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The D.L.R. Conditions for Translation Invariant Gaussian Measures on $\mathscr{S}'(R^d)$

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Introduction

Let $\mathscr{S}(R^d)$ be the Schwartz space of real-valued C^∞ functions on R^d which, together with all their derivatives, are rapidly decreasing, and denote by $\mathscr{S}'(R^d)$, the dual of $\mathscr{S}(R^d)$, the space of tempered distributions. If $\mathscr{G}$ is an open set in R^d, let $\mathscr{F}_{\mathscr{G}}$ be the σ-algebra generated by the functions $\varphi \to \varphi(f)$ as f runs through $C_0^\infty(\mathscr{G})$; and if S is an arbitrary non-empty subset of R^d, define $\mathscr{A}_S = \bigcap_{\varepsilon>0} \mathscr{F}_{S^\varepsilon}$, where $S^\varepsilon = \{x \in R^d : |x-S| < \varepsilon\}$. A probability measure μ on $\mathscr{S}'(R^d)$ is called a *Markov random field* if for all bounded open $\mathscr{G} \subseteq R^d$ and all bounded $\mathscr{F}_{\mathscr{G}}$-measurable $\Phi: \mathscr{S}'(R^d) \to C$, $E^\mu[\Phi \mid \mathscr{A}_{\mathscr{G}^c}] = E^\mu[\Phi \mid \mathscr{A}_{\partial\mathscr{G}}]$ (a.s., μ).

There are many known examples of Markov random fields in this context (cf. [4], [8], and Theorem (1.5) below). Simplest among these are those which are Gaussian and have conditional marginals which are translation invariant. It is with such fields that we will be dealing in this paper. Indeed, the problem that we want to solve is that of describing the set of Markov random fields ν which have the same conditional marginals as a given Gaussian translation invariant Markov random field μ. That is, given μ, let $\mathscr{M}_\mu$ be the set of all ν such that $\nu|_{\mathscr{F}_{\mathscr{G}}} \ll \mu|_{\mathscr{F}_{\mathscr{G}}}$ for all bounded open $\mathscr{G}$'s and $E^\nu[\Phi \mid \mathscr{A}_{\mathscr{G}^c}] = E^\mu[\Phi \mid \mathscr{A}_{\partial\mathscr{G}}]$ (a.s., ν) for all bounded open $\mathscr{G}$'s and all bounded $\mathscr{F}_{\mathscr{G}}$-measurable Φ's. We want to describe $\mathscr{M}_\mu$. (It is hard to miss the analogy between this problem and the problem of phase transition for a lattice gas as formulated by Dobrushin, Lanford, and Ruelle. Indeed, the problem is the same, the only change is in the context.) What we will show is that if the covariance of μ is given by $(\varphi, (-L)^{-1}\varphi)$, where L is a constant coefficient differential operator, then (under mild conditions on L) the extreme elements of $\mathscr{M}_\mu$ coincide with the translates of μ by tempered distributions H satisfying $LH = 0$. As a consequence we see that if $L1 = 0$ then there are many translation invariant Markov random fields with the specified con-

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ditional expectations. This situation should be compared with those in [1], [4], and [7]. Also the analogy between our results and those of Dobrushin [2] should be noted. Qualitatively the results are the same; however, in [2] Dobrushin concerns himself with random fields over the integer lattice.

Finally, once we have obtained the description of $\mathcal{M}_\mu$, we have been able to isolate an analytically prescribed subspace S of $\mathcal{S}'(R^d)$ such that $\mu(S)=1$ and the only extreme $\nu\in\mathcal{M}_\mu$ with $\nu(S)>0$ is $\nu=\mu$.

We take this opportunity to thank L. Accardi for mentioning this problem to us. Our only regret is that he did not have time to tell us for what he wanted to use the solution. We hope to eventually find out.

Section (1)

Up through Theorem (1.5), the contents of this section are simply our interpretation of Nelson's ideas. We have put in the details mostly to satisfy ourselves that Nelson's scheme works without a hitch even in the "mass free" case. Furthermore, we will need the notation introduced along the way.

Let $\sigma: R^d\to R^1$ be an even non-negative polynomial and denote by L the real constant coefficient differential operator whose symbol is σ. That is $Lf=(\sigma\hat{f})^\vee$ for $f\in\mathcal{S}(R^d)$, where "$\hat{\ }$" and "$\check{\ }$" are, respectively, the symbols denoting Fourier and inverse Fourier transform. Throughout we will assume that

$$\int_{\{x:\,\sigma(x)\leqq 1\}} \frac{1}{\sigma(x)}\,dx<\infty. \tag{1.1}$$

Next, define $A^{1/2}: \mathcal{S}(R^d)\to L^2(R^d)$ by $A^{1/2}f=\left(\frac{1}{\sigma^{1/2}}\hat{f}\right)^\vee$ and introduce on $\mathcal{S}(R^d)$ the inner product $(\cdot,\cdot)_A$ given by $(f,g)_A=(A^{1/2}f, A^{1/2}g)$ (throughout $(\cdot,\cdot)$ stands for the usual Hermitian $L^2(R^d)$-inner product). Complete $\mathcal{S}(R^d)$ with respect to the norm $\|\cdot\|_A$ determined by $(\cdot,\cdot)_A$, and let $\mathcal{H}_A$ denote this completion. It will be convenient for us to identify $\mathcal{H}_A$ with the space of $\chi\in\mathcal{S}'(R^d)$ such that $\hat{\chi}\in L^1_{\text{loc}}(R^d)$ and $\int\frac{1}{\sigma(x)}|\hat{\chi}(x)|^2\,dx<\infty$. That is, we will think of $\mathcal{H}_A$ as a subspace of $\mathcal{S}'(R^d)$. Clearly, the action of $\chi\in\mathcal{H}_A$ as an element of $\mathcal{S}'(R^d)$ is given by $\chi(f)=\int\hat{\chi}(x)\overline{\hat{f}(x)}\,dx$. Observe that if $\rho\in C_0^\infty(R^d)$ with $\int\rho(x)\,dx=1$ and $\rho_\varepsilon(x)=\varepsilon^{-d}\rho(x/\varepsilon)$, $\varepsilon>0$, then for any $\varphi\in\mathcal{H}_A$:

