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Probability Theory and Related Fields

On the Gaussian Approximation of Convolutions under Multidimensional Analogues of S.N. Bernstein's Inequality Conditions

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1. Introduction

In this paper we study the rate of approximation of distributions of sums of independent random vectors by corresponding Gaussian distributions. For summands we suppose the validity of multidimensional conditions which in one-dimensional case coincide with those of the well-known S.N. Bernstein's inequality (see [13], p. 55).

Throughout the paper we use the following notations. Let $\mathfrak{B}_k$ be the σ-field of the Borel subsets of the Euclidean space $\mathbb{R}^k$, $\mathfrak{F}_k$ be the set of probability measures on $\mathfrak{B}_k$, $\mathfrak{D}_k$ be the set of infinitely divisible distributions in $\mathfrak{F}_k$. The writing $x\in\mathbb{R}^k$ will further denote that $x=(x_1, \dots, x_k)$. For the scalar product of $x, y\in\mathbb{R}^k$ we use the notation $(x, y)=x_1 y_1+\dots+x_k y_k$. Besides the Euclidean norm $\|x\|=(x, x)^{\frac{1}{2}}$ we need the norm $|x|=\max\limits_{1\leqq j\leqq k}|x_j|$. For ε-neighbourhoods of a set $X\subset\mathbb{R}^k$ we use the notations

$$X^{\varepsilon}=\{y\in\mathbb{R}^k\colon \inf_{x\in X}\|x-y\|<\varepsilon\},$$
$$X^{(\varepsilon)}=\{y\in\mathbb{R}^k\colon \inf_{x\in X}|x-y|<\varepsilon\}.$$

For $x\in\mathbb{R}^k (k\geqq 2)$ we denote by $x'=(x_1, \dots, x_{k-1})\in\mathbb{R}^{k-1}$ a vector obtained by omitting the last coordinate of x. Similarly, the matrix $D'((k-1)\times(k-1))$, composed of the first $k-1$ rows and $k-1$ columns of a matrix $D(k\times k)$, will be also denoted by a prime.

The Lévy-Prohorov distance, generated by the Euclidean norm, is defined for $F, G\in\mathfrak{F}_k$ by

$$\pi(F, G)=\inf\{\varepsilon\colon F\{X\}\leqq G\{X^{\varepsilon}\}+\varepsilon,\ G\{X\}\leqq F\{X^{\varepsilon}\}+\varepsilon \text{ for any } X\in\mathfrak{B}_k\}. \tag{1.1}$$

As it is shown, e.g., in [6], the Lévy-Prohorov distance may be defined in other way:

$$\pi(F, G)=\inf\{\varepsilon\colon F\{X\}\leqq G\{X^{\varepsilon}\}+\varepsilon \text{ for any closed set } X\}. \tag{1.2}$$

We shall also consider the following characteristic of proximity of probability distributions, closely connected with the Lévy-Prohorov distance and depending on a parameter $\lambda > 0$:

$$\pi(F, G; \lambda) = \sup_{X \in \mathfrak{B}_k} \max \{F\{X\} - G\{X^\lambda\}, G\{X\} - F\{X^\lambda\}\}.$$

It was introduced by Zolotarev [25] and also considered in [4, 5, 21, 23]. Obviously, if we evaluate the characteristic $\pi(F, G; \lambda)$ for all $\lambda > 0$, then we get much more information than by the Lévy-Prohorov distance evaluation. In particular,

$$\pi(F, G) = \inf\{\lambda \colon \pi(F, G; \lambda) \leqq \lambda\}. \tag{1.3}$$

The symbols $c, c_1, c_2, \ldots$ will be used to denote absolute positive constants where c may stand for different values. Similarly, $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot), \ldots$ will denote positive constants depending only on the indicated argument. In the following text θ means quantities for which $|\theta| \leqq 1$; E_a is a probability measure concentrated at a point $a \in \mathbb{R}^k$; $E = E_0$ where 0 is the zero vector, $\mathscr{L}(\xi)$ means a distributions of a random vector ξ;

$$\hat{F}(t) = \int_{\mathbb{R}^k} e^{i(t,x)} F\{dx\}$$

denotes a characteristic function of $F \in \mathfrak{F}_k$. Products and powers of measures will be understood in the convolution sense: $FG = F * G$, $F^n = F^{*n}$.

For $\tau > 0$ we denote by $\mathscr{B}_1(\tau)$ the union $\bigcup_k \mathscr{B}_1(k, \tau)$ where

$$\mathscr{B}_1(k, \tau) = \{\mathscr{L}(\xi) \in \mathfrak{F}_k \colon \mathbf{E}\xi = 0 \quad \text{and}$$
$$|\mathbf{E}(\xi, t)^2 (\xi, u)^{m-2}| \leqq \tfrac{1}{2} m! \, \tau^{m-2} \|u\|^{m-2} \mathbf{E}(\xi, t)^2$$
$$\text{for every } m = 3, 4, \ldots \text{ and for all } t, u \in \mathbb{R}^k\}.$$

It can be easily seen that $F = \mathscr{L}(\xi) \in \mathscr{B}_1(1, \tau)$ if and only if ξ satisfies S.N. Bernstein's inequality conditions. It should be noted that $F \in \mathscr{B}_1(\tau)$ is actually a form of Cramér's condition of existence of exponential moments.

The following theorem is the main result of the paper.

Theorem 1.1. *Let $\tau > 0$ and $\xi_1, \ldots, \xi_n \in \mathbb{R}^k$ be independent random vectors such that $\mathscr{L}(\xi_i) \in \mathscr{B}_1(k, \tau)$ for $i = 1, \ldots, n$. Let $S = \xi_1 + \ldots + \xi_n$, $F = \mathscr{L}(S)$. Denote by Φ the Gaussian distribution with the zero mean and the same covariance operator as that of F. Then*

$$\pi(F, \Phi) \leqq c_1(k) \tau (|\ln \tau| + 1) \tag{1.4}$$

and for all $\lambda \geqq 0$

$$\pi(F, \Phi; \lambda) \leqq c_2(k) \exp\left(-\frac{\lambda}{c_3(k)\tau}\right). \tag{1.5}$$

Moreover, the constants $c_j(k)$ $(j = 1, 2, 3)$ may be taken in the form $c_j(k) = c_j k^{\frac{5}{2}}$.

This theorem is especially interesting because the right-hand sides of (1.4) and (1.5) are expressed only in terms of τ and are independent of any other characteristics of $\mathscr{L}(S)$ or $\mathscr{L}(\xi_i)$, including covariance operators. It should be also noted that the right side of (1.5) decreases exponentially when $\lambda \to \infty$. It is necessary to emphasize that, in general, Φ is not the standard Gaussian distribution because its covariance operator can be non-unit (it must coincide with that of F). Finally, the summands ξ_i are non-identically distributed and the constants depend only on the dimension k.

It can be easily seen that the inequality (1.4) may be derived from (1.5) with the help of (1.3). Moreover, (1.5) seems to be essentially more general in comparison with (1.4). Note that (1.5) gives meaningful information about the closeness of F to Φ for any $\tau > 0$ (1.4) being trivial for $\tau \geqq (c_1(k))^{-1}$. But at first we prove (1.4) and then deduce (1.5) by means of variation of a normalizing constant. In this connection we use the fact that if $\mathscr{L}(\xi) \in \mathscr{B}_1(k, \tau)$, $\alpha \in \mathbb{R}^1$ then $\mathscr{L}(\alpha\xi) \in \mathscr{B}_1(k, |\alpha|\tau)$ and the independence of the right-hand side of (1.4) with respect to the covariance operator of F.

The conditions of Theorem 1.1 are fulfilled for a sufficiently large class of distributions with exponentially decreasing tails. It is easy to see that these conditions are satisfied for zero mean probability measures concentrated on the ball $A_\tau = \{x \in \mathbb{R}^k \colon \|x\| \leqq \tau\}$. In Sect. 2 we show that infinitely divisible distributions whose Lévy-Khintchine spectral measures are concentrated on A_τ may be considered as shifted convolutions of distributions from $\mathscr{B}_1(c\tau)$. Hence the following result holds.

