedge t}$ is an additive functional with bounded 1-potential for the right process obtained from **X** by killing it at time T_n. Moreover, this killed process is $(\mathscr{F}_t)$-adapted. Applying (a) we obtain that Y^n is indistinguishable from a $(\mathscr{F}_t)$-predictable process, and since $\lim_n T_n = \infty$ a.s. we obtain the same property for Y.

3f) Relative Densities of Additive Increasing Processes

First, we state a generalized version of Motoo's theorem.

(3.55) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. Let $B, B' \in \mathscr{V}_{\mathrm{ad}}(\mathscr{H}_t)$ be both continuous and satisfy the following condition:*

$$f \in b\mathscr{E}, \quad f(X) \cdot B = 0 \text{ a.s.} \Rightarrow f(X) \cdot B' = 0 \text{ a.s.}$$

Then there exists $h \in \mathscr{E}_0$ such that $B' = h(X) \cdot B$ up to an evanescent set.

Note that because of (3.26, ii) we have $B, B' \in \mathscr{V}_{\mathrm{ad}}(\mathscr{F}_t)$. This theorem is proved in [1] for B, B' increasing, but only when **X** is a right process, an assumption which is explicitly used in the proof. The idea of computing the relative densities by means of Lebesgue's differentiation theorem originates with Kunita (unpublished) who described the procedure to Getoor who in turn communicated the proof to Meyer. The following auxiliary result may be of interest, since it does not require the strong Markov property.

(3.56) **Proposition.** *Suppose* $(\mathcal{M}_t)$ *is a Markov filtration. Let* $B\in\mathcal{V}_{ad}^{+}(\mathcal{H}_t')$ *satisfy* $dB_t \ll dt$ a.s. *Then there exists* $h\in p\mathcal{E}_0$ *such that* $B_t=\int_0^t h(X_s)\,ds$ *a.s.*

Proof. Set $Z_t=\liminf_{s\downarrow 0, s\in Q}(B_{t+s}-B_t)/s$ and $h(x)=E^x[Z_0]$. Since $Z_0\in p\mathcal{H}_0'$, we have $h\in p\mathcal{E}_0$. For every ω such that $dB_t(\omega)\ll dt$ the Lebesgue derivation theorem implies that $B_t(\omega)=\int_0^t Z_s(\omega)\,ds$ for all $t\geqq 0$. The additivity of B yields $Z_t=Z_0\circ\theta_t$ a.s., hence the Markov property implies $Z_t=h(X_t)$ a.s. for all $t\geqq 0$. That is, if $D=\{(\omega,t)\colon Z_t(\omega)\neq h(X_t(\omega))\}$ and $D_\omega=\{t\colon(\omega,t)\in D\}$ and $D_t=\{\omega\colon(\omega,t)\in D\}$, we have $P^x[D_t]=0$ for all $t\geqq 0$. If we can apply the Fubini Theorem to D relative to $P^x(d\omega)\otimes dt$, it follows that the Lebesgue measure of D_ω is zero for P^x-almost all ω, and since $B_t=\int_0^t Z_s\,ds$ a.s., we obtain the result.

Since $(\omega,t)\to B_t(\omega)$ is $\mathcal{F}\times\mathcal{R}_+$-measurable, $(\omega,t)\to Z_t(\omega)$ is $\mathcal{F}\times\mathcal{R}_+$-measurable as well. Let $\nu(\cdot)=U^1(x,\cdot)$ where $x\in E$ and $(U^\alpha)_{\alpha>0}$ is the resolvent of $\mathbf{X}$. There exists $h',h''\in\mathcal{E}$ such that $h'<h<h''$ and $\nu(h''-h')=0$. Since $\nu(h''-h')=\int_0^\infty e^{-t}E^x[(h''-h')(X_t)]\,dt$, the Fubini Theorem implies that $\{(\omega,t)\colon(h''-h')(X_t(\omega))>0\}$ is $P^x(d\omega)\otimes dt$-null and therefore the function $(\omega,t)\to h(X_t(\omega))$ is measurable with respect to the $P^x(d\omega)\otimes dt$-completion of $\mathcal{F}\times\mathcal{R}_+$. Hence D is measurable with respect to the same completion, and we can apply Fubini's Theorem to D.

Proof of (3.55). By (3.26, ii) we can replace B,B' by indistinguishable processes still denoted by B,B' and belonging to $\mathcal{V}_{ad}(\mathcal{H}_t')$. Let B^+, B^- (resp. B'^+,B'^-) be the positive and negative variation processes of B (resp. B'), and let $C=B^++B^-$, $C'=B'^++B'^-$: all these processes are in $\mathcal{V}_{ad}^{+}(\mathcal{H}_t')$ and by (3.21) applied to $\mathcal{M}_t^0=\mathcal{F}_t^0$ they are strongly additive. Set $F_t=t+C_t+C_t'$ and $\tau_t=\inf\{s\colon F_s>t\}$. Each τ_t is a finite $(\mathcal{H}_t')$-stopping time. Since F, B^+ are strongly additive and $(\mathcal{M}_t)$ is a strong Markov filtration, $\hat{B}_t^+=B_{\tau_t}^+$ is an increasing additive functional of the time-changed Markov process $\hat{X}_t=X_{\tau_t}$ and B^+ is adapted to the filtration $(\hat{\mathcal{H}}_t)$ associated to $\hat{X}$ by the same convention (3.23) as $(\mathcal{H}_t')$ to X. Moreover $d\hat{B}_t^+\ll dt$. By (3.56) there exists $h^+\in p\mathcal{E}_0$ such that $\hat{B}_t^+=\int_0^t h^+(\hat{X}_s)\,ds$ and changing time back yields $B_t^+=\int_0^t h^+(X_s)\,dF_s$ a.s. Similarly we obtain $h^-,h'^+,h'^-\in p\mathcal{E}_0$ such that $B_t^-=\int_0^t h^-(X_s)\,dF_s$, $B_t'^+=\int_0^t h'^+(X_s)\,dF_s$ and $B_t'^-=\int_0^t h'^-(X_s)\,dF_s$. The hypothesis on B, B' implies that $\{h^+=h^-\}\subset\{h'^+=h'^-\}$ up to a set of F-potential zero, so if we set $h=[(h'^+-h'^-)/(h^+-h^-)]\,1_{\{h^+\neq h^-\}}$ we obtain $B_t'=\int_0^t h(X_s)\,dB_s$ a.s.

Motoo's Theorem generalizes as follows in the discontinuous case.

(3.57) **Theorem.** *Suppose* $(\mathcal{M}_t)$ *is a strong Markov filtration. Let* $B, B' \in \mathcal{V}_{\mathrm{ad}}(\mathcal{H}_t)$ *be strongly additive and satisfy the following condition:*

Z strongly homogeneous $(\mathcal{H}_t)$*-optional,* $Z \cdot B = 0$ a.s. $\Rightarrow Z \cdot B' = 0$ a.s.

Then, there exists a strongly homogeneous $(\mathcal{H}_t)$*-optional process H such that* $B' = H \cdot B$ *up to an evanescent set.*

Proof. Set $B_t^c = B_t - B_0 - \sum_{0<s\leq t} \Delta B_s$, and define B'^c similarly. B^c and B'^c satisfy the conditions of (3.55), so there exists $h \in \mathcal{E}_0$ such that $B'^c = h(X) \cdot B^c$. Let $D = \{\Delta B = 0\}$. The process 1_D is strongly homogeneous and $1_D \cdot B = 0$, so $1_D \cdot B' = 0$ a.s. and we have up to an evanescent set: $\{\Delta B' \neq 0\} \subset \{\Delta B \neq 0\}$ (with the conventions $\Delta B_0 = B_0$ and $\Delta B'_0 = B'_0$). Then obviously the conclusion of our theorem is met by setting

$$H = h(X)\,1_D + \frac{\Delta B'}{\Delta B}\,1_{D^c}.$$

(3.58) *Remark.* Assume **X** is a right process and $\mathcal{M}_t^0 = \mathcal{F}_t^0$.

(a) Recall that for every 1-excessive function f the process $f(X)$ admits a.s. left-hand limits. Then one can prove ([41]) that a strongly homogeneous $(\mathcal{F}_t^e)$-optional process H, considered as a function on $\Omega \times \mathbb{R}_+$, is measurable with respect to the σ-field generated by the processes X and $f(X)_-$, where f goes through the set of all 1-excessive functions. Hence one can always choose an H as such in (3.57), since every $(\mathcal{F}_t)$-adapted increasing additive functional of **X** is indistinguishable from a $(\mathcal{F}_t^e)$-adapted functional. See [1] or [41].

(b) When in addition hypothesis (L) holds, Glover [16] has shown that in (3.57) one can choose an H of the form $H = h(X_-^*, X)$ where X_-^* is the left-hand limit of X in a suitable compactification E^* of E and where h is a Borel function on $E^* \times E$.

(c) When $B, B' \in (\mathcal{P} \cap \mathcal{V}_{\mathrm{ad}}^+)$ in (3.57), one can choose H measurable with respect to the σ-field on $\Omega \times \mathbb{R}_+$ generated by all processes $f(X)_-$, where f is any 1-excessive function.

4. Semimartingale Functions of a Markov Process

It will be assumed throughout this section that the underlying process $\mathbf{X} = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, P^x)$ is a right process. The state space E is a topological space which is homeomorphic to a universally measurable subset of a compact metrizable space (that is, a U-space); see [13] or [41]. We investigate here the following problem:

(4.1) For which real functions f on E is the process $f(X)$ a semimartingale over $(\Omega, \mathcal{M}, \mathcal{M}_t, P^x)$ for all $x \in E$?

A function f satisfying the condition of (4.1) will be called a *semimartingale function* for **X**. Let us note to begin with that, because of (3.24), and since a process $f(X)$ is $(\mathcal{F}_t)$-adapted as soon as it is $(\mathcal{M}_t)$-adapted, a function f is a semimartingale function if and only $f(X)$ is a semimartingale over $(\Omega, \mathscr{F}, \mathscr{F}_t, P^x)$ for every $x \in E$. So until the end of this section, we work only with the natural filtration $\mathscr{M}_t = \mathscr{F}_t = \mathscr{H}_t$.

Before stating the main result (4.6) we make some observations about the problem (4.1). If $f(X)$ is to be in $\mathscr{S}$, f must be $\mathscr{E}^*$-measurable, and the almost sure right-continuity of $f(X)$ implies that f must be finely continuous and $\mathscr{E}^e$-measurable; see [3], II-(4.8) and [41], (9.8). If T is a terminal time for $\mathbf{X}$, then $(\mathbf{X}, T)$ denotes the process $\mathbf{X}$ killed at time T: it is constructed on the same space $(\Omega, \mathscr{F}, P^x)$ by adjoining a death point Δ to E and by setting

$$\bar{X}_t(\omega) = \begin{cases} X_t(\omega) & \text{if } t < T(\omega), \\ \Delta & \text{if } t \geqq T(\omega). \end{cases}$$

If one restricts x to the set G of irregular points for T, that is, $G = \{x \in E: P^x[T > 0] = 1\}$, it is a standard fact that $(\mathbf{X}, T)$ restricted to $G \cup \{\Delta\}$ is a right process if the terminal time T is exact; see [41], (12.20) for example. If $T = T_K$ is the hitting time of a set $K \in \mathscr{E}^e$, then T_K is an exact terminal time and $G = E \setminus K^r$, where K^r is the set of regular points for K. The set K^r is finely closed and in $\mathscr{E}^e$, because it may be expressed in the form $K^r = \{\phi = 1\}$ where $\phi(x) = E^x[e^{-T_K}]$ is 1-excessive. Let P_K^α denote the α-order hitting operator for K, defined by $P_K^\alpha f(x) = E^x[e^{-\alpha T_K} f(X_{T_K})]$, $f \in p\mathscr{E}^*$. One has the following standard facts relating excessive functions for $(\mathbf{X}, T_K)$ to excessive functions for $\mathbf{X}$. For $f \in p\mathscr{E}^*$ and $\alpha > 0$, set

$$V^\alpha f(x) = E^x\Big[\int_{[0, T_K)} e^{-\alpha t} f(X_t)\, dt\Big], \qquad x \in E.$$

The restriction of V^α to $G = E \setminus K^r$ is the α-potential kernel for the process $(\mathbf{X}, T_K)$. Note that if $x \in K^r$, then $V^\alpha f(x) = 0$ for every $f \in p\mathscr{E}^*$. If (U^α) denotes the resolvent for $\mathbf{X}$, Dynkin's formula ([3], II-(1.2)) gives the relation

(4.2) $$V^\alpha f = U^\alpha f - P_K^\alpha U^\alpha f, \qquad f \in b\mathscr{E}^*, \quad \alpha > 0.$$

An $f \in p\mathscr{E}^*$ is called α-$(\mathbf{X}, T_K)$-excessive if

(4.3) $e^{-\alpha t} E^x[f(X_t) 1_{\{t < T_K\}}]$ increases to $f(x)$ when decreases to 0, for every $x \in E$.

It is easy to see that an α-$(\mathbf{X}, T_K)$-excessive function vanishes on K^r and its restriction to $G = E \setminus K^r$ is α-excessive for $(\mathbf{X}, T_K)$. Conversely, any function defined on G which is α-excessive for $(\mathbf{X}, T_K)$ is an α-$(\mathbf{X}, T_K)$-excessive function if it is set equal to zero on K^r. In particular, if $\alpha > 0$, every α-$(\mathbf{X}, T_K)$-excessive function is equal to the limit of an increasing sequence $(V^\alpha h_n)$ with $h_n \in b\,p\,\mathscr{E}^*$. Using (4.2) and the fact that if f is α-excessive then so is $P_K^\alpha f$, one derives the following:

(4.4) **Proposition.** *If $\alpha > 0$, every α-$(\mathbf{X}, T_K)$-excessive function on E is $\mathscr{E}^e$-measurable. Hence the σ-field $\mathscr{E}^e(\mathbf{X}, T_K)$ generated by all α-excessive functions of $(\mathbf{X}, T_K)$ is the trace of $\mathscr{E}^e$ on $G = E \setminus K^r$.*

It is also easy to see that a subset of G is finely open for $(\mathbf{X}, T_K)$ if and only if it is finely open for $\mathbf{X}$.

We recall the following simple way of forming α-$(\mathbf{X}, T_K)$-excessive functions.

(4.5) **Proposition.** *If $f \in p\mathscr{E}^*$, then $P_K^\alpha f \cdot 1_G$ is α-$(\mathbf{X}, T_K)$-excessive.*

Proof. We have, letting $h = 1_G \cdot P_K^\alpha f$,

$$e^{-\alpha t} E^x[h(X_t) 1_{\{t < T_K\}}] = e^{-\alpha t} E^x[1_G(X_t) 1_{\{t < T_K\}} E^{X_t}[e^{-\alpha T_K} f(X_{T_K})]].$$

However, $X_t \in G$ a.s. on $\{t < T_K\}$, and $T_K = t + T_K \circ \theta_t$ a.s. on $\{t < T_K\}$, so the last expression above reduces by the Markov property to

$$\begin{aligned} &E^x[e^{-\alpha(t + T_K \circ \theta_t)} f(X_{T_K}) 1_{\{t < T_K\}}] \\ &\quad = E^x[e^{-\alpha T_K} f(X_{T_K}) 1_{\{t < T_K\}}], \end{aligned}$$

which increases to $E^x[e^{-\alpha T_K} f(X_{T_K})] 1_G(x)$ as $t \downarrow 0$.

We may now described the solution to (4.1).

(4.6) **Theorem.** *A real function f on E is a semimartingale function for $\mathbf{X}$ if and only if the following condition is satisfied: there exists a sequence (E_n) of finely open $\mathscr{E}^e$-measurable sets with $E = \bigcup E_n$ such that if $T_n = T_{E_n^c}$ is the hitting time of E_n^c then* $\sup T_n = \infty$ a.s., *and for each n there exist bounded 1-$(\mathbf{X}, T_n)$-excessive functions g_n, h_n on E such that $f 1_{E - (E_n^c)^r} = g_n - h_n$. If f is bounded, the sets E_n may be constructed so that they are increasing, and hence the stopping times T_n are increasing.*

It should be remarked that the sufficiency of the condition is a simple consequence of a theorem of Meyer ([34], IV-T33 and the footnote on p. 313, or [22] (2.17)), since then the process $f(X)$ coincides with the semimartingale $g_n(X) - h_n(X)$ on the stochastic interval $[\![0, T_n[\![$.

The technical lemma used for producing the sets E_n is similar to a result of Revuz [38] based on an exponential formula of Meyer [31].