$$\|\rho_\varepsilon*\chi-\chi\|_A^2=\int\frac{1}{\sigma(x)}|\hat{\rho}(\varepsilon x)\,\hat{\chi}(x)-\hat{\chi}(x)|^2\,dx\to 0$$

as $\varepsilon\downarrow 0$, by Lebesgue's dominated convergence theorem. Hence if $\chi\in\mathcal{H}_A$ and $\operatorname{supp}(\chi)\subset\subset\mathcal{G}$ ($\subset\subset$ means "compactly contained"), then we can choose $\{f_n\}_1^\infty\subseteq C_0^\infty(\mathcal{G})$ so that $\|f_n-\chi\|_A\to 0$.

Now let μ be the probability measure on $\mathcal{S}'(R^d)$ such that

$$E^\mu[e^{i\varphi(f)}]=e^{-1/2(f,f)_A},\quad f\in\mathcal{S}(R^d). \tag{1.2}$$

Clearly μ is Gaussian and translation invariant. Denote by $\mathscr{H}_\mu$ the closure in $L^2(\mu)$ of the set of random variables $\varphi(f)$, $f\in\mathscr{S}(R^d)$. Thinking of $\mathscr{S}(R^d)$ as a dense subspace of $\mathscr{H}_A$ and noting that $f\to\varphi(f)$ is an isometry taking $\mathscr{S}(R^d)$ into a dense subspace of $\mathscr{H}_\mu$, we see that $\mathscr{H}_A$ is isometrically isomorphic to $\mathscr{H}_\mu$. Given $\chi\in\mathscr{H}_A$, we will use X_χ to stand for the element of $\mathscr{H}_\mu$ into which χ is sent by this isomorphism.

If $S\neq\varnothing$ is a subset of R^d, we use $\mathscr{F}_S$ and $\tilde{\mathscr{F}}_S$ to denote, respectively, the σ-algebras over $\mathscr{S}'(R^d)$ generated by $\{\varphi(f): f\in\mathscr{S}(R^d) \text{ and } \operatorname{supp}(f)\subset\subset S\}$ and $\{X_\chi: \chi\in\mathscr{H}_A \text{ and } \operatorname{supp}(\chi)\subset\subset S\}$. Clearly $\mathscr{F}_S\subseteq\tilde{\mathscr{F}}_S$. On the other hand, if S is open and $\chi\in\mathscr{H}_A$ satisfies $\operatorname{supp}(\chi)\subset S$, then we can find $\{f_n\}_1^\infty\subseteq C_0^\infty(S)$ such $\|\varphi(f_n)-X_\chi\|_{L^2(\mu)}=\|f_n-\chi\|_A\to 0$ as $n\to\infty$. Thus

$$(1.3)\qquad \mathscr{F}_S=\tilde{\mathscr{F}}_S(\text{a.s.}, \mu),\qquad S \text{ open in } R^d.$$

Next set $\mathscr{A}_S=\bigcap_{\varepsilon>0}\mathscr{F}_{S^\varepsilon}$ where $S^\varepsilon=\{x\in R^d: |x-S|<\varepsilon\}$. Because of (1.3)

$$\mathscr{A}_S=\tilde{\mathscr{A}}_S\equiv\bigcap_{\varepsilon>0}\tilde{\mathscr{F}}_{S^\varepsilon}\qquad(\text{a.s.}, \mu).$$

(1.4) **Lemma.** *Let $\mathscr{G}$ be a bounded open subset of R^d and denote by π the orthogonal projection in $\mathscr{H}_A$ onto the subspace $\{\chi\in\mathscr{H}_A: \operatorname{supp}(\chi)\subseteq\mathscr{G}^c\}$. Then for any $g\in C_0^\infty(\mathscr{G})$, $X_{\pi g}$ is $\mathscr{F}_{\partial\mathscr{G}}$-measurable and $Z_g\equiv\varphi(g)-X_{\pi g}$ is a Gaussian random variable which has mean 0 and is independent of $\tilde{\mathscr{F}}_{\mathscr{G}^c}$.*

Proof. To see that $X_{\pi g}$ is $\tilde{\mathscr{F}}_{\partial\mathscr{G}}$-measurable we need only check that $\operatorname{supp}(\pi g)\subseteq\partial\mathscr{G}$. Certainly $\operatorname{supp}(\pi g)\subseteq\mathscr{G}^c$. At the same time, if $\psi\in C_0^\infty((\bar{\mathscr{G}})^c)$, then $L\psi\in C_0^\infty((\bar{\mathscr{G}})^c)$ and so:

$$\pi g(\psi)=(\pi g, L\psi)_A=(g, L\psi)_A=(g,\psi)=0.$$

Thus $\operatorname{supp}(\pi g)\subseteq\partial\mathscr{G}$.

To prove the desired properties of Z_g, note that $\mathscr{H}_\mu$ under μ is a Gaussian family of mean 0 random variables and therefore Z_g is certainly a mean 0 Gaussian random variable. Furthermore, if $\chi\in\mathscr{H}_A$ with $\operatorname{supp}(\chi)\subset\subset\mathscr{G}^c$, then

$$E^\mu[Z_g\bar{X}_\chi]=((I-\pi)g,\chi)_A=0.$$

Hence Z_g is independent of $\tilde{\mathscr{F}}_{\mathscr{G}^c}$. Q.E.D.

