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Existence of Compatible Families of Proper Regular Conditional Probabilities*

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Summary. Let $(\Omega, \mathscr{F}, \mu)$ be a perfect probability space with $\mathscr{F}$ countably generated, and let $\mathbb{B}$ be a family of sub-σ-fields of $\mathscr{F}$. Under a countability condition on the family $\mathbb{B}$, I show that there exists a family $\{\pi_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ of regular conditional probabilities which are everywhere compatible. Under a more stringent condition on $\mathbb{B}$, I show that the $\pi_{\mathscr{B}}$ can furthermore be chosen to be everywhere proper. It follows that in the Dobrushin-Lanford-Ruelle formulation of the statistical mechanics of classical lattice systems, every (perfect) probability measure is a Gibbs measure for some specification.

1. Introduction

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space, and let $\mathscr{B}$ be a sub-σ-field of $\mathscr{F}$. A *regular conditional probability* (r.c.p.) *for* $(\Omega, \mathscr{F}, \mu)$ *given* $\mathscr{B}$ is a map $\pi\colon \Omega\times\mathscr{F}\to[0,1]$ such that:

(a) $\pi(\omega,\cdot)$ is a probability measure on $\mathscr{F}$, for each $\omega\in\Omega$;
(b) $\pi(\cdot, F)$ is $\mathscr{B}$-measurable, for each $F\in\mathscr{F}$;[1] and
(c) $\int \chi_B(\omega)\,\pi(\omega, F)\,d\mu(\omega)=\mu(B\cap F)$ for all $B\in\mathscr{B}$ and $F\in\mathscr{F}$.

π is said to be *proper at* ω_0 (on $\mathscr{B}$) if, in addition,

(d) $\pi(\omega_0, B)=\chi_B(\omega_0)$ for all $B\in\mathscr{B}$;

it is *(everywhere)proper* (on $\mathscr{B}$) if it is proper at ω_0 for all $\omega_0\in\Omega$.

If now $\mathscr{B}_1\subset\mathscr{B}_2$ are sub-σ-fields of $\mathscr{F}$, with $\mathscr{F}$ countably generated, and $\pi_{\mathscr{B}_1}$ [resp. $\pi_{\mathscr{B}_2}$] is an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}_1$ [resp. given $\mathscr{B}_2$], then it is not

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[1] In some previous works [1–4], it is required only that $\pi(\cdot, F)$ be $\bar{\mathscr{B}}_\mu$-measurable, where $\bar{\mathscr{B}}_\mu$ is the σ-field generated by $\mathscr{B}$ *and* the μ-null sets. I shall always adhere to the strict definition given above.

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hard to show (Lemma 3.1) that $\pi_{\mathscr{B}_1}$ and $\pi_{\mathscr{B}_2}$ are "almost compatible" in the sense that

(e) $\pi_{\mathscr{B}_1}(\omega, F) = \int \pi_{\mathscr{B}_1}(\omega, d\omega') \pi_{\mathscr{B}_2}(\omega', F)$ for all $F \in \mathscr{F}$

for μ-almost-every ω. It is natural to ask, therefore, whether $\pi_{\mathscr{B}_1}$ and $\pi_{\mathscr{B}_2}$ can be chosen so that (e) holds for *all* ω; and if so, whether $\pi_{\mathscr{B}_1}$ and $\pi_{\mathscr{B}_2}$ can also be chosen to be proper. It is the purpose of this paper to show that, under suitable conditions, the answer to these questions is yes. More generally, I shall show (Theorems 3.2 and 3.3) that if $\mathbb{B}$ is a family of sub-σ-fields of $\mathscr{F}$ satisfying suitable conditions, then (proper) regular conditional probabilities $\{\pi_{\mathscr{B}}\}_{\mathscr{B} \in \mathbb{B}}$ can be chosen so that (e) holds, for all ω, whenever $\mathscr{B}_1, \mathscr{B}_2 \in \mathbb{B}$ with $\mathscr{B}_1 \subset \mathscr{B}_2$.

Since the subject of this paper may seem rather pedantic – replacing "almost everywhere" by "everywhere" – it is perhaps worthwhile to indicate an application in which the result of this paper is of interest. In the Dobrushin-Lanford-Ruelle [5–9] formulation of the classical statistical mechanics of infinite systems, the basic mathematical object is a *specification* [10, 11]: one is given a measurable space $(\Omega, \mathscr{F})$, a family $\mathbb{B}$ of sub-σ-fields of $\mathscr{F}$ (directed downwards), and a family $\{\pi_{\mathscr{B}}\}_{\mathscr{B} \in \mathbb{B}}$ of mappings $\pi_{\mathscr{B}}$: $\Omega \times \mathscr{F} \to [0, 1]$ satisfying:

(a′) $\pi_{\mathscr{B}}(\omega, \cdot)$ is a probability measure on $\mathscr{F}$, for each $\omega \in \Omega$;

(b′) $\pi_{\mathscr{B}}(\cdot, F)$ is $\mathscr{B}$-measurable, for each $F \in \mathscr{F}$;

(d′) $\pi_{\mathscr{B}}(\omega, B) = \chi_B(\omega)$ for all $B \in \mathscr{B}$ and all $\omega \in \Omega$; and

(e′) $\pi_{\mathscr{B}}(\omega, F) = \int \pi_{\mathscr{B}_1}(\omega, d\omega') \pi_{\mathscr{B}_2}(\omega', F)$ for *all* $\omega \in \Omega$ and $F \in \mathscr{F}$ whenever $\mathscr{B}_1 \subset \mathscr{B}_2$.

The resemblance of these conditions to (a)–(e) is, of course, not accidental. Indeed, the $\pi_{\mathscr{B}}$ are to be interpreted as specified conditional probabilities (whence the name) for which one seeks a measure μ satisfying

(c′) $\int \chi_B(\omega) \pi_{\mathscr{B}}(\omega, F) \, d\mu(\omega) = \mu(B \cap F)$ for all $B \in \mathscr{B}$, $F \in \mathscr{F}$ and $\mathscr{B} \in \mathbb{B}$;

any such measure μ is called a *Gibbs measure* (or DLR measure) for the specification $\{\pi_{\mathscr{B}}\}_{\mathscr{B} \in \mathbb{B}}$. The crucial fact, however, is that the measure μ is not given in advance; hence, the notion "almost everywhere" has initially no meaning, and the compatibility condition (e′) must be assumed to hold for *all* ω. Indeed, the Gibbs measure μ may very well not be unique ("phase transition"), and in this case it is known that distinct Gibbs measures are extremely mutually singular [10, p. 21], making the notion of "almost everywhere" even more problematic.

