Z. Wahrscheinlichkeitstheorie verw. Gebiete
56, 427–441 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Conditional Expectations of Stationary Processes

Bruno Scarpellini

Math. Institute, University of Basel, Rheinsprung 21, CH-4051 Basel, Switzerland

Introduction

The present paper is a supplement to an earlier one [Sc] on the nonlinear prediction of continuous, stationary processes. In order to describe its content we briefly recall § 6 of [Sc]. To this end, let f_t, $t \in R$ (R = reals) be a continuous stationary process on a probability space $(\Omega, \mathfrak{B}, \mu)$, $(f_t \in L_2(\Omega, \mu))$. Let $\mathfrak{B}(f_s, s \leqq 0)$ be the Borel field induced by the functions f_s, $s \leqq 0$. For $g \in L_2(\Omega, \mu)$ let $E(\mathfrak{B}(f_s, s \leqq 0)/g)$ be the conditional expectation of g with respect to $\mathfrak{B}(f_s, s \leqq 0)$. The purpose of § 6 was to answer the following question: I) if f_t, $t \in R$ is a deterministic process, and if for a particular $\omega \in \Omega$ we know the past $f_t(\omega)$, $t \leqq 0$, is it then possible to reconstruct the future $f_t(\omega)$, $t > 0$ in terms of the past alone? The affirmative answer consisted in the following. A doubly infinite sequence of formulas $A^{\tau}_{mn}(\omega)$, $(m, n = 1, 2, \ldots, \tau > 0, \omega \in \Omega)$ was constructed with the properties: 1) $A^{\tau}_{mn}(\omega)$ depends only on the finitely many values $f_i(\omega)$, $i = -\tau k 2^{-m}$, $0 \leqq k \leqq m n 2^m$, 2) for almost all $\omega \in \Omega$ the following holds for almost all $\tau > 0$: $F^{\tau}_m(\omega) = \lim_n A^{\tau}_{mn}(\omega)$ exists and $\lim_m F^{\tau}_m(\omega) = f_\tau(\omega)$. It was stated but not proved that for an arbitrary process 2) has to be replaced by: 2*) for almost all ω the following holds for almost all $\tau > 0$: $\lim_n A^{\tau}_{mn}(\omega) = F^{\tau}_m(\omega)$ exists and $\lim_m F^{\tau}_m(\omega) = E(\mathfrak{B}(f_s, s \leqq 0)/f_\tau)(\omega)$.

In the first, preparatory part of the present paper we prove 2*). The second and main part of the paper has its root in the following observation. As is seen from 2*), an iterated double limit is involved in the computation of $E(\mathfrak{B}(f_s, s \leqq 0)/f_\tau)(\omega)$, which forces a computer to run infinitely often into the past. The important problem arises as to whether one can find another computational schema which computes $E(\mathfrak{B}(f_s, s \leqq 0)/f_\tau)(\omega)$ with the aid of a single passage to the limit. In [Or], D. Ornstein solves precisely this problem for the case of a discrete, ergodic process f_n, $n \in Z$ (Z = integers), having as range the set $\{0, 1\}$. He describes an algorithm which computes $E(\mathfrak{B}(f_n, n \leqq 0)/f_1)(\omega)$ with the aid of a single passage to the limit. In the second part we show that the method of Ornstein can be combined with the formulas obtained in the first

0044-3719/81/0056/0427/$03.00

part in order to obtain an algorithm of the above kind for continuous, stationary processes $f_t, t \in R$, which are required to be bounded but which are otherwise arbitrary. The extension of Ornstein's method is not completely straightforward, but depends on a finer analysis of his construction. The difficulty is thereby rather the arbitrary range than the continuous time. As a particularity, we mention that a value is effectively determined for the crucial constant β whose existence is stated in Lemma 5 of [Or].

Chapter I: Formulas for Conditional Expectations

§ 1. Summary of Notions

We briefly summarize those notions from [Sc], which will be used throughout the sequel. A flow is a quadruple $(\Omega, \mathfrak{B}, S_t, t \in R, \mu)$ with $\mathfrak{B}$ a Borel field over Ω, μ a probability measure on $\mathfrak{B}$, and S_t, $t \in R$ a measure preserving automorphism group on Ω, such that the mapping $(\omega, t) \to S_t(\omega)$ is measurable; Ω is a compact metric space and $\mathfrak{B}$ is induced by the topology. We put $\omega_t = S_t(\omega)$. With $f \in L_2(\Omega, \mu)$ we associate the stationary process f_t, $t \in R$ given by $f_t(\omega) = f(\omega_t)$. In the present chapter, use is made of the ergodic decomposition theory of Krylov-Bogoljubov (see [N-S]); for reasons of place we have to refer to [Sc], §2 for a summary of those of its properties which we need. The notation is the same as in [Sc]. Thus U_T is the set of transitive points. With each $\omega \in U_T$ there is associated an invariant ergodic Borel measure μ_ω having properties 1)–6) listed in §2 of [Sc]. Likewise there is for each $\tau \in R$ a corresponding set U_T^τ of transitive points; for each $\omega \in U_T^\tau$ there is a Borel measure μ_ω^τ having properties 1')–6') listed in §2 of [Sc]. We need the index sets $I_N^\tau = \{-\tau k 2^{-N}/0 \leq k \leq N 2^N, k \in Z\}$, and for $-1 \leq j \leq k_N$ (where $k_N = N 2^N$) we introduce the numbers α_j^N by: a) $\alpha_{-1}^N = -N - 2^{-N}$, b) if $-1 \leq j < k_N$ then $\alpha_{j+1}^N - \alpha_j^N = 2^{-N}$, c) $\alpha_{k_N}^N = N$. With $f \in L_2(\Omega, \mu)$ we associate the function f^N given by: d) $f^N(\omega) = \alpha_{-1}^N$ iff $f(\omega) < -N$, e) if $0 \leq j < k_N$ then $f^N(\omega) = \alpha_j^N$ iff $\alpha_j^N \leq f(\omega) < \alpha_{j+1}^N$, f) $f^N(\omega) = \alpha_{k_N}^N$ iff $N \leq f(\omega)$. Since the two maps $f \to f^N$ and $f \to f_t$ commute with each other we are entitled to introduce the abbreviation f_t^N for $(f^N)_t$ $(=(f_t)^N)$. We shall use repeatedly the Borel fields $\mathfrak{B}_M^\tau = \mathfrak{B}(f_i^N, i \in I_M^\tau)$, $\mathfrak{B}_- = \mathfrak{B}(f_s, s \leq 0)$ and $\mathfrak{B}_-^\tau = \mathfrak{B}(f_i, i \in \bigcup_N I_N^\tau)$ (where $\mathfrak{B}(f_i, i \in I)$ is the smallest Borel field with respect to which all f_i, $i \in I$ are measurable). If $\mathfrak{B}'$ is any one of these Borel fields, $E_\omega^\tau(\mathfrak{B}'/h)$ is the conditional expectation of $h \in L_2(\Omega, \mu)$ with respect to $\mathfrak{B}'$ and the measure μ_ω^τ, while $E_\omega(\mathfrak{B}'/h)$ is the conditional expectation with respect to $\mathfrak{B}'$ and the measure μ_ω. Finally we write $(A)\int f\,d\mu$ in place $\int_A f\,d\mu$.

§ 2. Formulas for Conditional Expectations

Henceforth a flow $(\Omega, \mathfrak{B}, S_t, t \in R, \mu)$ and an $f \in L_2(\Omega, \mu)$ are given in a fixed way. In order to prove statement 2*) mentioned in the introduction we recall a few expressions used in §6 of [Sc]. Δ_n is the $2n$-place function defined as follows:

$\Delta_n(x_1,\dots,x_n,y_1,\dots,y_n)=1$ iff $x_i=y_i$ for $i=1,\dots,n$ and $=0$ otherwise. If $J=\{i_1,\dots,i_n\}$ is a finite set of reals, if α_i, $i\in J$ and $\beta_i, i\in J$ are finite sets of reals indexed by the elements of J we simply write $\Delta(\alpha_i,\beta_i,i\in J)$ instead of $\Delta_n(\alpha_{i_1},\dots,\alpha_{i_n},\beta_{i_1},\dots,\beta_{i_n})$. Let $\lambda_1,\lambda_2,\dots$ be the possibly empty list of nonzero eigenvalues of the flow. A $\tau\in R$ is called noncritical if $\tau\lambda_k\neq 2\pi m$ for $k\geqq 1$, $m\in Z$. Finally, the following expressions are important in our considerations. For ω, $\zeta\in\Omega$ we set:

α) $H_{0\tau}^M(\omega,\zeta)=\Delta(f_i^M(\omega),f_{i-\tau}^M(\zeta),i\in I_M^\tau)\,f(\zeta)$,

β) $H_{1\tau}^M(\omega,\zeta)=\Delta(f_i^M(\omega),f_i^M(\zeta),i\in I_M^\tau)$.