(1.5) **Theorem** (Nelson). *Let $\mathscr{G}$ be a bounded open set in R^d and $\Phi: \mathscr{S}'(R^d)\to\mathbb{C}$ a bounded $\mathscr{F}_{\mathscr{G}}$-measurable function. Then $E^\mu[\Phi\,|\,\mathscr{A}_{\mathscr{G}^c}]=E^\mu[\Phi\,|\,\mathscr{A}_{\partial\mathscr{G}}]$ (a.s., μ). In particular, there is an $\mathscr{A}_{\partial\mathscr{G}}$-measurable version of $E^\mu[\Phi\,|\,\mathscr{A}_{\mathscr{G}^c}]$.*

Proof. Because μ is a Gaussian measure, we need only check that for each $g\in C_0^\infty(\mathscr{G})$: $E^\mu[\varphi(g)\,|\,\mathscr{A}_{\mathscr{G}^c}]$ admits an $\mathscr{A}_{\partial\mathscr{G}}$-measurable version. To this end, set $\mathscr{G}_\varepsilon=(\overline{(\mathscr{G}^c)^\varepsilon})^c$ and define π_ε accordingly (as in Lemma (1.4)) with $\mathscr{G}_\varepsilon$ replacing $\mathscr{G}$. Then, by Lemma (1.4), $E^\mu[\varphi(g)\,|\,\tilde{\mathscr{F}}_{(\mathscr{G}_\varepsilon)^c}]=X_{\pi_\varepsilon g}$ (a.s., μ); and so, by the remarks preceding Lemma (1.4), $E^\mu[\varphi(g)\,|\,\tilde{\mathscr{F}}_{(\mathscr{G}_{1/n})^c}]$ admits a $\mathscr{F}_{(\partial\mathscr{G})^{2/n}}$-measurable version Φ_n. On the other hand, by the martingale convergence theorem, $E^\mu[\varphi(g)\,|\,\tilde{\mathscr{F}}_{(\mathscr{G}_{1/n})^c}]$

$\to E^\mu[\varphi(g)\,|\,\tilde{\mathscr{A}}_{\mathscr{G}^c}]$ as $n\to\infty$; and, again by the remarks preceding Lemma (1.4), $E^\mu[\varphi(g)\,|\,\tilde{\mathscr{A}}_{\mathscr{G}^c}]=E^\mu[\varphi(g)\,|\,\mathscr{A}_{\mathscr{G}^c}]$ (a.s., μ). Hence, $\Phi_n\to E^\mu[\varphi(g)\,|\,\mathscr{A}_{\mathscr{G}^c}]$ (a.s., μ), and therefore $E^\mu[\varphi(g)\,|\,\mathscr{A}_{\mathscr{G}^c}]$ admits an $\mathscr{A}_{\partial\mathscr{G}}=\bigcap_{n\geq 1}\mathscr{F}_{(\partial\mathscr{G})^{2/n}}$-measurable version. Q.E.D.

We now know that μ is a Markov random field. Define the set $\mathscr{M}_\mu$ as in the introduction. Our next result shows that in general $\mathscr{M}_\mu$ will contain elements besides μ itself.

(1.6) **Theorem.** *Let $H\in\mathscr{S}'(R^d)\cap C^\infty(R^d)$ satisfy $LH=0$. Then the translate μ_H of μ by H (i.e. μ_H is the distribution of $\varphi+H$ under μ) is an element of $\mathscr{M}_\mu$.*

Proof. We must first show that $\mu_H|_{\mathscr{A}_\mathscr{G}}\ll\mu|_{\mathscr{A}_\mathscr{G}}$ for bounded open $\mathscr{G}$'s, and clearly it is enough to do this when $\mathscr{G}=B(0,R)$ for $R>0$. Given R, choose $\eta\in C_0^\infty(B(0,2R))$ so that $\eta\equiv 1$ on $B(0,3R/2)$, and set $f=L(\eta H)$ and

$$X(\varphi)=\exp[\varphi(f)-1/2(f,f)_A],\qquad \varphi\in\mathscr{S}'(R^d).$$

Then for any $g\in C_0^\infty(B(0,5R/4))$:

$$\begin{aligned}E^\mu[e^{i\varphi(g)}X(\varphi)]&=\exp[-1/2(g,g)_A+i(g,f)_A]\\&=\exp[-1/2(g,g)_A+i(g,AL(\eta H))]\\&=\exp[-1/2(g,g)_A+iH(\eta g)]\\&=\exp[-1/2(g,g)_A+iH(g)]=E^{\mu_H}[e^{i\varphi(g)}].\end{aligned}$$

Thus $X\,d\mu$ equals $d\mu_H$ on $\mathscr{A}_{B(0,R)}$, and so $\mu_H|_{\mathscr{A}_{B(0,R)}}\ll\mu|_{\mathscr{A}_{B(0,R)}}$.

To prove that $E^{\mu_H}[\Phi\,|\,\mathscr{A}_{\mathscr{G}^c}]=E^\mu[\Phi\,|\,\mathscr{A}_{\partial\mathscr{G}}]$ (a.s., μ_H) for bounded $\mathscr{F}_\mathscr{G}$-measurable Φ's, choose $R_0>0$ so that $\bar{\mathscr{G}}\subset\subset B(0,R_0)$ and let $R>R_0$ be arbitrary. Define η, f, and X as in the preceding paragraph relative to R. Then for any $B\in\mathscr{A}_{\mathscr{G}^c\cap B(0,R)}$:

$$\begin{aligned}E^{\mu_H}[\Phi,B]&=E^\mu[X\Phi,B]=E^\mu[XE^\mu[\Phi\,|\,\mathscr{A}_{\mathscr{G}^c}],B]\\&=E^\mu[XE^\mu[\Phi\,|\,\mathscr{A}_{\partial\mathscr{G}}],B]=E^{\mu_H}[E^\mu[\Phi\,|\,\mathscr{A}_{\partial\mathscr{G}}],B]\end{aligned}$$

since $\operatorname{supp}(f)\subseteq B(0,R)^c$ and therefore X is $\mathscr{A}_{\mathscr{G}^c}$-measurable. Q.E.D.