Theorem 1.2. *Let $H \in \mathfrak{D}_k$ be an infinitely divisible distribution with a characteristic function*

$$\hat{H}(t) = \exp\left\{ i(\alpha, t) - \tfrac{1}{2}(Bt, t) \right.$$
$$\left. + \int_{A_\tau} \left(e^{i(x,t)} - 1 - \frac{i(x,t)}{1+\|x\|^2} \right) \frac{1+\|x\|^2}{\|x\|^2} G\{dx\} \right\} \tag{1.6}$$

where $\alpha \in \mathbb{R}^k$, $B\colon \mathbb{R}^k \to \mathbb{R}^k$ is a non-negative linear operator and G is a bounded Borel measure concentrated on $A_\tau \setminus \{0\}$. Let Φ be the Gaussian distribution with its mean and its covariance operator coinciding with those of H. Then

$$\pi(H, \Phi) \leqq c_4(k)\tau(|\ln \tau| + 1) \tag{1.7}$$

and

$$\pi(H, \Phi; \lambda) \leqq c_5(k) \exp\left(-\frac{\lambda}{c_6(k)\tau} \right) \tag{1.8}$$

for every $\lambda > 0$. Here $c_j(k)$ $(j = 4, 5, 6)$ can be taken in the form $c_j(k) = c_j k^{\frac{5}{2}}$.

In one-dimensional case this theorem may be considered as a quantitative estimate of the stability of the characterization of Gaussian distributions as infinitely divisible distributions with their Lévy-Khintchine spectral measures concentrated at zero.

Earlier, results similar to Theorem 1.1 were obtained by Yurinskiĭ [19]. He has shown that under additional conditions ($|\xi_i| \leqq \tau$ almost surely for $i=1, \ldots, n$ and $\mathbf{E}(S, t)^2 \leqq \|t\|^2$ for all $t \in \mathbb{R}^k$) the following inequality holds:

$$\pi(F, \Phi) \leqq c(k)\tau(|\ln \tau|+1)^3 \tag{1.9}$$

(the characteristic $\pi(F, \Phi; \lambda)$ and the dependence of $c(k)$ on k were not studied in [19]). To prove Theorem 1.1 we shall apply some of the methods from [19] such as the use of Gaussian smoothing distributions, the induction on k, the study of one-dimensional conditional densities $p(x_k|x')$, $x \in \mathbb{R}^k$, the application of conjugate distributions.

We mention the following refinements of the methods of [19]. Firstly, in Sect. 3 we obtain a generalization of an inequality of Esseén [8] for characteristic functions. It will be used in Sect. 5 to prove an uniform bound for the closeness of densities of smoothed distributions. Secondly, for conjugate distributions we systematically apply the results having been obtained earlier for underlying distributions (see Lemmas 4.1, 7.1 and 8.1). We also use more exact bounds for the quantities connected with conjugate distributions.

The conjugate distributions are usually applied to estimate probabilities of large deviations. It can be easily seen that Theorem 1.1 implies inequalities that may be interpreted as bounds for such probabilities. For example, let us consider a triangular array $\{\{\xi_{lm}\}_{m=1}^{m_l}\}_{l=1}^{\infty}$ of row-wise independent random vectors (meaning that they are independent for each fixed value of l). Suppose $\xi_{lm} \in \mathscr{B}_1(\tau_l)$ for $m=1,2,\ldots,m_l$, $l=1,2,\ldots$. Let $S_l=\xi_{l1}+\ldots+\xi_{lm_l}$, $F_l=\mathscr{L}(S_l)$, let Φ_l be the Gaussian distributions with their means and their covariance operators coinciding with those of F_l and assume $X_l \in \mathfrak{B}_k$, $l=1,2,\ldots$. It follows from (1.5) that for the inequality

$$\limsup \frac{F_l\{X_l\}}{\Phi_l\{X_l\}} \leqq 1 \qquad (l \to \infty)$$

being true it is sufficient to require the validity of

$$\inf_{\lambda}\left(\frac{\Phi_l\{X_l^{\lambda} \setminus X_l\}}{\Phi_l\{X_l\}}+\frac{c_2(k)\exp\left(-\dfrac{\lambda}{c_3(k)\tau_l}\right)}{\Phi_l\{X_l\}}\right) \to 0 \tag{1.10}$$

when $l \to \infty$. Similarly, for the inequality

$$\liminf \frac{F_l\{X_l\}}{\Phi_l\{X_l\}} \geqq 1 \qquad (l \to \infty)$$

being valid it is sufficient to suppose

$$\inf_{\lambda}\left(\frac{\Phi_l\{X_l \setminus (X_l)_{-\lambda}\}}{\Phi_l\{X_l\}}+\frac{c_2(k)\exp\left(-\dfrac{\lambda}{c_3(k)\tau_l}\right)}{\Phi_l\{X_l\}}\right) \to 0 \tag{1.11}$$

when $l \to \infty$. Here $(X_l)_{-\lambda} \in \mathfrak{B}_k$ denotes an arbitrary set such that $((X_l)_{-\lambda})^{\lambda} \subset X_l$. It should be pointed out that the conditions (1.10), (1.11) are convenient for application because they are expressed in terms of Gaussian distributions. Moreover, we do not require that X_l belongs to more special set classes (convex, separated from zero etc.) as was done, e.g., in [1, 2, 12, 16].

The one-dimensional versions of Theorems 1.1 and 1.2 have been obtained in [20, 21]. It should be noted that from the results of Sahanenko [15] (see also [5]) it follows that if $k=1$ and the conditions of Theorem 1.1 are satisfied then

$$\pi(F, \Phi; \lambda) \leqq c\left(1+\frac{\sigma}{\tau}\right) \exp\left(-c\frac{\lambda}{\tau}\right)$$

for all $\lambda>0$ where $\sigma^2=\mathbf{D}S$.

Define the multidimensional Lévy distance by the formula

$$L(F, G)=\inf\{\varepsilon\colon F(x-\varepsilon\mathbb{1})-\varepsilon \leqq G(x) \leqq F(x+\varepsilon\mathbb{1})+\varepsilon \text{ for all } x\in\mathbb{R}^k\}$$

(here $F(x)$, $G(x)$ are corresponding multidimensional distribution functions, $\mathbb{1}=(1,1,\ldots,1)\in\mathbb{R}^k$.

The inequality (1.4) is optimal with respect to order. This can be derived from the following lemma due to Arak [20].

Lemma 1.1. *For any* $\tau\in(0,1]$ *there exist a distribution* $F\in\mathfrak{F}_1$ *and a positive integer* n *such that*

$$F\{[-\tau,\tau]\}=1, \qquad \int_{-\infty}^{\infty} xF\{dx\}=0$$

and for all $D\in\mathfrak{D}_1$

$$\pi(F^n, D) \geqq L(F^n, D) \geqq c\tau(|\ln\tau|+1).$$

Another simple example showing the unimprovability of the result of Theorem 1.1 is given by the distribution $F\in\mathfrak{F}_1$ with the density $f(x)=(2\tau)^{-1}\times\exp(-|x|/\tau)$. It may be easily proved that $F\in\mathscr{B}_1(1, c\tau)$ and $\pi(F,\Phi)\geqq L(F,\Phi)\geqq c\tau(|\ln\tau|+1)$ if $0<\tau<c$ where Φ is the corresponding Gaussian distribution.

The various estimates for tails of convolutions of distributions from classes similar to $\mathscr{B}_1(\tau)$ were earlier obtained, for example, in [12-18].

The results of this paper have been announced in [22, 23]. Note that Theorems 1.1 and 1.2 allow to get bounds for the rate of approximation of distributions of sums of independent random vectors by various approximating distributions (see [24]). Our results imply the following lemma that was essentially used in [24].