According to Lemmas 12, 13 in [Sc] the following holds:

γ) the limits

$$A_M^\tau(\omega)=\lim N^{-1}\sum_0^N H_{0\tau}^N(\omega,\omega_{-p\tau})$$

and

$$B_M^\tau(\omega)=\lim_N N^{-1}\sum_0^N H_{1\tau}^M(\omega,\omega_{-p\tau})$$

exist for almost all ω, and $B_M^\tau(\omega)\neq 0$ for almost all ω.

This gives rise to the difinition:

δ) $F_M^\tau(\omega)=\lim_N\left(\sum_0^N H_{0\tau}^M(\omega,\omega_{-p\tau})\left(\sum_0^N H_{1\tau}^M(\omega,\omega_{-p\tau})\right)^{-1}\right)$ if $A_M^\tau(\omega)$, $B_M^\tau(\omega)$ both exist and $B_M^\tau(\omega)\neq 0$, and $F_M^\tau(\omega)=0$ otherwise.

From δ) we obtain: ε) $F_M^\tau(\omega)=A_M^\tau(\omega)\,B_M^\tau(\omega)^{-1}$ for almost all ω. According to Theorem 4 and Corollary 3 in [Sc] the following holds:

*) if f_t, $t\in R$ is deterministic and $\tau>0$ then $\lim F_M^\tau(\omega)=f(\omega_\tau)$ for almost all ω.

Our first aim is to prove the following generalisation of *):

Theorem 1. *Assume* $\tau>0$. *Then* $\lim_M F_M^\tau(\omega)=E(\mathfrak{B}^\tau/f_\tau)(\omega)$ *for almost all* ω.

We split the proof into a few simple lemmas, from which the theorem easily follows.

Lemma 1. *The limit* $g(\omega)=\lim_M F_M^\tau(\omega)$ *exists and is finite for almost all* ω.

Proof. The proof is very similar to the proof of Theorem 4 in [Sc] and will be kept short. Let L be the set of ω's such that $\lim_M F_M^\tau(\omega)$ exists and is finite. According to 1′)–6′) in §2 of [Sc] and since L is measurable, it suffices to show: 1) $\mu_\omega^\tau(L)=1$ for almost all $\omega\in U_T^\tau$. Now let $E'\subseteq U_T^\tau$ be a set of measure 1 such that $\omega\in E'$ implies $f\in L_2(\Omega,\mu_\omega^\tau)$ (§2, [Sc]). According to Lemma 12 in [Sc] there is for every M a set $E_M\subseteq E'$ with $\mu(E_M)=1$ such that $\omega\in E_M$ implies: 2) $F_M^\tau(\xi)=E_\omega^\tau(\mathfrak{B}_M^\tau/f_\tau)(\xi)$ for μ_ω^τ-almost all ξ. From 2) we infer: 3) if $\omega\in E''=\bigcap_M E_M$ then $F_M^\tau(\xi)=E_\omega^\tau(\mathfrak{B}_M^\tau/f_\tau)(\xi)$ for μ_ω^τ-almost all ξ $(M=1,2,\dots)$. Now assume $\omega\in E''$. Since $\mathfrak{B}_-^\tau$ is the smallest Borel field containing all $\mathfrak{B}_M^\tau$, and since $\mathfrak{B}_M^\tau\subseteq\mathfrak{B}_{M+1}^\tau$, we infer from 3) and the martingale theorem: 4) $\lim_M F_M^\tau(\xi)=E_\omega^\tau(\mathfrak{B}_-^\tau/f_\tau)(\xi)$ for μ_ω^τ-almost all ξ. Since f and thus f_τ are in $L_2(\Omega,\mu_\omega^\tau)$ for $\omega\in E''$, 4) implies $\mu_\omega^\tau(L)=1$. As $\mu(E'')=1$, the lemma follows. □

Our next aim is to show that $g=\lim_M F_M^\tau$ is in $L_2(\Omega,\mu)$. To this end we need the

Lemma 2. *Fix $\tau>0$. Let $G\geqq 0$ be measurable and assume:* 1) $G\in L_1(\Omega,\mu_\omega^\tau)$ *for almost all* $\omega\in U_T^\tau$, 2) *there is a* $\varphi\in L_1(\Omega,\mu)$, $\varphi\geqq 0$, *such that* $\int G d\mu_\omega^\tau\leqq\varphi$ *almost everywhere. Then* $G\in L_1(\Omega,\mu)$ *and* $\int G d\mu=\int d\mu\int G d\mu_\omega^\tau$.

Proof. Put $G_N(\omega)=G(\omega)$ if $G(\omega)\leqq N$, $G_N(\omega)=N$ otherwise. Evidently we have:
1) $\int d\mu\int G_N d\mu_\omega^\tau\leqq\int d\mu\int G d\mu_\omega^\tau\leqq\infty$ and $\lim_N\int d\mu\int G_N d\mu_\omega^\tau=\int d\mu\int G d\mu_\omega^\tau$. According to decomposition theory (§2 in [Sc]) the following holds:
2) $\int d\mu\int G_N d\mu_\omega^\tau=\int G_N d\mu$. From assumption 2 of the lemma on the other hand we get $\int d\mu\int G_N d\mu_\omega^\tau\leqq\int\varphi d\mu$, which, together with 2) implies:
3) $\int G d\mu=\lim_N\int G_N d\mu\leqq\int\varphi d\mu$. Thus $G\in L_1(\Omega,\mu)$. By combining 1)–3) we obtain $\int G d\mu=\int d\mu\int G d\mu_\omega^\tau$, which proves the lemma. □

Lemma 3. $g=\lim F_M^\tau$ *belongs to* $L_1(\Omega,\mu)$ *and* $\int|g|^2 d\mu\leqq\int|f|^2 d\mu$.

Proof. The proof of Lemma 1 shows that g has the following property: 1) for almost all $\omega\in U_T^\tau$, $g(\xi)=E_\omega^\tau(\mathfrak{B}_-^\tau/f_\tau)(\xi)$ for μ_ω^τ-almost all ξ. Now let $E\subseteq U_T^\tau$ be a subset with $\mu(E)=1$ such that 1) holds for $\omega\in E$ and in addition $f\in L_2(\Omega,\mu_\omega^\tau)$. Now assume $\omega\in E$. Since $E_\omega^\tau(\mathfrak{B}_-^\tau/f_\tau)$ is the projection of f onto the subspace $L_2(\mathfrak{B}_-^\tau,\mu_\omega^\tau)$ of all $\mathfrak{B}_-^\tau$-measurable functions in $L_2(\Omega,\mu_\omega^\tau)$, we get: 3) $\int|g|^2 d\mu_\omega^\tau\leqq\int|f_\tau|^2 d\mu_\omega^\tau=\int|f|^2 d\mu_\omega^\tau$. But $\int|f|^2 d\mu=\int d\mu\int|f|^2 d\mu_\omega^\tau$ according to properties 1′)–6′) in §2 of [Sc], whence $\int|g|^2 d\mu\leqq\int|f|^2 d\mu$ follows from Lemma 2. □

We can now proceed to the proof of Theorem 1:

Proof of Theorem 1. Let χ_A be the characteristic function of the set $A\in\mathfrak{B}_-^\tau$. Since $\chi_A g\in L_1(\Omega,\mu)$ by Lemma 3, we have: 1) $(A)\int g d\mu=\int d\mu\int\chi_A g d\mu_\omega^\tau$. As noted in the proof of Lemma 1, for almost all ω the equation $g(\xi)=E_\omega^\tau(\mathfrak{B}_-^\tau/f_\tau)(\xi)$ holds for μ_ω^τ-almost all ξ. From this and 1) we get: 2) $(A)\int g d\mu=\int d\mu\int d\chi_A E_\omega^\tau(\mathfrak{B}_-^\tau/f_\tau)d\mu_\omega^\tau=\int d\mu\int\chi_A f_\tau d\mu_\omega^\tau=(A)\int f_\tau d\mu$. Since $A\in\mathfrak{B}_-^\tau$ was arbitrary, this implies $g=E(\mathfrak{B}_-^\tau/f_\tau)$ almost everywhere, which proves the theorem. □

Corollary 1. *Theorem 1 remains true if* $E(\mathfrak{B}_-^\tau/f_\tau)$ *is replaced by* $E(\mathfrak{B}_-/f_\tau)$.

Proof. This follows from Lemma 8 in [Sc], according to which $E(\mathfrak{B}_-^\tau/f_\tau)=E(\mathfrak{B}_-/f_\tau)$ holds almost everywhere. □

By a straightforward Fubini argument we infer from Corollary 1:

Corollary 2. *For almost all* ω, $\lim_M F_M^\tau(\omega)=E(\mathfrak{B}_-/f_\tau)(\omega)$ *holds for almost all* $\tau>0$.