It is now clear that the problem being considered here is the "inverse problem" in the theory of specifications: given a measure μ, is there a specification $\{\pi_{\mathscr{B}}\}_{\mathscr{B} \in \mathbb{B}}$ for which it is a Gibbs measure? To be sure, this problem is nowhere near as interesting, either physically or mathematically, as the "direct problem" of studying the existence, uniqueness, and properties of Gibbs measures for a given specification. It is, nevertheless, a natural question to pose; it is pleasant, therefore, that it has, at least in some cases, a satisfactory solution. Goldstein [12] and Preston [13] have resolved affirmatively the case of the discrete-state lattice model: here Ω is a countable Cartesian product of finite or countable sets, and the σ-fields in $\mathbb{B}$ are the cylinder sets over cofinite-dimensional bases. Theorem 3.3 of the present paper extends their result to general (continuous-state) lattice models. However, continuous models (point processes) [10, Sect. 6] are excluded by virtue of the countability condition on $\mathbb{B}$.

2. Regular Conditional Probabilities

It is useful to review the known theorems concerning the existence of a (proper) regular conditional probability given a single σ-field $\mathcal{B}$. The main theorem, due to Jiřina [14], employs the notion of a *perfect measure* [15]: a probability measure μ on $(\Omega, \mathcal{F})$ is called perfect if, for each real-valued $\mathcal{F}$-measurable function f and each subset $E \subset \mathbb{R}$ such that $f^{-1}[E] \in \mathcal{F}$, there exists a Borel set B such that $B \subset E$ and $\mu(f^{-1}[E \setminus B]) = 0$. Equivalently, μ is perfect if and only if for each such f, there exists a Borel set B such that $B \subset f[\Omega]$ and $\mu(\Omega \setminus f^{-1}[B]) = 0$. Also equivalently, μ is perfect if and only if its restriction to each countably generated sub-σ-field is compact. Jiřina's theorem then states that if $\mathcal{F}$ is countably generated and μ is perfect (equivalently, compact), then there exists a regular conditional probability for $(\Omega, \mathcal{F}, \mu)$ given any sub-σ-field $\mathcal{B}$.[2]

It is also useful to know simple sufficient criteria for the perfectness of large classes of measures. A measurable space $(\Omega, \mathcal{F})$ is called *perfect* if every probability measure μ on $(\Omega, \mathcal{F})$ is perfect. Then [15, 16] if $\mathcal{F}$ is countably generated, $(\Omega, \mathcal{F})$ is perfect if and only if $\mathcal{F}$ is isomorphic to the Borel σ-field of a universally measurable subset of a complete separable metric space. In particular, it suffices that $(\Omega, \mathcal{F})$ be a Blackwell space[3] [17] (isomorphic to the Borel σ-field on an analytic subset of a complete separable metric space), or more restrictively, a standard Borel space [18] (isomorphic to the Borel σ-field on (a Borel subset of) a complete separable metric space). Most spaces encountered in applications are standard Borel spaces, hence perfect. A countable Cartesian product of standard spaces is standard; an arbitrary Cartesian product of perfect spaces is perfect.

We now consider the conditions under which a regular conditional probability is proper. We first note the following [19][4]:

Lemma 2.1. *If $\mathcal{B}$ is countably generated, then every r.c.p. for $(\Omega, \mathcal{F}, \mu)$ given $\mathcal{B}$ is $\mu \upharpoonright \mathcal{B}$-almost-everywhere proper. That is, if π is an r.c.p., there exists a set $N \in \mathcal{B}$ with $\mu(N) = 0$ such that π is proper at ω_0 for all $\omega_0 \notin N$.*

Proof. By definition of r.c.p.,

$$\int \chi_B(\omega)\, \pi(\omega, B')\, d\mu(\omega) = \mu(B \cap B') = \int \chi_B(\omega)\, \chi_{B'}(\omega)\, d\mu(\omega)$$

for all $B, B' \in \mathcal{B}$; hence $\pi(\omega, B') = \chi_{B'}(\omega)$ μ-a.e. by the uniqueness part of the Radon-Nikodým theorem. Let $\mathcal{B}_0$ be a countable generating subfield of $\mathcal{B}$, and let

$$N = \bigcup_{B' \in \mathcal{B}_0} \{\omega \colon \pi(\omega, B') \neq \chi_{B'}(\omega)\}.$$

Then $N \in \mathcal{B}$, $\mu(N) = 0$, and it is not hard to show that π is proper at ω_0 for all $\omega_0 \notin N$. $\square$

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[2] Note that in Corollary I of [14] (English translation, p. 82), the Russian word "sovershenna" is mistranslated as "complete"; it should read "perfect". See e.g. the English summary accompanying the original version of [14] and reprinted in the translation.

[3] In [17], such spaces are called Lusin spaces.

[4] The main result of [19] (but not the result quoted) is, unfortunately, false.

The conditions for the existence of an *everywhere* proper r.c.p. are slightly more complicated.

Definition. If $\mathscr{B}$ is a sub-σ-field of $\mathscr{F}$, a *selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$* is a σ-homomorphism $\psi\colon \mathscr{F}\to\mathscr{B}$ which leaves $\mathscr{B}$ pointwise fixed, i.e. $\psi(B)=B$ for all $B\in\mathscr{B}$.

Remarks. (1) A selection homomorphism typically arises from a *selection function*, that is, a measurable mapping $f\colon (\Omega,\mathscr{B})\to(\Omega,\mathscr{F})$ such that $\omega\in B\in\mathscr{B}$ implies $f(\omega)\in B$. Then by looking at both B and B^c, we deduce that $\omega\in B$ if and only if $f(\omega)\in B$; that is, $f^{-1}[B]=B$. So $\psi=f^{-1}$ is a selection homomorphism.

(2) If $\mathscr{F}$ contains all one-point sets $\{\omega\}$, $\omega\in\Omega$, and $\psi=f^{-1}=g^{-1}$, then $f=g$. Indeed, if $f\neq g$, there is some ω such that $f(\omega)\neq g(\omega)$; but then $f^{-1}[\{f(\omega)\}]\neq g^{-1}[\{f(\omega)\}]$.

(3) Here is the standard example of a selection homomorphism: Let $(\Omega,\mathscr{F})=(\Omega_1\times\Omega_2,\mathscr{F}_1\times\mathscr{F}_2)$, and let $\mathscr{B}=\mathscr{F}_1$ considered as a σ-field on Ω in the obvious way. Fix any element $\omega_2^0\in\Omega_2$. Then the mapping $f(\omega)=\omega_1\times\omega_2^0$ is a selection function for $\mathscr{F}_1$ with respect to $\mathscr{F}$. Hence $\psi=f^{-1}$ is a selection homomorphism; in detail, $\psi(F)=\{(\omega_1,\omega_2)\in\Omega\colon (\omega_1,\omega_2^0)\in F\}$.

(4) If ψ is a selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$, note that $\mathscr{B}=\psi[\mathscr{B}]=\psi[\mathscr{F}]$. Thus, if $\mathscr{F}$ is countably generated (or more generally, if there exists a countably generated σ-field $\mathscr{F}'$ with $\mathscr{B}\subset\mathscr{F}'\subset\mathscr{F}$), then $\mathscr{B}$ is also countably generated. In general, therefore, there exist many sub-σ-fields which do not possess any selection homomorphism.

