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Probability Theory and Related Fields

Convergence Determining Sets in the Central Limit Theorem

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Summary. A complete characterisation is given of the class of all doublets which determine the rate of convergence in the central limit theorem. This enables a number of important properties of convergence determining sets to be deduced. In particular, it is shown that no singleton can be convergence determining, and any set consisting of four or more distinct points is convergence determining. Numerical and analytic methods are used to derive the geometry of the class of all convergence determining doublets.

1. Introduction

The notion of a "(rate of) convergence determining set" was introduced in [1]. These sets describe properties of rates of convergence in the central limit theorem, and may be defined in the following way. Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables with zero mean and unit variance, and set

$$\Delta_n(\mathscr{S}) = \sup_{x \in \mathscr{S}} \left| P\left(\sum_1^n X_i \leq n^{\frac{1}{2}} x\right) - \Phi(x) \right|,$$

where Φ is the standard normal distribution function and $\mathscr{S} \subseteq \mathbb{R}$. We say that $\mathscr{S}$ is convergence determining to order $n^{-\frac{1}{2}}$, if the ratio

$$\{\Delta_n(\mathscr{S}) + n^{-\frac{1}{2}}\} / \{\Delta_n(\mathbb{R}) + n^{-\frac{1}{2}}\}$$

is bounded away from zero and infinity as $n \to \infty$, for all choices of the distribution of X. Therefore the rate of convergence on a convergence determining set is the same as the rate of convergence on the whole real line, up to terms of order $n^{-\frac{1}{2}}$. (Recall that $n^{-\frac{1}{2}}$ is the order of the Berry-Esseen bound, and cannot be surpassed without additional restrictions on the distribution of X.)

The importance of the concept of convergence determining sets is that it permits an explicit connection between classical "uniform rates" of convergence, and rates of convergence on very small sets. For reasons which are partly technical, and which have their roots in the history of the subject, most investigators have concentrated on establishing rates of convergence uniformly on the whole real line. These are "worst possible" rates, in the sense that they describe the largest possible error in the central limit theorem. It is of practical as well as theoretical interest to know whether these rates are unduly pessimistic, in the sense that rates of convergence on much smaller sets might be much faster.

Our principal achievement in this paper is to provide a complete characterisation of the class of all convergence determining doublets; see Theorem 2.1 and Fig. 3.1. This immediately gives considerable information about convergence determining sets consisting of one or more elements. For example, a corollary to Theorem 2.1 is that there exists no such thing as a convergence determining singleton:

Theorem 1.1. *No singleton* $\{x\}$ *can be convergence determining to order* $n^{-\frac{1}{2}}$.

All known results on convergence determining sets (see [1, Theorem 1.4, p. 12, and Theorem 2.4, p. 46] and [2]) are trivial consequences of results proved here. In particular, it follows from the geometry of the class of convergence determining doublets that any set consisting of four or more distinct points, contains a doublet which is convergence determining, and so is convergence determining itself. On the other hand, it is possible to have a triplet of distinct points $\{x_1, x_2, x_3\}$, of which no doublet is convergence determining. The class of all convergence determining triplets has a surprisingly complex geometry, and will be studied elsewhere. Suffice to say here that there do exist triplets which are not convergence determining; an example is given in Theorem 2.1. Therefore distinct quadruplets represent the smallest sets which can be guaranteed to be convergence determining.

Section 2 presents a simple analytic formula which characterises convergence determining doublets. The consequences of this formula are described in Sect. 3, using a mixture of numerical and analytic methods. All of the theorems in this paper are stated in the context of rates of convergence to order $n^{-\frac{1}{2}}$, although they remain valid for rates of convergence to order n^{-1}; see [2, p. 356] for a definition. The proofs require only minor modification. A slightly different definition of convergence determining sets has been given by Heyde and Nakata [3].

By way of notation, we let $\phi(x)=(2\pi)^{-\frac{1}{2}}e^{-x^2/2}$ denote the standard normal density function, and define

$$\Phi(x)=\int_{-\infty}^{x}\phi(y)\,dy.$$

2. Convergence Determining Doublets and Quadruplets

It has been shown by Hall [1, Chap. 2] that

$$\sup_{-\infty<x<\infty} \left| P\left(\sum_{1}^{n} X_i \leq n^{\frac{1}{2}} x\right) - \Phi(x) - L_n(x) \right| = o(\delta_n) + O(n^{-\frac{1}{2}}),$$

and: $\{\sup_{-\infty<x<\infty} |L_n(x)| + n^{-\frac{1}{2}}\}/(\delta_n + n^{-\frac{1}{2}})$ is bounded away from zero and infinity as $n \to \infty$, where the "leading term" $L_n(x)$ is given by

$$L_n(x) = nE\{\Phi(x - X/n^{\frac{1}{2}}) - \Phi(x)\} - (1/2)\phi'(x)$$

and

$$\delta_n = E\{X^2 I(|X| > n^{\frac{1}{2}})\} + n^{-\frac{1}{2}} |E\{X^3 I(|X| \leq n^{\frac{1}{2}})\}| + n^{-1} E\{X^4 I(|X| \leq n^{\frac{1}{2}})\}.$$

Therefore the convergence determining nature (to order $n^{-\frac{1}{2}}$) of any set $\mathscr{S}$ depends only on the behaviour of L_n on $\mathscr{S}$. In fact

$$\Delta_n(\mathscr{S}) = \sup_{x \in \mathscr{S}} |L_n(x)| + o(\delta_n) + O(n^{-\frac{1}{2}}).$$

It follows that the doublet $\{x_1, x_2\}$ is convergence determining to order $n^{-\frac{1}{2}}$, if and only if, for all random variables X satisfying $EX=0$ and $EX^2=1$, the ratio

$$\{|L_n(x_1)| + |L_n(x_2)| + n^{-\frac{1}{2}}\}/\delta_n \tag{2.1}$$

is bounded away from zero as $n \to \infty$.

We shall characterize the class of all convergence determining doublets in terms of the function H, defined by

$$H(x_1, x_2, y) = \phi''(x_1) G(x_2, y) - \phi''(x_2) G(x_1, y), \tag{2.2}$$

for $-\infty < y < \infty$, where

$$G(x, y) = \Phi(x+y) - \Phi(x) - y\phi(x) - (1/2) y^2 \phi'(x).$$

Note that $L_n(x) = nE\{G(x, -X/n^{\frac{1}{2}})\}$.

Theorem 2.1. (a) *The triplet* $\mathscr{T} \equiv \{-\sqrt{3}, 0, \sqrt{3}\}$ *is not convergence determining.*

(b) *If* $\{x_1, x_2\}$ *is not a subset of* $\mathscr{T}$, *then* $\{x_1, x_2\}$ *is convergence determining to order* $n^{-\frac{1}{2}}$ *if and only if the only solution in* y *of the equation*

$$H(x_1, x_2, y) = 0, \tag{2.3}$$

is $y=0$.

Note that $G(x, 0)=0$, and so $H(x_1, x_2, 0)=0$ for all x_1, x_2. Since $H(x, x, y) = 0$ for all x and y, then an immediate corollary of Theorem 2.1 is that no singleton is convergence determining; see Theorem 1.1.