Lemma 1.2. *Suppose that* $F_1,\ldots,F_n\in\mathfrak{F}_k$, $F_i\{\{x\colon \|x\|\leqq\tau\}\}=1$, $\int xF_i\{dx\}=0$, $F=\prod_{i=1}^{n} F_i$ *and* D *is the accompanying infinitely divisible distribution with characteristic function*

$$\hat{D}(t)=\exp\left(\sum_{i=1}^{n}(\hat{F}_i(t)-1)\right).$$

Then

$$\pi(F, D) \leqq c(k)\tau(|\ln\tau|+1).$$

To prove this lemma it is sufficient to apply Theorems 1.1, 1.2 and the triangle inequality. Lemma 1.2 is actually one of the steps, that are necessary for proving the following theorem.

Theorem 1.3. *Suppose that the distributions* $F_i \in \mathfrak{F}_k$ *are represented in the form* $F_i = (1-p_i)U_i + p_i V_i$, *where* $0 \leqq p_i \leqq 1$; U_i, $V_i \in \mathfrak{F}_k$ *and*

$$U_i\{\{x\colon \|x\| \leqq \tau\}\} = 1, \qquad \int_{-\infty}^{\infty} x U_i\{dx\} = 0, \qquad i = 1, \ldots, n.$$

Let

$$F = \prod_{i=1}^{n} F_i, \qquad p = \max_{1 \leqq i \leqq n} p_i,$$

let $D \in \mathfrak{D}_k$ *be the accompanying infinitely divisible distribution with the characteristic function*

$$\hat{D}(t) = \exp\left(\sum_{i=1}^{n} (\hat{F}_i(t) - 1)\right).$$

Then

$$L(F, D) \leqq c(k)(p + \tau(|\ln \tau| + 1)).$$

Corollary 1.1. *Suppose that* $F_i \in \mathfrak{F}_k$ *and* $L(F_i, E) \leqq \varepsilon$, $i = 1, \ldots, n$. *Then there exists a distribution* $D \in \mathfrak{D}_k$ *such that*

$$L\left(\prod_{i=1}^{n} F_i, D\right) \leqq c(k)\varepsilon(|\ln \varepsilon| + 1).$$

Theorem 1.3 and Corollary 1.1 give a multidimensional generalization of the main results of a paper by Zaĭtsev and Arak [20]. In [20] we have obtained a definitive solution of an old problem stated by Kolmogorov in 1956 (the history of this problem may be found in [20]). The proof of Theorem 1.3 need the use of some new methods and will be published in another author's work.

2. Connection of $\mathscr{B}_1(\tau)$ with Other Classes of Probability Distributions

It is easy to check that if $\mathscr{L}(\xi) \in \mathscr{B}_1(\tau)$, $\tau > 0$ then

$$\mathbf{E}(\xi, t)^2 |(\xi, u)|^{m-2} \leqq (4/3)^{\frac{1}{2}} m! \, \|u\|^{m-2} \tau^{m-2} \mathbf{E}(\xi, t)^2$$

for any $t, u \in \mathbb{R}^k$ and for any $m = 3, 4, \ldots$ (it is sufficient to consider only odd numbers m and to use Hölder inequality). On the other hand, if for any $u, t \in \mathbb{R}^k$, $m = 3, 4, \ldots$ we have

$$\mathbf{E}\,\xi = 0, \qquad |\mathbf{E}(\xi, t)^2 (\xi, u)^{m-2}| \leqq \alpha m! \, \|u\|^{m-2} \tau^{m-2} \mathbf{E}(\xi, t)^2 \tag{2.1}$$

where $\alpha \geqq \frac{1}{2}$ then $\mathscr{L}(\xi) \in \mathscr{B}_1(2\alpha\tau)$.

Let $\mathscr{B}_2(\tau)=\bigcup\limits_k \mathscr{B}_2(k,\tau)$ where

$$\begin{aligned}\mathscr{B}_2(k,\tau)=\{&\mathscr{L}(\xi)\in\mathfrak{F}_k\colon \mathbf{E}\xi=0,\\ &\mathbf{E}(\xi,t)^2 e^{|(\xi,u)|}\leqq 4\mathbf{E}(\xi,t)^2 \text{ for any } t,u\in\mathbb{R}^k\\ &\text{such that } \|u\|\leqq\tau^{-1}\}.\end{aligned}$$

It is clear that $\mathscr{B}_1(\tau)$ and $\mathscr{B}_2(\tau)$ are increasing families of distributions when τ increases. It is easy to prove that if $\mathscr{L}(\xi)\in\mathscr{B}_j(\tau)$ $(j=1,2)$ then $\mathscr{L}(\alpha\xi)\in\mathscr{B}_j(|\alpha|\tau)$, $\mathscr{L}(U\xi)\in\mathscr{B}_j(\tau)$, $\mathscr{L}(\xi_0)\in\mathscr{B}_j(\tau)$ where $\alpha\in\mathbb{R}^1(\alpha\neq 0)$, U is an arbitrary unitary transformation of $\mathbb{R}^k$, ξ_0 is a vector composed from any subset of coordinates of a vector ξ (in particular, $\xi'\in\mathscr{B}_j(\tau)$).

Remark 2.1. In order to prove $\mathscr{L}(\xi)\in\mathscr{B}_2(\tau)$, it is sufficient to verify that $\mathbf{E}\xi=0$ and

$$\mathbf{E}(\xi,t)^2 e^{(\xi,u)}\leqq 2\mathbf{E}(\xi,t)^2$$

for any $t,u\in\mathbb{R}^k$ such that $\|u\|\leqq\tau^{-1}$ (this follows from the elementary inequality $e^{|x|}\leqq e^x+e^{-x}$).

Lemma 2.1. *There exist c_7, c_8 such that $\mathscr{B}_1(\tau)\subset\mathscr{B}_2(c_7\tau)$, $\mathscr{B}_2(\tau)\subset\mathscr{B}_1(c_8\tau)$ for any $\tau>0$.*

Proof. Let $\mathscr{L}(\xi)\in\mathscr{B}_1(\tau)$. Then for any $t,u\in\mathbb{R}^k$ such that $\|u\|\leqq(c_7\tau)^{-1}$ we have

$$\begin{aligned}|\mathbf{E}(\xi,t)^2 e^{(\xi,u)}|&=\left|\sum_{m=0}^{\infty}(m!)^{-1}\mathbf{E}(\xi,t)^2(\xi,u)^m\right|\\ &\leqq\tfrac12\mathbf{E}(\xi,t)^2\left(2+\sum_{m=1}^{\infty}(m+1)(m+2)(\|u\|\tau)^m\right)\\ &\leqq 2\mathbf{E}(\xi,t)^2 \qquad (2.2)\end{aligned}$$

if c_7 is large enough. According to Remark 2.1, (2.2) implies $\mathscr{L}(\xi)\in\mathscr{B}_2(c_7\tau)$.

Let now $\mathscr{L}(\xi)\in\mathscr{B}_2(\tau)$. Then for any $m=3,4,\ldots$; t, $u\in\mathbb{R}^k$ such that $\|u\|=\tau^{-1}$ we obtain

$$\begin{aligned}|\mathbf{E}(\xi,t)^2(\xi,u)^{m-2}|&\leqq(m-2)!\,\mathbf{E}(\xi,t)^2 e^{|(\xi,u)|}\\ &\leqq 4(m-2)!(\|u\|\tau)^{m-2}\mathbf{E}(\xi,t)^2\\ &\leqq\tfrac23 m!(\|u\|\tau)^{m-2}\mathbf{E}(\xi,t)^2. \qquad (2.3)\end{aligned}$$

Obviously, the validity of the inequality

$$|\mathbf{E}(\xi,t)^2(\xi,u)^{m-2}|\leqq\tfrac23 m!(\|u\|\tau)^{m-2}\mathbf{E}(\xi,t)^2$$

for all $t,u\in\mathbb{R}^k$ follows from the same fact obtained in (2.3) under the restriction $\|u\|=\tau^{-1}$. Therefore $\mathscr{L}(\xi)\in\mathscr{B}_1(4\tau/3)$ (see (2.1)).