We call a map $\pi\colon \Omega\times\mathscr{F}\to[0,1]$ a *$\mathscr{B}$-measurable probability kernel*[5] *on* $(\Omega,\mathscr{F})$ if it satisfies conditions (a) and (b) of the Introduction. It is said to be *proper at* ω_0 if (d) holds, and (*everywhere*) *proper* if it is proper at ω_0 for all $\omega_0\in\Omega$. Obviously, every r.c.p. for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$ is a $\mathscr{B}$-measurable probability kernel on $(\Omega,\mathscr{F})$.

Lemma 2.2. (a) [20, 21] *A sufficient condition for the existence of an everywhere proper $\mathscr{B}$-measurable probability kernel on $(\Omega,\mathscr{F})$ is the existence of a selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$. If $(\Omega,\mathscr{F})$ is a standard Borel space, this condition is also necessary. More weakly, if $\mathscr{F}$ is countably generated, it is necessary that $\mathscr{B}$ be countably generated.*

(b) [22] *An everywhere proper r.c.p. for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$ exists if and only if:* (1) *there exists a $\mu\restriction\mathscr{B}$-almost-everywhere proper r.c.p. for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$;* and (2) *there exists an everywhere proper $\mathscr{B}$-measurable probability kernel on $(\Omega,\mathscr{F})$. In particular, it suffices that:* (1) *$\mathscr{F}$ is countably generated;* (2) *μ is perfect;* and (3) *there exists a selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$.*

Proof. (a) Let ψ be a selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$. Then $\pi_1(\omega,F)=\chi_{\psi(F)}(\omega)$ is obviously a $\mathscr{B}$-measurable probability kernel on $(\Omega,\mathscr{F})$; and since $\psi(F)=F$ for $F\in\mathscr{B}$, by definition of selection homomorphism, it is everywhere proper. The converse for $(\Omega,\mathscr{F})$ standard Borel is considerably more difficult; see [20]. For the last statement we note that if π is proper, then $\mathscr{B}=$ the σ-field generated by $\{\pi(\cdot,F)\colon F\in\mathscr{B}\}=$ the σ-field generated by

[5] Also called stochastic kernel, Markov kernel, or transition probability.

$\{\pi(\cdot, F): F \in \mathscr{F}\}$. Thus, if $\mathscr{F}$ is countably generated (or more generally, if there exists a countably generated σ-field $\mathscr{F}'$ with $\mathscr{B} \subset \mathscr{F}' \subset \mathscr{F}$), then so is $\mathscr{B}$.

(b) The "only if" is trivial. For the "if" part, let π_0 be an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$ which is proper outside N, where $N \in \mathscr{B}$ and $\mu(N)=0$; and let π be an everywhere proper $\mathscr{B}$-measurable probability kernel on $(\Omega, \mathscr{F})$. Then

$$\tilde{\pi}(\omega, F) = \begin{cases} \pi_0(\omega, F) & \text{for } \omega \notin N \\ \pi(\omega, F) & \text{for } \omega \in N \end{cases}$$

is easily seen to be an everywhere proper r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$. The last statement follows from the foregoing and Lemma 2.1. □

Remark. It is not known (to me) whether the condition of (a) is necessary in general.

3. Families of Regular Conditional Probabilities

Regular conditional probabilities are always *almost* compatible:

Lemma 3.1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be sub-σ-fields of $\mathscr{F}$, with $\mathscr{B}_1 \subset \mathscr{B}_2$; and let $\pi_{\mathscr{B}_1}$ [resp. $\pi_{\mathscr{B}_2}$] be an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}_1$ [resp. given $\mathscr{B}_2$]. Then for each $F \in \mathscr{F}$, the set*

$$M(F) = \{\omega: \pi_{\mathscr{B}_1}(\omega, F) \neq \int \pi_{\mathscr{B}_1}(\omega, d\omega')\, \pi_{\mathscr{B}_2}(\omega', F)\}$$

is $\mathscr{B}_1$-measurable and $\mu(M(F))=0$. Moreover, if $\mathscr{F}$ is countably generated, then $M = \bigcup_{F \in \mathscr{F}} M(F)$ is $\mathscr{B}_1$-measurable and $\mu(M)=0$.

Proof. Both sides of the defining relation for $M(F)$ define $\mathscr{B}_1$-measurable functions of ω, so $M(F) \in \mathscr{B}_1$. To show that $\mu(M(F))=0$, it suffices to show that

$$\int \chi_B(\omega)\, \pi_{\mathscr{B}_1}(\omega, F)\, d\mu(\omega) = \int \chi_B(\omega) \left(\int \pi_{\mathscr{B}_1}(\omega, d\omega')\, \pi_{\mathscr{B}_2}(\omega', F)\right) d\mu(\omega)$$

for all $B \in \mathscr{B}_1$. Now the left side is $\mu(B \cap F)$, by definition of r.c.p. To evaluate the right side, interchange the order of integration (justified by a generalized Fubini theorem [23]) and perform the ω integral first; using the definition of r.c.p., we get $\int d\mu(\omega')\, \chi_B(\omega')\, \pi_{\mathscr{B}_2}(\omega', F)$, which again equals $\mu(B \cap F)$ since $B \in \mathscr{B}_1 \subset \mathscr{B}_2$.

Now let $\mathscr{F}_0$ be a countable generating subfield of $\mathscr{F}$. Since both sides of the defining relation for $M(F)$ define, for each fixed ω, probability measures in $F \in \mathscr{F}$, it follows from the monotone class theorem that $M = \bigcup_{F \in \mathscr{F}} M(F) = \bigcup_{F \in \mathscr{F}_0} M(F)$. Since $\mathscr{F}_0$ is countable, the result follows. □

Of course, it is not true in general that the compability equation holds for *all* ω. Nevertheless, there *exists* a choice of $\pi_{\mathscr{B}_1}$ and $\pi_{\mathscr{B}_2}$ for which this does occur. More generally:

Theorem 3.2. *Let $\mathscr{F}$ be countably generated, and let $\mathscr{B}$ be a family of sub-σ-fields of $\mathscr{F}$ such that for each $\mathscr{B}_1 \in \mathbb{B}$, the set $\{\mathscr{B}_2 \in \mathbb{B}: \mathscr{B}_1 \subset \mathscr{B}_2\}$ is countable.*

Then for any perfect measure μ on $(\Omega, \mathscr{F})$, there exists a family $\{\pi_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ of regular conditional probabilities for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$, such that

$$\pi_{\mathscr{B}_1}(\omega, F)=\int \pi_{\mathscr{B}_1}(\omega, d\omega')\, \pi_{\mathscr{B}_2}(\omega', F) \quad \textit{for all } \omega\in\Omega \textit{ and } F\in\mathscr{F} \tag{3.1}$$

whenever $\mathscr{B}_1, \mathscr{B}_2\in\mathbb{B}$ with $\mathscr{B}_1\subset\mathscr{B}_2$.

