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Two-Parameter Harnesses and the Wiener Process*

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1. Introduction

Harness properties of stochastic processes have been studied by J.M. Hammersley [1] and D. Williams [2, 3]. In [3] a simple harness is defined as a process $(X_t;\, t>0)$ such that for any $s<t<u$

$$(*)\qquad E[X_t|\mathbf{F}_s\vee\mathbf{G}_u]=\frac{(u-t)X_s+(t-s)X_u}{u-s},$$

where $\mathbf{F}_s\supset\sigma(X_r;\, r\leqq s)$ and $\mathbf{G}_u\supset\sigma(X_v;\, v\geqq u)$. It is shown that, within the class of processes with continuous trajectories and such that $EX_t^2<\infty$ for any $t>0$, the property $(*)$ is a characterization of brownian motion.

Here we consider the relations between different types of two-parameter reversed martingales and study the harness properties of the Wiener process in the plane $(W_z;\, z\in\mathbb{R}_+^2)$. Then, using recent results on two-parameter martingales, we give a characterization of W_z analogous to that given by Williams.

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2. Notations and Preliminaries

We use the following notations:

$(\Omega,\mathbf{F},P)$ is a complete probability space; $(\mathbf{F}_s^1;\, s\geqq 0)$, $(\mathbf{F}_t^2;\, t\geqq 0)$ (respectively $(\mathbf{G}_s^1;\, s\geqq 0)$, $(\mathbf{G}_t^2;\, t\geqq 0)$) are increasing (respectively decreasing) families of sub-σ-fields of $\mathbf{F}$; $z=(s,t)\in\mathbb{R}_+^2$; $z<z'$ iff $s\leqq s'$ and $t\leqq t'$; $z\ll z'$ iff $s<s'$ and $t<t'$; $[z,z']:=\{\zeta;\, z<\zeta<z'\}$; $|[z,z']|:=(s'-s)(t'-t)$; $|z|:=st$. We shall often write $\mathbf{F}_z^1$, $\mathbf{F}_z^2$, $\mathbf{G}_z^1$, $\mathbf{G}_z^2$ instead of $\mathbf{F}_s^1$, $\mathbf{F}_t^2$, $\mathbf{G}_s^1$, $\mathbf{G}_t^2$. Let us define $\mathbf{F}_z=\mathbf{F}_z^1\cap\mathbf{F}_z^2$, $\mathbf{G}_z=\mathbf{G}_z^1\cap\mathbf{G}_z^2$, $\mathbf{F}_z^3=\mathbf{F}_z^1\vee\mathbf{F}_z^2$, $\mathbf{G}_z^3=\mathbf{G}_z^1\vee\mathbf{G}_z^2$, $\mathbf{G}_\infty^1=\bigcap_{s\geqq 0}\mathbf{G}_s^1$, $\mathbf{G}_\infty^2=\bigcap_{t\geqq 0}\mathbf{G}_t^2$, $\mathbf{G}_{s,\infty}^1=\mathbf{G}_s^1\cap\mathbf{G}_\infty^2$, $\mathbf{G}_{\infty,t}=\mathbf{G}_\infty^1\cap\mathbf{G}_t^2$, $\mathbf{G}_\infty=\mathbf{G}_\infty^1\cap\mathbf{G}_\infty^2$, $\mathbf{G}_{s,\infty}^3=\bigcap_{t\geqq 0}\mathbf{G}_{s,t}^3$, $\mathbf{G}_{\infty,t}^3=\bigcap_{s\geqq 0}\mathbf{G}_{s,t}^3$.

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Let $(Y(z);\ z\in\mathbb{R}^2_+)$ be a process. For $z \ll z'$ let

$$Y[z,z'] = Y(z') - Y(z'\otimes z) - Y(z\otimes z') + Y(z), \qquad \text{where } z\otimes z' = (s,t'),$$
$$\bar{Y}[z,z'] = Y[z,z']/|[z,z']|.$$

In the following definition we consider some types of reversed martingales. They are useful in the study of harness properties.