(4.7) **Lemma.** *Let $B \in \mathscr{V}_{ad}^+$ with $\Delta B \leqq c$ for some constant c. Then, there exists an increasing sequence $(E_n)_{n \geqq 1}$ of finely open $\mathscr{E}^e$-measurable sets with $E = \bigcup E_n$, such that if $T_n = T_{E_n^c}$ is the hitting time of E_n^c then $\lim\limits_{(n)} \uparrow T_n = \infty$* a.s., *and for all $n \geqq 1$,*

$$\sup_{x \in E} E^x\Big[\int_{(0, T_n]} e^{-t}\, dB_t\Big] < \infty.$$

Proof. We may suppose that $c < 1$. By (3.27, i) we may also suppose that B is a perfect additive functional, that is, we have $B_{t+s} = B_t + B_s \circ \theta_t$ everywhere outside a null set not depending on (s, t). If M is defined by

$$M_t = e^{-B_t} \Pi_{0 < s \leqq t} [(1 - \Delta B_s) e^{\Delta B_s}],$$

then M is a decreasing right-continuous perfect multiplicative functional of $\mathbf{X}$ with $M_0 = 1$ a.s. and $M_t > 0$ a.s. for all $t \geqq 0$. Moreover, M satisfies

$$M = 1 - M_- \cdot B \tag{4.8}$$

The kernels V^α, $\alpha>0$, defined on $b\mathscr{E}^*$ by

$$V^\alpha f(x)=E^x\left[\int_0^\infty e^{-\alpha t}f(X_t)\,M_t\,dt\right],\qquad f\in b\mathscr{E}^*,$$

form a resolvent on E, (which turns out to be just the resolvent of the process $\mathbf{X}$ killed at rate $-dM_t/M_{t-}$, see [3] Chap. III, a fact which will not be used later). Let U_B^α be the α-potential kernel for B, defined by

$$U_B^\alpha f(x)=E^x\left[\int_0^\infty e^{-\alpha t}f(X_t)\,dB_t\right],\qquad f\in p\mathscr{E}^*.$$

It is well known ([3], Chap. IV) that $U_B^\alpha f$ is α-excessive on E if $f\in p\mathscr{E}^*$. The kernels V^α and U_B^α are related to each other by

$$U_B^\alpha V^\alpha=U^\alpha-V^\alpha,\qquad \alpha>0. \tag{4.9}$$

To prove (4.9) start with the formula

$$\begin{aligned}&\int_0^\infty e^{-\alpha t}\,dB_t\int_t^\infty e^{-\alpha(s-t)}f(X_s)\,M_{s-t}\circ\theta_t\,ds\\&\quad=\int_0^\infty e^{-\alpha s}f(X_s)\int_0^s M_s M_t^{-1}\,dB_t,\end{aligned} \tag{4.10}$$

valid a.s. since M is perfect, for all $f\in b\,p\,\mathscr{E}^*$. Using (4.8),

$$\begin{aligned}\int_0^s M_s M_t^{-1}\,dB_t&=\int_0^s M_s M_t^{-1}\,M_{t-}^{-1}(-dM_t)\\&=M_s\int_0^s d(M_t^{-1})=M_s(M_s^{-1}-1)=1-M_s.\end{aligned}$$

We obtain thus from (4.10)

$$\begin{aligned}&E^x\left[\int_0^\infty e^{-\alpha t}\,dB_t\int_t^\infty e^{-\alpha(t-s)}f(X_s)\,M_{s-t}\circ\theta_t\,ds\right]\\&=U^\alpha f(x)-V^\alpha f(x).\end{aligned} \tag{4.11}$$

However, the process

$$\int_t^\infty e^{-\alpha(s-t)}f(X_s)\,M_{s-t}\circ\theta_t\,ds=\left(\int_0^\infty e^{-\alpha u}f(X_u)\,M_u\,du\right)\circ\theta_t$$

admits the following optional projection for all measures P^x:

$$E^{X_t}\left[\int_0^\infty e^{-\alpha u}f(X_u)\,M_u\,du\right]=V^\alpha f(X_t),$$

by the strong Markov property. Using this in (4.11) proves (4.9).

Now let $\phi=V^1 1$. Then $0<\phi\leqq 1$ and by (4.9), $\phi=U^1 1-V^1 1$ is a difference of two bounded 1-excessive functions. In particular, ϕ is finely continuous and

$\mathscr{E}^e$-measurable. Let $G_n=\{x\in E\colon \phi(x)>1/n\}$. Each E_n is a finely open set in $\mathscr{E}^e$, and $E_n\uparrow E$. If $T_n=T_{E_n^c}$, since E_n^c is finely closed, $\phi(X_{T_n})\leqq\frac{1}{n}$ a.s. on $\{T_n<\infty\}$. It follows by definition of ϕ that for all $x\in E$

$$E^x\left[\int_{T_n}^{\infty} e^{-t}M_t\,dt\,\middle|\,\mathscr{F}_{T_n}\right]\leqq\frac{1}{n}e^{-T_n}M_{T_n}.$$

If $T=\lim T_n$, since $\int_{T_n}^{\infty} e^{-t}M_t\,dt\to\int_T^{\infty} e^{-t}M_t\,dt$ boundedly, taking expectations in the above inequality yields

$$E^x\left[\int_T^{\infty} e^{-t}M_t\,dt\right]=0.$$

Since $e^{-t}M_t>0$ a.s. for all $t\geqq 0$, we deduce that $T=\infty$ a.s.

Since $\Delta B\leqq c<1$, one has

$$E^x\Big[\int_{(0,T_n]} e^{-t}dB_t\Big]\leqq E^x\Big[\int_{(0,T_n)} e^{-t}dB_t\Big]+1$$

and since $n\,\phi(X_t)>1$ for all $t<T_n$,

$$\begin{aligned}E^x\Big[\int_{(0,T_n]} e^{-t}dB_t\Big]&\leqq 1+E^x\Big[\int_{(0,T_n)} e^{-t}n\,\phi(X_t)\,dB_t\Big]\\&\leqq 1+n\,U_B^1\,\phi(x)\leqq 1+n\end{aligned}$$

because $U_B^1\,\phi\leqq U^1\,1\leqq 1$ using (4.9).

We now apply (4.7) to obtain a more precise formulation of (3.41). This is the key step in the proof of (4.6).

(4.12) **Lemma.** *Let $Y\in\mathscr{S}_{\mathrm{ad}}$ be such that the jump process ΔY is bounded by a constant c. There exists $M\in\mathscr{L}_{\mathrm{ad}}$ and $A\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}$ such that $Y=M+A$, and there exists an increasing sequence (E_n) of finely open sets in $\mathscr{E}^e$ such that $\bigcup E_n=E$ and such that if T_n is the hitting time of E_n^c, then $\lim\limits_{(n)}\uparrow T_n=\infty$* a.s., *and for all n,*

$$\sup_{x\in E}E^x\Big[\int_{(0,T_n]} e^{-t}(d[Y,Y]_t+d[M,M]_t+|dA_t|)\Big]<\infty. \tag{4.13}$$

Proof. Theorem (3.18) implies the existence of $M\in\mathscr{L}_{\mathrm{ad}}$ and $A\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}$ such that $Y=M+A$, and the existence of the quadratic variation processes $[Y,Y]$ and $[M,M]$ which are in $\mathscr{V}_{\mathrm{ad}}^+$. It is well known that the total variation process $A'_t=\int_0^t|dA_s|$ is in $\mathscr{V}_{\mathrm{ad}}^+$. Since $|\Delta Y|\leqq c$, we have $\Delta[Y,Y]\leqq c^2$ and (3.41) implies $|\Delta M|\leqq 2c$ and $|\Delta A|\leqq c$, so $\Delta[M,M]\leqq 4c^2$ and $\Delta A'\leqq c$. Hence the final part of the lemma follows from (4.7) applied to $B=[Y,Y]+[M,M]+A'$.

Proof of (4.6). (i) Suppose first that f is a bounded semimartingale function. Let $Y=f(X)-f(X_0)$, so that $Y\in\mathscr{S}_{\mathrm{ad}}$ has uniformly bounded jumps. Let (E_n) be the sequence of $\mathscr{E}^e$-measurable finely open sets constructed in (4.12). Let $\overline{Y}_t$

$=\int_0^t e^{-s}\,dY_s$, where the stochastic integral has been constructed independently of the measure P^x, as in (3.12, vi). Using the formula for integration by parts,

$$e^{-t}f(X_t)=f(X_0)+\bar{Y}_t-\int_0^t e^{-u}f(X_u)\,du. \tag{4.14}$$

By (4.12), we have $Y=M+A$ with $M\in\mathscr{L}_{\mathrm{ad}}$ and $A\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}$. Set $\bar{M}_t=\int_0^t e^{-s}\,dM_s$ and $\bar{A}_t=\int_0^t e^{-s}\,dA_s$, so that $\bar{Y}=\bar{M}+\bar{A}$. We rewrite (4.14) as

$$f(X_0)=e^{-t}f(X_t)-\bar{M}_t-\bar{A}_t+\int_0^t e^{-s}f(X_s)\,ds \tag{4.15}$$

Take $t=T_n$, the hitting time of E_n^c, and take expectations in (4.15), to obtain

$$\begin{aligned} f(x)=&E^x[e^{-T_n}f(X_{T_n})]-E^x[\bar{M}_{T_n}]-E^x[\bar{A}_{T_n}]\\ &+E^x[\int_{(0,T_n]} e^{-s}f(X_s)\,ds]; \end{aligned} \tag{4.16}$$

since by (4.13) the process $(\bar{M}_{t\wedge T_n})_{t\geqq 0}$ is a square-integrable martingale for every P^x, (4.16) is valid even if T_n takes infinite values, and $E^x[\bar{M}_{T_n}]=0$. Write $A=A^+-A^-$ where A^+ and A^- are the positive and negative variation of A. As we pointed out in the proof of (4.12), the total variation A' of A is an additive functional and so therefore are $A^+=(A'+A)/2$ and $A^-=(A'-A)/2$.

From (4.16) we may write

$$\begin{aligned} f(x)=&E^x[e^{-T_n}f(X_{T_n})]-E^x[\int_{(0,T_n]} e^{-s}\,dA_s^+]\\ &+E^x[\int_{(0,T_n]} e^{-s}\,dA_s^-]+E^x[\int_{(0,T_n]} e^{-s}f(X_s)\,ds]. \end{aligned} \tag{4.17}$$

It follows from (4.5) that if G_n is the complement of the set of regular points for E_n^c, then $x\to 1_{G_n}(x)E^x[e^{-T_n}f(X_{T_n})]$ is a difference of two bounded functions, each of which is 1-($\mathbf{X}$, T_n)-excessive. The remaining part of the right-hand side of (4.17) is a sum of bounded terms of the form $h(x)=E^x[\int_{(0,T_n]} e^{-s}\,dB_s]$ where $B\in\mathscr{V}_{\mathrm{ad}}^+$. Each such expression defines a bounded 1-($\mathbf{X}$, T_n)-excessive function, for

$$\begin{aligned} e^{-t}E^x[h(X_t)1_{\{t<T_n\}}]&=e^{-t}E^x[E^{X_t}[\int_{(0,T_n]} e^{-s}\,dB_s]1_{\{t<T_n\}}]\\ &=e^{-t}E^x[1_{\{t<T_n\}}\int_{(0,T_n\circ\theta_t)} e^{-s}\,dB_s\circ\theta_t]\\ &=E^x[\int_{(t,t+T_n\circ\theta_t]} e^{-u}\,dB_u\,1_{\{t<T_n\}}]\\ &=E^x[\int_{(t,T_n]} e^{-u}\,dB_u\,1_{\{t<T_n\}}], \end{aligned}$$

which obviously increases to $h(x)$ as $t\downarrow 0$. Thus there exist bounded 1-$(\mathbf{X}, T_n)$-excessive functions g_n, h_n such that $f 1_{G_n}=g_n-h_n$. This proves the necessary condition in (4.6) for a bounded f. We remark that it is tempting to let $n\to\infty$ in (4.17): each term on the right-hand side of (4.17) converges, the last term converges to $E^x\left[\int_0^\infty e^{-s} f(X_s)\,ds\right]$, which is a difference of 1-excessive functions for $\mathbf{X}$, however, it may happen that $E^x\left[\int_0^\infty e^{-s}\,dA_s\right]$ does not have meaning.

(ii) We turn now to the general case where f is a not necessarily bounded semimartingale function. As we pointed out at the beginning of this section, f is finely continuous and $\mathscr{E}^e$-measurable. Let $K_n=\{x\colon |f(x)|\geqq n\}$ so that $K_n\in\mathscr{E}^e$ and K_n is finely closed. Let $H_n=K_n^r$, the set of regular points for K_n. Then H_n is finely closed, $H_n\subset K_n$, and $H_n\in\mathscr{E}^e$ ([3], II-(2.13)). Let S_n denote the hitting time of K_n. Then, since $|f(X_{S_n})|\geqq n$ a.s. on $\{S_n<\infty\}$, we have $\lim_{(n)}\uparrow S_n=\infty$ a.s.

Recalling the remarks on killed processes at the beginning of this section, each killed process $(\mathbf{X}, S_n)$ is a right process with state space $E\setminus H_n$. We denote by X_t^n the value at time t of the process X killed at S_n. Fine continuity of f implies that $\{|f|<n\}\subset E\setminus H_n\subset\{|f|\leqq n\}$ so the restriction of f to $E\setminus H_n$ is bounded by n. With the convention $f(\Delta)=0$, one has $f(X_t^n)=f(X_t)\,1_{[0,S_n)}(t)$ and so $f(X^n)$ is a semimartingale over $(\Omega,\mathscr{F},\mathscr{F}_t,P^x)$ for every $x\in E\setminus H_n$. That is, the restriction of f to $E\setminus H_n$ is a bounded semimartingale function of $(\mathbf{X}, S_n)$. Applying (i) we obtain an increasing sequence $(E_n^m)_{m\geqq 1}$ of subsets of $E\setminus H_n$ having the properties

(4.18) $$E_n^m\uparrow E\setminus H_n \quad \text{as } n\to\infty;$$

(4.19) E_n^m is finely open relative to $(\mathbf{X}, S_n)$ and E_n^m belongs to the σ-field on $E\setminus H_n$ generated by the α-excessive functions of $(\mathbf{X}, S_n)$;

(4.20) if S_n^m is the first exit time from $E_n^m\cup\{\Delta\}$ by X^n, then $\lim_{m}\uparrow S_n^m=\infty P^x$-a.s. for every $x\in E\setminus H_n$.

As we noted in (4.4) and the subsequent remark, (4.19) implies that $E_n^m\in\mathscr{E}^e$ and E_n^m is finely open for $\mathbf{X}$. The definition of S_n^m in (4.20) is equivalent to

$$S_n^m=\inf(t>0\colon\ t<S_n,\ X_t\notin E_n^m).$$

The process $(\mathbf{X}, S_n^m)$ may therefore be identified with $(\mathbf{X}, S_n^m\wedge S_n)$. Moreover,

$$S_n^m\wedge S_n\leqq R_n^m=\inf(t>0\colon X_t\notin E_n^m),$$

and so for all $x\in E\setminus H_n$, P^x-a.s. one has

$$\sup_{(m)} R_n^m\geqq\sup_{(m)} S_n^m\wedge S_n=S_n.$$

Therefore $\sup_{(n)}\sup_{(m)} R_n^m=\infty P^x$-a.s. for every $x\in E$. Letting (E_n) be an enumeration of the sets E_n^m, the necessary condition stated in (4.6) has been proved. Finally, the sufficient condition has been proved right after the statement of the theorem.

(4.21) *Remark.* It should be pointed out that the results of this section do not take account of the lifetime of the underlying process **X**. That is, the death point, if there is one, has been treated as an ordinary point of the state space. The reason for this is that the usual theory of semimartingales does not take account of the presence of a distinguished lifetime. It is undoubtedly the case that such a theory can be developed without too much difficulty, and it may be quite useful in discussing processes such as Brownian motion in the unit disc. There is a theory of local martingales with a distinguished lifetime, see [15] and [28]. In this framework, the result in (4.6) would change to $\sup_{(m)} T_n \geqq \zeta$ a.s., instead of $\sup_{(m)} T_n = \infty$ a.s.

5. Brownian Motion and Linear Diffusions

We shall prove in this section that every semimartingale function of a Brownian motion on the line is locally a difference of convex functions. For reference we record the following facts from real analysis. The proofs are standard.

(5.1) **Proposition.** *Let f be a real function defined on $\mathbb{R}$. Then, the following conditions on f are equivalent:*

(i) *for every compact interval I there exists a pair g, h of convex functions on I such that $f_{|I} = g - h$ (where $f_{|I}$ denotes the restriction of f to I);*

(ii) *the weak second derivative of f may be identified on each compact interval as a signed measure;*

(iii) *the function f has a right-continuous right-hand derivative which has finite variation on every compact interval.*

Instead of using (4.6) we shall return to (3.18) and base the arguments on some well known facts about linear Brownian motion. Let **X** denote the linear Brownian motion, so $E = \mathbb{R}$, and assume $\mathscr{M}_t = \mathscr{F}_t$. Let $L^x = (L^x_t)_{t \geqq 0}$ be a local time at x for **X**, normalized so that $x \to L^x_t$ is a density for the occupation time relative to the Lebesgue measure. One may select $(L^x)_{x \in E}$ so that for all ω, $(t, x) \to L^x_t(\omega)$ is jointly continuous (see [3], V-(3.30)) and the normalization amounts to the fact that for every $\Lambda \in \mathbb{R}$

$$\int_0^t 1_\Lambda(X_s(\omega))\, ds = \int_\Lambda L^x_t(\omega)\, dx. \tag{5.2}$$

The following closely related results are well known. See for example [50].