The virtue of Theorem 2.1 is that it transfers a property that was hitherto only expressable in terms of the cumbersome ratio (2.1), into one that can be represented by an explicit analytic formula. Nevertheless, Eq. (2.3) is not easy to interpret. In the next section, we shall use a mixture of analytic and numerical methods to construct from (2.3) the geometry of the class of all convergence determining doublets. Fortunately, the most important property of H is easy to describe, and so we state it here.

Theorem 2.2. *If* $1<x_2<x_1$; *or if* $x_2<x_1<-1$; *or if* $-1<x_2<x_1<1$; *then the only solution in* y *of Eq.* (2.3), *is* $y=0$.

It is known that any doublet containing just one of the points $+1$ or -1 is convergence determining [1, Theorem 1.4]. Thus if any set of four distinct points includes of the points $+1$ or -1 then it is convergence determining, and if not then it must contain a subset $\{x_1, x_2\}$ satisfying the conditions of Theorem 2.2. Therefore we have the following result:

Theorem 2.3. *Any set of four distinct points is convergence determining to order* $n^{-\frac{1}{2}}$.

We shall prove Theorem 2.2 at the end of Sect. 3, and give the proof of Theorem 2.1 below.

Proof of Theorem 2.1. The counterexample on p. 360 of [2] shows that $\mathcal{T}$ is not convergence determining; this proves (a). The proof of (b) is in two parts.

(i) Assume first that the only solution in y of (2.3), is $y=0$. We shall show by contradiction that $\{x_1, x_2\}$ is convergence determining. If it is not, then there exists a sequence $\mathcal{S}$ of positive integers diverging to $+\infty$, such that

$$\{|L_n(x_1)|+|L_n(x_2)|\}/\delta_n \to 0 \tag{2.4}$$

as $n\to\infty$ through $\mathcal{S}$. Write

$$\delta_n=\delta_{n2}+\delta_{n3}+\delta_{n4},$$

where

$$\delta_{n2}=E\{X^2 I(|X|>n^{\frac{1}{2}})\}, \qquad \delta_{n3}=n^{-\frac{1}{2}}|E\{X^3 I(|X|\leqq n^{\frac{1}{2}})\}|$$

and

$$\delta_{n4}=n^{-1}E\{X^4(|X|\leqq n^{\frac{1}{2}})\}.$$

By passing to a subsequence of $\mathcal{S}$ if necessary, we may assume that

$$\delta_{n3}/(\delta_{n2}+\delta_{n4})\to l$$

as $n\to\infty$ through $\mathcal{S}$, where $0\leqq l\leqq\infty$. In the case $l=\infty$, choose $x\in\{x_1, x_2\}$ with $\phi''(x)\neq 0$. (Note that the case $|x_1|=|x_2|=1$ is excluded by hypothesis.) Expanding $\Phi(x+y)$ in a Taylor series about x, we obtain:

$$\begin{aligned}
|L_n(x)| &= n|E\{G(x, -X/n^{\frac{1}{2}})\}| \\
&\geqq n|E\{G(x, -X/n^{\frac{1}{2}})I(|X|\leqq n^{\frac{1}{2}})\}| - n|E\{G(x, -X/n^{\frac{1}{2}})I(|X|>n^{\frac{1}{2}})\}| \\
&\geqq (1/6)|\phi''(x)|\,\delta_{n3} - (1/24)\sup_{-\infty<y<\infty}|\phi'''(y)|\,\delta_{n4} \\
&\quad - \sup_{-\infty<y<\infty}|\phi'(y)|\,\delta_{n2} \\
&\geqq (1/7)|\phi''(x)|\,\delta_{n3} \\
&\geqq (1/8)|\phi''(x)|\,\delta_n,
\end{aligned}$$

infinitely often as $n \to \infty$. This contradicts (2.4). In consequence $l < \infty$, and so for some $k > 0$,

$$k\delta_n < \delta_{n2} + \delta_{n4} \tag{2.5}$$

for all large $n \in \mathscr{S}$.

Define

$$\psi_{rs} \equiv \psi_{rs}(x_1, x_2) = \phi^{(r)}(x_1)\,\phi^{(s)}(x_2) - \phi^{(r)}(x_2)\,\phi^{(s)}(x_1).$$

Lemma 2.1. *If $\{x_1, x_2\}$ is not a subset of $\mathscr{T}$, and if the only solution of $H(x_1, x_2, y) = 0$ is at $y = 0$, then $\psi_{12}\psi_{23} > 0$.*

Proof. Observe the following asymptotic results:

(2.6) $\quad H(x_1, x_2, y) = (y^2/2)\psi_{12} + y\psi_{02} + O(1) \quad$ as $|y| \to \infty$, and

$\quad H(x_1, x_2, y) = (y^4/24)\psi_{23} + (y^5/120)\psi_{24} + O(y^6) \quad$ as $y \to 0$.

We shall show first that $\psi_{12} = \psi_{23} = 0$ is impossible. In this case we would have

$$O = \psi_{23} - 2\psi_{12} = (x_1 - x_2)\,\phi''(x_1)\,\phi''(x_2).$$

Since H is not identically zero, the cases $x_1 = x_2$ and $\phi''(x_1) = \phi''(x_2) = 0$ are excluded. But if, say, $\phi''(x_1) = 0 \neq \phi''(x_2)$ then it follows immediately from $\psi_{12} = 0$ that $\phi'(x_1) = 0$ as well, which is the required contradiction.

Thus if $\psi_{12} = 0$ then $\psi_{23} \neq 0$, so that H takes the same sign as ψ_{23} for small positive and negative y. If also $\psi_{02} \neq 0$, then $H \sim y\psi_{02}$ as $|y| \to \infty$, so that H takes different signs for large positive and large negative y. However H is a continuous function of y which is only supposed to cross the horizontal axis at $y = 0$. This contradiction implies that if $\psi_{12} = 0$ then $\psi_{02} = 0$ as well, so that $\{x_1, x_2\} = \{1, -1\}$ and $H \equiv 0$. Therefore $\psi_{12} \neq 0$.

Now suppose that $\psi_{23} = 0$ so that $\psi_{12} \neq 0$ and H takes the same sign as ψ_{12} for large positive and negative y. It follows as before that $\psi_{24} = 0$. Observe that

$$\phi^{(4)}(x) + x\,\phi^{(3)}(x) + 3\,\phi''(x) \equiv 0,$$

so that

$$0 = \psi_{24} = -\phi''(x_1)\,x_2\,\phi^{(3)}(x_2) + \phi''(x_2)\,x_1\,\phi^{(3)}(x_1).$$

Solving this equation simultaneously with $\psi_{23} = 0$ yields

$$(x_1 - x_2)\,\phi''(x_1)\,\phi^{(3)}(x_2) = (x_1 - x_2)\,\phi''(x_2)\,\phi^{(3)}(x_1) = 0.$$

Now $x_1 = x_2$ or $\phi''(x_1) = \phi''(x_2) = 0$ would give $H \equiv 0$, and $\phi^{(3)}(x_1) = \phi^{(3)}(x_2) = 0$ is excluded because $\{x_1, x_2\}$ is not a subset of $\mathscr{T}$. Thus we must have $\phi''(x) = \phi^{(3)}(x) = 0$ for some x, which is impossible. Therefore $\psi_{23} \neq 0$.