Further it will be necessary to use some properties of Gaussian distributions. Let $\Phi=\mathscr{L}(\eta)\in\mathfrak{F}_k$ be a Gaussian distribution with $\mathbf{E}\eta=0$ and a covariance operator (covariance matrix) D (we identify covariance operators with corresponding covariance matrices). It is well known that for any $t,u\in\mathbb{R}^k$ the following identities hold:

$$\mathbf{E}(\eta, t)^2 = (Dt, t) = \|D^{\frac{1}{2}}t\|^2, \tag{2.4}$$

$$\mathbf{E}e^{(\eta, u)} = \exp(\tfrac{1}{2}(Du, u)), \tag{2.5}$$

$$\mathbf{E}(\eta, t)e^{(\eta, u)} = (Du, t)\exp(\tfrac{1}{2}(Du, u)), \tag{2.6}$$

$$\mathbf{E}(\eta, t)^2 e^{(\eta, u)} = ((Dt, t) + (Du, t)^2)\exp(\tfrac{1}{2}(Du, u)). \tag{2.7}$$

Here $D^{\frac{1}{2}}$ is the non-negative linear operator such that $D^{\frac{1}{2}}D^{\frac{1}{2}} = D$. For the determinant of the matrix D we have the formula

$$\int_{\mathbb{R}^k} \exp(-\tfrac{1}{2}(Dt, t))dt = (2\pi)^{\frac{k}{2}}(\det D)^{-\frac{1}{2}}. \tag{2.8}$$

Lemma 2.2. *If the largest eigenvalue of a covariance operator D of a Gaussian distribution $\Phi = \mathscr{L}(\eta) \in \mathfrak{F}_k$ is equal to d^2 $(d>0)$ then $\Phi \in \mathscr{B}_1(cd)$.*

Proof. Let $t, u \in \mathbb{R}^k$. Then (2.4), (2.7) implies

$$\begin{aligned}|\mathbf{E}(\eta, t)^2 e^{(\eta, u)}| &\leqq (Dt, t)(1 + (Du, u))\exp(\tfrac{1}{2}(Du, u)) \\ &\leqq (Dt, t)(1 + d^2\|u\|^2)\exp(\tfrac{1}{2}d^2\|u\|^2) \\ &\leqq 2(Dt, t) = 2\mathbf{E}(\eta, t)^2\end{aligned}$$

if $\|u\|^2 d^2 \leqq c$ where c is small enough. By Remark 2.1 and Lemma 2.1 we obtain the statement of Lemma 2.2.

Remark 2.2. It is clear that any Gaussian distribution $\Phi \in \mathfrak{F}_k$ may be always represented as a convolution of Gaussian distributions with arbitrarily small eigenvalues of covariance operators. In view of Lemma 2.2 we can deal with Φ as if it were of class $\mathscr{B}_1(\tau)$ with arbitrarily small $\tau > 0$ as Φ can be replaced by a finite convolution of distrivutions from $\mathscr{B}_1(\tau)$.

A similar situation occurs for infinitely divisible distribution with their Lévy-Khintchine spectral measures concentrated on a bounded set.

Lemma 2.3. *Let $H = \mathscr{L}(\xi) \in \mathfrak{D}_k$ be an infinitely divisible distribution with $\mathbf{E}\xi = 0$ and characteristic function*

$$\hat{H}(t) = \exp\left\{i(\alpha, t) + \int_{A_\tau}\left(e^{i(x,t)} - 1 - \frac{i(x, t)}{1 + \|x\|^2}\right)\frac{1 + \|x\|^2}{\|x\|^2}G\{dx\}\right\} \tag{2.9}$$

where $\alpha \in \mathbb{R}^k$, G is a bounded Borel measure concentrated on the set $A_\tau \setminus \{0\}$, $A_\tau = \{x \in \mathbb{R}^k : \|x\| \leqq \tau\}$. There exist absolute constants c_9, c_{10} such that if $G\{\mathbb{R}^k\} \leqq c_9 \min\{1, \tau^2\}$ then $H \in \mathscr{B}_1(c_{10}\tau)$.

Proof. It is easy to show that $\mathbf{E}\xi = 0$ if and only if

$$(\alpha, t) = -\int_{A_\tau}(x, t)G\{dx\} \tag{2.10}$$

for all $t \in \mathbb{R}^k$. In view of (2.9), (2.10) we have

$$\mathbf{E}e^{(\xi, v)} = \exp\left\{\int_{A_\tau}(e^{(x, v)} - 1 - (x, v))\frac{1 + \|x\|^2}{\|x\|^2}G\{dx\}\right\} \tag{2.11}$$

for every $v \in \mathbb{R}^k$. Put $v = \beta t + u$ where $\beta \in \mathbb{R}^1$, $u, v \in \mathbb{R}^k$ and twice differentiate the identity obtained from (2.11) with respect to β. By substituting $\beta = 0$ we get

$$\begin{aligned}\mathbf{E}(\xi,t)^2 e^{(\xi,u)} = &\left(\left(\int_{A_\tau} (x,t)(e^{(x,u)}-1)\frac{1+\|x\|^2}{\|x\|^2} G\{dx\}\right)^2\right.\\ &\left.+\int_{A_\tau} (x,t)^2 e^{(x,u)} \frac{1+\|x\|^2}{\|x\|^2} G\{dx\}\right)\\ &\times \exp\left\{\int_{A_\tau} (e^{(x,u)}-1-(x,u))\frac{1+\|x\|^2}{\|x\|^2} G\{dx\}\right\}. \qquad (2.12)\end{aligned}$$

Taking into account the elementary inequalities $|e^y - 1| \leqq |y| e^{|y|}$, $|e^y - 1 - y| \leqq y^2 e^{|y|}/2$ we obtain

$$\begin{aligned}&\left(\int_{A_\tau} (x,t)(e^{(x,u)}-1)\frac{1+\|x\|^2}{\|x\|^2} G\{dx\}\right)^2\\ &\quad\leqq e^{2\|u\|\tau}\mathbf{E}(\xi,t)^2 \int_{A_\tau} (x,u)^2 \frac{1+\|x\|^2}{\|x\|^2} G\{dx\}\\ &\quad\leqq e^{2\|u\|\tau}\|u\|^2(1+\tau^2)G\{\mathbb{R}^k\}\,\mathbf{E}(\xi,t)^2, \qquad (2.13)\end{aligned}$$

$$\int_{A_\tau} (e^{(x,u)}-1-(x,u))\frac{1+\|x\|^2}{\|x\|^2} G\{dx\} \leqq \tfrac{1}{2} e^{\|u\|\tau}\|u\|^2(1+\tau^2)G\{\mathbb{R}^k\}. \qquad (2.14)$$

In addition,

$$\int_{A_\tau} (x,t)^2 e^{(x,u)} \frac{1+\|x\|^2}{\|x\|^2} G\{dx\} \leqq e^{\|u\|\tau}\mathbf{E}(\xi,t)^2. \qquad (2.15)$$

From (2.12)–(2.15) we deduce

$$\begin{aligned}\mathbf{E}(\xi,t)^2 e^{(\xi,u)} \leqq &(e^{\|u\|\tau} + e^{2\|u\|\tau}\|u\|^2(1+\tau^2)G\{\mathbb{R}^k\})\\ &\times \exp(\tfrac{1}{2} e^{\|u\|\tau}\|u\|^2(1+\tau^2)G\{\mathbb{R}^k\})\,\mathbf{E}(\xi,t)^2.\end{aligned}$$

It is evident that by means of a suitable choice of constants we may ensure the validity of an inequality $\mathbf{E}(\xi,t)^2 e^{(\xi,u)} \leqq 2\mathbf{E}(\xi,t)^2$ for any $t, u \in \mathbb{R}^k$, $\|u\| \leqq c\tau^{-1}$ provided that $G\{\mathbb{R}^k\} \leqq c \min\{1, \tau^2\}$. According to Remark 2.1 and Lemma 2.1 we obtain the statement of Lemma 2.3.