(5.3) **Proposition.** *If $A \in \mathscr{V}^+_{\mathrm{ad}}$, then A is continuous and there exists a Radon measure ν on $\mathbb{R}$ such that $A_t = \int L^x_t \, \nu(dx)$* a.s.

(5.4) **Proposition.** *If f is a convex function on $\mathbb{R}$ with weak second derivative identified with the Radon measure ν and righthand derivative f', then*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \tfrac{1}{2} \int_0^t L^x_t \, \nu(dx).$$

The result (5.4) is a routine consequence of the formula of Tanaka, which corresponds to the special case $f(x)=|x|$, see for example [34]. Here is our main result.

(5.5) **Theorem.** *Let f be a semimartingale function for linear Brownian motion* $\mathbf{X}$. *Then f is locally a difference of bounded convex functions.*

Proof. According to (3.18), there exists $A\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}$ such that $f(X)-f(X_0)-A\in\mathscr{L}_{\mathrm{ad}}$. The positive and negative variations A^+ and A^- of A are in $\mathscr{V}_{\mathrm{ad}}^+$, so by (5.3) there exist Radon measures ν^+, ν^- such that $A_t^+=\int L_t^x\,\nu^+(dx)$ and $A_t^-=\int L_t^x\,\nu^-(dx)$. Let g be the function on $\mathbb{R}$, determined up to an additive affine function, such that for every compact interval I the weak second derivative of g may be identified on I with $(\nu^+-\nu^-)_{|I}$. It follows from (5.4) that $g(X)-g(X_0)-A\in\mathscr{L}_{\mathrm{ad}}$. Therefore, if $h=f-g$, then $h(X)$ is a P^x-local martingale for every $x\in E$.

Outside a null set, we have the following: the process $h(X)$ is right-continuous, and each point $x\in\mathbb{R}$ is reached by the process X from above at some finite time and from below at some other finite time. It follows that h is a continuous function. If $a>0$, set $T_a=\inf\{t\colon |X_t-X_0|>a\}$. Since h is bounded over $[x-a,x+a]$, the process $(h(X_{t\wedge T_a}))_{t\geqq 0}$ is a bounded P^x-martingale, and thus $h(x)=E^x[h(X_{T_a})]$, which obviously equals $(h(x+a)+h(x-a))/2$. These relationships, valid for all $x\in\mathbb{R}$, $a>0$, and the continuity of h, imply that h is an affine function. Recalling (5.1) and the definition of g, the theorem is proven.

The recurrence of linear Brownian motion allows one to strengthen (5.5) in appearance.

(5.6) **Theorem.** *Suppose that there exists an initial law μ for the linear Brownian motion* $\mathbf{X}$ *such that $f(X)$ is a P^μ-semimartingale. Then $f(X)$ is a P^x-semimartingale for every $x\in\mathbb{R}$ and so, by* (5.5), *f is locally a difference of bounded convex functions.*

Proof. Let $x\in\mathbb{R}$ and $T=\inf\{t>0\colon X_t=x\}$. Then $P^\mu[T<\infty]=1$. Since $f(X)$ is a semimartingale over $(\Omega,\mathscr{F},\mathscr{F}_t,P^\mu)$, the process $Y_t=f(X_{T+t})$ is a semimartingale over $(\Omega,\mathscr{F},\mathscr{K}_t,P^\mu)$, where $\mathscr{K}_t=\mathscr{F}_{T+t}$: this is not hard to show directly, but is also a simple consequence of Kazamaki's theorem [25] to the effect that semimartingales are preserved under right-continuous time changes (namely, here: $\tau_t=T+t$). Since $Y_t\in\theta_T^{-1}(\mathscr{F}_t)$ and since $\theta_T^{-1}(\mathscr{F}_t)\subset\mathscr{K}_t$, by Stricker's theorem [45], Y is also a semimartingale over $(\Omega,\theta_T^{-1}(\mathscr{F}),\theta_T^{-1}(\mathscr{F}_t),P^\mu)$. As $X_T=x$ a.s., the strong Markov property implies that the filtered spaces $(\Omega,\mathscr{F},\mathscr{F}_t,P^x)$ and $(\Omega,\theta_T^{-1}(\mathscr{F}),\theta_T^{-1}(\mathscr{F}_t),P^\mu)$ can be identified via the mapping θ_T. Since $Y_t=f(X_t)\circ\theta_T$, a classical "change of space" theorem (see for example [21], § X-2-a) shows that $f(X)$ is a semimartingale over $(\Omega,\mathscr{F},\mathscr{F}_t,P^x)$.

(5.7) *Remark.* Examples of functions which are not semimartingale functions of the linear Brownian motion $\mathbf{X}$ were noted by Wang [49] and Yor [52], who remarked that $f(X)=|x|^\gamma$ $(0<\gamma<1)$ is not a semimartingale function. In the notes of Meyer [34] it is shown that if f is locally a difference of convex functions then for every semimartingale Y, $f(Y)$ is also a semimartingale and an expansion extending that in (5.4) was derived. The results (5.5) and (5.6) show that no more general class of functions operates on real-valued semimartingales.

There are trivial extensions of (5.5) and (5.6) to processes related to linear Brownian motion.

(5.8) **Theorem.** *A function f is a semimartingale function for the reflecting Brownian motion on $\mathbb{R}_+$ if and only if its weak second derivative may be identified on every finite interval of the form $[0,a]$ with a signed measure.*

Proof. It is sufficient to prove the result on a specific realization of the reflecting Brownian motion. We choose the one constructed in (3.3, i): $(\Omega, \mathscr{B}, \mathscr{B}_t, \theta_t, B_t, P^x)$ is a linear Brownian motion, $X_t = |B_t|$, $\mathscr{M}_t = \mathscr{B}_t$, $\mathscr{F}_t^0 = \sigma(X_s : s \leqq t)$. By (3.24, i) we have $f(X) \in \mathscr{S}(\mathscr{F}_t)$ if and only if $f(X) \in \mathscr{S}(\mathscr{M}_t)$. Since $f(X) = g(B)$ if $g(x) = f(|x|)$, the result is immediate from (5.5).

Note that, above, f must have a finite right-hand derivative at 0. The situation is slightly different in the case of a killed Brownian motion.

(5.9) **Theorem.** *Let* **X** *be a Brownian motion on $\mathbb{R}_{++} = (0, \infty)$ killed at 0. Then a function f on $\mathbb{R}_{++}$, which is defined arbitrarily at the death state Δ, is a semimartingale function for* **X** *if and only if the following conditions are satisfied:*

(i) *$f(0+)$ exists and is finite;*

(ii) *the right-hand derivative f' of f is right-continuous and of finite variation on every compact subinterval of $\mathbb{R}_{++}$;*

(iii) *the positive Radon measure v generated by the total variation of f' satisfies $\int_0^\varepsilon x\, v(dx) < \infty$ for some, and hence all, $\varepsilon > 0$.*

Proof. We shall show only the necessity of the conditions (their sufficiency easily follows from Ito's formula extended to convex functions, see [34], and from the forthcoming lemma (5.10)). Let ζ be the killing time. Since $\zeta < \infty$ a.s., (i) is clearly necessary. Since the value of f at Δ is irrelevant to the problem, we may suppose $f(\Delta) = f(0+)$.

Using (3.18) we may write $f(X) - f(X_0) = M + A$ where $M \in \mathscr{L}_{\text{ad}}$, $A \in \mathscr{V}_{\text{ad}}$. Since $f(X)$ is continuous at time ζ and constant on $[\zeta, \infty)$, it follows that A is the difference of two additive increasing functionals A^1, A^2 of **X** which do not charge $[\zeta, \infty)$. By a result of [3], VI-(4.21), and just as in the proof of (5.5), there exist positive Radon measures v^1, v^2 on $\mathbb{R}_{++}$ such that for $j = 1, 2$, $A_t^j = \int L_t^x v^j(dx)$ where L_t^x is local time for **X** at x. The necessity of condition (ii) is then proved just as in (3.5). Since $\zeta < \infty$ a.s., A_ζ^j must be finite a.s. for $j = 1, 2$. The condition (iii) will then follow from the following lemma.

(5.10) **Lemma.** *Let* **X** *be the linear Brownian motion. Let v be a positive Radon measure on $\mathbb{R}_{++}$. Let L_t^x be jointly continuous local times for* **X** *normalized as in* (5.2). *If T denotes the hitting time of $\{0\}$ then $\int L_{t \wedge T}^x v(dx) < \infty P^x$*-a.s. *for all $x > 0$ if and only if $\int_0^\varepsilon x\, v(dx) < \infty$ for some $\varepsilon > 0$,*

Proof. Because $P^x[T < \infty] = 1$ for all $x > 0$, $\int L_{t \wedge T}^x v(dx) < \infty$ a.s. if and only if $\int_0^\varepsilon L_{t \wedge T}^x v(dx) < \infty$ a.s. for some $\varepsilon > 0$. We may in particular assume that v is carried by $(0,1)$. Then $A_t = \int L^x_{t\wedge T}\, \nu(dx)$ defines a continuous increasing additive functional of $(\mathbf{X}, T)$ which is finite for all $t<T$. Applying (4.7), one obtains an increasing sequence (E_n) of finely open subsets of $\mathbb{R}_{++}\cup\{\Delta\}$ such that if T_n is the exit time from E_n, $\lim_n \uparrow T_n = \infty$ a.s. and $E^x[\int_{[0,T_n]} e^{-t} dA_t]$ is bounded in x on $\mathbb{R}_{++}$ for each n. However, the fine topology for Brownian motion is the same as the Euclidian topology, and the condition $\lim_n \uparrow T_n = \infty$ a.s. implies that for some n, E_n includes an interval of the form $(0,\varepsilon)$. If $S=\inf\{t\colon X_t \notin (0,\varepsilon)\}$, $E^x\left[\int_0^S e^{-t} dA_t\right]$ is bounded. If ε is chosen sufficiently small, $E^x[e^{2S}]$ is bounded in x: this is a simple consequence of the known distribution of S under P^x, or of estimates on the BMO-martingale $X_{t\wedge S}$ whose quadratic variation process is $t\wedge S$, [34] p. 348. For $0<x<\varepsilon$, the Cauchy-Schwarz inequality gives

$$E^x[A_S] = E^x[e^S e^{-S} A_S] \leqq E^x\left[e^S \int_0^S e^{-t} dA_t\right]$$
$$\leqq \left\{E^x[e^{2S}]\, E^x\left[\left(\int_0^S e^{-t} dA_t\right)^2\right]\right\}^{\frac{1}{2}}.$$

By Meyer's energy inequality, [30], $E^x\left[\left(\int_0^S e^{-t} dA_t\right)^2\right]$ is bounded by $2c^2$ where $c = \sup_{0<x<\varepsilon} E^x\left[\int_0^S e^{-t} dA_t\right] < \infty$. Since $E^x[A_S]=0$ for $x\geqq\varepsilon$, and $x\leqq 0$, we obtain $\sup_{(x)} E^x[A_S]<\infty$. But $E^x[A_S] = E^x[\int L^y_S\, \nu(dy)] = \int E^x[L^y_S]\, \nu(dy)$, and since $E^x[L^y_S]$ is equal to the Green's kernel for the interval $[0,\varepsilon)$ which, as a function of y, is proportional to y near 0, the result follows.

It should be noted that the conditions (5.9, i) and (5.9, iii) are not needed if the notion of semimartingale functions is changed according to (4.27). In this instance, we could say that $f(X)$ is a P^x-semimartingale on $[0,\zeta)$ if and only if, letting $T_n = \inf\left\{t\colon X_t \leqq \frac{1}{n}\right\}$, $f(X_t)\, 1_{[0,T_n]}(t)$ is a P^x-semimartingale for all n. In this sense, for example, $f(x)=x^\gamma$ is a semimartingale function (localized to $[0,\zeta)$) for all real γ, but $f(x)$ is a semimartingale function for $\mathbf{X}$ in the usual sense if and only if $\gamma\geqq 0$.

The theorems above permit us to describe those regular diffusions on an interval $J\subset\mathbb{R}_+$ which are semimartingales (see Sect. 7 for another point of view on this matter). To avoid going into a tedious analysis of boundary conditions, we shall confine our attention to one specific case. Suppose $\mathbf{X}$ is a conservative regular diffusion on an interval J and that $\mathbf{X}$ has scale function s and speed measure m. Consult [20] for the properties of diffusions used below. Let $\mathbf{Y}$ denote Brownian motion on the interval $s(J)$ with reflection at finite endpoints of $s(J)$, and let A be the additive functional of $\mathbf{Y}$ given by $A_t = \int L^x_t\, \tilde{m}(dx)$, where L^x_t are jointly continuous local times for $\mathbf{Y}$ and $\tilde{m}$ is the image of m under s. If $\tau_t = \inf\{s\colon A_s > t\}$, then $s^{-1}(Y_{\tau_t})$ is a model of $\mathbf{X}$ (i.e. a Markov process with the same transition semigroup). Thus, determining whether X is a semimartingale relative to every P^x is the same as determining whether s^{-1} is a semimartingale function of (Y_{τ_t}).

Case 1. $s(J)=\mathbb{R}$. Here $\mathbf{Y}$ is a Brownian motion on $\mathbb{R}$ so $A_\infty=\infty$ a.s. and so $\tau_t<\infty$ a.s. for all $t\geqq 0$. Kazamaki's Theorem [25] on changes of time shows that s^{-1} is a semimartingale function of (Y_{τ_t}) if and only if s^{-1} is a semimartingale function of $\mathbf{Y}$. By (5.5), $\mathbf{X}$ is a semimartingale relative to every P^x if and only if s^{-1} is locally a difference of convex functions.

The other cases are analyzed in a similar manner. We shall not discuss the details, but in brief, on every subinterval strictly interior to $s(J)$, s^{-1} must be equal to a difference of convex functions, while at finite endpoints of $s(J)$, s^{-1} must satisfy the condition corresponding to (5.9, i) and (5.9, iii) if the corresponding endpoint of J is a reflection point for $\mathbf{X}$. If the endpoint of J is not an exit point for $\mathbf{X}$, no endpoint condition on s^{-1} need be imposed.

Finally, we mention that if $\mathbf{X}$ is a Brownian motion on an open subset of $\mathbb{R}^m$ $(m\geqq 1)$, Theorem (4.6) shows that every semimartingale function is locally a difference of 1-excessive functions, each of which may be represented locally as the sum of a finely harmonic function and the 1-potential of a measure. This shows that one may not expect to have generalizations of the formulas of Ito and Tanaka beyond those given by Brosamler [4]; see also Meyer [35]. It would be of interest to delineate the class of functions on $\mathbb{R}^m$ such that for every m-dimensional semimartingale Y, the process $f(Y)$ is also a semimartingale: such a function f is obviously of the form described above; using (5.5) one also sees easily that for every C^2-map $\varphi\colon \mathbb{R}_+\to\mathbb{R}^m$, $f\circ\varphi$ is locally a difference of convex functions.

6. Additive Random Measures and Semimartingales

Throughout this section, all the assumptions, conventions and notations of Sect. 3 are in force. In addition, we consider an auxiliary topological space G and let $\mathscr{G}$ be its Borel σ-field. We denote by $\mathcal{O}(\mathscr{H}_t)$ and $\mathscr{P}(\mathscr{H}_t)$ the σ-fields of all $(\mathscr{H}_t)$-optional and $(\mathscr{H}_t)$-predictable subsets of $\Omega\times\mathbb{R}_+$ respectively, and let $\hat{\mathscr{P}}$ denote the product σ-field $\mathscr{P}(\mathscr{H}_t)\times\mathscr{G}$ on the space $\hat{\Omega}=\Omega\times\mathbb{R}_+\times G$.

6a) Random Measures

We start with the following convention supplementing (3.6).

(6.1) **Convention.** *Concerning G we assume that*

(i) *either G is a Borel subset of a compact metric space, i.e., a Lusin space,*

(ii) *or G is a universally measurable subset of a compact metric space (a U-space in the terminology of* [12]), *and in this case we suppose that* (3.6, i) *holds.*

We state the following well-known lemma without proof, see [12]. Here $(A,\mathscr{A})$ is a measurable space and M is an arbitrary family of positive σ-finite measures on it.