To complete the proof of Lemma 2.1 it remains to show that ψ_{12} and ψ_{23} are of the same sign; but since H does not cross the axis in $y > 0$, this follows from the asymptotic relations (2.6).

Continuing the proof of Theorem 2.1, it follows from the above lemma that ψ_{12} and ψ_{23} are either both strictly positive, or both strictly negative. We shall assume for definiteness that they are positive. Then the results (2.6) imply that

$$H(x_1, x_2, y) \geqq C(x_1, x_2)\,y^2 \min\{1, y^2\} \qquad \text{for all } y,$$

where $C(x_1, x_2) > 0$. Thus

$$\begin{aligned}\phi''(x_1) L_n(x_2) - \phi''(x_2) L_n(x_1) &= n E\{H(x_1, x_2, -X/n^{\frac{1}{2}})\}\\ &\geqq C(x_1, x_2) E[X^2 \min\{1, (X/n^{\frac{1}{2}})^2\}]\\ &= C(x_1, x_2)(\delta_{n2} + \delta_{n4})\\ &\geqq k\, C(x_1, x_2)\, \delta_n\end{aligned}$$

for all large $n \in \mathscr{S}$, using (2.5). This contradicts (2.4), and so completes the proof of part (i) of Theorem 2.1(b).

(ii) Assume next that for some $y_0 \neq 0$, $H(x_1, x_2, y_0) = 0$. We shall construct a random variable X with $E(X) = 0$ and $E(X^2) = 1$, and a sequence $\mathscr{S}$ of positive integers diverging to $+\infty$, such that

$$\{|L_n(x_1)| + |L_n(x_2)| + n^{-\frac{1}{2}}\}/\delta_n \to 0 \tag{2.7}$$

as $n \to \infty$ through $\mathscr{S}$.

It is known that the doublet $\{+1, -1\}$ is not convergence determining (see [1, Theorem 2.7, p. 63]), and so we may assume that at least one of $|x_1|, |x_2|$ – say $|x_1|$ – is not unity. In this case, $\phi''(x_1) \neq 0$, and

$$\begin{aligned}|L_n(x_1)| + |L_n(x_2)| &\leqq |\phi''(x_1)|^{-1}\, |\phi''(x_1) L_n(x_2) - \phi''(x_2) L_n(x_1)|\\ &\quad + \{|\phi''(x_1)|^{-1}\, |\phi''(x_2)| + 1\}\, |L_n(x_1)|\\ &= |\phi''(x_1)|^{-1}\, n\, |E\{H(x_1, x_2, -X/n^{\frac{1}{2}})\}|\\ &\quad + \{|\phi''(x_1)|^{-1}\, |\phi''(x_2)| + 1\}\, |L_n(x_1)|.\end{aligned}$$

Furthermore

$$\begin{aligned}&|L_n(x_1)| + n\, |E\{H(x_1, x_2, -X/n^{\frac{1}{2}})\}|\\ &\quad \leqq \{1 + |\phi''(x_2)|\}\, |L_n(x_1)| + |\phi''(x_1)|\, |L_n(x_2)|.\end{aligned}$$

Therefore (2.7) is equivalent to

$$[|L_n(x_1)| + n\, |E\{H(x_1, x_2, -X/n^{\frac{1}{2}})\}| + n^{-\frac{1}{2}}]/\delta_n \to 0 \tag{2.8}$$

as $n \to \infty$ through $\mathscr{S}$.

Define $y_1 \in \mathbb{R}$ by the equation

$$G(x_1, y_0) + (1/6)(y_0^3 - 2y_1)\, \phi''(x_1) = 0. \tag{2.9}$$

Now construct a discrete-valued random variable X, whose atoms include the points

$$a_m = -y_0\, 2^{m^2}, \qquad \text{of mass } p_m = 2^{-3m^2 + m},$$

and

$$b_m = \operatorname{sgn}(y_1)\, 2^{m^2 - m}, \qquad \text{of mass } q_m = |y_1|\, 2^{-3m^2 + 4m},$$

for $m \geqq m_0 > 1$. (If $y_1 = 0$, then $b_m = 0$ for all m.) We choose $m_0 > 0$, and two additional atoms, in such a way as to ensure that $E(X) = 0$ and $E(X^2) = 1$. Let $\mathscr{S}$ denote the subsequence $\{n = n(k) = 2^{2k^2},\ k \geqq 1\}$. We shall prove that (2.8) holds for this choice of X and $\mathscr{S}$.

Let $k_0 > m_0$ be so large that $2^{-k_0} < |y_0| < 2^{k_0}$. Then for $k \geqq k_0$,

$$|a_k| \leqq n^{\frac{1}{2}}(k) < |a_{k+1}| \quad \text{if } |y_0| \leqq 1,$$
$$|a_{k-1}| < n^{\frac{1}{2}}(k) < |a_k| \quad \text{if } |y_0| > 1.$$

Thus if $k \geqq k_0$ and $|y_0| \leqq 1$,

$$\begin{aligned}\delta_{n2} &= \sum_{m \geqq k+1} (a_m^2 p_m + b_m^2 q_m) + o(2^{-k^2}) \\ &= \sum_{m \geqq k+1} (y_0^2 2^{-m^2+m} + |y_1| 2^{-m^2+2m}) + o(2^{-k^2}) \\ &= |y_1| 2^{-k^2-1} + o(2^{-k^2}),\end{aligned}$$

(2.10)
$$\begin{aligned}n^{-\frac{1}{2}} E\{X^3 I(|X| \leqq n^{\frac{1}{2}})\} &= 2^{-k^2} \sum_{m \leqq k} (a_m^3 p_m + b_m^3 q_m) + o(2^{-k^2+k}) \\ &= \sum_{m \leqq k} (-y_0^3 2^{-k^2+m} + y_1 2^{-k^2+m}) + o(2^{-k^2+k}) \\ &= 2(y_1 - y_0^3) 2^{-k^2+k} + o(2^{-k^2+k})\end{aligned}$$

and

$$\delta_{n4} \sim 2^{-2k^2} \sum_{m \leqq k} (a_m^4 p_m + b_m^4 q_m) \sim y_0^4 2^{-k^2+k}.$$

If $|y_0| > 1$, the corresponding results are $\delta_{n2} \sim y_0^2 2^{-k^2+k}$,

(2.11)
$$n^{-\frac{1}{2}} E\{X^3 I(|X| \leqq n^{\frac{1}{2}})\} = (2y_1 - y_0^3) 2^{-k^2+k} + o(2^{-k^2+k}),$$

and $\delta_{n4} = |y_1| 2^{-k^2} + o(2^{-k^2})$. In both cases

(2.12)
$$\delta_n = \delta_{n2} + \delta_{n3} + \delta_{n4} \sim c 2^{-k^2+k}$$

as $k \to \infty$, where $c > 0$.