Definition 1. An integrable process $(M(z);\ \mathbf{G}^1_z, \mathbf{G}^2_z;\ 0<z)$ is

a) a *reversed strong martingale* if for any $z \ll z'$ $E[M[z,z']\,|\,\mathbf{G}^3_z] = 0$ a.s.

b_1) a *reversed 1-martingale* if for any $t \geqq 0$ and $s' > s \geqq 0$

$$E[M(s,t) - M(s',t)\,|\,\mathbf{G}^1_{s'}] = 0 \ \text{ a.s.}$$

b_2) a *reversed 2-martingale* if for any $s \geqq 0$ and $t' > t \geqq 0$

$$E[M(s,t) - M(s,t')\,|\,\mathbf{G}^2_{t'}] = 0 \ \text{ a.s.}$$

c) a *reversed martingale* if for any $z < z'$ $E[M(z) - M(z')\,|\,\mathbf{G}_{z'}] = 0$ a.s.

In order to establish relations between the different types of reversed martingales, we suppose in the first part of the following lemma that for any $s,t \geqq 0$ the limits $M(s,\infty)$ and $M(\infty,t)$ exist and "close" the reversed martingale, forming a new reversed martingale $(M(z);\ \mathbf{G}^1_z, \mathbf{G}^2_z;\ z\in[0,\infty])$.

Lemma 1. a) *Let* $(M(z);\ \mathbf{G}^1_z, \mathbf{G}^2_z;\ z\in[0,\infty])$ *be a reversed strong martingale. If for any* $s,t\in[0,\infty]$ $M(s,\infty) = M(\infty,t) = 0$ *a.s.,* M *is a reversed 1- and 2-martingale.*

b) *Every reversed 1- and 2-martingale is a reversed martingale. Conversely, if* $(M(z);\ \mathbf{G}^1_z, \mathbf{G}^2_z;\ 0<z)$ *is a reversed martingale adapted to* $(\mathbf{G}_z;\ 0<z)$ *and if* $\mathbf{G}^1_z \coprod_{\mathbf{G}_z} \mathbf{G}^2_z$ *for any* $z>0$, M *is a reversed 1- and 2-martingale.*

Proof. a) Let $t \geqq 0$ and $s' > s \geqq 0$. Then

$$E[M(s,t) - M(s',t)\,|\,\mathbf{G}^1_{s'}] = E[E[M[(s,t),(s',\infty)]\,|\,\mathbf{G}^3_{s'\infty}]\,|\,\mathbf{G}^1_{s'}] = 0 \ \text{ a.s.}$$

Therefore M is a reversed 1-martingale and the same argument shows that M is a reversed 2-martingale.

b) Let $0 < z < z'$. Then

$$\begin{aligned} E[M(z) - M(z')\,|\,\mathbf{G}_{z'}] &= E[E[M(z) - M(z\otimes z')\,|\,\mathbf{G}^2_{z\otimes z'}]\,|\,\mathbf{G}_{z'}] \\ &\quad + E[E[M(z\otimes z') - M(z')\,|\,\mathbf{G}^1_{z'}]\,|\,\mathbf{G}_{z'}] = 0 \ \text{ a.s.} \end{aligned}$$

Therefore M is a reversed martingale. Conversely, by the conditional independence, for any $t \geqq 0$ and $s' > s \geqq 0$

$$E[M(s,t) - M(s',t)\,|\,\mathbf{G}^1_{s'}] = E[M(s,t) - M(s',t)\,|\,\mathbf{G}_{s',t}] = 0 \ \text{ a.s.}$$

3. Harness Properties of the Wiener Process

In the following proposition we consider some harness properties of the Wiener process.