§ 3. A Generalisation

We conclude with a remark which will be of importance in the next chapter. Let $f,g\in L_2(\Omega,\mu)$ and set

$$H_{0\tau}^M(f,g/\omega,\zeta)=\Delta(f_i(\omega),\ f_{i-\tau}(\zeta),i\in I_M^\tau)\,g(\zeta),$$
$$H_{1\tau}^M(f/\omega,\zeta)=\Delta(f_i(\omega),\ f_i(\zeta),\ i\in I_M^\tau).$$

A straightforward inspection of §6 in [Sc] shows that the arguments used there can be taken over literally in order to obtain generalisations of the results in §6 of [Sc], in which the conditional expectations $E(\mathfrak{B}^\tau_-/f_\tau)$, $E^\tau_\omega(\mathfrak{B}^\tau_M/f_\tau)$, $E_\omega(\mathfrak{B}^\tau_M/f_\tau)$ are replaced by the conditional expectations $E(\mathfrak{B}^\tau_-/g_\tau)$, $E^\tau_\omega(\mathfrak{B}^\tau_M/g_\tau)$ and $E_\omega(\mathfrak{B}^\tau_M/g_\tau)$ respectively, whereby the Borel fields $\mathfrak{B}^\tau_-$, $\mathfrak{B}^\tau_M$, are as before defined in terms of f according to the definitions in §1. We have in particular:

γ') the limits $A^\tau_M(f, g/\omega) = \lim_N N^{-1} \sum_0^N H^M_{0\tau}(f, g/\omega, \omega_{-p\tau})$ and $B^\tau_M(f/\omega) = \lim_N N^{-1} \sum_0^N H^M_{1\tau}(f/\omega, \omega_{-p\tau})$ exist almost everywhere and again $B^\tau_M(f/\omega) \neq 0$ for almost all ω.

This gives rise to the definition:

δ') $F^\tau_M(f, g/\omega) = \lim_N \left(\sum_0^N H^M_{0\tau}(f, g/\omega, \omega_{-p\tau})\right)\left(\sum_0^N H^M_{1\tau}(f/\omega, \omega_{-p\tau})\right)^{-1}$ if $A^\tau_M(f, g/\omega)$, $B^\tau_M(f/\omega)$ both exist and $B^\tau_M(f/\omega) \neq 0$, and $F^\tau_M(f, g/\omega) = 0$ otherwise.

The generalisation of Lemma 12, b) in [Sc] eg. states:

*) for almost all $\omega \in U^\tau_T$ we have $F^\tau_M(f, g/\zeta) = E^\tau_\omega(\mathfrak{B}^\tau_M/g_\tau)(\zeta)$ for μ^τ_ω-almost all ζ.

Likewise, an inspection of the proof of Theorem 1 and its corollaries shows that they can be used without change in order to prove corresponding generalisations. Thus the generalisation of Theorem 1 now states:

Theorem 1*. *Assume* $\tau > 0$. *Then* $\lim_M F^\tau_M(f, g/\omega) = E(\mathfrak{B}^\tau_-/g_\tau)(\omega)$ *for almost all* ω.

We omit the evident reformulations of all lemmas and theorems proved so far; we will refer to them simply as the corresponding f, g-generalisation.

Chapter II: Guessing Schemes

§ 1. Preliminary Remarks

If a computer has to compute $E(\mathfrak{B}_-/f_\tau)(\omega)$ on the basis of Corollary 1 then it has to evaluate a double limit of the form $\lim_M \lim_N G^\tau_{MN}(\omega)$, where $G^\tau_{MN}(\omega)$ is given by the expression on the right hand side in clause δ) of Chap. I, §2. Thus the computer is forced to run infinitely often into the past, which is impossible. The question arises if we can find integers M_i, N_i with $M_i, N_i \to \infty$ as $i \to \infty$, such that $\lim_i G^\tau_{M_i N_i}(\omega) = \lim_M \lim_N G^\tau_{MN}(\omega)$. Thereby, M_i, N_i should depend only on a finite portion $f(\omega_s)$, $s \in I^\tau_{L_i}$ of the past (for some suitable L_i). Such an algorithm has been described by a student of D. Ornstein, D.H. Bailey, in his unpublished thesis [Ba], and, in a different way, by D. Ornstein in [Or]; it is the latter paper which serves as basis for what follows. The processes to which the algorithms in [Ba] and [Or] apply are ergodic, discrete time processes whose range consists of the two elements 0,1. Our aim is to generalize Ornstein's method to continuous processes f_t, $t \in R$ with arbitrary range but subject to the condition $|f_t| \leqq K$ almost everywhere for some K. With the exception of the last paragraph we make

Assumption 1. *The flow* $(\Omega, \mathfrak{B}, S_t, t\in R, \mu)$ *is ergodic.*

The nonergodic case can easily be reduced to the ergodic case by means of a standard decomposition argument, to be described in the last paragraph. By Lemma 9 in [Sc] and Assumption 1 it follows that the discrete system $(\Omega, \mathfrak{B}, S_\tau, \mu)$ is ergodic for all $\tau>0$ except at most denumerably many. Since subsets $L\subseteq R_+=\{t/t>0\}$ of measure zero do not count, we impose

Assumption 2. $\tau>0$ *is such that the discrete system* (Ω, B, S_τ, μ) *is ergodic.*

A number of lemmas and statements in [Or] carry over to the present case without change in their proofs; we will cite them without proof. However, the proofs of our counterparts of Lemmas 3, 5 in [Or] require a finer analysis; these proofs are given in detail.

§ 2. Guessing Schemes

Our first aim is to formulate our form of Ornstein's algorithm, or "guessing scheme", as it is called in [Or], and state the main result. Assume $f\in L_2(\Omega, \mu)$, let $L, m\geqq 0$ be integers and put $J_{Lm}^\tau=\{-k\tau 2^{-m}/0\leqq k\leqq L2^m, k\in Z\}$. With $\omega\in\Omega$, L, m and $\tau>0$ we associate the function whose domain is the index set J_{Lm}^τ and whose value for $i\in J_{Lm}^\tau$ is $f^m(\omega_i)$ (with f^m as in §1, Chap. I). This function is completely determined by L, m, τ, ω and f and will be denoted by $W_\tau(L, m/\omega)$; the dependence on f is supressed since f is kept fixed. $W_\tau(L, m/\omega)$ is our counterpart of the "word" or "string" used in [Or]. We say that $W_\tau(d, m/\omega)$ occurs at (place) p in $W_\tau(L, m/\omega)$ if $p+d\leqq L$ and if $f^m(\omega_{i-p\tau})=f^m(\omega_i)$ for $i\in J_{dm}^\tau$ (where $p\in Z$, $0\leqq p\leqq L$); likewise we speak of an occurrence at p. Next, let in addition to f, a further function $h\in L_2(\Omega, \mu)$ be given. We need three auxiliary functions $\sigma_\tau, Z_\tau, D_\tau$ depending on f, h, whose definition is as follows:

a) $\sigma_\tau(L, m, p/\omega)=1$ if $W_\tau(m, m/\omega)$ occurs at p in $W_\tau(L, m/\omega)$, and $=0$ otherwise,

b) $Z_\tau(L, m,/h/\omega)=\sum_{p=1}^{L-m} \sigma_\tau(L, m, p/\omega)\, h(\omega_{-(p-1)\tau})$,

c) $D_\tau(L, m/\omega) \quad =\sum_{p=0}^{L-m} \sigma_\tau(L, m, p/\omega)=$ number of occurrences of $W_\tau(m, m/\omega)$ in $W_\tau(L\, m/\omega)$.

Definition 1. For $L>m$ we set $g_\tau(L, m/h/\omega)=Z_\tau(L, m/h/\omega)\, D_\tau(L, m/\omega)^{-1}$ if $D_\tau(L, m/\omega)\neq 0$, and $=0$ otherwise

The functions g_τ in Definition 1 are our counterparts of the functions g in [Or]. An important notion associated with them is given by

Definition 2. Let $N\geqq 1$, $K\geqq 0$ be integers and assume $\varepsilon>0$. An integer $L>0$ is called (N, K, ε)-acceptable with respect to ω, h and the sequence $K=n_0<n_1<\ldots<n_N=L$ if: a) $|g_\tau(n_j, i/h/\omega)-g_\tau(n_t, s/h/\omega)|\leqq\varepsilon$ for $n_0\leqq i\leqq n_{j-1}, n_0\leqq s\leqq n_{t-1}$, $1\leqq j$, $t\leqq N$, b) for $n_0\leqq m\leqq n_{j-1}$ the word $W_\tau(m, m/\omega)$ occurs at least n_{j-1}^2 many times in the word $W_\tau(n_j, m/\omega)$ (that is

$D_\tau(n_j, m/\omega) \geqq n_{j-1}^2$). We call L (N, K, ε)-acceptable with respect to ω, h if there is a sequence $n_0, \dots, n_N$ with the above properties.

Remark. The small difference between Definition 2 and the definition of ε-acceptability in [Or] will be justified later and has no bearing on the proofs of the lemmas in the next paragraph.

