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A Note on the Convergence of Sequences of Conditional Expectations of Random Variables

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Summary. We disprove two theorems on the convergence of sequences of conditional expectations of random variables in [1] by providing a counter-example.

Let $(x_n)_{n=1,2,\ldots}$ be a sequence of random variables which are uniformly integrable. It is well known that

$$\overline{\lim_{n\to\infty}} E[x_n] \leqq E[\overline{\lim_{n\to\infty}} x_n] \tag{1}$$

but, if we replace the symbol $E[.]$ by that of the conditional expectation $E[.|\mathscr{G}]$ where $\mathscr{G}$ is some σ-field, will the inequality (1) still remain true? Such a proposition needs a proof. But Liptser and Shiryayev state it as Theorem 1.2 of their book [1] without giving a proof. They also state without proof (their Theorem 1.3) that if $0 \leqq x_n \to x$ a.s., $E[x_n] < \infty$, then

$$E[X_n|\mathscr{G}] \xrightarrow{n\to\infty} E[X|\mathscr{G}] \quad \text{a.s.} \tag{2}$$

if and only if (X_n) are uniformly integrable. Both theorems are not true. It is the purpose of this note to disprove the validity of the two theorems by proposing a counter-example as follows.

Let $\Omega = (0,1] \times (0,1]$ be the unit square in the Cartesian plane, $\mathscr{B}$ the Borel field on that square, and $p = \mu \times \mu$ the product measure, where μ is the Lebesque measure on (0,1].

Thus, $(\Omega, \mathscr{B}, p)$ forms a probability space. For every positive integer n, write $n = 2^k + j - 2$, where k is a positive integer, with $j = 1, 2, \ldots, 2^{k+1} - 2^k$. Then, to every n there corresponds a pair of positive integers (k,j) which we shall denote as functions of n by $(k,j) = (K(n), J(n))$.

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Let

$$A_n=\left\{s\colon \frac{J(n)-1}{2^{K(n)}}<s\leqq\frac{J(n)}{2^{K(n)}}\right\},$$

$$B_n=\left\{t\colon 0<y\leqq\frac{1}{2^{K(n)}}\right\},$$

and

$$X_n(s,t)=2^{K(n)}I_{A_n\times B_n}(s,t),$$

$$\mathscr{G}=\mathscr{B}_0\times(0,1],$$

where $\mathscr{B}_0$ is the Borel field on (0,1]. Evidently $\mathscr{G}$ is a sub-σ-field of $\mathscr{B}$. For every $B\in\mathscr{G}$, we have $B=B_0\times(0,1]$ where $B_0\in\mathscr{B}_0$. Furthermore,

$$\begin{aligned}\int_B x_n\,dP&=\int_{B_0\times(0,1]}2^{K(n)}I_{A_n\times B_n}(s,t)\,dP\\&=2^{K(n)}\mu[A_n\cap B_0]\,\mu(B_n)\\&=\mu[A_n\cap B_0]\\&=\int_{B_0\times(0,1]}I_{A_n\times(0,1]}(s,t)\,dP\\&=\int_B I_{A_n\times(0,1]}(s,t)\,dP.\end{aligned}$$

Hence

$E[x_n|\mathscr{G}]=I_{A_n\times(0,1]}$ a.s.

Since $x_n\to0$ and $E[x_n]\to0$, (x_n) are uniformly integrable. They satisfy the conditions of Theorems 1.2 and 1.3 of [1]. But $\overline{\lim_{n\to\infty}}E[x_n|\mathscr{G}]=\overline{\lim_{n\to\infty}}I_{A_n\times(0,1]}=1>0=E[\overline{\lim_{n\to\infty}}x_n|\mathscr{G}]$ a.s. leads to a contradiction.

Note of a referee: The example above also shows that the Corollary to Theorem 1.2 and 1.3 of [1] is wrong.

Reference

1. Липцер, Р. Щ., Ширяев, А. Н.: Статистика Случайных процессов. Москва, 1974. (English translation: Liptser, R.S., Shiryayev, A.N.: Statistics of Random Processes I. New York: Springer 1978.)

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