Before proving Theorem 1.1 we shall show that Theorem 1.2 may be easily derived from Theorem 1.1 and Lemma 2.3.

Proof of Theorem 1.2. Let $H = \mathscr{L}(\xi) \in \mathfrak{F}_k$ be an infinitely divisible distribution with a characteristic function (1.6). In view of the invariance of the Lévy-Prohorov distance with respect to a shift transformation of the distributions to be compared we can suppose $\mathbf{E}\xi = 0$ without loss of generality. For any natural number n the distribution H may be represented in the form

$$H = H_{1n}^n H_{2n}^n \qquad (2.16)$$

where H_{1n} and H_{2n} are infinitely divisible distributions with characteristic functions

$$\hat{H}_{1n}(t)=\exp\left\{i\left(\frac{\alpha}{n},t\right)+\int_{A_\tau}\left(e^{i(x,t)}-1-\frac{i(x,t)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}G_n\{dx\}\right\},$$
$$\hat{H}_{2n}(t)=\exp\{-\tfrac{1}{2}(B_n t,t)\}$$

where $G_n=n^{-1}G$, $B_n=n^{-1}B$. By the choice of sufficiently large n we can ensure that

$$H_{1n}, H_{2n}\in\mathscr{B}_1(c_{10}\tau) \tag{2.17}$$

(see Lemmas 2.2, 2.3 and Remark 2.2). Now the inequalities (1.7), (1.8) follow from (2.16), (2.17), (1.4), (1.5).

3. The Generalization of an Inequality of Esseen for Characteristic Functions

Here we shall prove an auxiliary inequality for characteristic functions of multidimensional distributions.

Lemma 3.1. *Let $\delta>0$ and $H=\mathscr{L}(\xi)$ be a symmetric distribution (this means $\mathscr{L}(\xi)=\mathscr{L}(-\xi)$) such that $\mathbf{E}\|\xi\|^3<\infty$. Let K be a compact convex set in $\mathbb{R}^k$ and $t_0\in K$ be a point for which*

$$\hat{H}(t_0)=\max_{t\in K}\hat{H}(t). \tag{3.1}$$

Then for all $t\in K$

$$\hat{H}(t)\leqq 1-\left(1-\frac{\delta^2}{2}\right)(1-\hat{H}(t_0))-\tfrac{1}{2}\mathbf{E}(u,\xi)^2+\left(\frac{1}{6}+\frac{1}{\delta}\right)\mathbf{E}|(u,\xi)|^3\leqq\exp\left\{-\left(1-\frac{\delta^2}{2}\right)(1-\hat{H}(t_0))-\tfrac{1}{2}\mathbf{E}(u,\xi)^2+\left(\frac{1}{6}+\frac{1}{\delta}\right)\mathbf{E}|(u,\xi)|^3\right\}, \tag{3.2}$$

where $u=t-t_0$.

If a probability measure $H\in\mathfrak{F}_k$ is non-symmetric then similar inequalities for $|\hat{H}(t)|$ can be obtained by estimating the characteristic function $|\hat{H}(t)|^2$ of the symmetrized distribution.

Lemma 3.1 may be considered as a generalization and a sharpening of Theorem 2, Chap. VII from the well known paper of Esseen [8]. Upper bounds have been obtained for the Lebesgue measure of those $t\in K$ for which $\hat{H}(t)>1-\varepsilon$ where K is an ellipsoid of a special form and ε is a small positive number. Esseen made use of his result to estimate integrals of the form

$$\int_K\prod_{i=1}^n|\hat{H}_i(t)|\,dt.$$

It should be noted that such integrals have been also estimated by other authors with the help of similar methods (see [7, 9, 10, 11]).

Proof of Lemma 3.1. In many respects it repeats the corresponding arguments of Esseen [8]. By expanding the cosine function in a Taylor series we obtain

$$\begin{aligned}\hat{H}(t) &= \mathbf{E}\cos((\xi, t_0)+(\xi, u))\\ &= \mathbf{E}\cos(\xi, t_0) - \mathbf{E}(\xi, u)\sin(\xi, t_0)\\ &\quad -\tfrac{1}{2}\mathbf{E}(\xi, u)^2\cos(\xi, t_0) + \frac{\theta}{6}\mathbf{E}|(\xi, u)|^3. \end{aligned} \tag{3.3}$$

The basic difference from [8] consists in the use of the inequality

$$-\mathbf{E}(\xi, u)\sin(\xi, t_0) \leqq 0 \tag{3.4}$$

which is valid for all $u = t - t_0$ such that $t \in K$. The left-hand side of (3.4) is actually the derivative of $\hat{H}$ at t_0 in direction u and hence $\leqq 0$ as t_0 is a maximum point on a segment between t_0 and t. Now (3.3), (3.4) imply that

$$\begin{aligned}\hat{H}(t) &\leqq \hat{H}(t_0) - \mathbf{E}(\xi, u)^2\cos(\xi, t_0)/2\\ &\quad + \theta\mathbf{E}|(\xi, u)|^3/6\\ &= 1 - (1-\hat{H}(t_0)) - \mathbf{E}(\xi, u)^2/2\\ &\quad + \mathbf{E}(\xi, u)^2(1-\cos(\xi, t_0))/2 + \theta\mathbf{E}|(\xi, u)|^3/6.\end{aligned} \tag{3.5}$$

The inequality (3.2) follows from (3.5) since

$$\begin{aligned}\mathbf{E}(\xi, u)^2(1-\cos(\xi, t_0))/2 &= \mathbf{E}(\xi, u)^2(1-\cos(\xi, t_0))\\ &\quad \times(\mathbb{1}_{\{|(\xi,u)|<\delta\}} + \mathbb{1}_{\{|(\xi,u)|\geqq\delta\}})/2\\ &\leqq \delta^2(1-\hat{H}(t_0))/2 + \delta^{-1}\mathbf{E}|(\xi, u)|^3.\end{aligned}$$

Remark 3.1. The right side of (3.2) may be easily minimized with respect to δ. Then this inequality takes the following form

$$\hat{H}(t) \leqq \hat{H}(t_0) - \tfrac{1}{2}\mathbf{E}(\xi, u)^2 + \tfrac{1}{6}\mathbf{E}|(\xi, u)|^3 + \tfrac{3}{2}(\mathbf{E}|(\xi, u)|^3)^{\frac{2}{3}}(1-\hat{H}(t_0))^{\frac{1}{3}}.$$

Corollary 3.1. *Suppose that the conditions of Lemma 3.1 are satisfied. If* $\mathbf{E}|(\xi, t_1 - t_2)|^3 \leqq \gamma\mathbf{E}(\xi, t_1 - t_2)^2$ *for all* $t_1, t_2 \in K$ *and for some* $\gamma > 0$ *then*

$$\hat{H}(t) \leqq \exp\left\{-\left(1-\frac{\delta^2}{2}\right)(1-\hat{H}(t_0)) - \left(\frac{1}{2} - \gamma\left(\frac{1}{6}+\frac{1}{\delta}\right)\right)\mathbf{E}(\xi, u)^2\right\}$$

for all $t \in K$ *where* $u = t - t_0$. *In particular, setting* $\delta = 6/5$, *we obtain*

$$\begin{aligned}\hat{H}(t) &\leqq \exp\{-0.28(1-\hat{H}(t_0)) - (\tfrac{1}{2}-\gamma)\mathbf{E}(\xi, u)^2\}\\ &\leqq \exp\{-(\tfrac{1}{2}-\gamma)\mathbf{E}(\xi, u)^2\}.\end{aligned} \tag{3.6}$$

Remark 3.2. Results similar to Lemma 3.1 may be obtained without the condition $\mathbf{E}\|\xi\|^3 < \infty$ since an arbitrary distribution $H \in \mathfrak{F}_k$ may be represented as

$H=(1-p)U+pV$ where $0\leqq p\leqq 1$, $U, V\in\mathfrak{F}_k$, $\int\limits_{\mathbb{R}^k}\|x\|^3 U\{dx\}<\infty$ (e.g., by a truncation). Then $|\hat{H}(t)|\leqq(1-p)|\hat{U}(t)|+p$ and it remains to use our results to $|\hat{U}(t)|$. It should be noted that the choice of a representation $H=(1-p)U+pV$ may be performed in different ways in accordance with our demands.