The notion of acceptability now leads immediately to the central notion of "guessing scheme", introduced in [Or]. In order to describe it, let ε_k, $k=1,2,\dots$ be a sequence such that $\varepsilon_k > \varepsilon_{k+1} > 0$ and $\lim_k \varepsilon_k = 0$. This sequence is arbitrary but fixed in the sequel.

Definition 3. Let $f, h \in L_2(\Omega, \mu)$ be given. With every ω and every integer $L>0$ we associate a number $\lambda_L^\tau(f, h/\omega)$ as follows. Let N_j, L_j, $j=1,\dots,k$ be the well determined, possibly empty sequence of pairs of integers such that: 1) L_1 is the smallest $L' \leqq L$ such that L' is (N, N, ε_1)-acceptable for some N and N_1 is the smallest such N, 2) L_{j+1} is the smallest L' with $L_j < L' \leqq L$ such that L' is $(N, N, \varepsilon_{j+1})$-acceptable for some $N > N_j$ and N_{j+1} is the smallest such N, 3) there is no L' with $L_k < L' \leqq L$ such that L' is $(N, N, \varepsilon_{k+1})$-acceptable for some $N > N_k$. If $N_j, L_j, j=1,\dots,k$ satisfying 1)–3) exist we put $\lambda_L^\tau(f, h/\omega) = g_\tau(L_k, N_k/h/\omega)$, otherwise we put $\lambda_L^\tau(f, h/\omega) = 0$.

One of the main results is

Theorem 2. *Let $f \in L_2(\Omega, \mu)$ be essentially bounded ($|f| \leqq K$ almost everywhere for some K). Then $\lim_L \lambda_L^\tau(f, f/\omega) = E(\mathfrak{B}_-^\tau / f_\tau)(\omega)$ for almost all ω.*

The rest of the paper is devoted to the proof of a generalisation of Theorem 2, which contains Theorem 2 as a corollary.

§ 3. Some Preparatory Lemmas

Before passing to the proof of Theorem 2 we collect those lemmas from [Or] which carry over without changes in proof. To this end, $f, h \in L_2(\Omega, \mu)$ are fixed; although fully used only in the next paragraph we impose already now on h the

Assumption 3. *There is a K such that $|h(\omega)| \leqq K$ for all ω.*

In the following $\mathfrak{B}_M^\tau, \mathfrak{B}_-^\tau, \mathfrak{B}_-$ are always the Borel fields $\mathfrak{B}(f_i^M, i \in I_M^\tau)$, $\mathfrak{B}(f_i, i \in \bigcup_M I_M^\tau)$ and $\mathfrak{B}(f_s, s \leqq 0)$ respectively. We denote by $a_M^\tau(\omega)$ the atom in $\mathfrak{B}_M^\tau$ which contains ω. Since by assumption the flow $(\Omega, B, S_t, t \in R, \mu)$ and the discrete system $(\Omega, \mathfrak{B}, S_\tau, \mu)$ are ergodic, it follows that for almost all ω the measure μ and the individual measures μ_ω^τ, μ_ω coincide: $\mu = \mu_\omega^\tau = \mu_\omega$. By Lemmas 12, 13 in [Sc] and the remarks in § 3 of Chap. I, this implies the existence of a set $M_0 \subseteq \Omega$ (kept fixed in the sequel) of measure 1 such that $\omega \in M_0$ has the properties:

1) $\lim_N N^{-1} \sum_0^N H_{0\tau}^M(f, h/\omega, \omega_{-p\tau}) = (a_M^\tau(\omega)) \int h_\tau d\mu$,

2) $\lim_N N^{-1} \sum_0^N H_{1\tau}^M(f/\omega, \omega_{-p\tau}) = \mu(a_M^\tau(\omega))$, (with $H_{0\tau}^M(f, h/)$, $H_{1\tau}^M(f/)$ as in § 3 of Chap. I),

3) $\mu(a_M^\tau(\omega)) \neq 0$ for $M = 1, 2, \ldots$.

Now $(a_M^\tau(\omega)) \int h_\tau d\mu \; \mu(a_M^\tau(\omega))^{-1}$, as a function of ω, is just a version of the conditional probability $E(B_M^\tau/h_\tau)$, and will henceforth be identified with it. Thus if $\omega \in M_0$ then:

4) $\lim_N \sum_0^N H_{0\tau}^M(f, h/\omega, \omega_{-p\tau}) \left(\sum_0^N H_{1\tau}^M(f/\omega, \omega_{-p\tau})\right)^{-1} = E(\mathfrak{B}_M^\tau/h_\tau)(\omega)$.

By the martingale theorem we can assume without loss of generality that $\omega \in M_0$ satisfies in addition:

5) $\lim_M E(\mathfrak{B}_M^\tau/h_\tau)(\omega) = E(\mathfrak{B}^\tau/h_\tau)(\omega)$.

From 1)–5), from the definition of the functions D_τ, g_τ in the last paragraph and since (Ω, B, S_τ, μ) is ergodic, one easily infers:

Lemma 4. *If* $\omega \in M_0$ *then*: 1) $\lim_L L^{-1} D_\tau(L, m/\omega) = \mu(a_M^\tau(\omega)) \neq 0$, 2) $\lim_L g_\tau(L, m/h/\omega) = E(\mathfrak{B}_m^\tau/h_\tau)(\omega)$, 3) $\lim_m \lim_L g_\tau(L, m/h/\omega) = E(\mathfrak{B}_-^\tau/h_\tau)(\omega)$.

The routine proof is omitted.

Lemma 5. *Assume* $\omega \in M_0$. *Then there is a sequence* $N_j, L_j, j = 1, 2, \ldots$ *of integers* > 0 *with* $N_j < N_{j+1}$, $L_j < L_{j+1}$ *such that* L_j *is* $(N_j, N_j, \varepsilon_j)$*-acceptable with respect to* ω *and* h.

The proof, based on Lemma 4, is the same as the proof of Lemma 2 in [Or] and thus omitted. Besides the notion of acceptability there is another important notion in [Or] which is described by

Definition 4. Assume $\omega \in M_0$. Let L, N, K be integers > 0; let α, $\lambda \in R$ satisfy $0 < \alpha < 1$ and $\lambda > 0$. *A*) L is said to be $\lambda - (N, K, \alpha)_+$-bad with respect to ω, h and the sequence $K = n_0 < n_1 < \ldots < n_N = L$ if:

1) $g_\tau(n_j, m/h/\omega) - E(\mathfrak{B}_m^\tau/h_\tau)(\omega) > \alpha$ whenever $n_0 \leqq m \leqq n_{j-1}$ and $1 \leqq j \leqq N$, 2) the word $W_\tau(m, m/\omega)$ occurs at least $\lambda n_{j-1} \alpha^{-2}$ many times in the word $W_\tau(n_j, m/\omega)$ whenever $n_0 \leqq m \leqq n_{j-1}$. *B*) L is said to be $\lambda - (N, K, \alpha)_-$-bad with respect to ω, h and $K = n_0 < n_1 < \ldots < n_N = L$ if:

1*) $E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - g_\tau(n_j, m/h/\omega) > \alpha$ whenever $n_0 \leqq m \leqq n_{j-1}$ and $1 \leqq j \leqq N$, 2) the same as 2) in *A*).

Remarks. We shall say that L is $\lambda - (N, K, \alpha)_\pm$-bad with respect to ω, h if it is so with respect to some sequence $n_0, \ldots, n_N$. Most of the time we have to consider the case $\lambda = 1$. We will therefore say that L is $(N, K, \alpha)_\pm$-bad if L is $1 - (N, K, \alpha)_\pm$-bad.

There are two lemmas in [Or] summarizing the properties of $(N, K, \alpha)_\pm$-badness which carry over to the present situation.

Lemma 6. *Assume* $\xi > 0$ *and* $\omega \in M_0$. *Then one of the following alternatives hold:* 1) $\lim_L \lambda_L^\tau(f, h/\omega) = E(\mathfrak{B}_-^\tau/h_\tau)(\omega)$, 2) *there is an* $\alpha > 0$ *and a sequence* $N_j, L_j, j = 1, 2, \ldots$ *with* $N_j < N_{j+1}$, $L_j < L_{j+1}$ *such that* L_j *is* $\xi - (N_j, N_j, \alpha)_+$*-bad with respect to* h, ω, 3) *there is an* $\alpha > 0$ *and a sequence* $N_j, L_j, j = 1, 2, \ldots$ *such that* L_j *is* $\xi - (N_j, N_j, \alpha)_-$*-bad with respect to* ω, h.