Proposition 2. *Let* $(W(z); z\in\mathbb{R}_+^2)$ *be a Wiener process on* $(\Omega, \mathbf{F}, P)$. *Let* $\zeta=(\sigma,\tau)\in\mathbb{R}_+^2$ *and let* $\mathbf{F}_s^1=\sigma(W(\zeta);\ \sigma\leqq s)$, $\mathbf{G}_s^1=\sigma(W(\zeta);\ \sigma\geqq s)$, $\mathbf{F}_t^2=\sigma(W(\zeta);\ \tau\leqq t)$, $\mathbf{G}_t^2=\sigma(W(\zeta);\ \tau\geqq t)$.

Consider for each fixed $z>0$ *the process* $(\bar{W}[z,z'];\ \mathbf{F}_z^3\vee\mathbf{G}_{z'}^1,\ \mathbf{F}_z^3\vee\mathbf{G}_{z'}^2;\ z\ll z')$. *It is adapted to* $(\mathbf{F}_z^3\vee\mathbf{G}_{z'};\ z\ll z')$ *and has the properties* a)–c) *of Definition 1, that is, for* $0<z\ll z'\ll z''$ *a.s.*

$$E[\bar{W}[z,z'']-\bar{W}[z,z''\otimes z']-\bar{W}[z,z'\otimes z'']+\bar{W}[z,z']\,|\,\mathbf{F}_z^3\vee\mathbf{G}_{z''}^3]=0, \tag{1}$$

$$E[\bar{W}[z,z']\,|\,\mathbf{F}_z^3\vee\mathbf{G}_{z''}^1]=\bar{W}[z,z''\otimes z'], \tag{2.1}$$

$$E[\bar{W}[z,z']\,|\,\mathbf{F}_z^3\vee\mathbf{G}_{z''}^2]=\bar{W}[z,z'\otimes z''], \tag{2.2}$$

$$E[\bar{W}[z,z']\,|\,\mathbf{F}_z^3\vee\mathbf{G}_{z''}]=\bar{W}[z,z'']. \tag{3}$$

Proof. Let $F=\{\zeta;\ \sigma\leqq s \text{ or } \tau\leqq t\}$ and $G=\{\zeta;\ \sigma\geqq s \text{ or } \tau\geqq t\}$. First we notice that in this case $\mathbf{F}_z^3=\sigma(W(\zeta);\ \zeta\in F)$ and $\mathbf{G}_z^3=\sigma(W(\zeta);\ \zeta\in G)$ and $\mathbf{G}_z=\sigma(W(\zeta);\ z<\zeta)$. Moreover $\mathbf{F}_z^3\vee\mathbf{G}_{z''}=(\mathbf{F}_z^3\vee\mathbf{G}_{z''}^1)\cap(\mathbf{F}_z^3\vee\mathbf{G}_{z''}^2)$. Here we only outline the proof of (1); the other equalities of the proposition are proved by the same method or follow by Lemma 1.

We have $\mathbf{F}_z^3\vee\mathbf{G}_{z''}^3=\sigma(A\cup B)$ where

$$A=\{W(\zeta);\ \zeta\in F\}\cup\{W(\sigma,t);\ s\leqq\sigma\leqq s''\}\cup\{W(s,\tau);\ t\leqq\tau\leqq t''\},$$

$$B=\{W[\zeta,\zeta'];\ \zeta,\zeta'\in G\}.$$

By Proposition 2.4 of [4] it follows that $\sigma(\{W(z')\}\cup A)$ is independent of $\sigma(B)$. Therefore $E[W(z')\,|\,\mathbf{F}_z^3\vee\mathbf{G}_{z''}^3]=E[W(z')\,|\,\sigma(A)]$. Let

$$E[W(z')\,|\,\sigma(A)]=\int_F f(\zeta)\,dW(\zeta)+\int_s^{s''} g(\sigma)\,dW(\sigma,t)+\int_t^{t''} h(\tau)\,dW(s,\tau)$$

where $f\in L^2(\mathbb{R}_+^2,\mathbf{B}(\mathbb{R}_+^2),\lambda_2)$ and $g,h\in L^2(\mathbb{R}_+,\mathbf{B}(\mathbb{R}_+),\lambda_1)$, $\mathbf{B}(\mathbb{R}_+^p)$ and λ_p being the Borel-σ-field and the Lebesgue measure on $\mathbb{R}_+^p$. The functions f, g and h are determined by the method of the moments and (1) follows by rearranging the terms. QED