4. The Properties of Conjugate Distributions

Let $h\in\mathbb{R}^k$ and $\mathscr{L}(\zeta)=H\in\mathfrak{F}_k$. The conjugate distribution $\overline{\mathscr{L}(\zeta)}=\bar{H}=\bar{H}(h)$ is defined by a formula

$$\bar{H}\{X\}=(\mathbf{E}e^{(\zeta,h)})^{-1}\int\limits_X e^{(x,h)}H\{dx\} \tag{4.1}$$

for all $X\in\mathfrak{B}_k$. It is clear that this definition has meaning only if $\mathbf{E}\exp((\zeta,h))<\infty$. A conjugate distribution essentially depends on the choice of a parameter h. A conjugate random vector having a distribution $\overline{\mathscr{L}(\zeta)}$ will be denoted by $\bar{\zeta}=\bar{\zeta}_h$. We shall also apply the notation $\zeta^*=\zeta_h^*=\bar{\zeta}-\mathbf{E}\,\bar{\zeta}$. In view of (4.1), for any $\mathfrak{B}_k$-measurable function φ such that $\mathbf{E}|\varphi(\zeta)e^{(\zeta,h)}|<\infty$ the following identity holds:

$$\mathbf{E}\varphi(\bar{\zeta})=(\mathbf{E}e^{(\zeta,h)})^{-1}\,\mathbf{E}\varphi(\zeta)e^{(\zeta,h)}. \tag{4.2}$$

Unless otherwise stated we shall always assume that all conjugate distributions are defined with the help of a parameter denoted by a letter h.

It is well known that the conjugate distribution for a convolution coincides with the convolution of conjugate distributions: if $U_1,\dots,U_n\in\mathfrak{F}_k$, $U=\prod\limits_{i=1}^n U_i$ then $\bar{U}=\prod\limits_{i=1}^n \bar{U}_i$. If $\Phi=\mathscr{L}(\eta)\in\mathfrak{F}_k$ is a Gaussian distribution with $\mathbf{E}\eta=0$ and a covariance operator D then the distribution $\bar{\Phi}$ is also Gaussian and for all $t\in\mathbb{R}^k$

$$\begin{gathered}\mathbf{E}(\bar{\eta},t)=(Dh,t),\\ \mathbf{D}(\bar{\eta},t)=\mathbf{D}(\eta,t)=\mathbf{E}(\eta,t)^2=\mathbf{E}(\eta^*,t)^2=(Dt,t)=\|D^{\frac{1}{2}}t\|^2\end{gathered} \tag{4.3}$$

(see (2.4)–(2.7), (4.2)).

Lemma 4.1. *Let $\tau>0$, $\mathscr{L}(\xi)\in\mathscr{B}_1(\tau)$ and let η be a Gaussian random vector with $\mathbf{E}\eta=0$ and covariance operator D coinciding with that of ξ. There exist $c_{11},\dots,c_{15}$ such that if $\|h\|\tau\leqq c_{11}$ then*

$$|\mathbf{E}(\bar{\xi},t)-\mathbf{E}(\bar{\eta},t)|\leqq c_{12}\|h\|\tau(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}}, \tag{4.4}$$

$$|\mathbf{E}(\xi^*,t)^2-\mathbf{E}(\eta^*,t)^2|\leqq c_{13}\|h\|\tau(Dt,t), \tag{4.5}$$

$$\left|\ln\frac{\mathbf{E}e^{(\xi,h)}}{\mathbf{E}e^{(\eta,h)}}\right|\leqq c_{14}\|h\|\tau(Dh,h) \tag{4.6}$$

for all $t\in\mathbb{R}^k$ and the distributions $\mathscr{L}(\xi^*)$ and $\mathscr{L}(\eta^*)=\mathscr{L}(\eta)$ belong to $\mathscr{B}_1(c_{15}\tau)$.

Proof. Let d^2 be the largest eigenvalue of the operator D and $u\in\mathbb{R}^k$, $\|u\|=1$, be a corresponding eigenvector. Since $\mathscr{L}(\xi)\in\mathscr{B}_1(\tau)$ we have

$$\mathbf{E}(\xi,u)^2\leqq(\mathbf{E}(\xi,u)^4)^{\frac{1}{2}}\leqq(12\tau^2\mathbf{E}(\xi,u)^2)^{\frac{1}{2}},$$
$$d^2=(Du,u)=\mathbf{E}(\eta,u)^2=\mathbf{E}(\xi,u)^2\leqq 12\tau^2.$$

By Lemma 2.2 it follows from this that $\mathscr{L}(\eta)=\mathscr{L}(\eta^*)\in\mathscr{B}_1(c\tau)$. In the sequel we shall need the inequality

$$(Dt,t)\leqq d^2\|t\|^2\leqq 12\tau^2\|t\|^2 \tag{4.7}$$

which is valid for all $t\in\mathbb{R}^k$. In particular, (4.7) implies $(Dh,h)\leqq c$ if $\|h\|\tau\leqq c$.

The derivation of (4.4)-(4.6) is similar to the proof of Lemma 3 from [19]. Throughout the proof we use the possibility to choose c_{11} as small as will be necessary for the validity of corresponding formulae.

Since $\mathscr{L}(\xi)\in\mathscr{B}_1(\tau)$, $\mathscr{L}(\eta)\in\mathscr{B}_1(c\tau)$, by expanding exponentials in Taylor series (see (2.2)) and by choosing c_{11} small enough we get

$$|\mathbf{E}(\xi,t)e^{(\xi,h)}|\leqq c(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}},\ \ |\mathbf{E}(\eta,t)e^{(\eta,h)}|\leqq c(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}}, \tag{4.8}$$
$$|\mathbf{E}(\xi,t)e^{(\xi,h)}-\mathbf{E}(\eta,t)e^{(\eta,h)}|\leqq c\|h\|\tau(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}}, \tag{4.9}$$
$$\mathbf{E}(\xi,t)^2e^{(\xi,h)}\leqq c(Dt,t),\ \ \mathbf{E}(\eta,t)^2e^{(\eta,h)}\leqq c(Dt,t), \tag{4.10}$$
$$|\mathbf{E}(\xi,t)^2e^{(\xi,h)}-\mathbf{E}(\eta,t)^2e^{(\eta,h)}|\leqq c\|h\|\tau(Dt,t), \tag{4.11}$$
$$|\mathbf{E}e^{(\xi,h)}-\mathbf{E}e^{(\eta,h)}|\leqq c\|h\|\tau(Dh,h). \tag{4.12}$$

In view of Jensen inequality

$$\mathbf{E}e^{(\xi,h)}\geqq e^{\mathbf{E}(\xi,h)}=1,\ \ \mathbf{E}e^{(\eta,h)}\geqq 1. \tag{4.13}$$