The proof, based on the definition of $\lambda_L^\tau(f, h/\omega)$ and on Lemma 5 is the same as the proof of Lemma 2 in [Or] and omitted. As to the last of the preparatory lemmas, let L be $(N, K, \alpha)_+$-bad with respect to ω, h and some sequence $K=n_0<\ldots<n_N=L$. Then n_j is $(j,K,\alpha)_+$-bad with respect to ω, h and the sequence $n_0,\ldots,n_j$. Thus there is a smallest $n'_j\leqq n_j$ such that n'_j is $(j,K,\alpha)_+$-bad with respect to ω, h. One easily verifies $n'_{j-1}<n'_j$. An analoguous statement holds of course for $(N,K,\alpha)_-$-badness. The properties of the numbers n'_j are described by

Lemma 7. A) *Let L be $(N,K,\alpha)_+$-bad with respect to ω, h. For $j=1,2,\ldots,N$ let n'_j be the smallest integer such that n'_j is $(j,K,\alpha)_+$-bad with respect to ω, h. Then n'_N is $(N,K,\alpha)_+$-bad with respect to ω, h and the sequence $K=n'_0<\ldots<n'_N$.*

B) *Likewise with $(N,K,\alpha)_-$-badness.*

The proof is the same as the proof of the corresponding Lemma 4 in [Or] and omitted. In order to have a simple expression at hand we introduce.

Definition 5. Let L be $(N,K,\alpha)_\pm$-bad with respect to ω, h. The sequence $n'_0=K$, $n'_1,\ldots,n'_N$ associated with ω according to Lemma 7 is called the canonical sequence of ω with respect to N,K,α.

§ 4. Proof of Theorem 2

We now proceed to the proof of Theorem 2. It is based on two lemmas, the first corresponding to Lemma 5 in [Or], and the second to Lemma 3 in [Or]. Once these lemmas are proved, Theorem 2 easily follows. In order to formulate the first of these lemmas it is convenient to introduce.

Definition 6. Let $M>m>0$ be integers, b an atom from $\mathfrak{B}^\tau_m$ and $\alpha\in R$ such that $0<\alpha<1$. A) By $F^+(M,\alpha,b/h)$ we denote the set of $\omega\in M_0$ for which there is an l with $m\leqq l\leqq M$ such that: 1) $g_\tau(l,m/h/\omega)-E(\mathfrak{B}^\tau_m/h_\tau)(\omega)>\alpha$, 2) the word $W_\tau(m,m/\omega)$ occurs at least $m\alpha^{-2}$ many times in the word $W_\tau(l,m/\omega)$. B) By $F^-(M,\alpha,b/h)$ we denote the set of $\omega\in M_0$ for which there is an l with $m\leqq l\leqq M$ such that: 1) $E(\mathfrak{B}^\tau_m/h_\tau)(\omega)-g_\tau(l,m/h/\omega)>\alpha$, 2) the same as 2) in A).

The basic lemma concerning the sets F^+, F^- is:

Lemma 8. *Assume $0<\alpha\leqq\frac{1}{4}$; put $\beta=1-\alpha+2\alpha^2$. Assume $0\leqq h\leqq 1$. Then:*

1) $\mu(F^+(M,\alpha,b/h))\leqq\beta\mu(b)$, 2) $\mu(F^-(M,\alpha,b/h))\leqq\beta\mu(b)$.

Remark. The constant β does not depend on M,b,h; use of this will be made later.

Proof. We split the proof into three steps, S1–S3, the first of which is rather routine and hence kept short.

S1. Assume $\omega\in M_0$, let $L>m>0$ be integers. We say that $W_\tau(m,m/\omega)$ occurs positively at p in $W_\tau(L,m/\omega)$ if there is an integer l with $m\leqq l\leqq M$ such that: 1) $p+l\leqq L$, 2) $W_\tau(m,m/\omega)$ occurs at p in $W_\tau(L,m/\omega)$, 3) $g_\tau(l,m/h/\omega_{-p\tau})-E(\mathfrak{B}^\tau_m/h_\tau)(\omega)>\alpha$, 4) $W_\tau(m,m/\omega)$ $(=W_\tau(m,m/\omega_{-p\tau}))$ occurs at least $m\alpha^{-2}$ many times in the part $W_\tau(l,m/\omega_{-p\tau})$ of $W_\tau(L,m/\omega)$. We call the occurrence in question negative if 1), 2), 4) remain true while 3) is replaced by 3*) $E(\mathfrak{B}^\tau_m/h_\tau)(\omega)-g_\tau(l,m/h/\omega_{-p\tau})>\alpha$. Now let $A^+(L,m/h/\omega)$ (resp. $A^-(L,m/h/\omega)$) be the number of positive (resp. negative) occurrences of $W_\tau(m,m/\omega)$ in $W_\tau(L,m/\omega)$; let, as in § 2, $D_\tau(L,m/\omega)$ be the number of occurrences of $W_\tau(m,m/\omega)$ in $W_\tau(L,m/\omega)$. Finally put

$Q^-(L,m/h/\omega)=A^{\pm}(L,m/h/\omega)\,D_\tau(L,m/\omega)^{-1}$. It then follows from a standard application of the ergodic theorem that there is a set $M_1\subseteq M_0$ with $\mu(M_1)=1$ such that $\omega\in M_1$ implies: *I)* $\lim_L Q^+(L,m/h/\omega)=\mu(F^+(M,\alpha,b/h))\,\mu(b)^{-1}$, *II)* $\lim_L Q^-(L,m/h/\omega)=\mu(F^-(M,\alpha,b/h))\,\mu(b)^{-1}$ ($\mu(b)\neq 0$ since $\omega\in M_0$). In order to prove the lemma we take an arbitrary but fixed $\omega\in M_1$ and try to estimate $Q^{\pm}(L,m/h/\omega)$. We thereby treat clauses 1), 2) of the lemma separately.

S2. In order to prove 1) we fix a $\delta>0$. According to S1 and Lemma 4 there is an L_0 such that $L\geqq L_0$ implies: α) $|Q^+(L,m/h/\omega)-\mu(F^+(M,\alpha,b/h))\,\mu(b)^{-1}|\leqq\delta$, β) $|g_\tau(L,m/h/\omega)-E(\mathfrak{B}^\tau_m/h_\tau)(\omega)|\leqq\delta$. We take an $L\geqq L_0$ arbitrary but fixed. We now proceed as in [Or] and split the word $W_\tau(L,m/\omega)$ into blocks $S_1,\dots,S_N$ as follows. A block S_j is determined by integers $p_j,\ l_j\geqq 0$ such that:

1) $p_j+l_j<p_{j+1}$ for $j<N$, 2) $W_\tau(m,m/\omega)$ occurs positively at p_j in $W_\tau(L,m/\omega)$ with p_j, l_j satisfying 1)–4) in S1, 3) p_{j+1} is the first integer $p>p_j+l_j$ such that $W_\tau(m,m/\omega)$ occurs positively at p in $W_\tau(L,m/\omega)$, 4) p_1 is the first integer $p\geqq 0$ such that $W_\tau(m,m/\omega)$ occurs positively at p in $W_\tau(L,m/\omega)$, 5) there is no $p>p_N+l_N$ such that $W_\tau(m,m/\omega)$ occurs positively at p in $W_\tau(L,m/\omega)$. With each block S_j we associate the numbers: a) $A_j=Z_\tau(l_j,m/h/\omega_{-p_j\tau})$ (with Z_τ as in § 2), b) D_j=number of p's with $p_j=p$, $p+m\leqq p_j+l_j$ such that $W_\tau(m,m/\omega)$ occurs at p in $W_\tau(L,m/\omega)$, c) B_j=number of p's with $p_j\leqq p$, $p+m\leqq p_j+l_j$ such that $W_\tau(m,m/\omega)$ occurs positively at p in $W_\tau(L,m/\omega)$. We also set: d) $D_L=D_\tau(L,m/\omega)$ (with D_τ as in § 2). We now proceed to the bookkeeping. By β) at the beginning of S2 we have: i) $g_\tau(L,m/h/\omega)\leqq E(\mathfrak{B}^\tau_m/h_\tau)(\omega)+\delta$. Now $g_\tau(L,m/h/\omega)=Z_\tau(L,m/h/\omega)\,D_L^{-1}$. Since $0\leqq h\leqq 1$ we also have $\sum A_j\leqq Z_\tau(L,m/h/\omega)$, which implies ii) $(\sum A_j)\,D_L^{-1}\leqq E(B^\tau_m/h_\tau)(\omega)+\delta$. Now let m_j be the number of p such that: *) $p_j\leqq p\leqq p_j+l_j<p+m$, **) $W_\tau(m,m/\omega)$ occurs positively at p. Evidently $m_j\leqq m$ and $\sum(B_j+m_j)=A^+(L,m/h/\omega)$=Total number of positive occurrences. Thus: iii) $Q^+(L,m/h/\omega)=\sum(B_j+m_j)\,D_L^{-1}\leqq(\sum A_j)\,D_L^{-1}\sup_j(B_j+m_j)\,A_j^{-1}$. According to the definition of positive occurrence we have:

iv) $A_j\geqq(E(\mathfrak{B}^\tau_m/h_\tau)(\omega)+\alpha)D_j$. Since $D_j\geqq B_j$ and $D_j\geqq m\alpha^{-2}>m_j\alpha^{-2}$, this yields: v) $\sup_j(B_j+m_j)\,A_j^{-1}\leqq(1+\alpha^2)\,(E(\mathfrak{B}^\tau_m/h_\tau)\,(\omega)+\alpha)^{-1}$. By combining ii), iii), v) with α) at the beginning of S2 we find:

vi) $\mu(F^+(M,\ \alpha,\ b/h))\ \ \mu(b)^{-1}-\delta\leqq(E(\mathfrak{B}^\tau_m/h_\tau)(\omega)+\delta)(1+\alpha^2)(E(\mathfrak{B}^\tau_m/h_\tau)(\omega)+\alpha)^{-1}$. Since $\delta>0$ was arbitrary and $0\leqq E(\mathfrak{B}^\tau_m/h_\tau)(\omega)\leqq 1$ this implies:

vii) $\mu(F^+(M,\alpha,b/h))\mu(b)^{-1}\leqq(1+\alpha^2)(1+\alpha)^{-1}\leqq 1-\alpha+2\alpha^2$. Thus 1) of the lemma holds.