Introduce

$$G^*(x, y) = \begin{cases} G(x, y) - (y^3/6)\phi''(x) & \text{if } |y| \leqq 1, \\ G(x, y) & \text{if } |y| > 1, \end{cases}$$

and note that

$$|G^*(x, y)| \leqq C_1(x) y^2 \min(1, y^2)$$

where $C_1(x)$ does not depend on y. Define

$$g_{km}(x) = n p_m G^*(x, -a_m/n^{\frac{1}{2}})$$

where $n = n(k) = 2^{2k^2}$. Taking $m = k$ we obtain

(2.13)
$$g_{mm}(x) = 2^{-k^2+k} G^*(x, y_0)$$

while for any $k \geqq k_0$ and $m \geqq m_0$,

(2.14)
$$\begin{aligned}|g_{km}(x)| &= 2^{2k^2-3m^2+m} |G^*(x, y_0 2^{m^2-k^2})| \\ &\leqq \begin{cases} 2^{2k^2-3m^2+m} C_1(x)(y_0 2^{m^2-k^2})^4 & \text{if } m \leqq k-1, \\ 2^{2k^2-3m^2+m} C_1(x)(y_0 2^{m^2-k^2})^2 & \text{if } m \geqq k+1 \end{cases} \\ &\leqq C_1(x) y_0^2 \max(1, y_0^2) 2^{-k^2-|k-m|} \quad \text{if } m \neq k.\end{aligned}$$

Similarly, if we define

$$g_{km}^{\dagger}(x)=n\,q_m\,G^*(x,\ -b_m/n^{\frac{1}{2}}),$$

we have

(2.15)
$$|g_{km}^{\dagger}(x)|\leqq 4\,C_1(x)\,|y_1|\,2^{-k^2-|k-m|}$$

for any $k\geqq k_0$ and $m\geqq m_0$.

Observe next that

$$n\,E\{G^*(x,\ -X/n^{\frac{1}{2}})\}=\sum_{m\geqq m_0}\{g_{km}(x)+g_{km}^{\dagger}(x)\}+O(n^{-1})$$

where the term $O(n^{-1})$ takes care of the extra pair of atoms introduced into the distribution of X to ensure that $E(X)=0$ and $E(X^2)=1$. Combining (2.13)–(2.15), we see that for a positive constant $C_2(x)$

$$\begin{aligned}&|\sum_{m\geqq m_0}\{g_{km}(x)+g_{km}^{\dagger}(x)\}-2^{-k^2+k}\,G^*(x,\,y_0)|\\&\quad\leqq C_2(x)\sum_{m\geqq m_0}2^{-k^2-|k-m|}\\&\quad\leqq 2^{-k^2}\,.\,2C_2(x)\sum_{m=0}^{\infty}2^{-m}=o(\delta_n),\end{aligned}$$

the last equality following from (2.12). Therefore

(2.16)
$$n\,E\{G^*(x,\ -X/n^{\frac{1}{2}})\}=2^{-k^2+k}\,G^*(x,\,y_0)+o(\delta_n),$$

whence

(2.17)
$$\begin{aligned}&n\,E\{H(x_1,x_2,\ -X/n^{\frac{1}{2}})\}\\&\quad=\phi''(x_1)\,n\,E\{G^*(x_2,\ -X/n^{\frac{1}{2}})\}-\phi''(x_2)\,n\,E\{G^*(x_1,\ -X/n^{\frac{1}{2}})\}\\&\quad=2^{-k^2+k}\,H(x_1,x_2,y_0)+o(\delta_n)\\&\quad=o(\delta_n).\end{aligned}$$

If $|y_0|\leqq 1$ then

(2.18)
$$\begin{aligned}&E\{G(x,\ -X/n^{\frac{1}{2}})\}\\&\quad=E\{G^*(x,\ -X/n^{\frac{1}{2}})\}-(1/6)\,\phi''(x)\,n^{-3/2}\,E\{X^3\,I(|X|\leqq n^{\frac{1}{2}})\}\\&\quad=E\{G^*(x,\ -X/n^{\frac{1}{2}})\}-(1/3)(y_1-y_0^3)\,\phi''(x)\,n^{-1}\,2^{-k^2+k}+o(\delta_n),\end{aligned}$$

the second equality following from (2.10) and (2.12). Thus,

$$\begin{aligned}&n\,E\{G(x,\ -X/n^{\frac{1}{2}})\}\\&\quad=2^{-k^2+k}\{G(x,\,y_0)-(y_0^3/6)\,\phi''(x)-(1/3)(y_1-y_0^3)\,\phi''(x)\}+o(\delta_n)\\&\quad=2^{-k^2+k}\{G(x,\,y_0)+(1/6)(y_0^3-2\,y_1)\,\phi''(x)\}+o(\delta_n),\end{aligned}$$

the first equality following from (2.16) and (2.18). If $|y_0|>1$ then $G(\cdot,y_0)=G^*(\cdot,y_0)$ whence

$$nE\{G(x, -X/n^{\frac{1}{2}})\} = 2^{-k^2+k} G(x, y_0) - (1/6)\phi''(x) E\{X^3 I(|X| \leqq n^{\frac{1}{2}})\} + o(\delta_n)$$
$$= 2^{-k^2+k}\{G(x, y_0) + (1/6)(y_0^3 - 2y_1)\phi''(x)\} + o(\delta_n),$$

the first equality following from (2.16) and the first line of (2.18), and the second equality coming from (2.11). Therefore, for a general x

$$L_n(x) = nE\{G(x, -X/n^{\frac{1}{2}})\}$$
$$= 2^{-k^2+k}\{G(x, y_0) + (1/6)(y_0^3 - 2y_1)\phi''(x)\} + o(\delta_n).$$

Taking $x = x_1$, and noting (2.9), we find that

$$L_n(x_1) = o(\delta_n). \tag{2.19}$$

The desired result (2.8) follows from (2.12), (2.17) and (2.19). (Note that $n^{-\frac{1}{2}} = 2^{-k^2}$.)

3. Properties of the Function H

The Euclidean plane $\mathbb{R}^2$ may be divided into two parts, the first consisting of those ordered pairs (x_1, x_2) which are convergence determining, and the second of those pairs which are not. Our aim in this section is to describe the geometry of this decomposition. Since the convergence determining nature of a doublet is described entirely by the function H, defined at (2.2), then our task is equivalent to that of deriving certain properties of H.

Observe that $H(x_1, x_2, y) = -H(x_2, x_1, y) = -H(-x_1, -x_2, -y)$. Therefore by Theorem 2.1, the ordered pairs (x_1, x_2), (x_2, x_1), $(-x_1, -x_2)$ and $(-x_2, -x_1)$ are either all convergence determining, or all not convergence determining. For this reason, we could confine attention to ordered pairs (x_1, x_2) in the quadrant $|x_2| \leqq x_1$.