Proof. By Jiřina's theorem [14], there exists for each $\mathscr{B}\in\mathbb{B}$ an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$ – call it $\tilde{\pi}_{\mathscr{B}}$. Then by Lemma 3.1, if $\mathscr{B}_1\subset\mathscr{B}_2$ we have $\tilde{\pi}_{\mathscr{B}_1}(\omega, F)=\int \tilde{\pi}_{\mathscr{B}_1}(\omega, d\omega')\, \tilde{\pi}_{\mathscr{B}_2}(\omega', F)$ for all $F\in\mathscr{F}$, for all ω except in a μ-null set $M_{\mathscr{B}_1, \mathscr{B}_2}\in\mathscr{B}_1$. The idea of the proof is to modify each $\tilde{\pi}_{\mathscr{B}}$ on a μ-null set $N_{\mathscr{B}}\in\mathscr{B}$ so that the modified version, which we shall call $\pi_{\mathscr{B}}$, is still an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$, and so that the compatibility Eq. (3.1) is now satisfied for *all* ω. In particular, we define $\pi_{\mathscr{B}}$ as follows:

$$\pi_{\mathscr{B}}(\omega, F)=\begin{cases}\tilde{\pi}_{\mathscr{B}}(\omega, F) & \text{for } \omega\notin N_{\mathscr{B}}\\ \mu(F) & \text{for } \omega\in N_{\mathscr{B}}.\end{cases} \tag{3.2}$$

Since $N_{\mathscr{B}}\in\mathscr{B}$ and $\mu(N_{\mathscr{B}})=0$, $\pi_{\mathscr{B}}$ is an r.c.p. for $(\Omega, \mathscr{F}, \mu)$ given $\mathscr{B}$. By definition of r.c.p., (3.1) now holds for $\omega\in N_{\mathscr{B}_1}$ (both sides equal $\mu(F)$). Thus, we only have to pick $\{N_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ so that (3.1) holds for $\omega\notin N_{\mathscr{B}_1}$. We proceed step-by-step to discover the required construction.

First of all, we demand that

$$N_{\mathscr{B}_1}\supset M_{\mathscr{B}_1, \mathscr{B}_2} \quad \text{whenever } \mathscr{B}_1\subset\mathscr{B}_2. \tag{$**$}$$

It follows from this that the "mixed" compatibility condition

$$\pi_{\mathscr{B}_1}(\omega, F)=\int \pi_{\mathscr{B}_1}(\omega, d\omega')\, \tilde{\pi}_{\mathscr{B}_2}(\omega', F) \quad \text{for all } F\in\mathscr{F} \tag{3.3}$$

holds for $\omega\notin N_{\mathscr{B}_1}$. However, this is not quite what we want, since the $\tilde{\pi}_{\mathscr{B}_2}(\omega', F)$ on the right side of (3.3) differs from $\pi_{\mathscr{B}_2}(\omega', F)$ for $\omega'\in N_{\mathscr{B}_2}$. This causes no harm if $\pi_{\mathscr{B}_1}(\omega, N_{\mathscr{B}_2})=0$, but may cause trouble otherwise. Therefore we define the set

$$\mathscr{P}_{\mathscr{B}_1}(N)=\{\omega\colon \tilde{\pi}_{\mathscr{B}_1}(\omega, N)>0\} \tag{3.4}$$

for any $N\in\mathscr{F}$. We note that $\mathscr{P}_{\mathscr{B}_1}(N)\in\mathscr{B}_1$; moreover, since $\int d\mu(\omega)\, \tilde{\pi}_{\mathscr{B}_1}(\omega, N) =\mu(N)$, it follows that $\mathscr{P}_{\mathscr{B}_1}(N)$ is μ-null whenever N is. Finally, we remark that $\mathscr{P}_{\mathscr{B}_1}$ is a σ-$\cup$-homomorphism, that is, $\mathscr{P}_{\mathscr{B}_1}\left(\bigcup_{i=1}^{\infty} N_i\right)=\bigcup_{i=1}^{\infty} \mathscr{P}_{\mathscr{B}_1}(N_i)$. This is because $\tilde{\pi}_{\mathscr{B}_1}\left(\omega, \bigcup_{i=1}^{\infty} N_i\right)>0$ if and only if $\tilde{\pi}_{\mathscr{B}_1}(\omega, N_i)>0$ for some i. We now demand that

$$N_{\mathscr{B}_1}\supset \mathscr{P}_{\mathscr{B}_1}(N_{\mathscr{B}_2}) \quad \text{whenever } \mathscr{B}_1\subset\mathscr{B}_2. \tag{$***$}$$

This suffices to exclude the potentially mischievous points, hence to make (3.1) hold also for $\omega\notin N_{\mathscr{B}_1}$.

The proof is now reduced to choosing μ-null sets $N_{\mathscr{B}} \in \mathscr{B}$ satisfying (**) and (***). First let us define

$$M_{\mathscr{B}_1} = \bigcup_{\substack{\mathscr{B}_2 \supset \mathscr{B}_1 \\ \mathscr{B}_2 \in \mathbb{B}}} M_{\mathscr{B}_1, \mathscr{B}_2} \tag{3.5}$$

and

$$\bar{M}_{\mathscr{B}_1} = \bigcup_{\substack{\mathscr{B}_2 \supset \mathscr{B}_1 \\ \mathscr{B}_2 \in \mathbb{B}}} M_{\mathscr{B}_2}. \tag{3.6}$$

Since $\{\mathscr{B}_2 \in \mathbb{B}: \mathscr{B}_1 \subset \mathscr{B}_2\}$ is countable, we have $M_{\mathscr{B}_1} \in \mathscr{B}_1$, $\bar{M}_{\mathscr{B}_1} \in \mathscr{F}$, and $\mu(M_{\mathscr{B}_1}) = \mu(\bar{M}_{\mathscr{B}_1}) = 0$. Clearly (**) is equivalent to $N_{\mathscr{B}_1} \supset M_{\mathscr{B}_1}$. Similarly, let us define

$$\bar{\mathscr{P}}_{\mathscr{B}_1}(N) = \bigcup_{\substack{\mathscr{B}_2 \supset \mathscr{B}_1 \\ \mathscr{B}_2 \in \mathbb{B}}} \mathscr{P}_{\mathscr{B}_2}(N) \tag{3.7}$$

for $N \in \mathscr{F}$. Clearly $\bar{\mathscr{P}}_{\mathscr{B}_1}(N) \in \mathscr{F}$ and is μ-null whenever N is; moreover, $\bar{\mathscr{P}}_{\mathscr{B}_1}$ is a σ-$\cup$-homomorphism.