The rest of this section is devoted to proving the converse of Theorem (1.6). For this purpose, we want to show that every $\nu\in\mathscr{M}_\mu$ is a stationary measure for a certain Ornstein-Uhlenbeck flow on $\mathscr{S}'(R^d)$. Since in an earlier paper [3], we classified all such stationary measures, we will be essentially done once we have shown this.

Define the semi-group $\{T_t\colon t\geq 0\}$ on $\mathscr{S}(R^d)$ by $T_t f=(e^{-t/2\sigma}\hat{f})^\vee$. For convenience, we will use f_t to denote $T_t f$. It is shown in [3] that $\{T_t\colon t\geq 0\}$ determines a unique transition probability function $P(t,\psi,\cdot)$ on $\mathscr{S}'(R^d)$ via the equation:

$$\int F(\varphi(f))\,P(t,\psi,d\varphi)=\int\gamma\left(\int_0^t\|f_s\|^2\,ds,\,y-\psi(f_t)\right)F(y)\,dy \tag{1.7}$$

for $F \in C_b(R^1)$ and $f \in \mathscr{S}(R^d)$, where $\gamma(\tau, \xi) = \frac{1}{(2\pi\tau)^{1/2}} e^{-\xi^2/2\tau}$. $P(t, \psi, \cdot)$ is the transition probability function for the Ornstein-Uhlenbeck flow alluded to in the preceding paragraph.

(1.8) **Lemma.** *Given $\vec{f} = (f_1, \ldots, f_n) \in (\mathscr{S}(R^d))^n$, let $\Gamma_{\vec{f}}(t, \cdot)$ denote the Gaussian measure on R^n having mean 0 and covariance*

$$\left(\left(\int_0^t ((f_i)_s, (f_j)_s)\, ds\right)\right)_{1 \leqq i, j \leqq n}.$$

Then for $F \in C_b(R^n)$:

$$\begin{aligned}(1.9) \qquad & \int F(\varphi(f_1), \ldots, \varphi(f_n))\, P(t, \psi, d\varphi) \\ &= (F * \Gamma_{\vec{f}}(t, \cdot))(\psi((f_1)_t), \ldots, \psi((f_n)_t)).\end{aligned}$$

In particular, if $F \in C_b^2(R^n)$ and we define

$$F^t(\xi_1, \ldots, \xi_n) = (F * \Gamma_{\vec{f}}(t, \cdot))(\xi_1, \ldots, \xi_n),$$

then

$$\begin{aligned}(1.10) \qquad & \int F(\varphi(f_1), \ldots, \varphi(f_n))\, P(t, \psi, d\varphi) - F(\psi(f_1), \ldots, \psi(f_n)) \\ &= \int_0^t 1/2 \left(\sum_{i,j=1} ((f_i)_s, (f_j)_s) \frac{\partial^2 F^s}{\partial \xi_i \partial \xi_j} \right. \\ &\quad \left. - \sum_{i=1}^n \psi(L(f_i)_s) \frac{\partial F^s}{\partial \xi_i} \right) (\psi((f_1)_s), \ldots, \psi((f_n)_s))\, ds;\end{aligned}$$

and so as $t \downarrow 0$:

$$\begin{aligned}(1.11) \qquad & \frac{1}{t} \left[\int F(\varphi(f_1), \ldots, \varphi(f_n))\, P(t, \psi, d\varphi) - F(\psi(f_1), \ldots, \psi(f_n))\right] \\ &\to 1/2 \left(\sum_{i,j=1}^n (f_i, f_j) \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} - \sum_{i=1}^n \psi(Lf_i) \frac{\partial F}{\partial \xi_i} \right) (\psi(f_1), \ldots, \psi(f_n))\end{aligned}$$

point-wise as well as in $L^1(\mu)$.

Proof. It is sufficient to prove (1.9) for $F(\xi_1, \ldots, \xi_n) = \exp\left[i \sum_{j=1}^n \lambda_j \xi_j\right]$, in which case (1.9) is a consequence of (1.7). Given (1.9), (1.10) follows by differentiating $(F * \Gamma_{\vec{f}}(t, \cdot))(\psi((f_1)_t), \ldots, \psi((f_n)_t))$ with respect to t. Finally, the point-wise convergence in (1.11) is evident from (1.10). To prove that the convergence takes place also in $L^1(\mu)$, observe that from (1.10) one obtains the estimate

$$\begin{aligned}& \left| \frac{1}{t} \left[\int F(\varphi(f_1), \ldots, \varphi(f_n))\, P(t, \psi, d\varphi) - F(\psi(f_1), \ldots, \psi(f_n))\right] \right|^2 \\ &\leqq C/t \int_0^t \left(1 + \sum_1^n |\psi(L(f_j)_s)|^2\right) ds\end{aligned}$$

where C depends only on the $L^2(R^d)$-norms of the f_j's and the $C_b^2(R^n)$-bounds on F. Hence, since μ is a Gaussian measure on $\mathscr{S}'(R^d)$ and $s \to L(f_j)_s$ is continuous, we see that

$$\sup_{0 \leqq t \leqq 1} E^\mu\left[\left|\frac{1}{t}\left[\int F(\varphi(f_1), \ldots, \varphi(f_n))\, P(t, \psi, d\varphi) - F(\psi(f_1), \ldots, \psi(f_n))\right]\right|^2\right] < \infty. \quad \text{Q.E.D.}$$