S3. It remains to prove clause 2) of the lemma. To start with, we take as in S2 a $\delta>0$ and an L_0 such that $L\geqq L_0$ implies:

α) $|Q^-(L,m/h/\omega)-\mu(F^-(M,\alpha,b/h))\,\mu(b)^{-1}|\leqq\delta$,

β) $|g_\tau(L,m/h/\omega)-E(\mathfrak{B}^\tau_m/h_\tau)\,(\omega)|\leqq\delta$.

An $L\geqq L_0$ is kept fixed. Next we split the word $W_\tau(L,m/\omega)$ into blocks $S_1,\dots,S_N$ in the same way as in step S2. Each block S_j is described by integers p_j, $l_j\geqq 0$, which satisfy a list of clauses 1′)–5′) which are the same as 1)–5) in step S2 with the following exceptions: the term "occurs positively" is replaced by "occurs negatively" and the numbers p_j, l_j are required to satisfy 1), 2), 3*),

4) in step S1. The bookkeeping which we apply to $W_\tau(L,m/\omega)$ now deviates in minor respects from that in S2. The numbers A_j, D_j, D_L introduced in S2 retain their meaning, while B_j is now the number of p with $p_j \leqq p$, $p+m \leqq p_j+l_j$ such that $W_\tau(m,m/\omega)$ occurs negatively at p in $W_\tau(L,m/\omega)$. In order to arrive at a first inequality, let E_j be the set of p with $p_j \leqq p$, $p+m \leqq p_j+l_j$ such that $W_\tau(m,m/\omega)$ occurs at p. Let S be the set of $p>0$ such that $W_\tau(m,m/\omega)$ occurs at p and such that p does not satisfy an inequality $p_j+1 \leqq p$, $p+m \leqq p_j+l_j$. Evidently $S \cap E_j$ contains at most p_j, and if $W_\tau(m,m/\omega)$ occurrs at p then $p \in S \cup \bigcup_j E_j$. Now put $C_L = \sum h(\omega_{-(p-l)\tau})$, $p \in S$ and $\hat{C}_L$ = number of elements in S. Since D_j is the number of elements in E_j, this implies: $\hat{C}_L + \sum(D_j - 1) \leqq D_L$. Since $0 \leqq h \leqq 1$ we also have $C_L \leqq \hat{C}_L$, which implies: a) $(\sum D_j) D_L^{-1} \leqq 1 + (\sum_j 1) D_L^{-1} - C_L D_L^{-1}$. On the other hand, C_L is evidently related to the numbers A_j according to: b) $C_L + \sum A_j = Z_\tau(L,m/h/\omega)$. From clause β) at the beginning of this step we find $g_\tau(L,m/h/\omega) \geqq E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \delta$. Since $g_\tau(L,m/h/\omega) = Z_\tau(L,m/h/\omega) D_L^{-1}$ we infer from this and b): c) $C_L D_L^{-1} \geqq E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \delta - (\sum A_j) D_L^{-1}$. On the other hand, according to the definition of S_j-block (clauses 1')-5') above) and the notion of negative occurrence we also have $g_\tau(l_j, m/h/\omega_{-p_j\tau}) \leqq E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \alpha$, which implies: d) $A_j \leqq (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \alpha) D_j$. By combining c) with d) we get: e) $C_L D_L^{-1} \geqq (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \delta) - (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \alpha)(\sum D_j) D_L^{-1}$. By combining a) with e) and observing that $\sum_j 1 = N$ is the number of blocks, we obtain:

f) $(1 - (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \alpha))(\sum D_j) D_L^{-1} \leqq 1 + N D_L^{-1} - (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \delta)$.

Now $D_j \geqq m\alpha^{-2}$ according to the definition of blocks and negative occurrence, thus $\sum D_j \geqq N m \alpha^{-2}$ and therefore $N D_L^{-1} \leqq N(\sum D_j)^{-1} \leqq m^{-1}\alpha^2 \leqq \alpha^2$ follows. From this, and since $\sum B_j \leqq \sum D_j$, we infer from f):

g) $(\sum B_j) D_L^{-1} \leqq (1 - E(\mathfrak{B}_m^\tau/h_\tau)(\omega) + \delta + \alpha^2)(1 - (E(\mathfrak{B}_m^\tau/h_\tau)(\omega) - \alpha))^{-1}$.

As in step S2, the total number of negative occurrences $A^-(L,m/h/\omega)$ is equal to $\sum(B_j + m_j)$ for some m_j with $0 \leqq m_j \leqq m$. On the other hand, since $D_j \geqq m\alpha^{-2} \geqq m_j \alpha^{-2}$ according to the definition of negative occurrence, we have $(\sum m_j) D_L^{-1} \leqq (\sum m_j)(\sum D_j)^{-1} \leqq \alpha^2$. Finally, by clause α) at the beginning of this step we also have $\mu(F^-(M,\alpha,b/h))\mu(b)^{-1} - \delta \leqq \sum(B_j + m_j) D_L^{-1}$. By combining these facts with inequality g) and by taking care of the fact that $\delta > 0$ was arbitrary, we obtain:

h) $\mu(F^-(M,\alpha,b/h))\mu(b)^{-1}$

$$\leqq (1 - E(\mathfrak{B}_m^\tau/h_\tau)(\omega) + \alpha^2)(1 + \alpha - E(\mathfrak{B}_m^\tau/h_\tau)(\omega))^{-1} + \alpha^2.$$

Now $\alpha \leqq E(\mathfrak{B}_m^\tau/h_\tau)(\omega) \leqq 1$, since we are concerned with negative occurrences, and since $0 \leqq h \leqq 1$. Using these facts and the condition $0 < \alpha \leqq \frac{1}{4}$ imposed on α by the lemma, one easily recognizes that the right hand side of inequality h) is always $\leqq 1 - \alpha + 2\alpha^2$. This proves clause 2) and hence the whole lemma. □

Evidently $F^+(M,\alpha,b/h) \subseteq F^+(M+1,\alpha,b/h)$ and likewise with F^-. Since β in lemma does not depend on M we get the

Corollary. *If* $0<\alpha\leqq\frac{1}{4}$ *then* $\mu(\bigcup_M F^{\pm}(M,\alpha,b/h))\leqq\beta\,\mu(b)$, *with* $\beta=1-\alpha+2\alpha^2$.

Notation. We put $F^+(\alpha,b/h)=\bigcup_M F^+(M,\alpha,b/h)$, $F^-(\alpha,b/h)=\bigcup_M F^-(M,\alpha,b/h)$.

We now come to the counterpart of Lemma 3 in [Or]. In order to formulate it we need

Definition 7. We denote by $S^+(N,K,\alpha/h)$ (resp. $S^-(N,K,\alpha/h)$) the set of $\omega\in M_0$ with the property: there is an $L>K$ such that L is $(N,K,\alpha)_+$-bad (resp. $(N,K,\alpha)_-$-bad) with respect to ω, h.

Prior to state Lemma 9, we recall that $\mathfrak{B}(f^m)$ is the smallest Borel field with respect to which f^m is measurable; f^m is thereby the function associated with f according to d), e), f) in § 1, Chapt. I. We then have

Lemma 9. *Let* $h\in L_2(\Omega,\mu)$ *and the integer* $m_0>0$ *satisfy:* a) $0\leqq h\leqq 1$, b) h *is* $\mathfrak{B}(f^{m_0})$*-measurable. Assume* $K\geqq m_0$ *and* $0<\alpha\leqq\frac{1}{4}$. *Then:*

a) $\mu(S^+(N,K,\alpha/h))\leqq\beta^N$, b) $\mu(S^-(N,K,\alpha/h))\leqq\beta^N$, *where* $\beta=1-\alpha+2\alpha^2$.

Proof. It suffices to discuss a); b) is treated in exactly the same way. Thus assume a) to hold for N; we prove a) for $N+1$. We proceed in three steps S1–S3.