Now define

$$\tilde{N}_{\mathscr{B}_1} = \bigcup_{j=0}^{\infty} \bar{\mathscr{P}}^j_{\mathscr{B}_1}(\bar{M}_{\mathscr{B}_1}) \tag{3.8}$$

where $\bar{\mathscr{P}}^0_{\mathscr{B}_1}(\bar{M}_{\mathscr{B}_1}) = \bar{M}_{\mathscr{B}_1}$ and $\bar{\mathscr{P}}^j_{\mathscr{B}_1}(\bar{M}_{\mathscr{B}_1}) = \bar{\mathscr{P}}_{\mathscr{B}_1}(\bar{\mathscr{P}}^{j-1}_{\mathscr{B}_1}(\bar{M}_{\mathscr{B}_1}))$. Clearly $\tilde{N}_{\mathscr{B}_1} \in \mathscr{F}$ and $\mu(\tilde{N}_{\mathscr{B}_1}) = 0$; moreover, by the σ-$\cup$-homomorphism property of $\bar{\mathscr{P}}_{\mathscr{B}_1}$,

$$\bar{\mathscr{P}}_{\mathscr{B}_1}(\tilde{N}_{\mathscr{B}_1}) = \bigcup_{j=1}^{\infty} \bar{\mathscr{P}}^j_{\mathscr{B}_1}(\bar{M}_{\mathscr{B}_1}) \subset \tilde{N}_{\mathscr{B}_1} \tag{3.9}$$

and indeed $\tilde{N}_{\mathscr{B}_1} = \bar{M}_{\mathscr{B}_1} \cup \bar{\mathscr{P}}_{\mathscr{B}_1}(\tilde{N}_{\mathscr{B}_1})$. In addition, $\tilde{N}_{\mathscr{B}_1} \supset \tilde{N}_{\mathscr{B}_2}$ whenever $\mathscr{B}_1 \subset \mathscr{B}_2$. Now let

$$N_{\mathscr{B}_1} = M_{\mathscr{B}_1} \cup \mathscr{P}_{\mathscr{B}_1}(\tilde{N}_{\mathscr{B}_1}) \tag{3.10}$$

for each $\mathscr{B}_1 \in \mathbb{B}$. Clearly $N_{\mathscr{B}_1} \in \mathscr{B}_1$ and $\mu(N_{\mathscr{B}_1}) = 0$. Moreover, $N_{\mathscr{B}_1} \supset M_{\mathscr{B}_1}$, so (**) is satisfied. Finally, $\tilde{N}_{\mathscr{B}_1} \supset M_{\mathscr{B}_2} \cup \mathscr{P}_{\mathscr{B}_2}(\tilde{N}_{\mathscr{B}_1}) \supset M_{\mathscr{B}_2} \cup \mathscr{P}_{\mathscr{B}_2}(\tilde{N}_{\mathscr{B}_2}) = N_{\mathscr{B}_2}$ whenever $\mathscr{B}_1 \subset \mathscr{B}_2$; hence $N_{\mathscr{B}_1} \supset \mathscr{P}_{\mathscr{B}_1}(N_{\mathscr{B}_2})$ whenever $\mathscr{B}_1 \subset \mathscr{B}_2$, so (***) is satisfied. This completes the proof. □

Remark. If, in addition to the hypotheses of Theorem 3.2, $\mathbb{B}$ is Noetherian (i.e. every nonempty subset of $\mathbb{B}$ has at least one maximal element), then the proof of Theorem 3.2 can be made conceptually simpler: instead of the roundabout groping for the sets $\{N_{\mathscr{B}}\}$ exemplified by the foregoing proof, the sets $\{N_{\mathscr{B}}\}$ can be constructed inductively [24] starting from the maximal elements of $\mathbb{B}$ and working downwards. The point is that when $N_{\mathscr{B}_1}$ is being chosen, $\{N_{\mathscr{B}_2}\}_{\mathscr{B}_2 \supset \mathscr{B}_1}$ will have already been chosen, so $N_{\mathscr{B}_1}$ can simply be defined as the set of ω where

$$\tilde{\pi}_{\mathscr{B}_1}(\omega, F) \neq \int \tilde{\pi}_{\mathscr{B}_1}(\omega, d\omega')\, \pi_{\mathscr{B}_2}(\omega', F)$$

for some $F \in \mathscr{F}$ and some $\mathscr{B}_2 \supset \mathscr{B}_1$.

The regular conditional probabilities $\pi_{\mathscr{B}}$ constructed in Theorem 3.2 will not in general be – and cannot in general be chosen to be – everywhere proper: for this it is necessary [Lemma 2.2(a)] that each $\mathscr{B}\in\mathbb{B}$ possess a selection homomorphism with respect to $\mathscr{F}$, at least if $\mathscr{F}$ is standard. However, we do have the following result:

Theorem 3.3. *Let $\mathscr{F}$ be countably generated, and let $\mathbb{B}$ be a family of sub-σ-fields of $\mathscr{F}$ such that for each $\mathscr{B}_1\in\mathbb{B}$, the set $\{\mathscr{B}_2\in\mathbb{B}\colon \mathscr{B}_1\subset\mathscr{B}_2$ or $\mathscr{B}_2\subset\mathscr{B}_1\}$ is countable. Assume further that $\mathbb{B}$ possesses a compatible family of selection homomorphisms, that is, a family of σ-homomorphisms $\{\psi_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ such that:*

(a) *for each $\mathscr{B}\in\mathbb{B}$, $\psi_{\mathscr{B}}$ is a selection homomorphism for $\mathscr{B}$ with respect to $\mathscr{F}$; and*

(b) *if $\mathscr{B}_1,\mathscr{B}_2\in\mathbb{B}$ with $\mathscr{B}_1\subset\mathscr{B}_2$, then $\psi_{\mathscr{B}_1}\circ\psi_{\mathscr{B}_2}=\psi_{\mathscr{B}_1}$.*

Then, if μ is any perfect measure on $(\Omega,\mathscr{F})$, there exists a family $\{\pi_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ of everywhere proper, regular conditional probabilities for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$ which satisfy (3.1) *whenever $\mathscr{B}_1,\mathscr{B}_2\in\mathbb{B}$ with $\mathscr{B}_1\subset\mathscr{B}_2$.*