Our preliminary investigation took the form of extensive numerical computation, leading to Fig. 3.1. Pairs (x_1, x_2) within the shaded regions, and on the curves forming their boundaries, are not convergence determining. All other pairs are convergence determining. Of course, the pattern is symmetric about the axes $x_1 = x_2$ and $x_1 = -x_2$.

It follows from Theorem 2.2 and [1, Theorem 1.4, p. 12] that the triangular shaped region A in Fig. 3.1 with the exception of points on $x_1 = x_2$, and the unit square B with the exception of points (0, 0) and (1, −1), consist entirely of convergence determining doublets. Several other aspects of the geometry may be deduced from properties of the interior curves represented by dotted lines in Fig. 3.1. These curves are the loci of solutions to the equations $u(x_1, x_2) = 0$ and $v(x_1, x_2) = 0$, where

$$u(x_1, x_2) = x_1^2 x_2^2 - x_1^2 - x_2^2 + 2x_1 x_2 + 3$$

and

$$v(x_1, x_2) = x_1 x_2 + 1.$$

Note that $\psi_{23}(x_1, x_2) = (x_1 - x_2)\, u(x_1, x_2)\, \phi(x_1)\, \phi(x_2)$ and $\psi_{12}(x_1, x_2) = (x_1 - x_2)\, v(x_1, x_2)\, \phi(x_1)\, \phi(x_2)$.

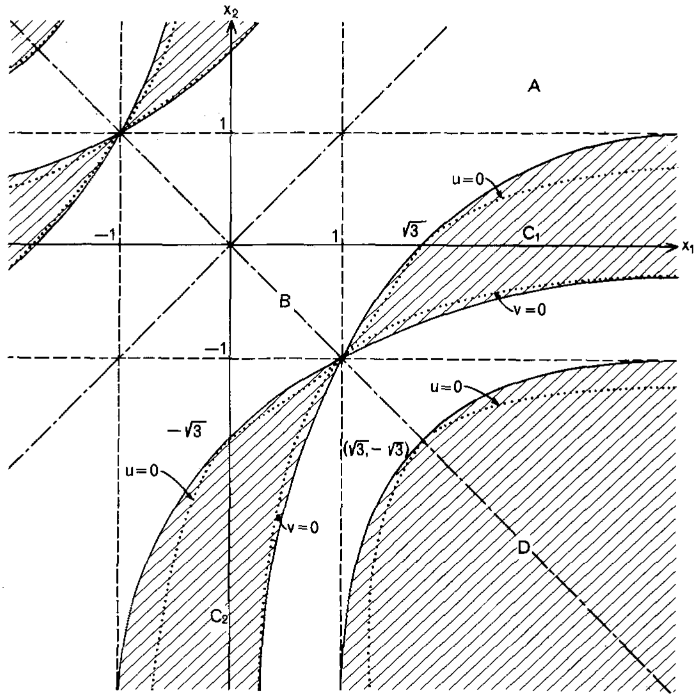

Fig. 3.1. Convergence determining doublets. *Shaded regions* (and their boundaries) represent doublets which are not convergence determining. *Dashed lines* are asymptotes, *dotted lines* are loci of solutions to $u=0$, or to $v=0$, and *dot-dashed lines* are axes of symmetry. An exception to this notation is that the line $x_1=x_2$ is both an axis of symmetry and a set of non-convergence determining doublets; see Theorem 1.1

Theorem 3.1. *If* $u(x_1, x_2)\, v(x_1, x_2)<0$, *then Eq.* (2.3) *has a solution* $y \neq 0$.

Points (x_1, x_2) within the area between the dotted lines in region $C_1 \cup C_2$ of Fig. 3.1, satisfy $u(x_1, x_2)<0$ and $v(x_1, x_2)>0$. That portion of region D below and to the right of the dotted lines, satisfies $u(x_1, x_2)>0$ and $v(x_1, x_2)<0$.

There are three important asymptotes in the quadrant $|x_2| \leqq x_1$: the lines $x_2=1$, $x_2=0$ and $x_2=-1$. In order to prove that the line $x_2=-1$ is an asymptote for the boundary of region D, we argue as follows. The boundary of D certainly lies above the curve of loci (x_1, x_2) satisfying $u(x_1, x_2)=0$ and $\min(x_1, -x_2)>1$. This curve lies below the line $x_2=-1$, and increases to the line as $x_1 \to +\infty$. Theorem 3.2 below shows that the boundary of D must lie beneath another curve which decreases to the line $x_2=-1$ as $x_1 \to \infty$. Therefore the boundary of D must converge to the asymptote $x_2=-1$, as $x_1 \to +\infty$.

In a like manner, Theorem 3.2 may be used to prove that the lower boundary of region C_1 lies below and converges towards the asymptote $x_2=0$ as $x_1 \to +\infty$. Similarly, it follows from Theorem 2.2 and properties of the function u that the upper boundary of C_1 lies below and converges towards the asymptote $x_2=1$, as $x_1 \to +\infty$.

Theorem 3.2. *For each* $\delta \in (0,1)$, *there exists* $x_0(\delta) > 0$ *such that*

$$H(x, -1+\delta, y) < 0$$

for all $x > x_0(\delta)$ *and all* $y \neq 0$.

Another salient feature of Fig. 3.1 is the intercept of the upper boundary of region C_1 with the x_1-axis. Since $u(\sqrt{3}, 0) = 0$, the intercept does not lie to the right of $\sqrt{3}$. It follows from Theorem 3.3 that the intercept equals $\sqrt{3}$.

Theorem 3.3. *For all* $0 < x < \sqrt{3}$ *and* $y \neq 0$, $H(x, 0, y) > 0$.

Likewise, it may be proved that the "vertex" of the region D is situated at $(\sqrt{3}, -\sqrt{3})$. (See [2].)

Perhaps the most intriguing property of the geometry of convergence determining doublets, is the way the regions C_1 and C_2 appear to meet in a single point at $(1, -1)$. (A similar junction occurs at $(-1, 1)$.) This is confirmed by Theorem 3.5, which shows in addition that the upper boundary of region C_1 passes smoothly into the lower boundary of region C_2. Indeed, let ε_1 be defined as in Theorem 3.4. Then the gradients of the upper boundary of C_1 and the lower boundary of C_2 at the point $(1, -1)$, are both equal to $1+\varepsilon_1$. Numerical methods show that $\varepsilon_1 \simeq 0.9111$.

Theorem 3.4. *Define functions* α *and* β *on* $(-\infty, \infty)$ *by*

$$\alpha(y) = \Phi(1+y) - \Phi(1-y) - 2y\phi(1)$$

and

$$\beta(y) = \Phi(1+y) - \Phi(1) - y\phi(1) - (1/2)y^2\phi'(1).$$

Then the number

$$\varepsilon_1 \equiv \inf\{\varepsilon > 0 \colon \alpha(y) + \varepsilon\beta(y) \geqq 0 \text{ for all } y > 0\},$$

is well-defined and satisfies $0 < \varepsilon_1 < 1$.