Remarks. a) As in the proof of Proposition 2, conditional expectations can also be calculated relative to σ-fields that represent other types of "past" and "future". For example one obtains under the assumptions of proposition 2 for $0<z\ll z'\ll z''$ $E[\tilde{W}[z',z'']|\mathbf{F}_z\vee\mathbf{G}_{z''}^3]=\tilde{W}[z,z'']$ where

$$\tilde{W}[z',z'']=|[z',z'']|^{-1}\left\{\frac{t'}{t''}[W(z'')-W(z'\otimes z'')]\right.$$
$$\left.+\frac{s'}{s''}[W(z'')-W(z''\otimes z')]-[W(z'')-W(z')]\right\}. \tag{4}$$

b) Equalities (1)–(4) remain true for the completion (in $(\Omega, \mathbf{F}, P)$) of the sigmafields generated by the Wiener process.

4. Characterization of the Wiener Process as a Harness

In this section we assume that for all $s, t \geqq 0$ the sub-σ-fields $\mathbf{F}^1_s$, $\mathbf{G}^1_s$, $\mathbf{F}^2_t$, $\mathbf{G}^2_t$ contain all the sets of probability zero of $\mathbf{F}$ and that the families $(\mathbf{F}^1_s \vee \mathbf{G}_\infty; s \geqq 0)$ and $(\mathbf{F}^2_t \vee \mathbf{G}_\infty; t \geqq 0)$ are right-continuous. Moreover we assume

$$\mathbf{G}^1_{s'} \underset{\mathbf{F}^3_z \vee \mathbf{G}_{z'}}{\perp\!\!\!\perp} \mathbf{G}^2_{t'} \quad \text{for} \quad 0 < z \ll z' < \infty, \tag{5}$$

$$\mathbf{F}^1_s \underset{\mathbf{F}_z \vee \mathbf{G}_\infty}{\perp\!\!\!\perp} \mathbf{F}^2_t \quad \text{for} \quad z > 0. \tag{6}$$

We note that (5) is equivalent to $\mathbf{F}^3_z \vee \mathbf{G}^1_{s'} \underset{\mathbf{F}^3_z \vee \mathbf{G}_{z'}}{\perp\!\!\!\perp} \mathbf{F}^3_z \vee \mathbf{G}^2_{t'}$ for $0 < z \ll z' < \infty$.

Definition 2. An integrable process $(Y(z);\, z \in \mathbb{R}^2_+)$ on $(\Omega, \mathbf{F}, P)$ is a *harness* if, for each fixed $z > 0$, the process $(\bar{Y}[z, z'];\, \mathbf{F}^3_z \vee \mathbf{G}^1_{z'}, \mathbf{F}^3_z \vee \mathbf{G}^2_{z'};\, z \ll z')$ is a reversed martingale.

Strong harnesses, 1- and 2-harnesses are defined analogously. In this section we show that, within the class of square integrable processes with continuous trajectories, the harnesses characterize (in the sense of the theorem) the Wiener process.

Theorem. *Let* $(Y(z),\, z \in \mathbb{R}^2_+)$ *be a harness such that for each* $z > 0$ $E[Y(z)^2] < \infty$ *and* $\sigma(Y(\zeta);\, \zeta < z) \subset \mathbf{F}_z$ *and* $\sigma(Y(\zeta);\, z < \zeta) \subset \mathbf{G}_z$. *If* Y *has continuous trajectories and vanishes on the axes a.s., there exist random variables* μ, $\mathbf{G}_\infty$*-measurable, and* $\sigma, \mathbf{F}_0 \vee \mathbf{G}_\infty$*-measurable, such that the regular conditional law of* Y *given* $\mathbf{F}_0 \vee \mathbf{G}_\infty$ *is the law of the process* $(\mu|z| + \sigma W(z);\, z \in \mathbb{R}^2_+)$.