Example (lattice model). Let $\mathscr{L}$ be a countable index set; for each $i\in\mathscr{L}$, let $(\Omega_i,\mathscr{F}_i)$ be a measurable space; and let $(\Omega,\mathscr{F})=(\bigtimes_{i\in\mathscr{L}}\Omega_i, \bigtimes_{i\in\mathscr{L}}\mathscr{F}_i)$. Then $\mathscr{F}$ is countably generated if and only if each $\mathscr{F}_i$ is; and $(\Omega,\mathscr{F})$ is perfect if and only if each $(\Omega_i,\mathscr{F}_i)$ is. For each subset $\Gamma\subset\mathscr{L}$, let $\mathscr{F}_\Gamma$ be the σ-field $\bigtimes_{i\in\Gamma}\mathscr{F}_i$ considered as a σ-field on Ω in the obvious way. Let $\mathscr{L}_1$ be any countable family of subsets of $\mathscr{L}$, and let $\mathbb{B}=\{\mathscr{F}_\Gamma\colon \Gamma\in\mathscr{L}_1\}$. Then choose any fixed element $\omega^0\in\Omega$, let $f_{\mathscr{F}_\Gamma}(\omega)=\omega_\Gamma\times\omega^0_{\Gamma^c}$ (in the obvious notation), and define $\psi_{\mathscr{F}_\Gamma}=f^{-1}_{\mathscr{F}_\Gamma}$. It is easy to verify that $\{\psi_{\mathscr{F}_\Gamma}\}_{\mathscr{F}_\Gamma\in\mathbb{B}}$ is a compatible family of selection homomorphisms. By taking $\mathscr{L}_1$ to be the family of all subsets of $\mathscr{L}$ with finite complement, we recover the Dobrushin-Lanford-Ruelle framework for lattice models in classical statistical mechanics [5–13]. In particular, Theorem 3.3 includes and generalizes the results of [12, 13].

Proof. By Lemma 2.2(b), there exists for each $\mathscr{B}\in\mathbb{B}$ an everywhere proper, regular conditional probability for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$ – call it $\tilde{\pi}_{\mathscr{B}}$. Then, by Lemma 3.1, if $\mathscr{B}_1\subset\mathscr{B}_2$ we have $\tilde{\pi}_{\mathscr{B}_1}(\omega,F)=\int\tilde{\pi}_{\mathscr{B}_1}(\omega,d\omega')\,\tilde{\pi}_{\mathscr{B}_2}(\omega',F)$ for all $F\in\mathscr{F}$, for all ω except in a μ-null set $M_{\mathscr{B}_1,\mathscr{B}_2}\in\mathscr{B}_1$. The philosophy of the proof is the same as that of Theorem 3.2; however, the choice of the null sets $N_{\mathscr{B}}\in\mathscr{B}$ and the definition of $\pi_{\mathscr{B}}$ are somewhat more delicate in the present case, since we must arrange that the modified kernels $\{\pi_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$ be not only everywhere compatible but also everywhere proper. First of all, we define $\pi_{\mathscr{B}}$ as follows:

$$\pi_{\mathscr{B}}(\omega,F)=\begin{cases}\tilde{\pi}_{\mathscr{B}}(\omega,F) & \text{for } \omega\notin N_{\mathscr{B}}\\ \chi_{\psi_{\mathscr{B}}(F)}(\omega) & \text{for } \omega\in N_{\mathscr{B}}.\end{cases}\tag{3.11}$$

Since $N_{\mathscr{B}}\in\mathscr{B}$ and $\mu(N_{\mathscr{B}})=0$, $\pi_{\mathscr{B}}$ is an r.c.p. for $(\Omega,\mathscr{F},\mu)$ given $\mathscr{B}$; and by the argument in Lemma 2.2, $\pi_{\mathscr{B}}$ is everywhere proper. To see what requirements are imposed upon the choice of $\{N_{\mathscr{B}}\}_{\mathscr{B}\in\mathbb{B}}$, we distinguish two cases:

Case 1. $\omega\in N_{\mathscr{B}_1}$. To make (3.1) hold in this case, we demand that

$$N_{\mathscr{B}_1}\subset N_{\mathscr{B}_2}\quad\text{whenever } \mathscr{B}_1\subset\mathscr{B}_2.\tag{$*$}$$

(Note that this is the *reverse* of the situation for the $\tilde{N}_{\mathscr{B}}$ in Theorem 3.2.) To see that this works, we compute the right side of (3.1) for $\omega \in N_{\mathscr{B}_1}$. Then $\pi_{\mathscr{B}_1}(\omega, \cdot) = \chi_{\psi_{\mathscr{B}_1}(\cdot)}(\omega)$ is a 0-1-valued probability measure on $\mathscr{F}$; hence, since $\mathscr{F}$ is countably generated, it is concentrated on an atom of $\mathscr{F}$. Indeed, $\pi_{\mathscr{B}_1}(\omega, \cdot)$ is concentrated on the unique atom A_ω of $\mathscr{F}$ such that $\omega \in \psi_{\mathscr{B}_1}(A_\omega)$. Now since $\omega \in N_{\mathscr{B}_1} = \psi_{\mathscr{B}_1}(N_{\mathscr{B}_1})$, it follows by definition of A_ω that $A_\omega \subset N_{\mathscr{B}_1}$. Thus we can restrict ourselves to $\omega' \in A_\omega \subset N_{\mathscr{B}_1} \subset N_{\mathscr{B}_2}$, so that $\pi_{\mathscr{B}_2}(\omega', F) = \chi_{\psi_{\mathscr{B}_2}(F)}(\omega')$. Then the right side of (3.1) equals 1 if $A_\omega \subset \psi_{\mathscr{B}_2}(F)$, 0 otherwise. Now if $A_\omega \subset \psi_{\mathscr{B}_2}(F)$, then $\psi_{\mathscr{B}_1}(A_\omega) \subset \psi_{\mathscr{B}_1}(\psi_{\mathscr{B}_2}(F))$ since $\psi_{\mathscr{B}_1}$ is a homomorphism; and the converse is true since A_ω is an atom and $\psi_{\mathscr{B}_1}(A_\omega)$ is nonempty.[6] Moreover, $\psi_{\mathscr{B}_1}(\psi_{\mathscr{B}_2}(F)) = \psi_{\mathscr{B}_1}(F)$ by the compatibility hypothesis. Hence the right side of (3.1) equals 1 if $\psi_{\mathscr{B}_1}(A_\omega) \subset \psi_{\mathscr{B}_1}(F)$, 0 otherwise. But since $\psi_{\mathscr{B}_1}(A_\omega)$ is an atom of $\mathscr{B}_1$ containing ω,[7] $\psi_{\mathscr{B}_1}(A_\omega) \subset \psi_{\mathscr{B}_1}(F)$ if and only if $\omega \in \psi_{\mathscr{B}_1}(F)$. That is, the right side of (3.1) equals $\chi_{\psi_{\mathscr{B}_1}(F)}(\omega)$, which is just what the left side equals.

Case 2. $\omega \notin N_{\mathscr{B}_1}$. As in the proof of Theorem 3.2, we demand that

$$N_{\mathscr{B}_1} \supset M_{\mathscr{B}_1, \mathscr{B}_2} \quad \text{whenever } \mathscr{B}_1 \subset \mathscr{B}_2 \tag{$**$}$$

and

$$N_{\mathscr{B}_1} \supset \mathscr{P}_{\mathscr{B}_1}(N_{\mathscr{B}_2}) \quad \text{whenever } \mathscr{B}_1 \subset \mathscr{B}_2, \tag{$***$}$$

where $\mathscr{P}_{\mathscr{B}_1}$ is defined by (3.4). $\mathscr{P}_{\mathscr{B}_1}$ possesses all the same properties as before; and as before, ($**$) and ($***$) suffice to exclude the potentially mischievous points from Case 2.