(6.2) **Lemma.** *Let $\{f(B)\colon B\in\mathscr{G}\}$ be a family of functions in $p\mathscr{A}$ such that $f(\bigcup B_n) = \sum f(B_n)$ m-a.e. for every $m\in M$ and every sequence (B_n) of pairwise disjoint sets in $\mathscr{G}$. Suppose (6.1, i) (resp. (6.1, ii)) holds. Then, there exists a positive kernel $K(a, dy)$ from $(A, \mathscr{A})$ (resp. from $(A, \mathscr{A}^*)$) into $(G, \mathscr{G})$ such that $K(\cdot, B) = f(B)$ m-a.e. for all $m\in M$ and $B\in\mathscr{G}$.*

Let $\Gamma(\omega; dt, dy)$ be a positive transition kernel from $(\Omega, \mathscr{H})$ into $(\mathbb{R}_+ \times G, \mathscr{R}_+ \otimes \mathscr{G})$, and for every measurable function W on $\hat{\Omega}$ set

(6.3) $$W*\Gamma_t(\omega) = \int W(\omega, s, y)\, 1_{[0,t]}(s)\, \Gamma(\omega; ds, dy)$$

whenever this integral makes sense, for instance when $W\geqq 0$. We will call Γ a *random measure* if the process $W*\Gamma$ defined by (6.3) is $(\mathscr{H}_t)$-optional for every positive $\mathcal{O}(\mathscr{H}_t)\otimes\mathscr{G}$ measurable function W on $\hat{\Omega}$. Among random measures, we distinguish the following classes:

$\hat{\mathscr{A}}_\sigma$ $= \{\Gamma\colon \Gamma$ is a random measure; there exists a $\hat{\mathscr{P}}$-measurable partition (D_n) of $\hat{\Omega}$ such that $1_{D_n}*\Gamma\in\mathscr{A}_{\text{loc}}$ for every $n\}$;

$\hat{\mathscr{P}}\cap\hat{\mathscr{A}}_\sigma = \{\Gamma\in\hat{\mathscr{A}}_\sigma\colon W*\Gamma$ is P^x-indistinguishable from a $(\mathscr{H}_t)$-predictable process, for every $W\in p\hat{\mathscr{P}}$, $x\in E\}$.

The subscript "σ" stands for σ-finite. Note the resemblance between the definition of $\hat{\mathscr{P}}\cap\hat{\mathscr{A}}_\sigma$ and that of $\mathscr{P}\cap\mathscr{V}$ in Sect. 3.

We extend the Definition (3.14) of the "big shifts" Θ_s to functions W on $\hat{\Omega}$ and to random measures Γ by

(6.4) $$\begin{aligned} \Theta_s W(\omega, t, y) &= W(\theta_s\omega, t-s, y)\, 1_{[s,\infty)}(t), \\ \Theta_s \Gamma(\omega; dt, dy) &= \Gamma(\theta_s\omega; dt-s, dy)\, 1_{[s,\infty)}(t). \end{aligned}$$

A simple computation shows that for every positive W

(6.5) $$\Theta_s(W*\Gamma) = (\Theta_s W)*(\Theta_s \Gamma).$$

For all facts about random measures we refer to [21]. Note that, if $Y\in\mathscr{A}^+_{\text{loc}}$, the formula $\Gamma(\omega; dt, \{0\}) = dY_t(\omega)$ defines a random measure $\Gamma\in\hat{\mathscr{A}}_\sigma$ on the singleton $G=\{0\}$; in this case, (6.4) and (6.5) reduce to the corresponding properties in Sect. 3. Accordingly, the following is an extension of (3.12, iii) and (3.15, iii).

(6.6) **Theorem.** *Let $\Gamma\in\hat{\mathscr{A}}_\sigma$.*

(i) There exists a $\tilde{\Gamma}\in\hat{\mathscr{P}}\cap\hat{\mathscr{A}}_\sigma$ which is a version of the dual predictable projection of Γ for every P^x, (that is, $\tilde{\Gamma}\in\hat{\mathscr{P}}\cap\hat{\mathscr{A}}_\sigma$ and $E^x[W*\tilde{\Gamma}_\infty] = E^x[W*\Gamma_\infty]$ for all $x\in E$ and $W\in p\hat{\mathscr{P}}$).

(ii) Suppose $(\mathscr{M}_t)$ is a Markov filtration and let $s\geqq 0$. Then, $\Theta_s\Gamma\in\hat{\mathscr{A}}_\sigma$, and $\Theta_s\tilde{\Gamma}$ is a version of the dual predictable projection of $\Theta_s\Gamma$. Moreover, if $(\mathscr{M}_t)$ is a strong Markov filtration, the same property holds when s is replaced by any finite $(\mathscr{H}_t)$-stopping time S.

Proof. (i) Let $(D_n)_{n\geqq 1}$ be a $\hat{\mathscr{P}}$-measurable partition of $\hat{\Omega}$ such that $C^n = 1_{D_n}*\Gamma\in\mathscr{A}_{\text{loc}}$ for every n. As usual, $\tilde{C}^n\in\mathscr{P}\cap\mathscr{V}^+$ denotes a version of the dual predictable projection of C^n, (see (3.12)).

For every $B \in \mathscr{G}$ the process $Y(n,B)=(1_B 1_{D_n})*\Gamma$ belongs to $\mathscr{A}_{\mathrm{loc}}$; thus, we may introduce its dual predictable projection $\tilde{Y}(n,B)$. Since $dY(n,B)_t \ll dC_t^n$ by construction, we have $d\tilde{Y}(n,B)_t \ll d\tilde{C}_t^n$ a.s. Hence, there exists a $(\mathscr{H}_t)$-optional process $f(n,B)$ such that $\tilde{Y}(n,B)=f(n,B)\cdot \tilde{C}^n$.

Next we apply Lemma (6.2) with $(A,\mathscr{A})=(\Omega\times\mathbb{R}_+,\mathcal{O}(\mathscr{H}_t))$ and $M=\{P^x(d\omega)\otimes dC_t^n(\omega)\colon x\in E\}$. For every pairwise disjoint sequence (B_q) we have $Y(n,\bigcup B_q)=\sum_q Y(n,B_q)$, and hence, $\tilde{Y}(n,\bigcup B_q)=\sum_q \tilde{Y}(n,B_q)$ up to an evanescent set, which implies that $f(n,\bigcup B_q)=\sum_q f(n,B_q)\,m$-a.e. for every $m\in M$. Thus, by (6.2), there exists a positive kernel $K^n(\omega,t;dy)$ such that $K^n(\cdot,B)=f(n,B)\,m$-a.e. for all $m\in M$ and $B\in\mathscr{G}$. We set $\tilde{\Gamma}^n(\omega;dt,dy)=d\tilde{C}_t^n(\omega)\,K^n(\omega,t;dy)$.

By definition, $1_B*\tilde{\Gamma}^n=\tilde{Y}(n,B)$ up to an evanescent set for all $B\in\mathscr{G}$. Hence, for every $D\in\mathscr{P}(\mathscr{H}_t)$, $1_{D\times B}*\tilde{\Gamma}^n=1_D\cdot(1_B*\tilde{\Gamma}^n)$ belongs to $\mathscr{P}\cap\mathscr{V}^+$, and

$$E^x[1_{D\times B}*\tilde{\Gamma}^n_\infty]=E^x[1_D\cdot\tilde{Y}(n,B)_\infty]$$
$$=E^x[1_D\cdot Y(n,B)_\infty]=E^x[(1_{D\times B}1_{D_n})*\Gamma_\infty].$$

Therefore, $\tilde{\Gamma}^n\in\tilde{\mathscr{P}}\cap\tilde{\mathscr{A}}_\sigma$, and by a monotone class argument we see that $E^x[W*\tilde{\Gamma}^n_\infty]=E^x[(W1_{D_n})*\Gamma_\infty]$ for all $x\in E$, $n\geqq 1$, $W\in p\tilde{\mathscr{P}}$. There remains to set $\tilde{\Gamma}=\sum\tilde{\Gamma}^n$, and the proof of (i) is finished.

(ii) Suppose $(\mathscr{M}_t)$ is a Markov filtration, and let $s\geqq 0$. We set $D_0'=\Omega\times[0,s)\times G$, and $D_n'=\{(\omega,t,y)\colon \Theta_s 1_{D_n}(\omega,t,y)=1\}$ for $n\geqq 1$. Then, $(D_n')_{n\geqq 0}$ is a $\tilde{\mathscr{P}}$-measurable partition of $\tilde{\Omega}$. We have $1_{D_0'}*(\Theta_s\Gamma)=0$, while for $n\geqq 1$, (6.5) implies that $1_{D_n'}*(\Theta_s\Gamma)=\Theta_s(1_{D_n}*\Gamma)$, which belongs to $\mathscr{A}_{\mathrm{loc}}$ because of (3.15, i). Thus $\Theta_s\Gamma\in\tilde{\mathscr{A}}_\sigma$, and we denote by $\widetilde{\Theta_s\Gamma}$ its dual predictable projection. Similarly, we have $\Theta_s\tilde{\Gamma}\in\tilde{\mathscr{P}}\cap\tilde{\mathscr{A}}_\sigma$. For all $n\geqq 1$, $B\in\mathscr{G}$, the process $(1_B 1_{D_n'})*\widetilde{\Theta_s\Gamma}$ which equals $\Theta_s[(1_B 1_{D_n})*\Gamma]$ because of (6.5). Then, (3.15, iii) implies that $(1_B 1_{D_n'})*\widetilde{\Theta_s\Gamma}=\Theta_s[(1_B 1_{D_n})*\tilde{\Gamma}]$, which equals $(1_B 1_{D_n'})*\Theta_s\tilde{\Gamma}$ because of (6.5) again. It is obvious that $(1_B 1_{D_0'})*\widetilde{\Theta_s\Gamma}=(1_B 1_{D_0'})*\Theta_s\tilde{\Gamma}=0$. So far, we have obtained: $(1_B 1_{D_n'})*\widetilde{\Theta_s\Gamma}=(1_B 1_{D_n'})*\Theta_s\tilde{\Gamma}$ for all $n\geqq 0$, $B\in\mathscr{G}$, from which we deduce that $\widetilde{\Theta_s\Gamma}=\Theta_s\tilde{\Gamma}$.

Finally, when $(\mathscr{M}_t)$ is a strong Markov filtration, the final assertion is proved exactly like the final assertion of (3.15).

The random measure Γ is called *additive* (resp. *strongly additive*) if

(6.7) (i) $\Gamma(\cdot,\{0\}\times G)=0$ a.s.

(ii) for all $s\geqq 0$, $(\Theta_s\Gamma)(\cdot,dt,dy)=\Gamma(\cdot,dt,dy)\,1_{[s,\infty)}(t)$ a.s. (resp. for all finite $(\mathscr{M}_t)$-stopping times S, $(\Theta_S\Gamma)(\cdot,dt,dy)=\Gamma(\cdot,dt,dy)\,1_{[S,\infty)}(t)$ a.s.).

(6.8) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a Markov (resp. strong Markov) filtration. Let $\Gamma\in\tilde{\mathscr{A}}_\sigma$ be additive (resp. strongly additive). Then there exists an additive (resp. strongly additive) $\tilde{\Gamma}\in\tilde{\mathscr{P}}\cap\tilde{\mathscr{A}}_\sigma$ which is a version of the P^x-dual predictable projection of Γ for every $x\in E$.*

Proof. It is immediate from (6.6) and (6.7), and by using the same argument as in the final part of the proof of (3.18) to obtain the strong additivity of $\tilde{\Gamma}$.

(6.9) *Remark.* Suppose $\mathscr{M}_t^0 = \mathscr{F}_t^0$, or more generally suppose that the function $s \to Z \circ \theta_s(\omega)$ on $\mathbb{R}_+$ is Borel for all $\omega \in \Omega$, $Z \in b\mathscr{M}^0$. Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. Then any additive random measures Γ satisfying the following condition is strongly additive:

there exists a $\mathcal{O}(H_t) \otimes \mathscr{G}$-measurable partition (D_n) of $\tilde{\Omega}$ such that $1_{D_n} * \Gamma \in \mathscr{V}_{\mathrm{ad}}$ for every n.

This follows easily form (3.27, ii): for each $B \in \mathscr{G}$, the process $1_{D_n \cap (\Omega \times \mathbb{R}_+ \times B)} * \Gamma$ is additive, so is strongly additive. The desired conclusion thus immediately follows from the separability of the σ-field $\mathscr{R}_+ \otimes \mathscr{G}$.

6b) *Integer-Valued Random Measures and Stochastic Integrals*

Except for the next definition, the content of this subsection will not be used in the sequel. A random measure Γ is called *integer-valued* if it has the form

(6.10) $$\Gamma(\omega; dt, dy) = \sum_{s \geq 0} 1_A(\omega, s)\, \varepsilon_{(s, Z_s(\omega))}(dt, dy),$$

where $A \in \mathcal{O}(\mathscr{H}_t)$, where Z is a G-valued $(\mathscr{H}_t)$-optional process, and where ε_a denotes the Dirac measure at point a.

Let $\Gamma \in \tilde{\mathscr{A}}_\sigma$ be an integer-valued random measure. Let $\tilde{\Gamma} \in \tilde{\mathscr{P}} \cap \tilde{\mathscr{A}}_\sigma$ be a version of its dual predictable projection. We will recall some facts about stochastic integrals with respect to $\Gamma - \tilde{\Gamma}$, see [21]. If $W \in \tilde{\mathscr{P}}$, we put

(6.11) $$\tilde{W}_t(\omega) = \int_G W(\omega, t, y)\, \Gamma(\omega; \{t\}, dy) - \int_G W(\omega, t, y)\, \tilde{\Gamma}(\omega; \{t\}, dy)$$

where this expression makes sense, and $\tilde{W}_t(\omega) = \infty$ where it does not. We denote by $G(\Gamma, P^x)$ the set of all $W \in \tilde{\mathscr{P}}$ such that the increasing process $[\sum_{s \leq \cdot} (\tilde{W}_s)^2]^{1/2}$ is P^x-locally integrable. Let $W \in G(\Gamma, P^x)$; then, there exists a P^x-local martingale N, unique up to a P^x-evanescent set, which is a compensated sum of jumps satisfying $N_0 = 0$ and

(6.12) $$\Delta N = \tilde{W} \quad \text{up to a } P^x\text{-evanescent set.}$$

This local martingale N is called the stochastic integral of W with respect to $\Gamma - \tilde{\Gamma}$. Note that $N = W * \Gamma - W * \tilde{\Gamma}$ whenever $W \in G(\Gamma, P^x)$ is such that $W * \Gamma$ admits a P^x-locally integrable variation.

(6.13) **Proposition.** *Let $\Gamma \in \tilde{\mathscr{A}}_\sigma$ be integer-valued. Let $\tilde{\Gamma} \in \tilde{\mathscr{P}} \cap \tilde{\mathscr{A}}_\sigma$ be its dual predictable projection. Let $W \in \bigcap_{x \in E} G(\Gamma, P^x)$.*

(i) *There exists a $W * (\Gamma - \tilde{\Gamma}) \in \mathscr{L}$ which is a version of the P^x-stochastic integral of W with respect to $\Gamma - \tilde{\Gamma}$ for every $x \in E$.*

(ii) *Suppose $(\mathscr{M}_t)$ is a Markov filtration, and let $s \geq 0$. We have $\Theta_s W \in \bigcap_{x \in E} G(\Theta_s \Gamma, P^x)$ and $(\Theta_s W) * (\Theta_s \Gamma - \Theta_s \tilde{\Gamma}) = \Theta_s(W * (\Gamma - \tilde{\Gamma}))$.*

(iii) *Suppose* $(\mathscr{M}_t)$ *is a strong Markov filtration. Then property* (ii) *holds if* s *is replaced by any finite* $(\mathscr{H}_t)$*-stopping time* S.

(iv) *Suppose* $(\mathscr{M}_t)$ *is a Markov (resp. strong Markov) filtration. If* Γ *is additive (resp. strongly additive), and if* $W1_{[s,\infty)} = \Theta_s W$ *for all* $s \geqq 0$, *then the process* $W * (\Gamma - \tilde{\Gamma})$ *is additive (resp. strongly additive).*

Proof. (i) Let N^x be a version of the P^x-stochastic integral of W with respect to $\Gamma - \tilde{\Gamma}$. Since $[\sum_{s \leqq \cdot} (\tilde{W}_s)^2]^{1/2} \in \mathscr{A}_{\text{loc}}$, the process

$$A(n)_t = \sum_{0 < s \leqq t} \tilde{W}_s 1_{\left\{|\tilde{W}_s| > \frac{1}{n}\right\}}$$

is in $\mathscr{A}_{\text{loc}}$, and we denote by $\tilde{A}(n)$ its dual predictable projection, as constructed in (3.12, iii). We know that $P^x\text{-}\lim_n (A(n)_t - \tilde{A}(n)_t) = N_t^x$ for all $t \geqq 0$, $x \in E$, and the result follows from (3.29).

(ii) The process associated to $\Theta_s W$, $\Theta_s \Gamma$ and $\Theta_s \tilde{\Gamma}$ by (6.10) is $\Theta_s \tilde{W}$, and we know that $\Theta_s \tilde{\Gamma} = \widetilde{\Theta_s \Gamma}$ from (6.6, ii). We have

$$[\sum_{r \leqq \cdot} (\Theta_s \tilde{W})_r^2]^{1/2} = \Theta_s([\sum_{r \leqq \cdot} (\tilde{W})_r^2]^{1/2}),$$

which is in $\mathscr{A}_{\text{loc}}$ because of (3.15, i). Therefore $\Theta_s W \in \bigcap_{x \in E} G(\Theta_s \Gamma, P^x)$, and $(\Theta_s W) * (\Theta_s \Gamma - \Theta_s \tilde{\Gamma}) \in \mathscr{L}$ is well defined by (i). Moreover, $\widetilde{\Theta_s A(n)} = \Theta_s A(n)$, which implies

$$P^x\text{-}\lim_n [(\Theta_s A(n))_t - (\Theta_s A(n))_t] = (\Theta_s W) * (\Theta_s \Gamma - \Theta_s \tilde{\Gamma})_t.$$

Then, the result follows readily from (3.31).