(1.12) **Lemma.** *Let Φ, $\Psi \in C_b(\mathscr{S}'(R^d))$. Then*

$$\int \Psi(\psi)\left(\int \Phi(\varphi)\, P(t, \psi, d\varphi)\right) \mu(d\psi) = \int \Phi(\psi)\left(\int \Psi(\varphi)\, P(t, \psi, d\varphi)\right) \mu(d\psi), \qquad t \geqq 0. \tag{1.13}$$

In particular, μ is $P(t, \psi, \cdot)$-invariant, and so if $f_1, \ldots, f_n \in \mathscr{S}(R^d)$ and $F \in C_b^2(R^n)$, then

$$\int \left[\sum_{i,j=1}^n (f_i, f_j) \frac{\partial^2 F}{\partial \xi_i\, \partial \xi_j} - \sum_{i=1}^n \psi(Lf_i) \frac{\partial F}{\partial \xi_j}\right] (\psi(f_1), \ldots, \psi(f_n))\, \mu(d\psi) = 0, \tag{1.14}$$

Proof. We need only prove (1.13) for $\Phi(\varphi) = e^{i\varphi(f)}$ and $\Psi(\varphi) = e^{i\varphi(g)}$ where $f, g \in \mathscr{S}(R^d)$. But

$$\begin{aligned}
&\int e^{i\psi(g)}\left(\int e^{i\varphi(f)}\, P(t, \psi, d\varphi)\right) \mu(d\psi) \\
&= \int e^{i\psi(g)}\, e^{i\psi(f_t) - 1/2 \int_0^t \|f_s\|^2\, ds}\, \mu(d\psi) \\
&= \exp\left[-1/2\,(g + f_t, g + f_t)_A - 1/2 \int_0^t \|f_s\|^2\, ds\right] \\
&= \exp\left[-1/2\,((g, g)_A + (f, f)_A + (g, f_t) + (f, g_t))\right].
\end{aligned}$$

The last of these equalities results from

$$\begin{aligned}
(f_t, f_t)_A + \int_0^t \|f_s\|^2\, ds &= \int \frac{1}{\sigma(x)}\, e^{-t\sigma(x)}\, |\hat{f}(x)|^2\, dx + \int_0^t \left(\int e^{-s\sigma(x)}\, |\hat{f}(x)|^2\, dx\right) ds \\
&= \int \frac{1}{\sigma(x)}\, |\hat{f}(x)|^2\, dx = (f, f)_A
\end{aligned}$$

plus

$$(g, f_t)_A = \int \frac{1}{\sigma(x)}\, e^{-t/2\,\sigma(x)}\, \hat{g}(x)\, \overline{\hat{f}(x)}\, dx = (f, g_t)_A.$$

Since the final expression is symmetric in f and g, (1.13) now follows.

The invariance of μ is obvious from (1.13) upon taking $\Psi \equiv 1$. Finally, combining the invariance of μ with the convergence result in (1.11), one easily arrives at (1.14). Q.E.D.

(1.15) **Lemma.** *Let $f, g \in C_0^\infty(R^d)$ have disjoint supports. Then for all $F \in C_b^2(R^1)$ und $G \in C_b(R^1)$:*

$$(1.16)\qquad \int (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, G(\psi(g))\, \mu(d\psi) = 0.$$

Proof. Clearly it is enough to prove (1.16) for $F, G \in C_0^\infty(R^1)$. Given such F and G and applying (1.14) to $F(\varphi(f)) \cdot G(\varphi(g))$, we obtain

$$\begin{aligned}&\int (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, G(\psi(g))\, \mu(d\psi)\\ &\quad = -\int (\|g\|^2 G''(\psi(g)) - \psi(Lg)\, G'(\psi(g)))\, F(\psi(f))\, \mu(d\psi),\end{aligned}$$

since $(f, g) = 0$. On the other hand, from (1.13) we know that

$$\begin{aligned}&\int (\int F(\varphi(f))\, P(t, \psi, d\varphi) - F(\psi(f)))\, G(\psi(g))\, \mu(d\psi)\\ &\quad = \int (\int G(\varphi(g))\, P(t, \psi, d\varphi) - G(\psi(g)))\, F(\psi(f))\, \mu(d\psi).\end{aligned}$$

Hence, if we apply (1.11) to both sides, we get:

$$\begin{aligned}&\int (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, G(\psi(g))\, \mu(d\psi)\\ &\quad = \int (\|g\|^2 G''(\psi(g)) - \psi(Lg)\, G'(\psi(g)))\, F(\psi(f))\, \mu(d\psi).\end{aligned}$$

After combining these two, we arrive at (1.16). Q.E.D.

(1.17) **Lemma.** *If $\mathcal{G}$ is a bounded open set in R^d and $f \in C_0^\infty(\mathcal{G})$, then for every $F \in C_b^2(R^1)$:*

$$(1.18)\qquad \int_B (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, \mu(d\psi) = 0, \qquad B \in \mathcal{A}_{\mathcal{G}^c}.$$

Proof. Since $f \in C_0^\infty(\mathcal{G})$, we can find $\varepsilon_0 > 0$ so that $\operatorname{supp}(f) \cap (\mathcal{G}^c)^{\varepsilon_0} = \varnothing$. Hence if $0 < \varepsilon < \varepsilon_0$ and $g \in C_0^\infty((\mathcal{G}^c)^\varepsilon)$, then, by (1.16), for all $G \in C_b(R^1)$:

$$\int (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, G(\psi(g))\, \mu(d\psi) = 0.$$

Clearly (1.18) is a consequence of this. Q.E.D.