Theorem 3.5. *Suppose* $x_{n1} \downarrow 1$ *and* $x_{n2} \downarrow -1$ *as* $n \to \infty$, *in such a way that*

$$(x_{n2}+1)/(x_{n1}-1) \to 1+\varepsilon_0,$$

where $-\infty < \varepsilon_0 < \infty$. *Consider the condition:*

$$\text{for all } y \neq 0, \quad H(x_{n1}, x_{n2}, y) \neq 0. \tag{3.1}$$

If $\varepsilon_0 > \varepsilon_1$ *or* $\varepsilon_0 < -\varepsilon_1/(1+\varepsilon_1)$, *then* (3.1) *holds for all sufficiently large* n. *If* $-\varepsilon_1/(1+\varepsilon_1) < \varepsilon_0 < \varepsilon_1$, *and* $\varepsilon_0 \neq 0$, *then* (3.1) *fails for all sufficiently large* n.

The cases $\varepsilon_0 = 0$, ε_1 or $-\varepsilon_1/(1+\varepsilon_1)$ are necessarily excluded from Theorem 3.5, since in those situations the validity of (3.1) depends on the relative rates at which $x_{n1} - 1 \to 0$ and $(x_{n2}+1)/(x_{n1}-1) - (1+\varepsilon_0) \to 0$.

Our last theorem shows that boundaries of the shaded regions in Fig. 3.1 cannot be convergence determining.

Theorem 3.6. *If* $x_{ni} \to x_i$ *for* $i = 1, 2$, *and if each doublet* $\{x_{n1}, x_{n2}\}$ *is not convergence determining, then neither is* $\{x_1, x_2\}$.

We conclude this section with outlines of the proofs of Theorems 3.1–3.6 and 2.2. Many of the proofs depend on sign properties of certain univariate functions. These properties will be stated without derivation. In most cases they may be deduced immediately by examining the first few derivatives of the function.

Proof of Theorem 3.1. The theorem is immediate from Lemma 2.1, except when $\{x_1, x_2\} \subseteq \mathcal{T}$. If $\{x_1, x_2\} \subseteq \mathcal{T}$ then $\psi_{23}(x_1, x_2)=0$, and so either $u(x_1, x_2)=0$ or $x_1=x_2$. The former contradicts $uv<0$, and the latter implies $H\equiv 0$.

Proof of Theorem 3.2. First we treat the case of positive y, and prove that for each $\delta \in (0,1)$, there exists $x_0(\delta)>0$ such that

$$H(x, -1+\delta, y)<0 \quad \text{for all } x>x_0(\delta) \text{ and all } y>0. \tag{3.2}$$

Let

$$\begin{aligned} A(x, y) &= 2\pi \exp\{x^2/2+(1-\delta)^2/2\}(\partial/\partial y)H(x, -1+\delta, y) \\ &= -y\{(x^2-1)(1-\delta)-x\delta(2-\delta)\}+(x^2-1)(e^{(1-\delta)y-y^2/2}-1) \\ &\quad +\delta(2-\delta)(e^{-xy-y^2/2}-1). \end{aligned}$$

If x is so large that

$$(x^2-1)(1-\delta)-x\delta(2-\delta)>0, \tag{3.3}$$

and if $y \geqq 2(1-\delta)$ (implying that $(1-\delta)y-y^2/2 \leqq 0$), then $A(x, y)<0$. Therefore $H(x, -1+\delta, y)$ is decreasing in $y \geqq 2(1-\delta)$, for each x satisfying (3.3). Furthermore, as $y \to +\infty$,

$$\begin{aligned} &H(x, -1+\delta, y) \\ &\quad \sim -(4\pi)^{-1} y^2 \exp\{-x^2/2-(1-\delta)^2/2\}\{(x^2-1)(1-\delta)-x\delta(2-\delta)\}<0, \end{aligned}$$

provided (3.3) holds, and also, $H(x, -1+\delta, 0)=0$. These three results tell us enough about the shape of $H(x, -1+\delta, \cdot)$ for us to conclude that (3.2) fails if and only if

(3.4) there exist sequences $x_n \to +\infty$ and $0<y_n \to l$, $0 \leqq l \leqq 2(1-\delta)$, such that $A(x_n, y_n)=0$ and $H(x_n, -1+\delta, y_n) \geqq 0$ for all n.

Suppose (3.4) holds. Since $A(x_n, y_n)=0$, then

$$x_n^2-1=\frac{\delta(2-\delta)\{\exp(-x_n y_n - y_n^2/2)-1+x_n y_n\}}{[1+(1-\delta)y_n-\exp\{(1-\delta)y_n-y_n^2/2\}]}. \tag{3.5}$$

If $l>0$, then the right hand side of (3.5) is asymptotic to

$$\delta(2-\delta)x_n l/[1+(1-\delta)l-\exp\{(1-\delta)l-l^2/2\}] \tag{3.6}$$

as $n \to \infty$, unless

$$1+(1-\delta)l-\exp\{(1-\delta)l-l^2/2\}=0. \tag{3.7}$$

However, the term in (3.6) cannot be asymptotic to the left hand side of (3.5), and so (3.7) must hold. Furthermore,

$$1+(1-\delta)\,l-\exp\{(1-\delta)\,l-l^2/2\}$$
$$>1+(1-\delta)\,l-\exp\{(1-\delta)\,l-(1-\delta)^2\,l^2/2\}>0,$$

the second inequality following from the fact that

(3.8) $$1+y-\exp(y-y^2/2)>0 \quad \text{for all } y>0.$$

Therefore (3.7) is false, and so our assumption that $l>0$ was wrong.

If $l=0$ then $y_n\to 0$, and then it follows from (3.5) that

$$(x_n y_n)^2\sim 2\{\exp(-x_n y_n-y_n^2/2)-1+x_n y_n\}$$

as $n\to\infty$. This entails $x_n y_n\to 0$ as $n\to\infty$, and in that case it may be proved that

(3.9) $$H(x_n,\,-1+\delta,\,y_n)\sim -(\pi/12)x_n^3\,y_n^4\,\delta(2-\delta)\exp\{-x_n^2/2-(1-\delta)^2/2\}<0$$

as $n\to\infty$. This contradicts (3.4), and so proves (3.2).

Next we show that for each $\delta\in(0,1)$, there exists $x_0(\delta)>0$ such that

(3.10) $$H(x,\,-1+\delta,\,y)<0 \quad \text{for all } x>x_0(\delta) \text{ and all } y<0.$$

From an argument similar to that leading to (3.4), we see that (3.10) fails to hold if and only if

(3.11) there exist sequences $x_n\to+\infty$ and $0>y_n\to l,\ -\infty\leqq l\leqq 0$, such that $A(x_n, y_n)=0$ and $H(x_n,\,-1+\delta,\,y_n)\geqq 0$ for all n.