(iii) It is proved as (ii), just replacing s by S.

(iv) It is immediate from (ii) and (iii): see the proof of (3.18).

6c) More on Additive Integer-Valued Random Measures

When $(\mathscr{M}_t)$ is a strong Markov filtration, we have much better results about dual predictable projections of additive random measures.

In order to avoid tedious difficulties, we concentrate on the class $\tilde{\mathscr{A}}^1_{\sigma,\text{ad}}$ of additive integer-valued random measures Γ for which there exists a $\tilde{\mathscr{P}}$-measurable partition (D_n) of $\tilde{\Omega}$ such that $1_{D_n} * \Gamma \in \mathscr{A}_{\text{loc,ad}}$. Obviously $\tilde{\mathscr{A}}^1_{\sigma,\text{ad}} \subset \mathscr{A}_\sigma$.

(6.14) **Theorem.** *Suppose* $(\mathscr{M}_t)$ *is a strong Markov filtration. Let* $\mathscr{O}$ *be the* σ*-field* $\mathscr{O}(\mathscr{H}_t')$ *(resp. its universal completion* $\mathscr{O}(\mathscr{H}_t')^*$*) when* (6.1, i) *(resp.* (6.1, ii)*) holds. If* $\Gamma \in \tilde{\mathscr{A}}^1_{\sigma,\text{ad}}$, *there exists* $F \in (\mathscr{P} \cap \mathscr{V}^+_{\text{ad}})(\mathscr{H}_t')$ *and a positive kernel* $\tilde{K}(\omega, t; dy)$ *from* $(\Omega \times \mathbb{R}_+, \mathscr{O})$ *into* $(G, \mathscr{G})$, *such that*

(6.15) $$\tilde{\Gamma}(\omega; dt, dy) = dF_t(\omega)\, \tilde{K}(\omega, t; dy)$$

is a version $\tilde{\Gamma} \in \tilde{\mathscr{P}} \cap \tilde{\mathscr{A}}_\sigma$ *of the* $(\mathscr{H}_t)$*-dual predictable projection of* Γ *for every* P^x.

Remark. By (3.27) and (6.9) applied to $\mathscr{M}_t^0=\mathscr{F}_t^0$, F and $\tilde{\Gamma}$ are strongly additive even when Γ is not so.

Proof. Let $(D_n)_{n\geqq 1}$ be a $\tilde{\mathscr{P}}$-measurable partition of $\tilde{\Omega}$ such that $B^n=1_{D_n}*\Gamma\in\mathscr{A}_{\mathrm{loc,ad}}$. Since B^n satisfies the conditions of (3.53), there exists a sequence $(C(n,m))_{m\geqq 1}$ of $\mathscr{E}_0$-measurable sets such that $\bigcup_m C(n,m)=E$ and that $H(n,m)=1_{C(n,m)}(X)\cdot B^n$ satisfies $a(n,m)=\sup_{x\in E} E^x[\int e^{-t}\,dH(n,m)_t]<\infty$. Let $H=\sum_{n,m\geqq 1} b(n,m)\,H(n,m)$, where $(b(n,m))$ is a sequence of positive numbers satisfying $\sum_{n,m\geqq 1} b(n,m)\,a(n,m)<\infty$. It is easy to check that $H\in\mathscr{A}^+_{\mathrm{loc,ad}}$, and (3.18) and (3.26) imply that one can find a $F\in(\mathscr{P}\cap\mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}'_t)$ which is a version of the $(\mathscr{H}_t)$-dual predictable projection of H.

From now on, we copy the proof of (6.6), with the following changes. By (3.26) again, we can choose $\tilde{Y}(n,B)$ in $(\mathscr{P}\cap\mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}'_t)$. Since $\bigcup_m C(n,m)=E$, and since $\bigcup_n D_n=\tilde{\Omega}$, it is easy to check that $dY(n,B)_t\ll dH_t$; therefore, we can find a $(\mathscr{H}'_t)$-optional process $f(n,B)$ such that $\tilde{Y}(n,B)=f(n,B)\cdot F$ a.s. We consider the kernel K^n constructed in (6.6), and we set $\tilde{\Gamma}^n(\omega;\,dt,\,dy)=dF_t(\omega)\,K^n(\omega,t;dy)$. Since one knows that $\tilde{\Gamma}=\sum_{n\geqq 1}\tilde{\Gamma}^n$ is the dual predictable projection of Γ, we obtain the factorization (6.14) by putting $\tilde{K}=\sum_{n\geqq 1}K^n$.

(6.16) *Remark.* The factorization (6.15) is by no means unique. However, for a given $F\in\mathscr{P}\cap\mathscr{V}^+_{\mathrm{ad}}$, $\tilde{K}$ is unique up to a set which is $P^x(d\omega)\otimes dF_t(\omega)$-null for every $x\in E$. Now, if $F'\in\mathscr{P}\cap\mathscr{V}^+_{\mathrm{ad}}$ satisfies $dF_t\ll dF'_t$ a.s., there exists another kernel $\tilde{K}'$ such that (6.15) holds with F' and $\tilde{K}'$: this property obviously follows from the previous proof.

(6.17) *Remark.* We know that $\tilde{\Gamma}$ is "predictable", but in general one cannot choose a "predictable" kernel $\tilde{K}$. However, this would be possible if $\mathscr{H}'_t=\mathscr{F}_t$ and if (6.1, i) holds and if **X** were a right process: see (3.22). But even when such a choice of $\tilde{K}$ is not possible, as far as computations are concerned, the kernel $\tilde{K}$ behaves as if it were predictable. For instance if $W\in p\tilde{\mathscr{P}}$ and if Z is a positive $(\mathscr{H}_t)$-predictable process such that $Z\cdot(W*\Gamma)\in\mathscr{A}_{\mathrm{loc}}$, then the dual predictable projection of this process is $(Z\tilde{K}(W))\cdot F$, where $\tilde{K}(W)_t(\omega)=\int\tilde{K}(\omega,t;dy)\,W(\omega,t,y)$. For various related matters, we refer to [42], (30.4).

(6.18) *Remark.* We know that $\tilde{\Gamma}$ is strongly additive. It follows from (3.57) that for every $W\in p\tilde{\mathscr{P}}$ satisfying identically $W1_{[s,\infty)}=\Theta_s W$ the process $\tilde{K}(W)$ defined in (6.17) is equal to a strongly homogeneous process, up to a $P^x(d\omega)\otimes dF_t(\omega)$-negligible set for all $x\in E$. However we do not know if we can choose a kernel $\tilde{K}$ which is strongly homogeneous in the sense that $K(\cdot,S(\cdot)+t;\cdot)=K(\theta_S(\cdot),t;\cdot)$ a.s. for every $t\geqq 0$, for all finite stopping times S.

We say that the random measure Γ is *quasi-left-continuous* if

$$\Gamma(\omega;\{T(\omega)\}\times G)=0 \qquad \text{a.s. in } \omega \text{ on } \{T<\infty\}$$

for every predictable $(\mathscr{H}_t)$-stopping time T. When $\Gamma\in\tilde{\mathscr{A}}^1_{\sigma,\mathrm{ad}}$ is given by (6.10), this is equivalent to saying that $\Lambda\cap[\![T]\!]$ is evanescent for every predictable stopping time T. This is also equivalent to saying that in (6.13) we can choose a F which is continuous. Then, it is possible to take advantage of Motoo's Theorem (3.56).

(6.19) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. If $\Gamma\in\tilde{\mathscr{A}}^1_{\sigma,\mathrm{ad}}$ is quasi-left continuous, there exist a continuous $F\in\mathscr{V}^+_{\mathrm{ad}}(\mathscr{H}'_t)$ and a positive kernel $K(x,dy)$ from $(E,\mathscr{E}_0)$ into $(G,\mathscr{G})$, such that*

$$\tilde{\Gamma}(\omega;dt,dy)=dF_t(\omega)\,K(X_t(\omega),dy) \tag{6.20}$$

is a version of the $(\mathscr{H}_t)$-dual predictable projection of Γ for every P^x.

About uniqueness of (F,K), the same comments as those in (6.16) can be made.

Proof. The beginning of the proof is like (6.14), but here, since Γ is quasi-left continuous, we find a continuous F. Applying (3.55), we obtain $g(n,B)\in p\mathscr{E}_0$ such that $\tilde{Y}(n,B)=g(n,B)(X)\cdot F$. We apply (6.2) to each family $(g(n,B))_{B\in\mathscr{G}}$, with $(A,\mathscr{A})=(E,\mathscr{E}_0)$, and M is the family of measures: $D\in\mathscr{G}\to E^x[1_D(x)\cdot F_\infty]$, for all $x\in E$. There exists a positive kernel $K^n(x,dy)$ from $(E,\mathscr{E}_0)$ into $(G,\mathscr{G})$ such that $\tilde{Y}(n,B)=K^n(x,B)\cdot F$ (when using (6.2), we recall that $\mathscr{E}_0=\mathscr{E}_0^*$ when (6.1, ii) holds). The proof is completed by setting $K=\sum_n K^n$, the justification of it being like in (6.14).

(6.21) *Remark.* We could prove (6.14) and (6.19) for every strongly additive $\Gamma\in\tilde{\mathscr{A}}_\sigma$, by using a result of Meyer [36] Théorème 3.

6d) Applications to Additive Semimartingales

First, we recall some more facts about semimartingales which are defined on $(\Omega, \mathscr{H}, \mathscr{H}_t, P)$, where P is an arbitrary probability measure on $(\Omega, \mathscr{H})$. We consider a m-dimensional semimartingale $Y=(Y^i)_{i\leqq m}$. We define the m-dimensional process $Y^e=((Y^e)^i)_{i\leqq m}$ by (3.20), in which formula $|\cdot|$ denotes the usual norm on $\mathbb{R}^m$. Now, $Y-Y_0-Y^e$ is a m-dimensional special semimartingale, whose canonical decomposition we denote by $Y-Y_0-Y^e=M+B$. We also define the following integer-valued random measure Γ on $G=\mathbb{R}^m$ by

$$\Gamma(\omega;dt,dy)=\sum_{s>0}1_{\{\Delta Y_s(\omega)\neq 0\}}\varepsilon_{(s,\Delta Y_s(\omega))}(dt,dy). \tag{6.22}$$

Γ is called the *jump measure* of Y. According to [17, 24], the *local characteristics* of Y consist of the following triplet $(B,C,\tilde{\Gamma})$:

(i) $B=(B^i)_{i\leqq m}$ is the process appearing above,

(ii) $C=(C^{ij})_{i,j\leqq m}$ where $C^{ij}=[(Y^i)^c,(Y^j)^c]$,

(iii) $\tilde{\Gamma}$ is the dual predictable projection of Γ.

This triplet is unique, up to a P-null set, and it does indeed characterize the distribution of Y in some particular cases (as we shall see in the next section). Moreover, one may choose a version of $(B, C, \tilde{\Gamma})$ which satisfies the following identically:

(6.23) (i) for all $t \geqq s \geqq 0$, $C_t - C_s$ is a nonnegative symmetric matrix.

(ii) $\tilde{\Gamma}(\omega; \mathbb{R}_+ \times \{0\}) = 0$

(iii) $\int_{\mathbb{R}^m} (|y|^2 \wedge 1)\, \tilde{\Gamma}(\omega; [0,t] \times dy) < \infty$ for every $t \geqq 0$.

Returning to our Markov process, we shall write $Y \in \mathscr{S}^m$ when $Y = (Y^i)_{i \leqq m}$ and $Y^i \in \mathscr{S}$ for every $i \leqq m$; we do similarly for the other classes of processes.

(6.24) **Theorem.** *Let* $Y \in \mathscr{S}^m$.

(i) *There exists a* $B \in (\mathscr{P} \cap \mathscr{V})^m$, *an a.s. continuous* $C \in \mathscr{V}^{m^2}$, *and a* $\tilde{\Gamma} \in \tilde{\mathscr{P}} \cap \tilde{\mathscr{A}}_\sigma$, *such that* $(B, C, \tilde{\Gamma})$ *is a version of the local characterisitcs of* Y *for every* P^x.

(ii) *Suppose* $(\mathscr{M}_t)$ *is a Markov filtration. If* $Y_0 = 0$, *then for every* $s \geqq 0$, $(\Theta_s B, \Theta_s C, \Theta_s \Gamma)$ *is a version of the local characteristics of* $\Theta_s Y$.

(iii) *Suppose* $(\mathscr{M}_t)$ *is a strong Markov filtration. When* $Y_0 = 0$ *the property* (ii) *holds if* s *is replaced by any finite* $(\mathscr{H}_t)$*-stopping time* S.

(iv) *Suppose* $(\mathscr{M}_t)$ *is a Markov (resp. strong Markov) filtration. If* Y *is additive, then* B, C, $\tilde{\Gamma}$ *are additive (resp. strongly additive).*

Remark. When $Y_0 \neq 0$, the local characterisitcs of $\Theta_s Y$ are easily expressed in terms of, but are not equal to, $(\Theta_s B, \Theta_s C, \Theta_s \tilde{\Gamma})$. We leave it to the reader.

Proof. (i) Let $D_0 = \Omega \times \mathbb{R}_+ \times \{0\}$ and, for $n \geqq 1$,

$$D_n = \left\{(\omega, t, y) \colon \omega \in \Omega,\ t \geqq 0,\ y \in \mathbb{R}^m,\ |y| \in \left[\frac{1}{n}, \frac{1}{n-1}\right)\right\}.$$

The jump measure Γ of Y obviously satisfies $1_{D_n} * \Gamma \in \mathscr{A}_{\mathrm{loc}}$ for every $n \geqq 0$, thus implying $\Gamma \in \tilde{\mathscr{A}}_\sigma$. The existence of $(B, C, \tilde{\Gamma})$ follows from (3.12) and (6.6).

(iii) Since $Y_0 = 0$, the jump measure of $\Theta_s Y$ is $\Theta_s \Gamma$, and the process associated to $\Theta_s Y$ by (3.20) is $\Theta_s Y^e$. Then, the result follows from (3.15) and (6.6).

(iii) It is immediate from (3.15) and (6.6).

(iv) Since Γ is additive when Y is additive, (3.18) and (6.8) imply that B, C, $\tilde{\Gamma}$ are additive. Moreover, $\Gamma \in \tilde{\mathscr{A}}^1_{\sigma,\mathrm{ad}}$ since $1_{D_n} * \Gamma \in \mathscr{A}_{\mathrm{loc,ad}}$. Now, (3.6) and (6.14) imply that B, C, $\tilde{\Gamma}$ are P^x-indistinguishable from some $(\mathscr{H}_t')$-predictable processes and measures, and their strong additivity follows from (3.21) and (6.9) applied to $\mathscr{M}_t^0 = \mathscr{F}_t^0$.

(6.25) **Theorem.** *Suppose* $(\mathscr{M}_t)$ *is a strong Markov filtration. Let* $Y \in \mathscr{S}^m_{\mathrm{ad}}$. *There exist*

(i) *a* $F \in (\mathscr{P} \cap \mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}_t')$;

(ii) *a* $(\mathscr{H}_t')$*-optional process* $\hat{b} = (\hat{b}^i)_{i \leqq m}$;

(iii) *a* $(\mathscr{H}_t')$*-optional process* $\tilde{c} = (\tilde{c}^{ij})_{i,j \leqq m}$ *with values in the set of all symmetric nonnegative matrices;*

(iv) *a positive kernel* $\hat{K}(\omega, t; dy)$ *from* $(\Omega \times \mathbb{R}_+, \mathscr{O}(\mathscr{H}_t'))$ *into* $(\mathbb{R}^m, \mathscr{R}^m)$, *satisfying* $\hat{K}(\{0\}) = 0$ *and* $\int (|y|^2 \wedge 1)\, \hat{K}(dy) < \infty$ *such that*

(6.26) $$B = \hat{b} \cdot F, \qquad C = \tilde{c} \cdot F, \qquad \tilde{\Gamma}(\omega, dt \times dy) = dF_t(\omega)\, \hat{K}(\omega, t; dy)$$

form a version of the $(\mathscr{H}_t)$*-local characteristics of* Y.

Proof. We have seen in the proof of (6.24) that B, C, Γ, $\hat{\Gamma}$ are strongly additive and that $\Gamma\in\tilde{\mathscr{A}}^1_{\sigma,\mathrm{ad}}$. If D_n is like in the proof of (6.24), we have $1_{D_n}*\Gamma\in\mathscr{A}_{\mathrm{loc}}$, thus $\Gamma\in\tilde{\mathscr{A}}^1_{\sigma,\mathrm{ad}}$. Since (6.1, i) holds (because $G=\mathscr{R}^m$), (6.14) implies the existence of $F'\in(\mathscr{P}\cap\mathscr{V}^+_{\mathrm{ad}})(\mathscr{H}'_t)$ and of a kernel $\tilde{K}'$ from $(\Omega\times\mathbb{R}_+,\mathcal{O}(\mathscr{H}'_t))$ into $(\mathbb{R}^m,\mathscr{R}^m)$ such that $\hat{\Gamma}$ admits the factorization (6.15) with F' and $\tilde{K}'$. *Set*

$$F_t=F'_t+\sum_{i\leq m}\int_0^t|dB^i_s|+\sum_{i\leq m}C^{ii}_t.$$

Because of (3.26), and up to a change on an evanescent set, we can suppose that B^i, C^{ij}, F are in $(\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}})(\mathscr{H}'_t)$. Since $dF_t\ll dF'_t$, $\hat{\Gamma}$ admits another factorization (6.15) based upon F, with some kernel $\tilde{K}$, see Remark (6.16). Since $dB^i_t\ll dF_t$ and $dC^{ij}_t\ll dF_t$, $(\mathscr{H}'_t)$-optional processes $\tilde{b}$ and $\tilde{c}$ such that $B=\tilde{b}\cdot F$ and $C=\tilde{c}\cdot F$ exist.