The proof is given by the following lemmas.

Lemma 3. a) $(Y(z)/|z|;\, \mathbf{G}^1_z, \mathbf{G}^2_z;\, 0 \ll z)$ *is a reversed martingale.*

b) $\lim_{z \in \mathbb{Q}^2, z \uparrow \infty} Y(z)/|z|$ *exists a.s. (and will be denoted by* μ*).*

c) *For* $0 < z \ll z' < \infty$, $\mu = E[\bar{Y}[z, z'] | \mathbf{F}^3_z \vee \mathbf{G}_\infty]$ *a.s.*

d) $(Y(z) - \mu|z|;\, \mathbf{F}^1_z \vee \mathbf{G}_\infty, \mathbf{F}^2_z \vee \mathbf{G}_\infty;\, z \in \mathbb{R}^2_+)$ *is a strong martingale, denoted by* $(X(z);\, z \in \mathbb{R}^2_+)$.

e) $(X(z)/|z|;\, \mathbf{G}^1_z, \mathbf{G}^2_z;\, 0 \ll z)$ *is a reversed martingale and*

$$\lim_{z \in \mathbb{Q}^2, z \uparrow \infty} X(z)/|z| = 0 \quad a.s.$$

f) X *is a harness.*

Proof. a) Let $0 \ll z < z'$. By the harness property

$$\begin{aligned} E[Y(z)/|z| \,|\, \mathbf{G}_{z'}] &= E[E[\bar{Y}[0, z] \,|\, \mathbf{F}^3_0 \vee \mathbf{G}_{z'}] \,|\, \mathbf{G}_{z'}] \\ &= E[\bar{Y}[0, z'] \,|\, \mathbf{G}_{z'}] = Y(z')/|z'| \end{aligned}$$

and a) follows.

b) By Theorem 2.3 of [5] and a) it follows that $(Y(z')/|z'|;\ z'\geqslant 0)$ converges a.s. as $z'\in\mathbb{Q}^2$, $z'\uparrow\infty$.

c) As in b) we consider, for z and z' fixed such that $0<z\ll z'$, the reversed martingale in $z''(E[\bar{Y}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_{z''}];\ z'\ll z'')$. It also converges a.s. to $E[\bar{Y}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_\infty]$ as $z''\in\mathbb{Q}^2$, $z''\uparrow\infty$. We show that this limit is a.s. equal to μ; by the harness property

$$E[\bar{Y}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_{z''}]=\bar{Y}[z,z'']$$
$$=\frac{Y(z'')}{|z''|}\frac{s''t''}{(s''-s)(t''-t)}-\frac{Y(s'',t)}{s''t}\frac{s''t}{(s''-s)(t''-t)}$$
$$-\frac{Y(s,t'')}{st''}\frac{st''}{(s''-s)(t''-t)}+\frac{Y(z)}{|[z,z'']|}\to\mu,$$

a.s. and L^1, as $z''\uparrow\infty$, the processes $(Y(s'',t)/s''t;\ \mathbf{G}_{s'',t};\ s''>0)$ and $(Y(s,t'')/st'';\ \mathbf{G}_{s,t''};\ t''>0)$ being reversed martingales with one-dimensional parameter. Therefore $E[\bar{Y}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_\infty]=\mu$ a.s.