S1. We start with an observation. Let φ be $\mathfrak{B}(f^M)$-measurable for some M, and let ω,ω' belong to the same atom from $\mathfrak{B}^\tau_m$. Then one easily verifies: 1) $\varphi(\omega_i)=\varphi(\omega'_i)$ for all $i\in I^\tau_M$. Next consider an $\omega\in S^+(N,K,\alpha/h)$. By definition of S^+ there is an $L>K$ such that L is $(N,K,\alpha)_+$-bad with respect to ω and h. With ω we associate its canonical sequence $n_0,\dots,n_N$ ($K=n_0$, $n_N\leqq L$) according to Lemma 7 and Definition 5. Let $b(\omega)$ be the atom from $\mathfrak{B}^\tau_{n_N}$ which contains ω. We claim: 2) if $\omega'\in M_0\cap b(\omega)$ then $\omega'\in S^+(N,K,\alpha/h)$ and $b(\omega)=b(\omega')$. In order to see this we first note that h and f^p are $\mathfrak{B}(f^{n_N})$-measurable, provided $p\leqq n_N$. By Remark 1) this implies $h(\omega_i)=h(\omega'_i)$ and $f^p(\omega_i)=f^p(\omega'_i)$ for all $i\in I^\tau_{n_N}$ and $p\leqq n_N$. Now $g_\tau(l,p/h/\omega)$ and $g_\tau(l,p/h/\omega')$ depend only on $h(\omega_i)$, $f^p(\omega_i)$, $i\in I^\tau_{n_N}$ and $h(\omega'_i)$, $f^p(\omega'_i)$, $i\in I^\tau_{n_N}$ respectively, provided $p\leqq l\leqq n_N$. Therefore $g_\tau(l,p/h/\omega)=g_\tau(l,p/h/\omega')$ for all l, p with $K\leqq p\leqq l\leqq n_N$. From this fact however, clause 2) easily follows.

S2. Next let J be the collection of atoms $b\in\bigcup_M\mathfrak{B}^\tau_M$ of the form $b=b(\omega)$ for some $\omega\in S^+(N,K,\alpha/h)$; that is, $b\in J$ iff there is an $\omega\in S^+(N,K,\alpha/h)$ with associated canonical sequence $n_0,\dots,n_N$ such that $b=b(\omega)$ is the atom from $\mathfrak{B}^\tau_{n_N}$ containing ω. From 2) in S1 we infer: 3) two atoms $b,b'\in J$ are either disjoint or coincide, 4) $M_0\cap(\bigcup_J b)=S^+(N,K,\alpha/h)$. Since $\mu(M_0)=1$, 3), 4) imply: 5) $\mu(\bigcup_J b)=\sum_J\mu(b)=\mu(S^+(N,K,\alpha/h))$. Now assume $b\in J$, say $b\in\mathfrak{B}^\tau_m$ for some m. Let $C(b)$ be the set of $\omega\in b$ such that: a) there is an $l\geqq m$ such that $g_\tau(l,m/h/\omega)-E(\mathfrak{B}^\tau_m/h_\tau)(\omega)>\alpha$, b) $W_\tau(m,m/\omega)$ occurs at least $m\alpha^{-2}$ often in $W_\tau(l,m/\omega)$. According to the last lemma we have: 6) $\mu(C(b))\leqq\beta\,\mu(b)$, $\beta=1-\alpha+2\alpha^2$. Moreover, if $b\neq b'$ $(b,b'\in J)$ then $C(b)\cap C(b')=0$. Thus: 7) $\mu(\bigcup_J C(b))\leqq(\sum_J\mu(b))\,\beta\leqq\beta\,\mu(S^+(N,K,\alpha/h))$.

S3. By 7) the induction step from N to $N+1$ is accomplished if we can show: *) $S^+(N+1, K, \alpha/h) \subseteq \bigcup_J C(b)$. Thus take an $\omega \in S^+(N+1, K, \alpha/h)$ and let $n_0, \ldots, n_N, n_{N+1}$ be its canonical sequence with respect to $N+1, K, \alpha$; $n_0, \ldots, n_N$ is then the canonical sequence of ω with respect to N, K, α according to Lemma 7 and Definition 5. But this implies $\omega \in b$ and $b \in J$ for some $b \in \mathfrak{B}^\tau_{n_N}$ and in addition: 8) $\omega \in C(b)$. This proves *) and thus the induction step. Since the induction basis $N=1$ is treated in exactly the same way, the lemma follows. □

We now come to the main result, which contains Theorem 2 as a corollary.

Theorem 3. *Let $h \in L_2(\Omega, \mu)$ be bounded, say $-A \leqq h \leqq B$, where $A \geqq 0$, $B > 0$. Assume that there is a sequence $h_m \in L_2(\Omega, \mu)$, $m = 1, 2, \ldots$, such that: 1) h_m is $\mathfrak{B}(f^m)$-measurable, 2) $-A \leqq h_m \leqq B$, 3) $\lim_m \sup_\xi |h_m(\xi) - h(\xi)| = 0$. Then $\lim_L \lambda^\tau_L(f, h/\omega) = E(\mathfrak{B}^\tau_-/h_\tau)(\omega)$ for almost all ω.*

Proof. Without loss of generality we assume $B \geqq 1$. We proceed in three steps.

S1. Let $0 < \alpha \leqq \frac{1}{4}$ be given. We introduce the functions $h' = (A+B)^{-1}(h+A)$, $h'_m = (A+B)^{-1}(h_m + A)$ which have the properties: a) $0 \leqq h' \leqq 1$, b) $0 \leqq h'_m \leqq 1$, c) h'_m is $\mathfrak{B}(f^m)$-measurable, d) $\lim_m \sup_\xi |h'(\xi) - h'_m(\xi)| = 0$. According to the last lemma, $\mu(S^+(m, m, \alpha/h'_m)) \leqq \beta^m$, where $\beta = 1 - \alpha + 2\alpha^2$. Therefore $\mu\left(\bigcup_{m=M}^\infty S^+(m, m, \alpha/h'_m)\right) \leqq \beta^N(1-\beta)$. Thus $\mu\left(\bigcap_{N=1}^\infty \bigcup_{m=N}^\infty S^+(m, m, \alpha/h'_m)\right) = 0$. Likewise, $\mu\left(\bigcap_{N=1}^\infty \bigcup_{m=N}^\infty S^-(m, m, \alpha/h'_m)\right) = 0$. Now let S be the set

$$\bigcup_{k=4}^\infty \left\{\bigcap_{N=1}^\infty \bigcup_{m=N}^\infty S^+(m, m, (2k)^{-1}/h'_m) \cup \bigcap_{N=1}^\infty \bigcup_{m=N}^\infty S^-(m, m, (2k)^{-1}/h'_m)\right\}.$$

Evidently, $\mu(S) = 0$. The theorem is proved if we can show: I) if $\lambda^\tau_L(f, h/\omega)$, $L = 1, 2, \ldots$ does not converge against $E(\mathfrak{B}^\tau_-/h_\tau)(\omega)$ for an $\omega \in M_0$, then $\omega \in S$.

S2. Before proceeding to the proof of I) we need two properties of the notion of λ-badness (Definition 4). In order to state the first, let $\varphi, \psi \in L_2(\Omega, \mu)$ with $0 \leqq \varphi, \psi \leqq 1$ be arbitrary but fixed for the moment. For $\omega \in M_0$ we then have:

1) $|E(\mathfrak{B}^\tau_m/\varphi_\tau)(\omega) - E(\mathfrak{B}^\tau_m/\psi_\tau)(\omega)| \leqq \sup_\xi |\psi(\xi) - \varphi(\xi)|$ for all $m \geqq 0$. This inequality follows immediately from our representation of $E(\mathfrak{B}^\tau_m/\varphi_\tau)$, $E(\mathfrak{B}^\tau_m/\psi_\tau)$ given in § 3 of this chapter and the fact that $\mu(a^\tau_m(\omega)) \neq 0$ for $\omega \in M_0$. In the same way we infer from the definition of g_τ: 2) $|g_\tau(l, p/\varphi/\omega) - g_\tau(l, p/\psi/\omega)| \leqq \sup_\xi |\varphi(\xi) - \psi(\xi)|$. Now assume $\alpha > 0$, $\sup_\xi |\varphi(\xi) - \psi(\xi)| \leqq \varepsilon$ and $\alpha - 2\varepsilon > \frac{1}{2}\alpha$. From 1), 2) and the definition of λ-badness (Definition 4) one easily infers the first property: 3) if L is $\lambda - (N, K, \alpha)_\pm$-bad with respect to ω, φ and the sequence $n_0, \ldots, n_N$ then L is $\frac{1}{4}\lambda - (N, K, \frac{1}{2}\alpha)_\pm$-bad with respect to ω, ψ and the same sequence. In order to state the second property, assume that for some $\lambda \geqq 1$, L is $\lambda - (N, K, \alpha)_+$-bad with respect to $\omega \in M_0$, h and the sequence $n_0, \ldots, n_N$. Thus $g_\tau(n_j, m/h/\omega) - E(\mathfrak{B}^\tau_m/h_\tau)(\omega) > \alpha$ for $n_0 \leqq m \leqq n_{j-1}$, $j = 1, \ldots, N$. A straightforward calculation based on the definition of g_τ and the fact that $(A$

$+B)h'-A=h$ then yields the inequality:

4) $g_\tau(n_j, m/h'/\omega) - E(\mathfrak{B}^\tau_m/h'_\tau)(\omega) > (A+B)^{-1}(\alpha - AD_\tau(n_j, m/\omega)^{-1})$.