By (4.2), (4.7)-(4.9), (4.12), (4.13) we have (if c_{11} is sufficiently small):

$$\begin{aligned}&|\mathbf{E}(\bar{\xi},t)-\mathbf{E}(\bar{\eta},t)|=|(\mathbf{E}e^{(\xi,h)})^{-1}\mathbf{E}(\xi,t)e^{(\xi,h)}\\&\quad-(\mathbf{E}e^{(\eta,h)})^{-1}\mathbf{E}(\eta,t)e^{(\eta,h)}|\leqq|\mathbf{E}(\xi,t)e^{(\xi,h)}-\mathbf{E}(\eta,t)e^{(\eta,h)}|\\&\quad+|(\mathbf{E}e^{(\xi,h)}-\mathbf{E}e^{(\eta,h)})\mathbf{E}(\eta,t)e^{(\eta,h)}|\\&\leqq c\|h\|\tau(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}}\end{aligned}$$

that is (4.4) holds. Similarly, it follows from (4.2), (4.7), (4.10)-(4.13) that

$$|\mathbf{E}(\bar{\xi},t)^2-\mathbf{E}(\bar{\eta},t)^2|\leqq c\|h\|\tau(Dt,t) \tag{4.14}$$

and from (4.2), (4.8), (4.13) that

$$|\mathbf{E}(\bar{\xi},t)|+|\mathbf{E}(\bar{\eta},t)|\leqq c(Dt,t)^{\frac{1}{2}}(Dh,h)^{\frac{1}{2}}. \tag{4.15}$$

Taking into account that $(Dh,h)\leqq c$, $\mathbf{E}(\xi^*,t)^2=\mathbf{E}(\bar{\xi},t)^2-(\mathbf{E}(\bar{\xi},t))^2$, $\mathbf{E}(\eta^*,t)^2=\mathbf{E}(\bar{\eta},t)^2-(\mathbf{E}(\bar{\eta},t))^2$ we obtain (4.5) from (4.4), (4.14), (4.15). If $\|h\|\tau\leqq c_{11}\leqq c$

then by (4.12), (4.13) we get

$$\max\left\{\frac{\mathbf{E}e^{(\xi,h)}}{\mathbf{E}e^{(\eta,h)}}, \frac{\mathbf{E}e^{(\eta,h)}}{\mathbf{E}e^{(\xi,h)}}\right\} \leqq 1+|\mathbf{E}e^{(\xi,h)}-\mathbf{E}e^{(\eta,h)}|$$
$$\leqq 1+c\,\|h\|\,\tau(Dh,h)$$
$$\leqq \exp(c\,\|h\|\,\tau(Dh,h)) \tag{4.16}$$

that is (4.6) is valid.

Let us show that $\mathscr{L}(\xi^*)\in\mathscr{B}_2(c\tau)$ if $\|h\|\,\tau\leqq c_{11}\leqq c$. By Remark 2.1, for this it suffices to check that $\mathbf{E}(\xi^*,t)^2 e^{(\xi^*,u)}\leqq 2\mathbf{E}(\xi^*,t)^2$ if $\|u\|\,\tau\leqq c$. Denoting $v=u+h$ and using (3.2) we obtain

$$\begin{aligned}\mathbf{E}(\xi^*,t)^2 e^{(\xi^*,u)} &= \mathbf{E}((\bar{\xi},t)-\mathbf{E}(\bar{\xi},t))^2 e^{(\bar{\xi},u)-\mathbf{E}(\bar{\xi},u)}\\ &= (e^{\mathbf{E}(\bar{\xi},u)}\mathbf{E}e^{(\xi,h)})^{-1}\{\mathbf{E}(\xi,t)^2 e^{(\xi,v)}\\ &\quad -2\mathbf{E}(\xi,t)e^{(\xi,v)}\mathbf{E}(\bar{\xi},t)+(\mathbf{E}(\bar{\xi},t))^2\mathbf{E}e^{(\xi,v)}\}.\end{aligned} \tag{4.17}$$

If $\|v\|\,\tau\leqq c$ where c is small enough we can use for v all relations earlier obtained for h. Further calculations will be performed for $\|v\|\,\tau\leqq c$, $\|u\|\,\tau\leqq c$, $\|h\|\,\tau\leqq c$ where constants c are as small as is necessary for the correctness of arguments. Thus, by (4.7), (4.15) $\mathbf{E}|(\bar{\xi},u)|\leqq c\,\|h\|\,\tau$, in view of (2.5), (4.7), (4.16) $\mathbf{E}e^{(\xi,h)}=\exp(c\theta\,\|h\|\,\tau)$ and from (2.7), (4.7) it follows that

$$\mathbf{E}(\eta,t)^2 e^{(\eta,v)}=(Dt,t)e^{c\theta\,\|v\|\,\tau}. \tag{4.18}$$

Further, by (4.11), (4.18) we get $\mathbf{E}(\xi,t)^2 e^{(\xi,v)}=(Dt,t)\exp(c\theta\,\|v\|\,\tau)$ and from (2.5), (4.7), (4.8), (4.15), (4.16) we deduce

$$-2\mathbf{E}(\xi,t)e^{(\xi,v)}\mathbf{E}(\bar{\xi},t)+(\mathbf{E}(\bar{\xi},t))^2\mathbf{E}e^{(\xi,v)}$$
$$=c\theta(\|h\|+\|v\|)\tau(Dt,t).$$

By substituting the relations just obtained in (4.17) we find

$$\mathbf{E}(\xi^*,t)^2 e^{(\xi^*,u)}=(Dt,t)\exp(c\theta\tau(\|v\|+\|h\|)). \tag{4.19}$$

Since $\mathbf{E}(\xi^*,t)^2=(Dt,t)\exp(c\theta\,\|h\|\,\tau)$ in view of (4.3), (4.5), we get from (4.19) that

$$\mathbf{E}(\xi^*,t)^2 e^{(\xi^*,u)}\leqq 2\mathbf{E}(\xi^*,t)^2 \tag{4.20}$$

if $\|h\|\,\tau\leqq c_{16}$, $\|u\|\,\tau\leqq c_{17}$, $\|v\|\,\tau\leqq c_{18}$. If $\|h\|\,\tau\leqq c_{11}\leqq\max\{c_{16}, c_{18}/2\}$, $\|u\|\,\tau\leqq\max\{c_{17}, c_{18}/2\}$, we have $\|v\|\,\tau\leqq c_{18}$. Hence (4.20) is valid and therefore $\mathscr{L}(\xi^*)\in\mathscr{B}_2(c\tau)$. It remains to use Lemma 2.1.

5. The Beginning of the Proof of Theorem 1.1

First we shall prove (1.4). Let us assume without loss of generality that $\tau\leqq e^{-1}$ so that $|\ln\tau|=\ln 1/\tau\geqq 1$. We shall use the smoothing inequality

$$\pi(F,\Phi)\leqq\pi(FG_0G,\Phi G_0G)+2\pi(G_0,E)+2\pi(G,E)$$

which is valid for any $G_0, G \in \mathfrak{F}_k$ and follows from the weak regularity of the Lévy-Prohorov distance ($\pi(V_1 V_3, V_2 V_3) \leqq \pi(V_1, V_2)$ for any $V_1, V_2, V_3 \in \mathfrak{F}_k$, see [25]). Finally, we shall choose G_0 and G to be Gaussian with zero means and with covariance operators all whose eigenvalues are equal to $ck^4\tau^2|\ln\tau|$ and $ck^3\tau^2|\ln\tau|$ respectively. We shall show that in this case

$$2\pi(G_0, E) + 2\pi(G, E) \leqq ck^{\frac{5}{2}}\tau|\ln\tau|$$

and it will remain to investigate the proximity of smoothed distributions $(FG_0)G$ and $(\Phi G_0)G$. According to Remark 2.2, G_0 may be represented in the convolution form: $G_0 = (G_{00})^m$ where $G_{00} \in \mathscr{B}_1(\tau)$. This yields the possibility to reduce FG_0 to a convolution of distributions from $\mathscr{B}_1(\tau)$, i.e. to the same form that F has itself. The role of G_0 is to make the smallest eigen-value σ^2 of the covariance operator D of distribution FG_0G sufficiently large. Thus, we may omit the distribution G_0 and study only $\pi(FG, \Phi G)$ assuming however that $\sigma^2 \geqq ck^4\tau^2|\ln\tau|$. But at first it will be required only that $\sigma^2 > 0$.