We are thus reduced to choosing μ-null sets $N_{\mathscr{B}} \in \mathscr{B}$ satisfying ($*$), ($**$) and ($***$); the difference from Theorem 3.2 is based on the need to satisfy ($*$). First recall that $\mathscr{B}_1$ and $\mathscr{B}_2$ are said to be *comparable* if $\mathscr{B}_1 \subset \mathscr{B}_2$ or $\mathscr{B}_2 \subset \mathscr{B}_1$. Now define a relation $\equiv$ on $\mathbb{B}$ by saying that $\mathscr{B} \equiv \mathscr{B}'$ if there is a finite sequence $\{\mathscr{B}_k\}_{0 \leq k \leq n}$ in $\mathbb{B}$ such that $\mathscr{B}_0 = \mathscr{B}$, $\mathscr{B}_n = \mathscr{B}'$, and $\mathscr{B}_k$ is comparable to $\mathscr{B}_{k+1}$ for each k. It is clear that $\equiv$ is an equivalence relation; and it follows easily from the hypothesis on $\mathbb{B}$ that each equivalence class is countable.

Now define $M_{\mathscr{B}_1}$ as in (3.5) and let

$$\bar{\bar{M}}_{\mathscr{B}_1} = \bigcup_{\substack{\mathscr{B}_2 \equiv \mathscr{B}_1 \\ \mathscr{B}_2 \in \mathbb{B}}} M_{\mathscr{B}_2}. \tag{3.12}$$

Since the equivalence classes are countable, we have $M_{\mathscr{B}_1} \in \mathscr{B}_1$, $\bar{\bar{M}}_{\mathscr{B}_1} \in \mathscr{F}$, and $\mu(M_{\mathscr{B}_1}) = \mu(\bar{\bar{M}}_{\mathscr{B}_1}) = 0$. Clearly ($**$) is equivalent to $N_{\mathscr{B}_1} \supset M_{\mathscr{B}_1}$. Similarly, let us define

$$\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}(N) = \bigcup_{\substack{\mathscr{B}_2 \equiv \mathscr{B}_1 \\ \mathscr{B}_2 \in \mathbb{B}}} \mathscr{P}_{\mathscr{B}_2}(N) \tag{3.13}$$

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6 In detail: if $A_\omega \not\subset \psi_{\mathscr{B}_2}(F)$, then $A_\omega \cap \psi_{\mathscr{B}_2}(F) = \emptyset$ since A_ω is an atom. Then $\psi_{\mathscr{B}_1}(A_\omega) \cap \psi_{\mathscr{B}_1}(\psi_{\mathscr{B}_2}(F)) = \emptyset$ since $\psi_{\mathscr{B}_1}$ is a homomorphism. But $\omega \in \psi_{\mathscr{B}_1}(A_\omega)$, so $\psi_{\mathscr{B}_1}(A_\omega)$ is nonempty. Hence $\psi_{\mathscr{B}_1}(A_\omega) \not\subset \psi_{\mathscr{B}_1}(\psi_{\mathscr{B}_2}(F))$.

7 In detail: we know that $\psi_{\mathscr{B}_1}(A_\omega)$ is nonempty. Now let $B \subset \psi_{\mathscr{B}_1}(A_\omega)$, $B \in \mathscr{B}_1$. Then $\psi_{\mathscr{B}_1}(B \cap A_\omega) = \psi_{\mathscr{B}_1}(B) \cap \psi_{\mathscr{B}_1}(A_\omega) = B \cap \psi_{\mathscr{B}_1}(A_\omega) = B$. Now $B \cap A_\omega$ equals either $\emptyset$ or A_ω, since A_ω is an atom; hence B equals either $\emptyset$ or $\psi_{\mathscr{B}_1}(A_\omega)$. Thus $\psi_{\mathscr{B}_1}(A_\omega)$ is an atom of $\mathscr{B}_1$

for $N\in\mathscr{F}$. Clearly $\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}(N)\in\mathscr{F}$ and is μ-null whenever N is; moreover, $\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}$ is a σ-$\cup$-homomorphism.

Now define

$$\tilde{\tilde{N}}_{\mathscr{B}_1}=\bigcup_{j=0}^{\infty}\bar{\bar{\mathscr{P}}}^j_{\mathscr{B}_1}(\bar{\bar{M}}_{\mathscr{B}_1}) \tag{3.14}$$

where $\bar{\bar{\mathscr{P}}}^0_{\mathscr{B}_1}(\bar{\bar{M}}_{\mathscr{B}_1})=\bar{\bar{M}}_{\mathscr{B}_1}$ and $\bar{\bar{\mathscr{P}}}^j_{\mathscr{B}_1}(\bar{\bar{M}}_{\mathscr{B}_1})=\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}(\bar{\bar{\mathscr{P}}}^{j-1}_{\mathscr{B}_1}(\bar{\bar{M}}_{\mathscr{B}_1}))$. Clearly $\tilde{\tilde{N}}_{\mathscr{B}_1}\in\mathscr{F}$ and $\mu(\tilde{\tilde{N}}_{\mathscr{B}_1})=0$; moreover, by the σ-$\cup$-homomorphism property of $\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}$,

$$\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}(\tilde{\tilde{N}}_{\mathscr{B}_1})=\bigcup_{j=1}^{\infty}\bar{\bar{\mathscr{P}}}^j_{\mathscr{B}_1}(\bar{\bar{M}}_{\mathscr{B}_1})\subset\tilde{\tilde{N}}_{\mathscr{B}_1} \tag{3.15}$$

and indeed $\tilde{\tilde{N}}_{\mathscr{B}_1}=\bar{\bar{M}}_{\mathscr{B}_1}\cup\bar{\bar{\mathscr{P}}}_{\mathscr{B}_1}(\tilde{\tilde{N}}_{\mathscr{B}_1})$. In addition, $\tilde{\tilde{N}}_{\mathscr{B}_1}=\tilde{\tilde{N}}_{\mathscr{B}_2}$ whenever $\mathscr{B}_1\equiv\mathscr{B}_2$, in particular whenever $\mathscr{B}_1\subset\mathscr{B}_2$. Now let

$$N_{\mathscr{B}_1}=\bigcup_{\substack{\mathscr{B}_0\subset\mathscr{B}_1\\ \mathscr{B}_0\in\mathbb{B}}}(M_{\mathscr{B}_0}\cup\mathscr{P}_{\mathscr{B}_0}(\tilde{\tilde{N}}_{\mathscr{B}_1})) \tag{3.16}$$