Finally, we consider the set A of all (ω,t) where, either $c(\omega,t)$ is not symmetric nonnegative, or $\tilde{K}(\omega,t;\{0\})>0$, or $\int|y|^2\wedge 1\,\tilde{K}(\omega,t;dy)=\infty$. We have $A\in\mathcal{O}(\mathscr{H}'_t)$, and $E^x[1_A\cdot F_\infty]=0$ for every $x\in E$, because of (6.23). Thus, replacing $\tilde{c}$ and $\tilde{K}$ by $1_{A^c}\tilde{c}$ and $1_{A^c}\tilde{K}$ completes the proof.

It follows from this proof and from (3.57) that we can choose strongly homogeneous versions for the processes $\tilde{b}$ and $\tilde{c}$.

If Y is quasi-left continuous, B is continuous and Γ is also quasi-left continuous. Using (3.55) and (6.19), we obtain similarly:

(6.27) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a strong Markov filtration. Let $Y\in\mathscr{S}^m_{\mathrm{ad}}$ be P^x-quasi-left-continuous for all $x\in E$. Then, there exist:*

(i) *a continuous $F\in\mathscr{V}^+_{\mathrm{ad}}(\mathscr{H}'_t)$;*

(ii) *a $\mathscr{E}_0$-measurable function $b=(b^i)_{i\leq m}$;*

(iii) *a $\mathscr{E}_0$-measurable function $c=(c^{ij})_{i,j\leq m}$ with values in the set of all symmetric nonnegative matrices;*

(iv) *a positive kernel $K(x,dy)$ from $(E,\mathscr{E}_0)$ into $(\mathbb{R}^m,\mathscr{R}^m)$ satisfying $K(x,\{0\})=0$ and $\int(|y|^2\wedge 1)K(x,dy)<\infty$; such that*

(6.28) $$B=b(X)\cdot F,\qquad C=c(X)\cdot F,\qquad \hat{\Gamma}(\omega;dt\times dy)=dF_t(\omega)\,K(X_t(\omega),dy)$$

are a version of the $(\mathscr{H}_t)$-local characteristics of Y.

About uniqueness of $(F;\,\tilde{b},\,\tilde{c},\,\tilde{K})$ in (6.25), or of $(F;\,b,\,c,\,K)$ in (6.27), the comments of (6.16) are still valid. In particular, when Y is quasi-left continuous, we can apply simultaneously (6.25) and (6.27); in this case, if $(F;\,\tilde{b},\,\tilde{c},\,\tilde{K})$ satisfies (6.26) and if $(F;\,b,\,c,\,K)$ satisfies (6.28), *with the same continuous F*, we have

(6.29) $$\tilde{b}_t(\omega)=b(X_t(\omega)),\qquad \tilde{c}_t(\omega)=c(X_t(\omega)),$$
$$\tilde{K}(\omega,t;dy)=K(X_t(\omega),dy)$$

except on a null F-potential set (that is, a $D\subset\Omega\times\mathbb{R}_+$ such that $E^x[1_D\cdot F_\infty]=0$ for every $x\in E$). However, note that except when $\mathscr{E}_0=\mathscr{E}$, we cannot have (6.29) holding true everywhere, in general, since $b(X_t(\omega)),\ldots$, are not necessarily $(\mathscr{H}_t)$-optional (for instance when b is $\mathscr{E}^*$-measurable, but not Borel).

7. Markov Processes That Are Semimartingales

In this section we suppose that $E=\mathbb{R}^m$ and that the underlying process $\mathbf{X}=(\Omega,\mathscr{F},\mathscr{F}_t,\theta_t,X_t,P^x)$ is a *strong Markov process*: in other word, $\mathscr{F}_t^0=\mathscr{M}_t^0$ and $(\mathscr{M}_t)$ is a strong Markov filtration. Since we are interested in the property for X to be a semimartingale and since X if $(\mathscr{F}_t^0)$-adapted, we would get no further generality by allowing the inclusions $\mathscr{F}_t^0\subset\mathscr{M}_t^0$ to be strict: see (3.24).

The process $\mathbf{X}$ is said to be a *Markov semimartingale* if $X\in\mathscr{S}^m$, or equivalently since $X_{t+s}=X_t\circ\theta_s$, if $X-X_0\in\mathscr{S}^m_{\text{ad}}$. In this case we can apply Theorem (6.25) to $X-X_0$. We say that $\mathbf{X}$ is a *Hunt semimartingale* if $X\in\mathscr{S}^m$ and if $\mathbf{X}$ is quasi-left continuous. If $X\in\mathscr{S}^m$, then $\mathbf{X}$ is a Hunt semimartingale if and only if we can choose F in (6.25) to be continuous. Note that we depart slightly from the usual definition of a Hunt process [3], since we assume neither normality for $\mathbf{X}$, nor Borel measurability for its transition semigroup. Finally we call $\mathbf{X}$ an *Ito process* if $X\in\mathscr{S}^m$ admits $F_t=t$ for the process F appearing in (6.25) applied to $X-X_0$: it turns out that these are exactly the processes introduced by Ito in [19] as solutions of certain stochastic differential equations, a fact that will be proved in the forthcoming paper [6].

Our purpose in this section is to characterize Markov and Hunt semimartingales and Ito processes in terms of their generators. Ito processes have nice and workable characterizations and we will show in (7.13) that every Hunt semimartingale is obtained by a random time change from an Ito process.

7a) Generators

We start by introducing various generators that will be used for characterizing Markov semimartingales. We are presenting them in increasing order of generality (and decreasing order of interest!). The notion of an extended generator we are putting next is due to Dynkin [10].

(7.1) *Definition.* An operator G with domain $\mathscr{D}_G$ is said to be an *extended generator* for $\mathbf{X}$ if $\mathscr{D}_G$ consists of those functions $f\in\mathscr{R}^m$ for which there exists a function $Gf\in\mathscr{E}_0$ such that the process

$$C_t^f=f(X_t)-f(X_0)-\int_0^t Gf(X_s)\,ds \tag{7.2}$$

is well defined and belongs to $\mathscr{L}$ (and thus to $\mathscr{L}_{\text{ad}}$).

(7.3) *Remarks. i)* For every $f\in\mathscr{D}_G$, Gf is uniquely defined up to a set of potential zero. Thus $(G,\mathscr{D}_G)$ is not a linear operator in the ordinary sense of the word: actually $\mathscr{D}_G$ is uniquely defined, but there exist various "versions" of G on the same domain $\mathscr{D}_G$, each version being "almost linear".

(ii) Suppose $\mathbf{X}$ is a right process, and let $(G^*,\mathscr{D}_{G^*})$ be its weak infinitesimal generator. Then, by Dynkin's formula, and since $f(X)$ is a.s. right-continuous for every $f\in\mathscr{D}_{G^*}$, we have $\mathscr{D}_{G^*}\subset\mathscr{D}_G$ and G^*f is a version of Gf for every $f\in\mathscr{D}_{G^*}$.

(iii) The condition that $Gf \in \mathscr{E}_0$ in Definition (7.2) is there to insure that C^f be adapted to $(\mathscr{H}_t)$. This can be weakened by requiring only that $Gf \in \mathscr{R}^{m^*} = \mathscr{E}^*$, a change which does not enlarge the domain $\mathscr{D}_G$: in fact, if for some $f \in b\mathscr{R}^m$ there is $Gf \in b\mathscr{R}^{m^*}$ such that C^f is a local martingale (necessarily additive) on every $(\Omega, \mathscr{F}, \mathscr{F}_t, P^x)$, then there exists $g \in \mathscr{E}_0$ such that $\{g \neq Gf\}$ is of potential zero (this follows from applying (3.18, ii) to $Y = f(X)$ and using (3.55)) thus, g can serve as Gf.

The next definition follows Kunita [26].

(7.4) *Definition.* Let $F \in \mathscr{V}_{\text{ad}}^+$ be continuous. Then, an operator G_F with domain $\mathscr{D}_{G_F}$ is said to be an *F-extended generator of* **X** if $\mathscr{D}_{G_F}$ consists of those $f \in b\mathscr{R}^m$ for which there exists a $G_F f \in \mathscr{E}_0$ such that the process $f(X) - f(X_0) - G_F f(X) \cdot F$ is well defined and belongs to $\mathscr{L}$ (and thus to $\mathscr{L}_{\text{ad}}$).

(7.5) *Definition.* Let $F \in \mathscr{P} \cap \mathscr{V}_{\text{ad}}^+$. Then an operator $\tilde{G}_F$ with domain $\mathscr{D}_{\tilde{G}_F}$ is said to be an *F-random generator* of **X** if $\mathscr{D}_{\tilde{G}_F}$ consists of those $f \in b\mathscr{R}^m$ for which there exists an $(\mathscr{H}_t)$-optional process $\tilde{G}_F f$ such that

$$(\tilde{G}_F f) \cdot F \in \mathscr{P} \cap \mathscr{V}, \quad f(X) - f(X_0) - (\tilde{G}_F f) \cdot F \in \mathscr{L}.$$

Remarks (7.2, i, ii, iii) apply to the F-extended generators and F-random generators, with the role of zero potential sets being played by zero F-potential sets. The following is an immediate consequence of Motoo's Theorem (3.55).

(7.6) **Lemma.** *If $F \in \mathscr{V}_{\text{ad}}^+$ is continuous, then $\mathscr{D}_{G_F} = \mathscr{D}_{\tilde{G}_F}$, and for every $f \in \mathscr{D}_{G_F}$, $\tilde{G}_F f = G_F f(X)$ except possibly on a set of F-potential zero.*

To complete this account of the preliminaries, we need two more notions. They are needed because we want to achieve the sharpest possible result; but the reader may as well consider $\mathbb{C}^2(\mathbb{R}^m)$ or $\mathbb{C}^\infty(\mathbb{R}^m)$ in lieu of the two classes being introduced next.

(7.7) *Definition.* A class $\mathscr{C}$ of functions is said to be a *full class* if for all $i, q \in \mathbb{N}$ with $i \leqq m$ there exists a finite family $\{f_1, \ldots, f_n\} \subset \mathscr{C}$ and $g \in \mathbb{C}^2(\mathbb{R}^m)$ such that

$$x^i = g(f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in \mathbb{R}^m$, $|x| \leqq q$.

(7.8) *Definition.* A class of Borel functions on $\mathbb{R}^m$ is said to be a *complete class* if it contains a countable subset $\mathbb{C} \subset \mathbb{C}^2(\mathbb{R}^m)$ with the property that, for every $x \in \mathbb{R}^m$, the countable collection of numbers

$$\sum_{i \leqq m} \beta^i D_i f(x) + \tfrac{1}{2} \sum_{i,j \leqq m} \gamma^{ij} D_{ij} f(x)$$
$$+ \int \rho(dy)[f(x+y) - f(x) - 1_{[0,1]}(|y|) \sum_{i \leqq m} y^i D_i f(x)],$$

$f \in \mathscr{C}$, completely determines the vector $\beta \in \mathbb{R}^m$, the symmetric nonnegative matrix γ, and the positive measure ρ satisfying $\rho(\{0\}) = 0$ and $\int (|y|^2 \wedge 1)\, \rho(dy) < \infty$.

For example, $\mathbb{C}^2(\mathbb{R}^m)$, $\mathbb{C}^\infty(\mathbb{R}^m)$, $\mathbb{C}_K^\infty(\mathbb{R}^m)$, and the class $\{x \to e^{i\langle u, x\rangle}: u \in \mathbb{R}^m\}$ are classes that are both full and complete.

7b) Characterization of Markov Semimartingales

Recall that our only basic assumption is that $\mathbf{X}$ is an $\mathbb{R}^m$-valued right-continuous strong Markov process. Here is the main result of this section, (that is, the most "fundamental", although probably useless as such).

(7.9) **Theorem.** (i) $\mathbf{X}$ *is a Markov semimartingale if and only if there exists* $F\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}^+$ *such that* $\mathscr{D}_{\tilde{G}_F}$ *is a full class.*

(ii) *Suppose* $\mathbf{X}$ *is a Markov semimartingale, and let* $(F, \tilde{b}, \tilde{c}, \tilde{K})$ *satisfy* (6.25) *applied to* $X-X_0$. *Then,* $\mathbb{C}^2(\mathbb{R}^m)\subset\mathscr{D}_{\tilde{G}_F}$, *and for every* $f\in\mathbb{C}^2(\mathbb{R}^m)$, *the process defined by*

$$\begin{aligned}\tilde{L}f_t(\omega)= &\sum_{i\leq m}\tilde{b}_t^i(\omega)D_i f(X_{t^-}(\omega))+\tfrac{1}{2}\sum_{i,j\leq m}\tilde{c}_t^{ij}(\omega)D_{ij}f(X_{t^-}(\omega))\\ &+\int\tilde{K}(\omega,t;dy)[f(X_{t^-}(\omega)+y)-f(X_{t^-}(\omega))\\ &-1_{[0,1]}(|y|)\sum_{i\leq m}y^i D_i f(X_{t^-}(\omega))]\end{aligned}$$

is a version of $\tilde{G}_F f$.

(iii) *Suppose* $\mathbf{X}$ *is a Markov semimartingale, and let* $F\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}^+$ *be such that* $\mathscr{D}_{\tilde{G}_F}$ *is a complete class. Then there exists* $\tilde{b}$, $\tilde{c}$, $\tilde{K}$, *such that* $(F, \tilde{b}, \tilde{v}, \tilde{K})$ *satisfies* (6.25) *applied to* $X-X_0$. *Hence,* $\mathbb{C}^2(\mathbb{R}^m)\subset\mathscr{D}_{\tilde{G}_F}$ *and* (7.10) *gives a version of* $\tilde{G}_F$ *on* $\mathbb{C}^2(\mathbb{R}^m)$.

At a first glance it may appear that the sufficient condition in (i) is the most interesting statement in Theorem (7.9); however, we shall see that it is completely obvious. In fact, (ii) and (iii) are far more interesting.

Proof. (ii) This statement is the key point for the whole theorem; its proof will be a simple application of Ito's formula and will not differ much from a proof by Kunita [26].

Let X^e be the process associated to X by (3.20). We denote by $X-X^e-X_0=M+B$ the canonical decomposition of $X-X^e-X_0$; we have $X^e\in\mathscr{V}^m$, $M\in\mathscr{L}^m$, $B\in(\mathscr{P}\cap\mathscr{V})^m$. Let Γ be the jump measure of X; its dual predictable projection is denoted by $\tilde{\Gamma}$. Finally, let $C=(C^{ij})_{i,j\leq m}$ with $C^{ij}=[(X^i)^c,(X^j)^c]$. By hypothesis, B, C, $\tilde{\Gamma}$ are given by (6.25).

Let $f\in\mathbb{C}^2(\mathbb{R}^m)$. We apply Ito's formula to $f(X)$, relative to the measure P^x (the result is independent of x):

$$\begin{aligned}f(X)-f(X_0)=&\sum_i D_i f(X_-)\cdot X^i+\tfrac{1}{2}\sum_{i,j}D_{ij}f(X_-)\cdot C^{ij}\\ &+\sum_{0<s\leq\cdot}[f(X_s)-f(X_{s^-})-\sum_i D_i f(X_{s^-})\Delta X_s^i].\end{aligned}$$

Using the definition of Γ, since $X=X_0+X^e+M+B$, we have

$$\begin{aligned}f(X)-f(X_0)=&\sum_i D_i f(X_-)\cdot M^i+\sum_i D_i f(X_-)\cdot B^i+\tfrac{1}{2}\sum_{i,j}D_{ij}f(X_-)\cdot C^{ij}\\ &+\int[f(X_{s^-}+y)-f(X_{s^-})-1_{[0,1]}(|y|)\sum_i y^i D_i f(X_{s^-})]\,\Gamma(ds,dy).\end{aligned}$$

Consider the right-hand side of this expression: The first sum is a local martingale. The second and third are continuous processes in $\mathscr{V}$, thus they belong to $\mathscr{A}_{\mathrm{loc}}$. The fourth term is in $\mathscr{V}$; but since $f(X)\in\mathscr{S}_p$ (because f is bounded), this term is in fact in $\mathscr{A}_{\mathrm{loc}}$, (see [34] for instance); hence its dual predictable projection is given by the same expression with $\tilde{\Gamma}$ replacing Γ. Thus, we have proved that the process

$$f(X)-f(X_0)-\sum_i D_i f(X_-)\cdot B^i-\tfrac{1}{2}\sum_{i,j} D_{ij} f(X_-)\cdot C^{ij}$$
$$-\int [f(X_{s-}+y)-f(X_{s-})-1_{[0,1]}(|y|)\sum_i y^i D_i f(X_{s-})]\,\tilde{\Gamma}(ds,dy)$$

is in $\mathscr{L}$, and each of the three last terms above is in $\mathscr{P}\cap\mathscr{V}$. Using (6.25) and (7.10), it is easy to check that the sum of these three last terms is the process $(\tilde{L}f)\cdot F$. We have proved that $(\tilde{L}f)\cdot F\in\mathscr{P}\cap\mathscr{V}$ and that $f(X)-f(X_0)-(\tilde{L}f)\cdot F\in\mathscr{L}$, thus obtaining (ii).