Now $D_\tau(n_j, m/\omega)$ is the number of occurrences of $W_\tau(m, m/\omega)$ in $W_\tau(n_j, m/\omega)$ which is by assumption $\geqq \lambda n_{j-1} \alpha^{-2} \geqq K\alpha^{-2}$. We thus find:

5) $g_\tau(n_j, m/h'/\omega) - E(\mathfrak{B}^\tau_m/h'_\tau)(\omega) > (A+B)^{-1}(\alpha - A\alpha^2 K^{-1})$.

From 5) and the definition of λ-badness we now immediately infer the second property: 6) if $\alpha - A\alpha^2 K^{-1} > \frac{1}{2}\alpha$ then L is $\frac{1}{4}\lambda(A+B)^{-2} - (N, K, \frac{1}{2}\alpha(A+B)^{-1})_+$-bad with respect to ω, h' and $n_0, \ldots, n_N$. The situation is simpler if L is $\lambda-(N, K, \alpha)_-$-bad with respect to h. Without assumption on K we find: 6*) L is $\lambda(A+B)^{-2} - (N, K, \alpha(A+B)^{-1})_-$-bad with respect to ω, h' and $n_0, \ldots, n_N$.

S3. We now come to the proof of I) in S1. Let $\omega \in M_0$ and assume that $\lambda^\tau_L(f, h/\omega)$, $L=1,2,\ldots$ does not converge to $E(\mathfrak{B}^\tau_-/h_\tau)(\omega)$. By Lemma 6 there is a sequence $L_j, N_j, j=1,2,\ldots$ with $N_j < N_{j+1}$, $L_j < L_{j+1}$ and an $\alpha > 0$ such that either L_j is $64(A+B)^2 - (N_j, N_j, \alpha)_+$-bad with respect to ω, h for $j=1,2,\ldots$ or else L_j is $64(A+B)^2 - (N_j, N_j, \alpha)_-$-bad with respect to ω, h for $j=1,2,\ldots$. Let e.g. the first be the case. According to 6) in S2 there is a j_0 with the property: 7) if $j \geqq j_0$ then L_j is $16 - (N_j, N_j, \frac{1}{2}\alpha(A+B)^{-1})_+$-bad with respect to h', ω. Now let $k \geqq 4$ be so large that $k^{-1} \geqq \frac{1}{2}\alpha(A+B)^{-1} > (k+1)^{-1}$. Since $(k+1)^{-1} \geqq \frac{1}{4}\alpha(A+B)^{-1}$ we infer from 7):

8) if $j \geqq j_0$ then L_j is $4 - (N_j, N_j, (k+1)^{-1})_+$-bad with respect to ω, h'.

Now let $\varepsilon > 0$ satisfy $(k+1)^{-1} - 2\varepsilon > \frac{1}{2}(k+1)^{-1}$; let $j_1 \geqq j_0$ be so large that $j \geqq j_1$ implies $\sup_\xi |h'_{N_j}(\xi) - h'(\xi)| \leqq \varepsilon$. From 3) in S2 and 8) we infer:

9) if $j \geqq j_1$ then L_j is $(N_j, N_j, \frac{1}{2}(k+1)^{-1})_+$-bad with respect to ω, h'_{N_j}.

But this implies $\omega \in S^+(N_j, N_j, \frac{1}{2}(k+1)^{-1}, h'_{N_j})$ for $j \geqq j_1$ and thus $\omega \in S$. The argument in case of $(N_j, N_j, \alpha)_-$-badness is exactly the same. Thus I) in S1 is proved, whence the theorem follows. □

We now come to the

Proof of Theorem 2. Assume $|f| \leqq K$ almost everywhere for some K. Thus there is an integer $N > 0$ so large that $-N \leqq f \leqq N$ almost everywhere. Put $h(\omega) = f(\omega)$ if $-N \leqq f(\omega) \leqq N$, $h(\omega) = -N$ if $f(\omega) < -N$, $h(\omega) = N$ if $f(\omega) > N$. It is not difficult to find a list of functions h_m, $m=1,2,\ldots$ which are $\mathfrak{B}(f^m)$-measurable, satisfy $-N \leqq h_m \leqq N$ and such that $\lim_m \sup_\xi |h(\xi) - h_m(\xi)| = 0$. By Theorem 3 we then have $\lim_L \lambda^\tau_L(f, h/\omega) = E(\mathfrak{B}^\tau_-/f_\tau)(\omega)$ for almost all ω. On the other hand there is a set $E \subseteq \Omega$ with $\mu(E) = 1$ such that $h(\omega_i) = f(\omega_i)$ for all $i \in \bigcup_M I^\tau_M$, provided $\omega \in E$. Since $\lambda^\tau_L(f, h/\omega)$ and $\lambda^\tau_L(f, f/\omega)$ depend only on $h(\omega_i)$, $f(\omega_i)$, $i \in \bigcup_M I^\tau_M$, they coincide on E, whence the theorem follows. □

§ 5. Ergodic Decomposition

Theorem 3 has been proved under the assumption that the underlying flows is ergodic. However, as has been pointed out in § 2, the nonergodic case can easily be reduced to the ergodic case by means of an ergodic decomposition argument which proceeds along similar lines to the decomposition argument used in [Sc]. We indicate briefly this argument. Let $h \in L_2(\Omega, \mu)$ satisfy the conditions of Theorem 3, let $\tau > 0$ be noncritical and let E be the set of ξ's such that $\lim_L \lambda_L^\tau(f, h/\xi) = E(\mathfrak{B}_-^\tau/h_\tau)(\xi)$. Since τ is noncritical and according to the results obtained in Chap. I one finds a set E_0 with $\mu(E_0) = 1$ such that $\omega \in E_0$ implies: 1) the discrete system $(\Omega, \mathfrak{B}, S_\tau, \mu_\omega)$ is ergodic, 2) $E_\omega(\mathfrak{B}^\tau/h_\tau)(\xi) = E(\mathfrak{B}_-/h_\tau)(\xi)$ for μ_ω-almost all ξ. To the discrete system $(\Omega, \mathfrak{B}, S_\tau, \mu_\omega)$ we can apply Theorem 3 and infer: 3) $\lim_L \lambda_L^\tau(f, h/\xi) = E_\omega(\mathfrak{B}_-^\tau/h_\tau)(\xi)$ for μ_ω-almost all ξ. This implies $\mu_\omega(E) = 1$ for $\omega \in E_0$, and thus $\mu(E) = 1$, since $\mu(E_0) = 1$. However, this is precisely Theorem 3 for an arbitrary flow $(\Omega, \mathfrak{B}, S_t, t \in R, \mu)$. The corollaries then follow as before.

§ 6. Open Questions

There are quite a number of parameters associated with stationary processes, such as spectrum, entropy etc. If one wants to compute any of these entities with the aid of the past $f(\omega_t)$, $t \leqq 0$ of a single trajectory one is immediately led to the evaluation of iterated double, triple or even higher limits. The question arises if there are algorithms which replace these iterated limits by a single limit in a similar way as we have done in this chapter for the computation of the conditional expectation, following the line of [Or]. For some of these parameters, in particular for the entropy, an affirmative answer has been given by D.H. Bailey ([Ba]) under the proviso that the process in question is a discrete time process whose range consists of finitely many values. For the entropy of a continuous time process with arbitrary range, as introduced [Sc] (Definition 2, § 5) the question still remains open; and for a large class of other parameters (eg. those related to the spectrum) the question is open even for discrete time processes with finite range.

Acknowledgments. We are indebted to the referees for the correction of a number of syntactical errors and to Urs Kaeser from the Math. Institute of University of Basel for having pointed out to us a gap in the proof of Theorem 3.

References

Ba Bailey, D.H.: Sequential schemes for classifying and predicting ergodic processes. Thesis, Stanford University 1976

N-S Nemitkii, V., Stepanov, V.: Qualitative theory of differential equations. Princeton, New York: Princeton University Press

Or Ornstein, D.S.: Guessing the next output of a stationary process. Israel J. Math. **30**, 292–296 (1978)

Sc Scarpellini, B.: Entropy and nonlinear prediction. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 165–178 (1979)

Received January 2, 1980; in revised form February 1, 1981