d) $Y(z)$ is $\mathbf{F}_z$-measurable by hypothesis, μ is $\mathbf{G}_\infty$-measurable. Therefore $X(z)$ is $\mathbf{F}_z\vee\mathbf{G}_\infty$-measurable. $X(z)$ vanishes a.s. on the axis. Moreover, for $0<z\ll z'$, $E[X[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_\infty]=E[Y[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_\infty]-\mu|[z,z']|=0$ a.s.

e) $X(z)$ is $\mathbf{G}_z$-measurable. For $0<z<z'$

$$E[X(z)/|z|\,|\mathbf{G}_{z'}]=E[Y(z)/|z|\,|\mathbf{G}_{z'}]-\mu=Y(z')/|z'|-\mu=X(z')/|z'| \quad \text{a.s.}$$

f) Let $0<z\ll z'<z''$. Then

$$E[\bar{X}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_{z''}]=E[\bar{Y}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_{z''}]-\mu$$
$$=\bar{Y}[z,z'']-\mu=\bar{X}[z,z''] \quad \text{a.s.} \quad \text{QED}$$

By Lemma 3, d) we have

Lemma 4 [6, p. 120]. *For* $0<z\ll z'$

$$E[(X[z,z'])^2\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]=E[X^2[z,z']\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty] \quad (i=1,2).$$

Lemma 5. *For* z, z', z'' *such that* $0<z\ll z'$ *and* $z\ll z''$

$$E[(X[z,z'])^2/|[z,z']|\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]$$
$$=E[(X[z,z''])^2/|[z,z'']|\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty] \quad (i=1,2).$$

Proof. We may assume $z'<z''$. If z' and z'' are not comparable, we have the equality of the lemma for $[z,z']$ and $[z,\sup(z',z'')]$ on the one hand and for $[z,z'']$ and $[z,\sup(z',z'')]$ on the other hand. Therefore we have the equality for $[z,z']$ and $[z,z'']$.

Let $z\ll z'<z''$. By Lemma 3, f)

$$E[(X[z,z''])^2/|[z,z'']|\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]=E[X[z,z'']\,E[\bar{X}[z,z']\,|\mathbf{F}_z^3\vee\mathbf{G}_{z''}]\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]$$
$$=E[X[z,z'']\,\bar{X}[z,z']\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]$$
$$=E[\bar{X}[z,z']\,\{X[z,z']+E[X[z\otimes z',z'\otimes z'']\,|\mathbf{F}_{z\otimes z'}^3\vee\mathbf{G}_\infty]$$
$$+E[X[z'\otimes z,z''\otimes z']\,|\mathbf{F}_{z'\otimes z}^3\vee\mathbf{G}_\infty]+E[X[z',z'']\,|\mathbf{F}_{z'}^3\vee\mathbf{G}_\infty]\}\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty]$$
$$=E[(X[z,z'])^2/|[z,z']|\,|\mathbf{F}_z^i\vee\mathbf{G}_\infty] \quad \text{a.s.} \quad \text{QED}$$

Corollary 6. *For z, z', z'' such that $0<z\ll z'$ and $z\ll z''$*

$$E[\overline{X^2}[z,z']\,|\,\mathbf{F}_z^i\vee\mathbf{G}_\infty]=E[\overline{X^2}[z,z'']\,|\,\mathbf{F}_z^i\vee\mathbf{G}_\infty]\qquad(i=1,2).$$

Proof. Application of the equality of lemma 4 on both sides of the equality of Lemma 5. QED

For each $z>0$ we choose now $\hat{z}=(\hat{s},\hat{t})\gg z$ and set

$$\tilde{\alpha}_i(z)=E[\overline{X^2}[z,\hat{z}]\,|\,\mathbf{F}_z^i\vee\mathbf{G}_\infty]\qquad(i=1,2).$$

Lemma 7. *There exist (one-parameter) martingales $(\alpha_1(s);\ \mathbf{F}_s^1\vee\mathbf{G}_\infty;\ s\geqq 0)$ (respectively $(\alpha_2(t);\ \mathbf{F}_t^2\vee\mathbf{G}_\infty;\ t\geqq 0)$ such that for each $z>0$ $\alpha_1(s)=\tilde{\alpha}_1(z)$ a.s. (respectively $\alpha_2(t)=\tilde{\alpha}_2(z)$ a.s.).*