(i) Since $\mathbb{C}^2(\mathbb{R}^m)$ is a full class, the necessary condition follows from (6.25) and (ii). Conversely, suppose that $\mathscr{D}_{\tilde{G}_F}$ is a full class for some $F\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}^+$. From Definition (7.5), $f(X)\in\mathscr{S}$ for every $f\in\mathscr{D}_{\tilde{G}_F}$. Applying Ito's formula to the function g appearing in Definition (7.7) shows that the process X^i coincides with a semimartingale on each stochastic interval $[\![0,T_q[\![$, where $T_q=\inf\{t:|X_t|\geqq q\}$. Since $\lim T_q=\infty$ a.s., X^i itself is in $\mathscr{S}$: see [34]. This proves the sufficiency of the condition (i).

(iii) Suppose $\mathbf{X}$ is a Markov semimartingale. Let $F\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}^+$ be such that $\mathscr{D}_{\tilde{G}_F}$ is a complete class. We associate to $\mathscr{D}_{\tilde{G}_F}$ a countable class $\mathscr{C}\subset\mathscr{D}_{\tilde{G}_F}\cap\mathbb{C}^2(\mathbb{R}^m)$ satisfying the property stated in (7.8). Because of (6.25) and of (ii), there also exist $F'\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}^+$ and $(\tilde{b}', \tilde{c}', \tilde{K})$ such that $\mathbb{C}^2(\mathbb{R}^m)\subset\mathscr{D}_{\tilde{G}_{F'}}$, and that $\tilde{G}_{F'}$ coincide on $\mathbb{C}^2(\mathbb{R}^m)$ with the operator $\tilde{L}'$ defined by (7.10) with $(\tilde{b}', \tilde{c}', \tilde{K}')$.

Let $f\in\mathscr{C}$. We have that $f(X)-f(X_0)-(\tilde{G}_F f)\cdot F\in\mathscr{L}$ and that $f(X)-f(X_0)-(\tilde{L}'f)\cdot F'\in\mathscr{L}$, while $(\tilde{G}_F f)\cdot F$ and $(\tilde{L}'f)\cdot F'$ are in $\mathscr{P}\cap\mathscr{V}$. Uniqueness of the canonical decomposition of $f(X)$ yields

(7.11) $$(\tilde{G}_F f)\cdot F=(\tilde{L}'f)\cdot F' \quad \text{up to an evanescent set.}$$

Set $D=\{(\omega,t)\colon (\tilde{L}'f)_t(\omega)=0 \text{ for every } f\in\mathscr{C}\}$. The characteristic property (7.8) of $\mathscr{C}$ implies that D is exactly the set where $\tilde{b}'=0$, $\tilde{c}'=0$, $\tilde{K}'=0$. Hence we may replace F' by $I_{D^c}\cdot F'$ without altering (6.25), that is, we can suppose that $1_D\cdot F'=0$. This property, together with (7.11), obviously implies that $dF'_t\ll dF_t$ a.s. Hence, we know that there exists a triplet $(\tilde{b}, \tilde{c}, \tilde{K})$ such that $(F;\, \tilde{b}, \tilde{c}, \tilde{K})$ satisfies all the conditions of (6.25), and we have proved (iii).

(7.12) *Remark.* More generally, let us turn back to the assumptions of Sect. 3, with an arbitrary state space E, and suppose $(\mathscr{M}_t)$ is a strong Markov filtration. For $Y\in\mathscr{S}_{\mathrm{ad}}^m$, consider a term $(F;\, \tilde{b}, \tilde{c}, \tilde{K})$ satisfying the conditions of (6.25) relative to Y. Then, the same proof as above shows that, if we define $\tilde{L}f$ for $f\in\mathbb{C}^2(\mathbb{R}^m)$ by (7.10) where $X_{t-}(\omega)$ is replaced by $Y_{t-}(\omega)$, we have $(\tilde{L}f)\cdot F\in\mathscr{P}\cap\mathscr{V}_{\mathrm{ad}}$ and $f(Y)-f(0)-(\tilde{L}f)\cdot F\in\mathscr{L}_{\mathrm{ad}}$.

7c) Characterization of Hunt Semimartingales

(7.13) **Proposition.** *Every Hunt semimartingale is a random time-changed Ito process.*

Proof. By hypothesis, the local characteristics $(B, C, \tilde{\Gamma})$ of $X - X_0$ (and of X as well) are given by (6.25), with $F \in \mathscr{V}_{\mathrm{ad}}^+$ continuous. We can suppose that F is strictly increasing and satisfies $\lim_t F_t = \infty$ (by replacing F by $F_t + t$, for instance).

Let $\tau_t = \inf\{s: F_s > t\}$. Since each τ_t is a finite $(\mathscr{H}_t)$-stopping time, we can put $\hat{X}_t = X_{\tau_t}$, $\hat{\theta}_t = \theta_{\tau_t}$, $\hat{\mathscr{H}}_t = \mathscr{H}_{\tau_t}$. It is well known that $\hat{\mathbf{X}} = (\Omega, \mathscr{H}, \hat{\mathscr{H}}_t, \hat{\theta}_t, \hat{X}_t, P^x)$ is a Hunt process. By Kazamaki's theorem [25], $\hat{X}$ is a semimartingale on $(\Omega, \mathscr{H}, \hat{\mathscr{H}}_t, P^x)$ for every $x \in E$, and its local characteristics are $\hat{B}_t = B_{\tau_t}$, $\hat{C}_t = C_{\tau_t}$, and $\tilde{\Gamma}'$ defined by $\tilde{\Gamma}'([0,t] \times D) = \tilde{\Gamma}([0, \tau_t] \times D)$ for all $t \geqq 0$, $D \in \mathscr{R}^m$, see [21], ch. X. In other words, $(\hat{B}, \hat{C}, \tilde{\Gamma}')$ are given by (6.25), where F is replaced by $\hat{F}_t = F_{\tau_t}$. Since $\hat{F}_t = t$, $\hat{\mathbf{X}}$ is an Ito process.

Characterizations of Hunt semimartingales and Ito processes are very easily obtained from (7.9).

(7.14) **Theorem.** (i) $\mathbf{X}$ *is a Hunt semimartingale if and only if there exist a continuous* $F \in \mathscr{V}_{\mathrm{ad}}^+$ *such that* $\mathscr{D}_{G_F}$ *is a full and complete class.*

(ii) *Suppose* $\mathbf{X}$ *is a Hunt semimartingale, and let* $(F;\, b,\, c,\, K)$ *satisfy* (6.27) *applied to* $X - X_0$. *Then,* $\mathbb{C}^2(\mathbb{R}^m) \subset \mathscr{D}_{G_F}$, *and the following operator* L *on* $\mathbb{C}^2(\mathbb{R}^m)$

$$Lf(x) = \sum_{i<m} b^i(x) D_i f(x) + \tfrac{1}{2} \sum_{i,j \leqq m} c^{ij}(x) D_{ij} f(x) + \int K(x, dy)[f(x+y) - f(x) - 1_{[0,1]}(|y|) \sum_{i,j \leqq m} y^i D_i f(x)] \tag{7.15}$$

is a version of the restriction of G_F *to* $\mathbb{C}^2(\mathbb{R}^m)$.

(iii) *Suppose* $\mathbf{X}$ *is a Hunt semimartingale, and let* $F \in \mathscr{V}_{\mathrm{ad}}^+$ *be continuous and such that* $\mathscr{D}_{G_F}$ *is a complete class. Then, there exist* b, c, K *such that* $(F;\, b,\, c,\, K)$ *satisfies* (6.27) *applied to* $X - X_0$. *Hence,* $C^2(\mathbb{R}^m) \subset \mathscr{D}_{G_F}$, *and* (7.15) *gives a version of* G_F *on* $\mathbb{C}^2(\mathbb{R}^m)$.

Proof. (ii) If $(F;\, b,\, c,\, K)$ satisfies (6.27) applied to $X - X_0$, there exists $\tilde{b}$, $\tilde{c}$, $\tilde{K}$ such that $(F;\, \tilde{b},\, \tilde{c},\, \tilde{K})$ satisfies (6.24) and that (6.29) holds. Then, if $\tilde{L}$ is given by (7.10), we have $(\tilde{L}f) \cdot F = Lf(X_-) \cdot F$ a.s. for every $f \in \mathbb{C}^2(\mathbb{R}^m)$, and (ii) follows from (7.9, ii).

(i) Since $\mathbb{C}^2(\mathbb{R}^m)$ is a full and complete class, the necessity of the condition follows from (ii). The sufficiency of the condition is deduced from (7.9, i), (7.9, iii), (7.6), and the definition of a Hunt semimartingale.

(iii) It is immediate from (7.9, iii).

(7.16) **Theorem.** (i) $\mathbf{X}$ *is an Ito process if and only if the domain* $\mathscr{D}_G$ *of its extended generator is a full and complete class.*

(ii) *In this case, suppose* $(F_t = t;\, b,\, c,\, K)$ *satisfies* (6.27) *applied to* $X - X_0$. *Then,* $\mathbb{C}^2(\mathbb{R}^m) \subset \mathscr{D}_G$, *and the operator* L *defined by* (7.15) *is a version of the restriction of* G *to* $\mathbb{C}^2(\mathbb{R}^m)$.

Proof. It is immediate from (7.14), once noticed that $(G, \mathscr{D}_G)$ is the F-extended generator $(G_F, \mathscr{D}_{G_F})$ for $F_t = t$.

Among other things, this theorem tells us that the Ito processes are also the "diffusion processes with Lévy generator" introduced by Stroock [47], except that here we do not make continuity, Borel measurability or boundedness assumptions on b, c, K; (but we do assume universal measurability).

A number of results related to (7.14, ii) and (7.16, ii) have been proved by many authors, under hypothesis (L), or when $\mathbf{X}$ is a Feller process: see in particular Kunita [26], Skorokhod [42], [43], and [44] where (7.16, ii) is implicitly proved when the process is Feller and has finite variation over finite intervals (in that case, $c=0$ and $\int (|y| \wedge 1) K(x, dy) < \infty$). See also Ikeda and Watanabe [18] for one of the first results related to these matters.

Unfortunately, we do not have such a nice characterization as (7.16, i) in terms of the weak infinitesimal generator $(G^*, \mathscr{D}_{G^*})$, because of the lack of conditions for functions in $\mathscr{D}_G$ to belong to $\mathscr{D}_{G^*}$. However, the following is obvious: see (7.3, ii).

(7.17) **Corollary.** *Suppose* $\mathbf{X}$ *is a right process. If the domain of its weak infinitesimal generator is a full and complete class, then* $\mathbf{X}$ *is an Ito process.*

The previous theorems may be complemented in several ways. For instance, if we make the additional assumption that $X \in \mathscr{V}$ (it is the case, for example, when the process is a.s. increasing), then $\tilde{c}=0$ (resp. $c=0$) in (7.10) (resp. (7.15)): it follows that in (7.9, ii) (resp. (7.14, ii), resp. (7.16, ii)) we have $\mathbb{C}^1(\mathbb{R}^m) \subset \mathscr{D}_{G_F}$ (resp. $\mathscr{D}_{G_F}$, resp. $\mathscr{D}_G$). That is the situation examined in [44].

Finally, let us state another immediate corollary, which extends a result of Roth [39].

(7.18) **Corollary.** *Let* $(\hat{G}, \mathscr{D}_{\hat{G}})$ *be a linear operator from* $\mathbb{C}^2(\mathbb{R}^m)$ *into the space of finite* $\mathscr{R}^{m*}$*-measurable functions such that*

(i) $\mathscr{D}_{\hat{G}}$ *is a full and complete class;*

(ii) *there exists (at least) one* $\mathbb{R}^m$*-valued, right-continuous, strong Markov process* $\mathbf{X}$ *with infinite lifetime, whose extended generator is an extension of* $(\hat{G}, \mathscr{D}_{\hat{G}})$.

Then, there exists a triplet (b, c, K) *satisfying conditions* (ii)–(iv) *of* (6.27) *and such that the operator* L *on* $\mathbb{C}^2(\mathbb{R}^m)$ *defined by* (7.15) *is an extension of* $(\hat{G}, \mathscr{D}_{\hat{G}})$.

8. Stochastic Differential Equations and Markov Processes

Let $\mathbf{X} = (\Omega, \mathscr{M}, \mathscr{M}_t, \theta_t, X_t, P^x)$ be an underlying Markov process. The assumptions and notational conventions that were established at the beginning of Sect. 3 are in effect throughout this section. In addition, we denote by $\mathscr{D}$ the class of all real-valued right continuous $(\mathscr{H}_t)$-adapted processes on Ω, which a.s. admit left-hand limits.

We will consider the stochastic differential equation

$$(8.1) \qquad Y = H + F(Y) \cdot Z,$$

where $H \in \mathscr{D}$ and $Z \in \mathscr{S}$ are given processes, and $F(Y)$ denotes the "coefficient", which depends on the solution Y, this solution belonging to $\mathscr{D}$ when it exists. Under some conditions on H, Z, F, to be described later, we show in Theorem (8.11) that the process (X, Y) is Markov (resp. strong Markov) when $(\mathscr{M}_t)$ is a Markov (resp. strong Markov) filtration. The results here are similar to those of [37] but they are sharper and more general.

In order to allow simultaneous consideration of all possible deterministic initial values $H_0 = y$, $y \in \mathbb{R}$, of the solution Y it is convenient to introduce the following enlargement of the space Ω: we define $\bar{\Omega} = \Omega \times \mathbb{R}$ with the σ-fields $\bar{\mathscr{M}}_t^0 = \mathscr{M}_t^0 \otimes \mathscr{R}$, $\bar{\mathscr{M}}^0 = \mathscr{M}^0 \otimes \mathscr{R}$, and with the probability measure $\bar{P}^{x,y} = P^x \otimes \varepsilon_y$, $(x,y) \in \bar{E} = E \times \mathscr{R}$. We also introduce the filtration $(\bar{\mathscr{H}}_t)_{t \geqq 0}$, which is the smallest right-continuous filtration satisfying $\mathscr{H}_t \otimes \mathscr{R} \subset \bar{\mathscr{H}}_t$ for all $t \geqq 0$ (that is, $\bar{\mathscr{H}}_t = \bigcap_{s>t} \mathscr{H}_s \otimes \mathscr{R}$).

There is a one-to-one correspondence between processes $\bar{V}$ on $\bar{\Omega}$, and families of processes $(V^y)_{y \in \mathbb{R}}$ on Ω, which is given by $V_t^y(\omega) = \bar{V}_t(\omega, y)$: we will use both notations $\bar{V}$ and $(V^y)_{y \in \mathbb{R}}$. Finally, let us denote by $\bar{\mathscr{D}}$ the set of all $(\bar{\mathscr{H}}_t)$-adapted right-continuous processes on $\bar{\Omega}$ which have left-hand limits $\bar{P}^{x,y}$-a.s. for every $(x,y) \in \bar{E}$. Of course, $\bar{V} \in \bar{\mathscr{D}}$ implies $V^y \in \mathscr{D}$ for all $y \in \mathbb{R}$.

We turn now to the definition of the coefficient F appearing in (8.2).

(8.2) *Definition.* A coefficient F is an application $Y \to F(Y)$ of $\mathscr{D}$ into the set of all $(\mathscr{H}_t)$-predictable processes, which satisfies

(i) if Y, $Y' \in \mathscr{D}$, $\omega \in \Omega$, $t > 0$ satisfy $Y_s(\omega) = Y'_s(\omega)$ for all $s < t$, then $F(Y)_t(\omega) = F(Y')_t(\omega)$;

(ii) if $\bar{Y} = (Y^y)_{y \in \mathbb{R}}$ belongs to $\bar{\mathscr{D}}$, then $(F(Y^y))_{y \in \mathbb{R}}$ is $(\bar{\mathscr{H}}_t)$-predictable.

(8.3) *Remark.* Let W be the set of all right-continuous functions on $\mathbb{R}_+$, with the usual filtration $(\mathscr{W}_t)_{t \geqq 0}$. In most cases, (see [21] for instance) F is constructed as follows: we consider a function $\hat{F}$ on $\Omega \times W \times \mathbb{R}$ which, considered as a process on $\Omega \times W$, is $(\mathscr{H}_t \otimes \mathscr{W}_t)$-predictable, and we set $F(Y)_t(\omega) = \hat{F}(\omega, Y_.(\omega), t)$ for $y \in \mathscr{D}$. In this case, conditions (i) and (ii) above are obviously satisfied.