for each $\mathscr{B}_1\in\mathbb{B}$. Clearly $N_{\mathscr{B}_1}\in\mathscr{B}_1$ and $\mu(N_{\mathscr{B}_1})=0$; moreover, $N_{\mathscr{B}_1}\subset N_{\mathscr{B}_2}$ whenever $\mathscr{B}_1\subset\mathscr{B}_2$, so (*) is satisfied. Furthermore, $N_{\mathscr{B}_1}\supset M_{\mathscr{B}_1}$, so (**) is satisfied. Finally, $N_{\mathscr{B}_2}\subset\bar{\bar{M}}_{\mathscr{B}_2}\cup\bar{\bar{\mathscr{P}}}_{\mathscr{B}_2}(\tilde{\tilde{N}}_{\mathscr{B}_2})=\tilde{\tilde{N}}_{\mathscr{B}_2}=\tilde{\tilde{N}}_{\mathscr{B}_1}$ whenever $\mathscr{B}_1\equiv\mathscr{B}_2$; hence $N_{\mathscr{B}_1}\supset\mathscr{P}_{\mathscr{B}_1}(\tilde{\tilde{N}}_{\mathscr{B}_1})\supset\mathscr{P}_{\mathscr{B}_1}(N_{\mathscr{B}_2})$ whenever $\mathscr{B}_1\equiv\mathscr{B}_2$, so (***) is satisfied. This completes the proof. $\square$

Remarks. (1) I do not know whether the countability hypothesis of Theorem 3.3 can be weakened to that of Theorem 3.2.

(2) The argument in Case 1 of the above proofs is considerably clearer in terms of selection functions (if they exist): If $\omega\in N_{\mathscr{B}_1}$, then $\pi_{\mathscr{B}_1}(\omega,\cdot)=\delta_{f_{\mathscr{B}_1}(\omega)}$, the unit mass concentrated on the atom of $\mathscr{F}$ which contains the point $f_{\mathscr{B}_1}(\omega)$. Hence the right side of (3.1) equals $\pi_{\mathscr{B}_2}(f_{\mathscr{B}_1}(\omega),F)$. But since $\omega\in N_{\mathscr{B}_1}\in\mathscr{B}_1$, we have $f_{\mathscr{B}_1}(\omega)\in N_{\mathscr{B}_1}\subset N_{\mathscr{B}_2}$ by definition of selection function, so $\pi_{\mathscr{B}_2}(f_{\mathscr{B}_1}(\omega),F)=\chi_F(f_{\mathscr{B}_2}(f_{\mathscr{B}_1}(\omega)))=\chi_F(f_{\mathscr{B}_1}(\omega))=\pi_{\mathscr{B}_1}(\omega,F)$ by the compatibility equation $f_{\mathscr{B}_2}\circ f_{\mathscr{B}_1}=f_{\mathscr{B}_1}$. (Note that $f_{\mathscr{B}_2}\circ f_{\mathscr{B}_1}=f_{\mathscr{B}_1}$ is slightly stronger than the homomorphism version $f^{-1}_{\mathscr{B}_1}\circ f^{-1}_{\mathscr{B}_2}=f^{-1}_{\mathscr{B}_1}$. However, if $\mathscr{F}$ contains all one-point sets $\{\omega\}$, $\omega\in\Omega$, the two formulations are equivalent.)

(3) If $\mathscr{B}_1\subset\mathscr{B}_2$, the *reversed* compatibility condition $\psi_{\mathscr{B}_2}\circ\psi_{\mathscr{B}_1}=\psi_{\mathscr{B}_1}$ is automatic, since $\psi_{\mathscr{B}_1}(F)\in\mathscr{B}_2\subset\mathscr{B}_2$ for all $F\in\mathscr{F}$, and $\psi_{\mathscr{B}_2}$ acts as the identity on $\mathscr{B}_2$.

(4) In one very special case (which does not, unfortunately, include the lattice models), the proof of Theorem 3.3 can be considerably simplified. Assume that $\mathbb{B}$ consists of a decreasing sequence $\mathscr{B}_1\supset\mathscr{B}_2\supset\ldots$ of sub-σ-fields of $\mathscr{F}$, each of which has a selection *function* $f_{\mathscr{B}_k}$. Then the $\pi_{\mathscr{B}_n}$ can be constructed inductively: First let $\pi_{\mathscr{B}_1}=\tilde{\pi}_{\mathscr{B}_1}$. Now assume that $\{\pi_{\mathscr{B}_k}\}_{1\leq k\leq n}$ have been constructed and are everywhere proper and everywhere compatible. Let

$$N_{n+1}=\{\omega:\ \tilde{\pi}_{\mathscr{B}_{n+1}}(\omega,F)\neq\int\tilde{\pi}_{\mathscr{B}_{n+1}}(\omega,d\omega')\,\pi_{\mathscr{B}_n}(\omega',F)\ \text{for some}\ F\in\mathscr{F}\}$$

and note that $N_{n+1}\in\mathscr{B}_{n+1}$ and $\mu(N_{n+1})=0$. Now define

$$\pi_{\mathscr{B}_{n+1}}(\omega,F)=\begin{cases}\tilde{\pi}_{\mathscr{B}_{n+1}}(\omega,F) & \text{for } \omega\notin N_{n+1}\\ \pi_{\mathscr{B}_n}(f_{\mathscr{B}_{n+1}}(\omega),F) & \text{for } \omega\in N_{n+1}.\end{cases}$$

Since $f_{\mathscr{B}_{n+1}}$ and N_{n+1} are $\mathscr{B}_{n+1}$-measurable, so is $\pi_{\mathscr{B}_{n+1}}(\cdot, F)$; and its properness follows from that of $\pi_{\mathscr{B}_n}$ and the definition of selection function. The compatibility equation $\pi_{\mathscr{B}_{n+1}}(\omega, F) = \int \pi_{\mathscr{B}_{n+1}}(\omega, d\omega')\, \pi_{\mathscr{B}_n}(\omega', F)$ obviously holds for $\omega \notin N_{n+1}$; and it holds for $\omega \in N_{n+1}$ because the properness of $\pi_{\mathscr{B}_n}$ implies that $\pi_{\mathscr{B}_n} \pi_{\mathscr{B}_n} = \pi_{\mathscr{B}_n}$. Finally, $\pi_{\mathscr{B}_{n+1}}$ is now compatible with $\pi_{\mathscr{B}_k}$ for all $k \leqq n$, since compatibility is transitive.

It might be throught that this case is not covered by Theorem 3.3, since the selection functions $f_{\mathscr{B}_k}$ need not satisfy any compatibility condition. However, it is easy to show, using remark (3) above, that $\tilde{\psi}_{\mathscr{B}_n} = f_{\mathscr{B}_n}^{-1} \circ f_{\mathscr{B}_{n-1}}^{-1} \circ \ldots \circ f_{\mathscr{B}_1}^{-1}$ defines a *compatible* family of selection homomorphisms, in the sense of hypothesis (b) of Theorem 3.3.

Note Added in Proof. A recent article of Kuznetsov [25] treats questions closely related to those treated here.

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