(1.19) **Lemma.** *If $\nu \in \mathcal{M}_\mu$, then for all $f \in \mathcal{S}(R^d)$ the random variable $\varphi(Lf)$ has the same distribution under ν and μ. Furthermore, for each $f \in \mathcal{S}(R^d)$ and $F \in C_b^2(R^1)$:*

$$(1.20)\qquad \int (\|f\|^2 F''(\psi(f)) - \psi(Lf)\, F'(\psi(f)))\, \nu(d\psi) = 0.$$

Proof. We need only prove the first part under the assumption that $f \in C_0^\infty(R^d)$. Given $f \in C_0^\infty(R^d)$, choose $R > 0$ so that $\operatorname{supp}(f) \subset\subset B(0, R)$. Then, by Lemma (1.4), we can write $\varphi(Lf) = X_{\pi Lf} + Z_{Lf}$ where Z_{Lf} is independent of $\tilde{\mathcal{F}}_{B(0,R)^c}$. Hence

$$E^\nu[e^{i\lambda Z_{Lf}}] = E^\nu[E^\mu[e^{i\lambda Z_{Lf}} \mid \mathcal{A}_{S(0,2R)}]] = E^\mu[e^{i\lambda Z_{Lf}}]$$

since $\mathcal{A}_{S(0,2R)} \subseteq \tilde{\mathcal{F}}_{B(0,R)^c}$ and $\nu \in \mathcal{M}_\mu$. Thus if we can show that $\pi Lf = 0$, then we will be done. But $\operatorname{supp}(\pi Lf) \subseteq B(0, R)^c$ and so

$$\|\pi Lf\|_A^2 = (\pi Lf, Lf)_A = \int \widehat{\pi Lf}(x)\, \overline{\hat{f}(x)}\, dx = (\pi Lf)(f) = 0$$

since $f \in C_0^\infty(B(0, R))$.

In proving (1.20), we can and will assume that $f \in C_0^\infty(B(0, R))$ for some $R > 0$. Given $F \in C_b^2(R^1)$, define $\Phi(\varphi) = \|f\|^2 F''(\varphi(f)) - \varphi(Lf) F'(f))$. Then by the preceding $\Phi \in L^1(\nu)$. Moreover, since Φ is $\mathscr{F}_{B(0,R)}$-measurable, we can combine Theorem (1.5) and (1.18) to conclude that

$$E^\mu[\Phi \mid \mathscr{A}_{S(0,R)}] = 0 \qquad \text{(a.s., } \mu).$$

Hence since $E^\nu[\Phi \mid \mathscr{A}_{B(0,R)^c}] = E^\mu[\Phi \mid \mathscr{A}_{S(0,R)}]$ (a.s., ν), we have $E^\nu[\Phi] = 0$. Q.E.D.

It should be noted that if A: $\mathscr{S}(R^d) \to \mathscr{S}(R^d)$, then the first part of Lemma (1.19) shows that $\mathscr{M}_\pi = \{\mu\}$ without any further ado, since we can in this case write every $f \in \mathscr{S}(R^d)$ as Lg with $g = Af$. Thus all our machinery involving $P(t, \psi, \cdot)$ is relevant only when A fails to map $\mathscr{S}(R^d)$ into itself.

(1.21) **Theorem.** *If $\nu \in \mathscr{M}_\mu$, then ν is $P(t, \psi, \cdot)$-invariant.*

Proof. Let $f \in \mathscr{S}(R^d)$ and $F \in C_b^2(R^1)$ be given and define F^t accordingly as in Lemma (1.8) (here $n = 1$). Then by (1.9):

$$\begin{aligned} &\int F(\varphi(f))\, P(t, \psi, d\varphi) - F(\psi(f)) \\ &\quad = 1/2 \int_0^t (\|f_s\|^2 (F^s)''(\psi(f)) - \psi(Lf_s)(F^s)'(\psi(F_s)))\, ds. \end{aligned}$$

Furthermore, by (1.20), for each s

$$\int (\|f_s\|^2 (F^s)''(\psi(f)) - \psi(Lf_s)(F^s)'(\psi(f_s)))\, \nu(d\psi) = 0.$$

Finally, but the first part of Lemma (1.19),

$$\sup_{0 \leq s \leq t} E^\nu[|\psi(Lf_s)|^2] = \sup_{0 \leq s \leq t} E^\mu[|\psi(Lf_s)|^2] < \infty.$$

Hence we can apply Fubini's theorem to complete the proof. Q.E.D.

In order to arrive at our final result, we must borrow the following fact about the structure of $P(t, \psi, \cdot)$-invariant measures from [3] (cf. Theorem (5.7) and Lemma (5.17)).

(1.22) **Theorem.** *If ν is $P(t, \psi, \cdot)$-invariant, then there is a unique probability measure m_ν on $\{H \in \mathscr{S}'(R^d)$: $LH = 0\}$ such that $\nu = \int \mu_H m_\nu(dH)$.*

Combining Theorems (1.6), (1.21), and (1.22) we arrive at our main result.

(1.23) **Theorem.** *If m is a probability measure on $\{H \in \mathscr{S}'(R^d) \cap C^\infty(R^d)$: $LH = 0\}$, then $\int \mu_M m(dH) \in \mathscr{M}_\mu$. Conversely, if $\nu \in \mathscr{M}_\mu$, then there is a unique probability measure m_ν on $\{H \in \mathscr{S}'(R^d)$: $LH = 0\}$ such that $\nu = \int \mu_H m_\nu(dH)$. Hence, if $\{H \in \mathscr{S}'(R^d)$: $LH = 0\} \subseteq C^\infty(R^d)$, then the mapping $m \to \int \mu_H m(dH)$ defines a one-to-one mapping from the set of probability measures on $\{H \in \mathscr{S}'(R^d)$: $LH = 0\}$ onto $\mathscr{M}_\mu$.*

(1.23) *Remark.* The condition $\{H \in \mathscr{S}'(R^d)$: $LH = 0\} \subseteq C^\infty(R^d)$ is not so restrictive as it may appear at first. Indeed, for many choices of $\sigma(\cdot)$, it is possible to check this condition by hand (e.g. when $\sigma(\cdot)=\sigma_0(\cdot)+\sigma_1(\cdot)$ where $\sigma_0(\cdot)$ is homogeneous polynomial of degree $2n$ such that $\sigma_0(x)>0$ for $x\neq 0$ and $\sigma_1(\cdot)$ is a non-negative polynomial of degree strictly less than $2n$). The best result that we know on this subject are due to L. Hörmander and can be found in section (4.1) of this book [4]. Perhaps the most useful sufficient condition is that $\sigma(\cdot)>0$ on $R^d\setminus\{0\}$.