A *solution* of Eq. (8.1) for the measure P^x is a process $Y \in \mathscr{D}$ such that $F(Y) \in L(Z, P^x)$ and that (8.1) holds true up to a P^x-evanescent set (note that by (8.2, i), $F(Y)$ and $F(Y')$ are P^x-indistinguishable whenever Y and Y' are such).

(8.4) *Definition.* The coefficient F is said to be *acceptable* if for all $x \in E$, $H \in \mathscr{D}$, $Z \in \mathscr{S}$ the following hold:

(i) a unique (up to a P^x-evanescent set) solution Y of (8.1) exists;

(ii) we can define by induction $Y(1) = H$,

(8.5) $$Y(n+1) = H + F(Y(n)) \cdot Z, \qquad n \geqq 1,$$

and then $P^x\text{-}\lim_n Y(n)_t = Y_t$ for every $t \geqq 0$.

There has been much recent progress in obtaining sufficient conditions for F to be acceptable. These conditions suppose that F is constructed via a $\hat{F}$ as in (8.3), and amount to a suitable local Lipshitz condition on the function $\hat{F}(\omega, \cdot, t)$: see [8, 11, 29, 21].

(8.6) *Remark.* For simplicity, we consider only one-dimensional processes. Obviously the same results would hold true when Z, H, Y are multidimensional and F is matrix valued, with the appropriate dimensions. In that case, the stochastic integral in (8.1) may be taken componentwise, or more generally "globally": see [22].

The following is a simple consequence of Theorem (3.12). For ease of referencing later, we state it for a "measurable" family of Eqs. (8.1) instead of a single one.

(8.7) **Theorem.** *Let $\bar{H}=(H^y)_{y\in\mathbb{R}}\in\bar{\mathscr{D}}$. Let $\bar{Z}=(Z^y)_{y\in\mathbb{R}}\in\bar{\mathscr{D}}$ be such that $Z^y\in\mathscr{S}$ for every $y\in\mathbb{R}$; let F be an acceptable coefficient. Then there exists $\bar{Y}=(Y^y)_{y\in\mathbb{R}}\in\bar{\mathscr{D}}$ such that for every $y\in\mathbb{R}$ and for every measure P^x, Y^y is the solution of the equation*

$$(8.8)\qquad Y^y=H^y+F(Y^y)\cdot Z^y.$$

Proof. Although it would be possible to use a direct argument based upon a mild extension of the lemmas of § 3c for processes "depending on a parameter", see [46], we will rather base our argument on an enlargement of $\mathbf{X}$ to which these lemmas apply directly.

On $\bar{\Omega}$ we define the operators $\bar{\theta}_t(\omega,y)=(\theta_t(\omega),y)$ and the $\bar{E}$-valued process $\bar{X}_t(\omega,y)=(X_t(\omega),y)$. It is immediate to check that $\bar{\mathbf{X}}=(\bar{\Omega},\bar{\mathscr{H}}^0,\bar{\mathscr{H}}^0_{t+},\bar{\theta}_t,\bar{X}_t,\bar{P}^{x,y})$ is a Markov process satisfying all the assumptions of § 3a, with $(\bar{\mathscr{H}}_t)$ satisfying (3.6) relative to $\bar{\mathbf{X}}$. Since $\bar{P}^{x,y}=P^x\otimes\varepsilon_y$, $\bar{Z}$ is a semimartingale over each space $(\bar{\Omega},\bar{\mathscr{H}},\bar{\mathscr{H}}_t,\bar{P}^{x,y})$. It is immediate to check that, for any $(\bar{\mathscr{H}}_t)$-predictable process $\bar{K}$ on $\bar{\Omega}$, $\bar{K}\in\bigcap_{(x,y)\in\bar{E}}L(\bar{Z},\bar{P}^{x,y})$ if and only if $K^y\in\bigcap_{x\in E}L(Z^y,P^y)$ for all $y\in\mathbb{R}$, and that in this case we can find a version of the stochastic integral $\bar{K}\cdot\bar{Z}$ valid for each space $(\bar{\Omega},\bar{\mathscr{H}},\bar{\mathscr{H}}_t,\bar{P}^{x,y})$ and such that $(\bar{K}\cdot\bar{Z})^y=K^y\cdot Z^y$ for all $y\in\mathbb{R}$.

In view of these remarks, and using (8.2, ii), (8.4, ii), and (3.12, vi) applied to $\bar{\mathbf{X}}$, we can define by induction $\bar{Y}(1)=\bar{H}$, $\bar{Y}(n+1)=\bar{H}+\bar{F}(\bar{Y}(n))\cdot\bar{Z}$, where $\bar{F}(\bar{V})^y=F(V^y)$ for all $y\in\mathbb{R}$, $\bar{V}\in\bar{\mathscr{D}}$. We have $\bar{Y}(n)\in\bar{\mathscr{D}}$ for all $n\geqq 1$. Let $Y^{x,y}$ be a version of the solution of (8.8) for the measure P^x. According to (8.4, ii) we have P^x-$\lim_{(n)} Y(n)^y_t=Y^{x,y}_t$ for all $x\in E$, $y\in\mathbb{R}$, $t\geqq 0$. Therefore $\bar{P}^{x,y}$-$\lim_{(n)} \bar{Y}(n)_t=\bar{Y}^{x,y}_t$ for all $(x,y)\in\bar{E}$, $t\geqq 0$, where $\bar{Y}^{x,y}_t(\omega,y')=Y^{x,y}_t(\omega)$. Applying Lemma (3.29) to $\bar{\mathbf{X}}$, we obtain a $\bar{Y}\in\bar{\mathscr{D}}$ that is $\bar{P}^{x,y}$-indistinguishable from $\bar{Y}^{x,y}$ for all $(x,y)\in\bar{E}$, and the theorem is proven.

We turn now to our desired Markov property. For this, we need further assumptions on H, Z, F. We suppose that the processes H and Z are additive or strongly additive. Concerning F we need the following

(8.9) *Definition.* The coefficient F is said to be *homogeneous* (resp. *strongly homogeneous*) if

(i) for all $Y\in\mathscr{D}$, $s\geqq 0$ (resp. S finite stopping time), the processes $\Theta_s(F(Y))$ and $F(\Theta_s Y)$ (resp. $\Theta_S(F(Y))$ and $F(\Theta_S Y)$) are indistinguishable on $]\!]s,\infty[\![$ (resp. on $]\!]S,\infty[\![$).

(ii) for all $\omega\in\Omega$, $s\geqq 0$, $t>s$, $Y, Y'\in\mathscr{D}$ such that $Y_r(\omega)=Y'_r(\omega)$ when $s<r<t$, we have $F(Y)_t(\omega)=F(Y')_t(\omega)$.

For example, the coefficient F defined by

$$F(Y)_t(\omega)=f(X_{t^-}(\omega), Y_{t^-}(\omega))\, 1_{(0,\infty)}(t)$$

is strongly homogeneous when X is left-hand-limited and f is a function on $\bar{E}$.

According to (8.7), we denote by Y^y a version of the solution of

$$(8.10) \qquad Y^y = y + H + F(Y^y)\cdot Z$$

that is valid for every P^x and such that $\bar{Y}=(Y^y)_{y\in\mathbb{R}}\in\bar{\mathscr{D}}$ (we apply (8.7) to $Z^y=Z$ and $H^y=y+H$; recall that here $H_0=0$ a.s.).

(8.11) **Theorem.** *Suppose $(\mathscr{M}_t)$ is a Markov (resp. strong Markov) filtration. Let $Z\in\mathscr{S}$, $H\in\mathscr{D}$ be additive (resp. strongly additive). Let F be an acceptable, homogeneous (resp. strongly homogeneous) coefficient. Let $\bar{Y}=(Y^y)_{y\in\mathbb{R}}\in\bar{\mathscr{D}}$ be such that Y^y is the solution of Eq.* (8.10) *for every $y\in\mathbb{R}$. Then*

$$(8.12) \qquad \bar{E}^{x,y}[f(X_{S+t}, \bar{Y}_{S+t})\,|\,\mathscr{H}_S] = \bar{E}^{X_S, \bar{Y}_S}[f(X_t, \bar{Y}_t)], \quad f\in b(\mathscr{E}\times\mathbb{R})$$

for all $(x,y)\in\bar{E}$, $S=s\in\mathbb{R}_+$ *(resp. S finite $(\mathscr{H}_t)$-stopping time).*

In other words, $(\bar{\Omega}, \mathscr{H}, \mathscr{H}_t, (X_t, \bar{Y}_t), \bar{P}^{x,y})$ is a (strong) right-continuous Markov process in the sense of [3], except that the shift operators are not present. We can define these operators as follows

$$\bar{\theta}_t(\omega, y) = (\theta_t(\omega), \bar{Y}_t(\omega, y)).$$

We have $X_{t+s}=X_t\circ\bar{\theta}_s$, on $\bar{\Omega}$, but $\bar{\theta}_{t+s}=\bar{\theta}_t\circ\bar{\theta}_s$ and $\bar{Y}_{t+s}=\bar{Y}_t\circ\bar{\theta}_s$ hold only almost surely.

We shall prove only the simple Markov property, since as usual the strong Markov property is proved exactly in the same way by just replacing $s\geqq 0$ by a finite stopping time S. We begin by two lemmas, in which $s\geqq 0$ is fixed. Set

$$(8.13) \qquad H'_t = H_t - H_{t\wedge s}, \qquad Z'_t = Z_t - Z_{t\wedge s}.$$

If G is a finite $\mathscr{H}_s$-measurable variable, we denote by $\tilde{Y}^G$ a version of the solution of the following equation, valid for all P^x (see (8.7)):

$$(8.14) \qquad \tilde{Y}^G = G\, 1_{[\![s,\infty[\![} + H' + F(\tilde{Y}^G)\cdot Z'.$$

(8.15) **Lemma.** *If $G=Y^y_s$, we have $Y^G_t=Y^y_t$* a.s. *for all $t\geqq s$.*

Proof. Set

$$\tilde{Y}_u = Y^y_u\, 1_{\{u<s\}} + \tilde{Y}^G_u\, 1_{\{u\geqq s\}},$$

which belongs to $\mathscr{D}$. If $t\geqq s$, by using (8.13) and (8.10) we obtain

$$\begin{aligned}\tilde{Y}_t = \tilde{Y}^G_t &= Y^y_s + H'_t + F(\tilde{Y}^G)\cdot Z'_t\\ &= y + H_t + F(Y^y)\cdot Z_s + F(\tilde{Y}^G)\cdot Z'_t.\end{aligned}$$

By (8.9, ii) we have $F(Y^y)_u = F(\tilde{Y})_u$ if $u\leqq s$, and $F(\tilde{Y}^G)_u = F(\tilde{Y})_u$ if $u>s$. Since $Z'_u=0$ if $u\leqq s$, we obtain

(8.16) $$\tilde{Y}_t = y + H_t + F(\tilde{Y}) \cdot Z_t$$

for all $t \geqq s$. Since (8.16) obviously holds for $t < s$, we obtain that $\tilde{Y}$ is a solution of (8.10), and the result follows from (8.4, i).

(8.17) **Lemma.** *The processes $\Theta_s Y^y$ and $\tilde{Y}^y$ are indistinguishable (recall that $\tilde{Y}^y$ is the solution of* (8.14) *with $G = y$).*

Proof. By the definition (3.14) and by (3.15, vi), we have

$$\Theta_s Y^y = y 1_{[\![s, \infty[\![} + \Theta_s H + (\Theta_s F(Y^y)) \cdot (\Theta_s Z), \text{ a.s.}$$

(3.19) implies $\Theta_s H = H'$ a.s. and $\Theta_s Z = Z'$ a.s. (8.9, i) implies that $(\Theta_s F(Y^y))_u = F(\Theta_s Y^y)_u$ a.s. if $u > s$; since $Z'_u = 0$ for $u \leqq s$, we obtain

$$\Theta_s Y^y = y 1_{[\![s, \infty[\![} + H' + F(\Theta_s Y^y) \cdot Z' \qquad \text{a.s.}$$

The result follows from the uniqueness (8.4, i) of the solution of (8.14).

Proof of (8.11). Since for each $(x, y) \in \bar{E}$ the σ-field $\bar{\mathscr{H}}_s$ is contained in the $\bar{P}^{x,y}$-completion of $\mathscr{H}_s \otimes \{\phi, \mathbb{R}\}$, we have only to prove that

(8.18) $$\bar{E}^{x,y}[Vf(X_{s+t}, \bar{Y}_{s+t})] = \bar{E}^{x,y}[V\bar{E}^{X_s, \bar{Y}_s}[f(X_t, \bar{Y}_t)]]$$

for all $s \geqq 0$, $t \geqq 0$, $V \in b\mathscr{H}_s$, $f \in b(\mathscr{E} \otimes \mathscr{R})$. Let $G = Y^y_s$. We have

$$\begin{aligned}\bar{E}^{x,y}[Vf(X_{s+t}, \bar{Y}_{s+t})] &= E^x[Vf(X_{s+t}, Y^y_{s+t})] \\ &= E^x[Vf(X_{s+t}, \tilde{Y}^G_{s+t})]\end{aligned}$$

by (8.15). But (8.7) implies that $(\omega, y) \to \tilde{Y}^y_{s+t}(\omega)$ is $\mathscr{H}_{s+t} \otimes \mathscr{R}$-measurable while $G \in \mathscr{H}_s$, so the preceding expression is equal to

$$\begin{aligned}&= \int P^x(d\omega)\, V(\omega)\, E^x[f(X_{s+t}(\cdot), \tilde{Y}^{G(\omega)}_{s+t}(\cdot)) | \mathscr{H}_s](\omega) \\ &= \int P^x(d\omega)\, V(\omega)\, E^x[f(X_t, Y^{G(\omega)}_t) \circ \theta_s(\omega) | \mathscr{H}_s](\omega)\end{aligned}$$

by (8.17). Applying the Markov property (3.2, ii) shows that this is equal to

$$\begin{aligned}&= \int P^x(d\omega)\, V(\omega)\, E^{X_s(\omega)}[f(X_t, Y^{G(\omega)}_t)] \\ &= \int P^x(d\omega)\, V(\omega)\, \bar{E}^{X_s(\omega), G(\omega)}[f(X_t, \bar{Y}_t)] \\ &= E^x[V\bar{E}^{X_s, Y^y_s}[f(X_t, \bar{Y}_t)]] \\ &= \bar{E}^{x,y}[V\bar{E}^{X_s, \bar{Y}_s}[f(X_t, \bar{Y}_t)]]\end{aligned}$$

Thus (8.18) is established, and the proof is complete.

Now, an obvious question arises: if one knows that the underlying Markov process **X** is a right process, is $(X, \bar{Y})$ also a right process? Of course $(X, \bar{Y})$ is not in general a right process in the usual sense, since for instance it does not admit the usual shifts $\bar{\theta}_t$. However it may happen that its canonical realization is a right process: this is a property of its transition semigroup, which will then be called a *right semi-group*.

(8.19) **Corollary.** *Suppose* **X** *is a right process. Let* $Z \in \mathcal{S}_{ad}$, $H \in \mathcal{D}$ *be additive and* $(\mathcal{F}^e_{t+})$*-adapted. Let F be an acceptable strongly homogeneous coefficient satisfying* (8.2) *with* $\mathcal{H}_t = \mathcal{F}^e_{t+}$. *Let* $\bar{Y} = (Y^y)_{y \in \mathbb{R}} \in \bar{\mathcal{D}}$ *be such that* Y^y *is the solution of Eq.* (8.10) *for every* $y \in \mathbb{R}$. *Then the process* $(X, \bar{Y})$ *admits a right transition semi-group.*

Because of (3.21), Z and H are strongly additive. The assumption that Z be $(\mathcal{F}^e_{t+})$-adapted is not a restriction since by a slight extension of [1], any $Z \in \mathcal{S}_{ad}$ is indistinguishable from a $Z' \in \mathcal{S}_{ad}$ which is $(\mathcal{F}^e_{t+})$-adapted.

Proof. We put $\bar{\mathcal{H}}_t = \mathcal{F}^e_{t+}$, and our assumptions imply that $\bar{Y}$ is $(\bar{\mathcal{H}}_t)$-adapted, see (8.7). If $(\bar{P}_t)$ denotes the transition semigroup of $(X, \bar{Y})$, it follows that $(x, y) \to \bar{P}_t \bar{f}(x, y) = E^x[\bar{f}(X_t, Y^y_t)]$ is $\mathcal{E}^e \times \mathcal{R}$-measurable for all $\bar{f} \in b\bar{\mathcal{E}}$. Since $\mathcal{E}^e$ is contained in the σ-field of nearly Borel subsets of E for **X**, it is obvious that $\bar{P}_t \bar{f}$ is nearly-Borel-measurable relative to $(X, \bar{Y})$ for every $\bar{f} \in b\bar{\mathcal{E}}$. By [41], (7.6), this property implies that $(\bar{P}_t)$ is a right semigroup.

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Received January 5, 1980