Again, x_n and y_n are connected by (3.5). If $l=-\infty$, then the right hand side of (3.5) equals

$$(1-\delta)^{-1}\,\delta(2-\delta)\,y_n^{-1}\,\{\exp(-x_n y_n-y_n^2/2)-1+o(1)\}+O(x_n).$$

But the factor within braces cannot tend to $-\infty$ as $n\to\infty$, and so the entire expression cannot be asymptotic to the left hand side of (3.5). Therefore $l>-\infty$. If $l=0$ then it may be proved as before that $x_n y_n\to 0$ and (3.9) holds, which contradicts (3.11). Finally, if $-\infty<l<0$ then it may be shown that

$$\begin{aligned}H(x_n,\,-1+\delta,\,y_n)&=\phi''(-1+\delta)\{1-\Phi(x_n+y_n)\}+O\{x_n^2\exp(-x_n^2/2)\}\\&\sim -\delta(2-\delta)\,\phi(-1+\delta)\,x_n^{-1}\exp(x_n|y_n|-x_n^2/2-l^2/2)\\&<0,\end{aligned}$$

which contradicts (3.11).

Proof of Theorem 3.3. Let

$$\begin{aligned}A(x,y)&=2\pi x^{-1}\exp(x^2/2+y^2/2)(\partial/\partial y)\,H(x,0,y)\\&=x(1-e^{y^2/2})+y\,e^{y^2/2}+x^{-1}(e^{-xy}-1).\end{aligned}$$

We shall prove that if $0<x\leqq\sqrt{3}$, then $A(x,y)>0$ for $y>0$ and $A(x,y)<0$ for $y<0$. Since $H(x,0,0)=0$, this implies Theorem 3.3. Observe that

$$B(x, y) \equiv \partial A/\partial x = 1 - e^{y^2/2} + x^{-2}(1 - e^{-xy} - xye^{-xy}),$$
$$C(x, y) \equiv \partial A/\partial y = (1 + y^2 - xy)\, e^{y^2/2} - e^{-xy},$$

and

$$D(x, y) \equiv \partial^2 A/\partial x\, \partial y = y(e^{-xy} - e^{y^2/2}).$$

To treat the case $y>0$, note that $B(x, 0)=0$, and $D(x, y)<0$ for all $x>0$ and $y>0$. Thus $B(x, y)<0$ for all $x, y>0$. So, for each $y>0$, $A(x, y)$ is decreasing in x, and we need only show that $A(\sqrt{3}, y)>0$. Now, it may be shown that

$$E(y) \equiv e^{-y^2/2}\, C(\sqrt{3}, y) = y^2 - \sqrt{3}\, y + 1 - e^{-y^2/2 - \sqrt{3}y}$$

is positive for all $y>0$. However $A(\sqrt{3}, 0)=0$, so we have $A(\sqrt{3}, y)>0$ for all $y>0$, as required.

Next we consider the case $y<0$. We shall show that $C(x, y)>0$ for all $0<x \leqq \sqrt{3}$ and $y<0$; since $A(x, 0)=0$ for all $x>0$, this implies that $A(x, y)<0$ for $0<x \leqq \sqrt{3}$ and $y<0$, as required. Now, if $y^2/2 \geqq -xy$ then clearly $C(x, y)>0$. Otherwise $D(x, y)<0$, so for fixed $y<0$, $C(x, y)$ is decreasing on $x \in (-y/2, \infty)$. The conditions $0<x \leqq \sqrt{3}$ and $y^2/2 < -xy$ imply that $-2\sqrt{3}<y<0$. Thus it only remains to show that $C(\sqrt{3}, y)>0$ for all y in this range; but it is easy to show that $E(y)>0$ for all such y, so the proof of Theorem 3.3 is complete.

Proof of Theorem 3.4. There exist positive constants $c_1, \ldots, c_4$ such that $\alpha(y) \sim -c_1 y^5$ and $\beta(y) \sim c_2 y^4$ as $y \to 0$, and $\alpha(y) \sim -c_3 y$ and $\beta(y) \sim c_4 y^2$ as $y \to \infty$. Therefore for each $\varepsilon>0$, $\alpha(y)+\varepsilon\beta(y)$ is positive for both small y and large y. It may be proved that $\alpha(y)<0$ for all $y>0$, and $\beta(y)>0$ for all $y \neq 0$. Consequently, ε_1 is well-defined and positive. It may be proved that $\alpha(y)+\beta(y)>0$ for all $y \neq 0$, and so $\varepsilon_1 \leqq 1$. We shall show last of all that

$$\alpha(y) + \varepsilon_1 \beta(y) = 0 \quad \text{for some } y>0. \tag{3.12}$$

It follows from this that $\varepsilon_1 < 1$.

For each $\varepsilon<\varepsilon_1$, there exists $y_\varepsilon>0$ such that $\alpha(y_\varepsilon)+\varepsilon\beta(y_\varepsilon)=0$. Choose a subsequence of ε-values along which $y_\varepsilon \to l$, $0 \leqq l \leqq \infty$, as $\varepsilon \uparrow \varepsilon_1$. The asymptotic relations given above for α and β, imply that $0<l<\infty$. By continuity, $\alpha(l)+\varepsilon_1\beta(l)=0$, proving (3.12).

Proof of Theorem 3.5. An argument similar to that just above may be used to prove that the number

$$\varepsilon_2 \equiv \inf\{\varepsilon>0 \colon \alpha(y) - \varepsilon\beta(y) \leqq 0 \text{ for all } y<0\}$$

is well-defined and satisfies $0<\varepsilon_2<1$, and that ε_2 is the unique number which satisfies

$$\alpha(y) - \varepsilon_2 \beta(y) \leqq 0 \quad \text{for all } y<0, \text{ and } =0 \text{ for some } y<0. \tag{3.13}$$

A little algebra shows that

(3.14) $$\alpha(y)+\varepsilon_1\beta(y)=-(1+\varepsilon_1)\{\alpha(-y)-\varepsilon_1(1+\varepsilon_1)^{-1}\beta(-y)\}$$

for all y. Combining (3.12)–(3.14), we may deduce that $\varepsilon_2=\varepsilon_1(1+\varepsilon_1)^{-1}$.

We shall analyse the behaviour of $H(x_{n1}, x_{n2}, y_n)$ as $n\to\infty$, for an arbitrary sequence $\{y_n\}$. To simplify matters we shall look along a subsequence $\mathscr{S}$, for which $y_n\to l$, $-\infty\leqq l\leqq\infty$, as $n\to\infty$ through $\mathscr{S}$. For ease of notation we shall write x_1, x_2 and y for x_{n1}, x_{n2} and y_n, with $n\in\mathscr{S}$, and define $\delta=x_{n1}-1>0$ and $\varepsilon=(x_{n2}+1)/(x_{n1}-1)-1$. Assume that $\varepsilon=\varepsilon(n)\to\varepsilon_0\neq 0$, as $n\to\infty$ through $\mathscr{S}$. Then it may be proved that

(3.15) $$H(x_1, x_2, y)\sim(2\pi e)^{-1}\varepsilon_0\delta y^2, \quad \text{if } |l|=\infty;$$

(3.16) $$H(x_1, x_2, y)\sim(12\pi e)^{-1}\varepsilon_0\delta y^4, \quad \text{if } l=0;$$

and

(3.17) $$H(x_1, x_2, y)=(2/\pi e)^{\frac{1}{2}}\delta\{\alpha(l)+\varepsilon_0\beta(l)\}+o(\delta), \quad \text{if } 0<|l|<\infty;$$

as $n\to\infty$ through $\mathscr{S}$.