(1.24) *Remark.* The introduction of $P(t,\psi,\cdot)$ may appear to be simply a device with which we have reduced the problem at hand to one which has already been solved. Indeed, a more direct route to Theorem (1.22) would run as follows. Starting from Lemma (1.19), we know that for any $v\in\mathscr{M}_\mu$ and $g\in\mathscr{S}(R^d)$ the distribution of $\varphi(Lg)$ under v is the same as it is under μ. Now suppose that for any $f\in\mathscr{S}(R^d)$ we could construct $\{g_n\}_1^\infty\subseteq\mathscr{S}(R^d)$ so that $\|f-Lg_n\|_A\to 0$ and $f-Lg_n$ vanishes on $B(\mathcal{O},n)\equiv(\{x\in R^d\colon |x|<n\})$. Setting $f_n=Lg_n$ and $h_n=f-f_n$, we would have: $\varphi(f)=\varphi(f_n)+\varphi(h_n)$ where $\varphi(f_n)$ under v is a mean 0 Gaussian with variance $(f_n, L^{-1}f_n)$. Furthermore, we would know that $\varphi(f_n)\xrightarrow{L^2(v)} X_f$ where X_f is a mean 0 Gaussian with variance $(f, L^{-1}f)$. Hence, it would follow that $\varphi(h_n)\xrightarrow{L^2(v)} Y_f$ where Y_f must be tail-measurable and therefore independent of X_f. Without much trouble, it would also be possible to show that $f\to X_f$ and $f\to Y_f$ can be chosen so that $X_\bullet$ and $Y_\bullet$ are tempered distribution-valued random variables. Finally, it is clear that Y_{Lg} would have to be 0 for all $g\in\mathscr{S}(R^d)$ and therefore that $Y_\bullet$ must be an L-harmonic distribution. The result of all this would therefore be that we could write $\varphi(f)=X_f+Y_f$, $f\in\mathscr{S}(R^d)$, where $f\to X_f$ and $f\to Y_f$ are independent $\mathscr{S}'(R^d)$-valued random variables, X_f under v is a mean 0 Gaussian with variance $(f, L^{-1}f)$, and $LY\equiv 0$. Obviously, this is just what is needed to prove part of Theorem (1.22) not covered by Theorem (1.6).

The preceding paragraph leaves the problem of constructing the sequence $\{g_n\}_1^\infty$. We have been able to construct $\{g_n\}_1^\infty$ in the case when $L=-\Delta$, but the technique that we used does not appear to be readily generalized. This is the main reason why we chose the route via $P(t,\psi,\cdot)$. A secondary reason is that the connection between the D.L.R. conditions and the flow determined by $P(t,\psi,\cdot)$ seems to us to be of independent interest in its own right.

Section 2

We now have a quite complete description of $\mathscr{M}_\mu$. In particular, we know that if the only $H\in\mathscr{S}'(R^d)$ satisfying $LH=0$ is $H=0$, then $\mathscr{M}_\mu=\{\mu\}$. This will of course be the case if A maps $\mathscr{S}(R^d)$ into itself. On the other hand, if for instance $\sigma(x)=|x|^2$, there are an infinity of non-trivial $H\in\mathscr{S}'(R^d)$ such that $LH=0$. Thus it is natural to ask if it is not possible to isolate an analytically describable Hilbert subspace S of $\mathscr{S}'(R^d)$ such that $\mu(S)=1$ and the only $H\in S$ satisfying $LH=0$ is $H=0$. If we can find such an S, then it is clear that $\mathscr{M}_\mu\cap\{v\colon v(S)=1\}=\{\mu\}$.

In order to prove that S exists, we will make the following additional assumptions about $\sigma(\cdot)$. Namely, we assume that there exists a $\delta>0$ such that

$$\int_{\{x:\sigma(x)\leqq 1\}} (\sigma(x))^{-(1+1/\delta)}dx<\infty. \tag{2.1}$$

To carry out our program, we need some notation. For $k \geqq 0$, let h_k denote the k^{th} Hermite function:

$$h_k(x) = (\pi^{1/2} 2^k k!)^{-1/2} (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}, \qquad x \in R^1.$$

Given a multiindex $\alpha \in I \equiv (\{0, 1, \ldots, n, \ldots\})^d$ and $x = (x_1, \ldots, x_d) \in R^d$, set

$$h_\alpha(x) = h_{\alpha_1}(x_1) \ldots h_{\alpha_d}(x_d).$$

Recall that $\{h_\alpha: \alpha \in I\}$ forms an ortho-normal basis in $L^2(R^d)$ and that $f \in L^2(R^d)$ is an element of $\mathscr{S}(R^d)$ if and only if $\{(f, h_\alpha): \alpha \in I\}$ is rapidly decreasing, in which case $\sum_{|\alpha| \leqq N} (f, h_\alpha) h_\alpha \to f$ as $N \uparrow \infty$ in $\mathscr{S}(R^d)$.