Proof. We show the existence and the martingale property of α_1; the proof for α_2 is identical. For $z>0$ we have a.s.

$$\begin{aligned}|[z,\hat{z}]|\,\tilde{\alpha}_1(z)&=E[X^2[z,\hat{z}]\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]\\&=E[X^2[(s,0),\hat{z}]\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]-E[X^2[(s,0),(\hat{s},t)]\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]\\&=|[(s,0),\hat{z}]|\,\tilde{\alpha}_1((s,0))-|[(s,0),(\hat{s},t)]|\,\tilde{\alpha}_1((s,0))\\&=|[z,\hat{z}]|\,\tilde{\alpha}_1((s,0)).\end{aligned}$$

For $s'>s\geqq 0$ we have a.s.

$$\begin{aligned}|[(s',0),\hat{z}']|\,E[\alpha_1(s')\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]&=E[X^2[(s',0),\hat{z}']\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]\\&=|[(s',0),\hat{z}']|\,\alpha_1(s),\end{aligned}$$

and therefore $E[\alpha_1(s')\,|\,F_s^1\vee\mathbf{G}_\infty]=\alpha_1(s)$ a.s. QED

Lemma 8. *For each $t\geqq 0$ $(X^2(s,t)-t\int_0^s\alpha_1(\sigma)\,d\sigma;\ \mathbf{F}_s^1\vee\mathbf{G}_\infty;\ s\geqq 0)$ and for each $s\geqq 0$ $(X^2(s,t)-s\int_0^t\alpha_2(\tau)\,d\tau;\ \mathbf{F}_t^2\vee\mathbf{G}_\infty;\ t\geqq 0)$ are martingales, adapted to $\mathbf{F}_{s,t}\vee\mathbf{G}_\infty$.*

Proof. First, we note that, by the right-continuity of the families $(\mathbf{F}_s^1\vee\mathbf{G}_\infty;\ s\geqq 0)$ and $(\mathbf{F}_t^2\vee\mathbf{G}_\infty;\ t\geqq 0)$, the martingales α_1 and α_2 have right-continuous modifications (also denoted by α_1 and α_2). Therefore the Lebesgue integrals with respect to these modifications exist.

Fix $t\geqq 0$. For each $s'>s\geqq 0$ we have a.s.

$$\begin{aligned}&E[X^2(s',t)-X^2(s,t)\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]\\&\quad=E[X^2[(s,0),(s',t)]\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty]=\alpha_1(s)\,t(s'-s).\end{aligned}$$

By the Lemma 7

$$E\left[\int_0^{s'}\alpha_1(\sigma)\,d\sigma\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty\right]=\int_0^s\alpha_1(\sigma)\,d\sigma+\alpha_1(s)\,(s'-s).$$

Therefore

$$E\left[X^2(s',t)-t\int_0^{s'}\alpha_1(\sigma)\,d\sigma\,|\,\mathbf{F}_s^1\vee\mathbf{G}_\infty\right]=X^2(s,t)-t\int_0^s\alpha_1(\sigma)\,d\sigma.$$

Let us show now that $\alpha_1(s)$ is $\mathbf{F}_{s,0} \vee \mathbf{G}_\infty$-measurable: Choose a sequence $(z_n = (s_n, t_n);\ n \in \mathbb{N})$ with $s_n > s$ and $t_n \downarrow 0$. By (6) we have a.s.