To complete the proof of Theorem 3.5, assume first that $\varepsilon_0>\varepsilon_1$ or $\varepsilon_0<-\varepsilon_2$. If condition (3.1) fails to hold for all sufficiently large n, then we may choose $y_n\neq 0$, and a subsequency $\mathscr{S}$, such that $H(x_{n1}, x_{n2}, y_n)=0$ for all $n\in\mathscr{S}$, and $y_n\to l$ as $n\to\infty$ through $\mathscr{S}$. It follows from (3.15) and (3.16) that we must have $0<|l|<\infty$. However, the sign properties of α and β given in the first paragraph of the proof of Theorem 3.4, and the fact that α is an odd function, imply that

$$\text{if } \varepsilon_0>\varepsilon_1, \quad \text{then } \alpha(z)+\varepsilon_0\beta(z)>0 \text{ for all } z\neq 0;$$
$$\text{if } \varepsilon_0<-\varepsilon_2, \quad \text{then } \alpha(z)+\varepsilon_0\beta(z)<0 \text{ for all } z\neq 0.$$

From this fact and (3.17), it follows that $H(x_{n1}, x_{n2}, y_n)$ is nonzero for all large n. This contradiction proves the result.

Finally, assume that $-\varepsilon_2<\varepsilon_0<\varepsilon_1$, and $\varepsilon_0\neq 0$. For definiteness, assume that $0<\varepsilon_0<\varepsilon_1$. By definition of ε_1, there exists $y^*>0$ such that $\alpha(y^*)+\varepsilon_0\beta(y^*)<0$. But $\alpha(y)+\varepsilon_0\beta(y)\sim\varepsilon_0\beta(y)>0$ as $y\to 0$, and so there exists $y^{**}\in(0, y^*)$ such that $\alpha(y^{**})+\varepsilon_0\beta(y^{**})>0$. We may now deduce from (3.17) that for all sufficiently large n, there exists $y^{***}=y^{***}(n)\in(y^{**}, y^*)$, such that $H(x_1, x_2, y^{***})=0$. Therefore condition (3.1) fails to hold for large n, as had to be proved.

Proof of Theorem 3.6. Choose $y_n\neq 0$ such that $H(x_{n1}, x_{n2}, y_n)=0$. We may assume without loss of gererality that $y_n\to l$, where $-\infty\leqq l\leqq\infty$, as $n\to\infty$. Asymptotic considerations such as those in the proof of Theorem 3.5, allow us to conclude that $0<|l|<\infty$. In that case,

$$0=H(x_{n1}, x_{n2}, y_n)\to H(x_1, x_2, l),$$

and so $\{x_1, x_2\}$ cannot be convergence determining.

Proof of Theorem 2.2. Let I denote the interval $(1, \infty)$ or $(-\infty, -1)$ or $(-1, 1)$, and J the interval $(0, \infty)$ or $(-\infty, 0)$. Observe that

(3.18) $$(\partial/\partial y)H(x_1, x_2, y)=\phi''(x_1)\phi''(x_2)\{A(x_2, y)-A(x_1, y)\},$$

where

$$A(x, y) = (x^2 - 1)^{-1}(e^{-y^2/2 - xy} - 1 + xy).$$

We shall prove that for all pairs (I, J),

(3.19) $\partial A/\partial x$ is of the one sign for all $x \in I$ and $y \in J$.

Then it will follow from (3.18) that $\partial H/\partial y$ is of the one sign for all $x_1, x_2 \in I$ with $x_1 > x_2$ and all $y \in J$, and hence that H is of the one sign for all such x_1, x_2 and y, since $H(x_1, x_2, 0) = 0$.

Condition (3.19) will follow if we prove that for all $x \in I$ and $y \in J$,

$$B(x, y) \equiv (x^2 - 1)^2 \partial A/\partial x = (x^2 - 1) y(1 - e^{-y^2/2 - xy}) - 2x(e^{-y^2/2 - xy} - 1 + xy)$$

is of the one sign. It may be proved that

(3.20) $$B(1, y) = -B(-1, -y) \begin{cases} < 0 & \text{for all } y > 0 \\ > 0 & \text{for all } y < 0; \end{cases}$$

the second inequality follows from (3.8). Furthermore, $B(x, y) \to \operatorname{sgn}(-y)\infty$ as $|x| \to \infty$, for fixed $y \neq 0$. These sign properties imply that if $B(x, y_0)$ changes sign in $x \in I$, for some $y_0 \in J$, then there must exist $x_0 \in I$ with

(3.21) $$\left.\frac{\partial B}{\partial x}\right|_{(x_0, y_0)} = 0 \quad \text{and} \quad \operatorname{sgn}\{B(x_0, y_0)\} = \operatorname{sgn}(y_0).$$

Put $C(x, y) = \partial B/\partial x$, and observe that

$$C(x, y) = e^{-y^2/2 - xy}\{(x^2 - 1) y^2 - 2\} + 2(1 - xy) = 0$$

only if

$$\{2 - (x^2 - 1) y^2\} e^{-y^2/2 - xy} = 2(1 - xy).$$

In this case, substitution back into the formula for B produces

$$\{2 - (x^2 - 1) y^2\} B(x, y) = (x^2 - 1)(x^2 + 1) y^3,$$

and so if (3.21) is to hold, we must have

$$\operatorname{sgn}\{2 - (x_0^2 - 1) y_0^2\} = \operatorname{sgn}(x_0^2 - 1).$$

This is clearly impossible if $x_0^2 < 1$, and so we may assume that $x_0^2 > 1$.

Now let

$$D(x, y) = e^{y^2/2 + xy} C(x, y) = (x^2 - 1) y^2 - 2 + 2(1 - xy) e^{y^2/2 + xy}.$$

Note that $D(x, y) \to \operatorname{sgn}(-xy)\infty$ as $|x| \to \infty$, for fixed $y \neq 0$. Furthermore D always has the same sign as C, and $C(1, y) = C(-1, -y) = B(1, y)$. Thus it follows from the inequalities (3.20) that $D(x, y_0)$ has the same sign for x near each end of I.

But $D(x_0, y_0) = 0$, so there must exist $x_1 \in I$ with

(3.22) $$\left.\frac{\partial D}{\partial x}\right|_{(x_1, y_0)} = 0 \quad \text{and} \quad \operatorname{sgn}\{C(x_1, y_0)\} = \operatorname{sgn}(x_1 y_0) \text{ or } 0.$$

Observe that $\partial D/\partial x = 2xy^2(1 - e^{y^2/2+xy})$, so

$$y_0^2/2 + x_1 y_0 = 0. \tag{3.23}$$

This is clearly impossible if $x_1 y_0 > 0$. Furthermore, (3.23) implies that $C(x_1, y_0) = (x_1 y_0)^2 > 0$, which contradicts (3.22) if $x_1 y_0 < 0$. It follows that $B(x, y_0)$ cannot change sign in $x \in I$, and the theorem is proved.

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