(2.2) **Lemma.** *For the $\delta > 0$ in* (2.1)*:*

$$\mu\left(\left\{\varphi: \sum_{n=0}^{\infty} \sum_{\alpha=1} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2 < \infty\right\}\right) = 1,$$

where $\{T_t: t \geqq 0\}$ is the semi-group on $\mathscr{S}(R^d)$ introduced before (1.7).

Proof. Certainly it suffices to show that

$$E^\mu\left[\sum_{n=0}^{\infty} \sum_{\alpha \in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2\right] < \infty.$$

But

$$\begin{aligned} &E^\mu\left[\left|\sum_{n=0}^{\infty} \sum_{\alpha \in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)\right|^2\right] \\ &= \sum_{\alpha \in I} (1+|\alpha|)^{-(3d+2)/2} \int \sum_{n=0}^{\infty} \frac{1}{\sigma(x)} e^{-n^\delta \sigma(x)} |h_\alpha(x)|^2 \, dx. \end{aligned} \tag{2.3}$$

Now write $G_\delta(\lambda) = \sum_{n=0}^{\infty} e^{-\lambda n^\delta} = \int_0^\infty e^{-\lambda t} dU_\delta(t)$, where $U_\delta(t) = \sum_{n^\delta \leqq t} 1 = [t^{1/\delta}] + 1$. By a standard Abelian theorem (cf. p. 420 of [1]), we see that $G_\delta(\lambda)$ is asymptotic to $\Gamma(1 + 1/\delta) \lambda^{-1/\delta}$ as $\lambda \downarrow 0$. Since $G_\delta(\lambda)$ is bounded for λ in each interval $[\varepsilon, \infty)$ with $\varepsilon > 0$, we now obtain:

$$G_\delta(\lambda) \leqq C_0(\delta) \left(\left(\frac{1}{\lambda}\right)^{1/\delta} \vee 1\right).$$

With this estimate, we get:

$$\begin{aligned} &\int \sum_{n=0}^{\infty} \frac{1}{\sigma(x)} e^{-n^\delta \sigma(x)} |h_\alpha(x)|^2 \, dx \\ &\leqq C_0(\delta) \Big[\int_{\{x: \sigma(x) \leqq 1\}} (\sigma(x))^{-(1+1/\delta)} |h_\alpha(x)|^2 \, dx \\ &\quad + \int_{\{x: \sigma(x) \geqq 1\}} |h_\alpha(x)|^2 \, dx \Big] \\ &\leqq C_0(\delta) \Big[\|h_\alpha\|^2_{L^\infty(R^d)} \int_{\{x: \sigma(x) \leqq 1\}} (\sigma(x))^{-(1+1/\delta)} \, dx + 1 \Big]. \end{aligned}$$

But $\|h_\alpha\|_{L^\infty(R^d)} \leqq C_1(d)(1+|\alpha|)^{d/4}$ (cf. (A.12) in [3]), and so there is a $C(d,\delta)<\infty$ such that

$$\int \sum_{n=0}^{\infty} \frac{1}{\sigma(x)} e^{-n^\delta \sigma(x)} |h_\alpha(x)|^2 \, dx \leqq C(d,\delta)(1+|\alpha|)^{d/2}.$$

Plugging this back into (2.3), we have:

$$E^\mu \left[\sum_{n=0}^{\infty} \sum_{\alpha\in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2 \right] \leqq C(d,\delta) \sum_{\alpha\in I} (1+|\alpha|)^{-(d+1)} < \infty. \quad \text{Q.E.D.}$$

We now define

$$S = \left\{ \varphi \in \mathscr{S}'(R^d) \colon \sum_{n=0}^{\infty} \sum_{\alpha\in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2 < \infty \right\}. \tag{2.4}$$

It is clear that S can be given a natural Hilbert space structure and that S is a dense F_σ-subset of $\mathscr{S}'(R^d)$.

(2.5) **Lemma.** *If $\varphi \in S$, then $\varphi(T_\delta f) \to 0$ as $n\to\infty$ for all $f\in\mathscr{S}(R^d)$. In particular, if $H\in S$ and $LH=0$, then $H=0$.*

Proof. Given $f\in\mathscr{S}(R^d)$, we have:

$$\begin{aligned} |\varphi(T_{n^\delta} f)| &= \Big|\sum_{\alpha\in I} (f, h_\alpha)\, \varphi(T_{n^\delta} h_\alpha)\Big| \\ &\leqq \Big|\sum_{\alpha\in I} (f, h_\alpha)^2 (1+|\alpha|)^{(3d+2)/2}\Big|^{1/2} \\ &\quad \cdot \Big|\sum_{\alpha\in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2\Big|^{1/2}, \end{aligned}$$

and, since $\varphi\in S$,

$$\sum_{\alpha\in I} (1+|\alpha|)^{-(3d+2)/2} |\varphi(T_{n^\delta} h_\alpha)|^2 \to 0 \qquad \text{as } n\to\infty.$$

Finally, if $H\in S$ and $LH=0$, then $H(T_t f)=H(f)$ for all $t\geqq 0$ and $f\in\mathscr{S}(R^d)$. Hence $H(f)=\lim_{n\to\infty} H(T_{n^\delta} f)=0$, $f\in\mathscr{S}(R^d)$. Q.E.D.

(2.6) **Lemma.** *If $H\in S'(R^d)$ and $LH=0$, then $\mu_H(S)>0$ implies $H=0$.*

Proof. If $\mu_H(S)>0$, then, since $\mu_H(S+H)=1$, $S\cap(S+H)\neq\varnothing$. Hence there exist $\varphi, \psi\in S$ such that $\varphi=\psi+H$. But this means that $0=LH=L(\varphi-\psi)$, and so $\varphi=\psi$. In other words $H=0$. Q.E.D.

(2.7) **Theorem.** *Let S be defined as in* (2.4). *Then $\mathscr{M}_\mu \cap \{\nu\colon \nu(S)=1\}=\{\mu\}$.*

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