$$\alpha_1(s) = E[\overline{X^2}[(s,0), z_n] \mid \mathbf{F}_s^1 \vee \mathbf{G}_\infty] = E[\overline{X^2}[(s,0), z_n] \mid \mathbf{F}_{s,t_n} \vee \mathbf{G}_\infty]$$

for each n. By (6) and the right-continuity of the family $(\mathbf{F}_t^2 \vee \mathbf{G}_\infty;\ t \geqq 0)$, $\alpha_1(s)$ is $\mathbf{F}_{s,0} \vee \mathbf{G}_\infty$-measurable. QED

Lemma 9. *There exists an* $\mathbf{F}_0 \vee \mathbf{G}_\infty$*-measurable random variable* α *such that* $(X^2(z) - \alpha|z|;\ \mathbf{F}_z \vee \mathbf{G}_\infty;\ z > 0)$ *is a martingale.*

Proof. Let us define

$$M^{(1)}(s,t) = X^2(s,t) - A^{(1)}(s,t), \qquad \text{where} \quad A^{(1)}(s,t) = t \int_0^s \alpha_1(\sigma)\, d\sigma,$$

$$M^{(2)}(s,t) = X^2(s,t) - A^{(2)}(s,t), \qquad \text{where} \quad A^{(2)}(s,t) = s \int_0^t \alpha_2(\tau)\, d\tau,$$

$$B(s,t) = M^{(1)}(s,t) - M^{(2)}(s,t) = A^{(2)}(s,t) - A^{(1)}(s,t).$$

By the Lemma 8, B is a weak martingale and a continuous process of finite variation. The Doléans-Föllmer-measure μ_B of B is zero on the σ-field of predictable sets.

Let Z be a bounded and measurable process, ΠZ its predictable projection [7]. By the results in [8] we have for each $z > 0$

$$\mu_B(Z) = E\left[\int_{[0,z]} Z\, dB\right] = E\left[\int_{[0,z]} \Pi Z\, dB\right] = \mu_B(\Pi Z) = 0.$$

Therefore

$$\begin{aligned} 1 &= P(B(s,t) = 0 \quad \text{for all} \quad s,t \geqq 0) \\ &= P\left(\int_0^s \int_0^t [\alpha_1(\sigma) - \alpha_2(\tau)]\, d\sigma\, d\tau = 0 \quad \text{for all} \quad s,t \geqq 0\right) \\ &= P(\alpha_1(s) = \alpha_2(\tau) \quad \text{for almost all} \quad (s,t) \in \mathbb{R}_+^2). \end{aligned}$$

Since α_1 and α_2 are a.s. right-continuous

$$P(\alpha_1(s) = \alpha_2(t) \quad \text{for each} \quad (s,t) \in \mathbb{R}_+^2) = 1.$$

Therefore there exists a $\mathbf{F}_0 \vee \mathbf{G}_\infty$-measurable random variable such that

$$P(\alpha_1(s) = \alpha_2(t) = \alpha \quad \text{for each} \quad (s,t) \in \mathbb{R}_+^2) = 1.$$

Therefore $(X^2(s,t) - \alpha s t;\ \mathbf{F}_{s,t} \vee \mathbf{G}_\infty;\ s,t \geqq 0)$ is a martingale. QED

We complete now the proof of the theorem:

By the Lemma 3, d) $(X(z);\ \mathbf{F}_z \vee \mathbf{G}_\infty;\ z > 0)$ is a continuous strong martingale and by the lemma 9 $(X^2(z) - \alpha|z|;\ \mathbf{F}_z \vee \mathbf{G}_\infty;\ z > 0)$ is a martingale. By Proposition 5.4 in [9] we conclude that the regular conditional law of $X(z)$ given $\mathbf{F}_0 \vee \mathbf{G}_\infty$ is the law of $\alpha^{1/2} W(z)$ where $W(z)$ is the two-parameter Wiener process. Therefore the regular conditional law of $Y(z) = \mu|z| + X(z)$ given $\mathbf{F}_0 \vee \mathbf{G}_\infty$ is the law of $\mu|z| + \alpha^{1/2} W(z)$, and the theorem is proved.

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