Beginning with this section we consider the following situation. There are independent random vectors $\xi_1, \ldots, \xi_n, \zeta_1$ and $\eta_1, \ldots, \eta_n, \zeta_2$ with zero means and such that $F_i = \mathscr{L}(\xi_i) \in \mathscr{B}_1(\tau)$ $(i = 1, \ldots, n)$; the distributions $G_i = \mathscr{L}(\eta_i)$ are Gaussian with the covariance operators coinciding with those of F_i; the vectors ζ_1, ζ_2 are also Gaussian with a common distribution G (the case $G = E$ is not excluded). Denote

$$F = \prod_{i=1}^{n} F_i, \qquad \Phi = \prod_{i=1}^{n} G_i,$$

$$S = \xi_1 + \ldots + \xi_n + \zeta_1, \qquad R = \eta_1 + \ldots + \eta_n + \zeta_2.$$

Let us introduce the independent conjugate random vectors $\bar{\xi}_1, \ldots, \bar{\xi}_n, \bar{\zeta}_1$ and $\bar{\eta}_1, \ldots, \bar{\eta}_n, \bar{\zeta}_2$ defined by means of a parameter h and let $\bar{S} = \bar{S}_h = \bar{\xi}_1 + \ldots + \bar{\xi}_n + \bar{\zeta}_1$, $\bar{R} = \bar{R}_h = \bar{\eta}_1 + \ldots + \bar{\eta}_n + \bar{\zeta}_2$. It is clear that $\mathscr{L}(S) = FG$, $\mathscr{L}(\bar{S}_h) = \bar{F}(h)\bar{G}(h)$, $\mathscr{L}(R) = \Phi G$, $\mathscr{L}(\bar{R}_h) = \bar{\Phi}(h)\bar{G}(h)$. Denote the covariance operators of distributions G, F, FG, $\bar{F}(h)$, $\bar{F}(h)\bar{G}(h)$ by B, D_0, $D = B + D_0$, $D_0(h)$, $D(h) = B + D_0(h)$ respectively. We denote corresponding minimal eigenvalues of this operators by b^2, σ_0^2, σ^2, $\sigma_0^2(h)$, $\sigma^2(h)$. Assume that $\sigma_0^2 > 0$. Hence $\det D \geqq \det D_0 > 0$, $\sigma^2 > 0$ and the operator D is invertible.

By Lemma 4.1 we know that $\mathscr{L}(\xi_i^*) \in \mathscr{B}_1(c_{15}\tau)$ if $\|h\|\tau \leqq c_{11}$ where $\xi_i^* = \bar{\xi}_i - \mathbf{E}\bar{\xi}_i$, $i = 1, \ldots, n$. Therefore, sometimes we shall replace the condition $F_i \in \mathscr{B}_1(\tau)$ by the condition $F_i \in \mathscr{B}_1(c_{15}\tau)$, keeping in mind the application of results obtained for $\mathscr{L}(S)$ to the centered conjugate distributions $\mathscr{L}(S_h^*)$ where $\|h\|\tau \leqq c_{11}$. As a rule, this will lead only to the change of several absolute constants.

Lemma 5.1. *There exist* c_{19}, c_{20}, c_{21} *such that for* $\|h\|\tau \leqq c_{19} \leqq c_{11}$ *and for all* $t \in \mathbb{R}^k$ *the following relations hold:*

$$|\mathbf{E}(\bar{S}_h, t) - \mathbf{E}(\bar{R}_h, t)| \leqq c_{12}\|h\|\tau(D_0 t, t)^{\frac{1}{2}}(D_0 h, h)^{\frac{1}{2}}, \tag{5.1}$$

$$|(D(h)t, t) - (Dt, t)| \leqq c_{13}\|h\|\tau(D_0 t, t), \tag{5.2}$$

$$\left|\ln \frac{\mathbf{E}e^{(S,h)}}{\mathbf{E}e^{(R,h)}}\right| \leqq c_{14}\|h\|\tau(D_0 h, h), \tag{5.3}$$

$$(\det D(h))^{\frac{1}{2}} = (\det D)^{\frac{1}{2}} \exp(c_{20}\theta k\tau \|h\|), \tag{5.4}$$

$$\sigma(h) = \sigma \exp(c_{21}\theta\tau \|h\|). \tag{5.5}$$

Proof. The inequalities (5.1)-(5.3) follow immediately from Lemma 4.1 (see (4.4)-(4.6)). The relation (5.4) may be easily derived from (5.2) with the help of (2.8) since $(D_0 t, t) \leqq (Dt, t)$ for all $t \in \mathbb{R}^k$. Finally, we get (5.5) from (5.2) by means of the identities

$$\sigma^2 = \inf_{\|t\|=1} (Dt, t), \qquad \sigma^2(h) = \inf_{\|t\|=1} (D(h)t, t).$$

The constant c_{19} should be chosen sufficiently small.

Remark 5.1. It follows from (5.4) that in the conditions of Lemma 5.1 we have $\det D(h) > 0$ and the operator $D(h)$ is invertible.

To compare the values of probability densities $p(x)$ and $q(x)$, corresponding to FG and ΦG, at a fixed point $x \in \Pi \subset \mathbb{R}^k$ (Π is defined below) we investigate the densities of conjugate distributions defined by the parameter $h = \tilde{h}$ which is the solution of the equation $\mathbf{E}\bar{S}_h = x$ (such choice of h is generally accepted, see, e.g., [2, 3, 12, 14]).

Lemma 5.2. *There exist c_{22}, c_{23} such that for $x \in \Pi = \{x \in \mathbb{R}^k \colon 6\tau\sigma^{-1} \|D^{-\frac{1}{2}}x\| \leqq c_{22}\}$ it is possible to find $\tilde{h} = \tilde{h}(x)$ for which*

$$\mathbf{E}\bar{S}_{\tilde{h}} = x, \tag{5.6}$$

$$\|\tilde{h}\|\tau \leqq c_{22} \leqq c_{19}, \tag{5.7}$$

$$\sigma\|\tilde{h}\| \leqq \|D^{\frac{1}{2}}\tilde{h}\| \leqq 6\|D^{-\frac{1}{2}}x\|, \tag{5.8}$$

$$\|D^{\frac{1}{2}}\tilde{h} - D^{-\frac{1}{2}}x\| \leqq 36 c_{12} \frac{\tau}{\sigma} \|D^{-\frac{1}{2}}x\|^2, \tag{5.9}$$

$$\mathbf{E}e^{(S,\tilde{h})-(x,\tilde{h})} = \exp\left(-\tfrac{1}{2}\|D^{-\frac{1}{2}}x\|^2 + c_{23}\frac{\theta\tau}{\sigma}\|D^{-\frac{1}{2}}x\|^3\right). \tag{5.10}$$

Proof. Set

$$c_{22} = \min\{c_{19}, (2c_{14})^{-1}\}. \tag{5.11}$$

It is clear that for $x = 0$ the parameter $\tilde{h}$ must be taken equal to zero. Now fix $x \in \Pi$, $x \neq 0$ and consider the ellipsoid M given by

$$M = \{h \in \mathbb{R}^k \colon \|D^{\frac{1}{2}}h\| \leqq 6\|D^{-\frac{1}{2}}x\|\}.$$

Since $x \in \Pi$ we have for $h \in M$:

$$\|h\|\tau \leqq \tau\sigma^{-1}\|D^{\frac{1}{2}}h\| \leqq 6\tau\sigma^{-1}\|D^{-\frac{1}{2}}x\| \leqq c_{22}. \tag{5.12}$$

Let us introduce the function

$$\varphi(h) = \ln \mathbf{E}e^{(S,h)-(x,h)}. \tag{5.13}$$

Obviously, $\varphi(0) = 0$. By (5.11), (5.12) for $h \in M$ we have $\|h\|\tau \leqq c_{19}$ and Lemma 5.1 is applicable. In particular, the inequality (5.1) may be rewritten in the form

$$|(D^{-\frac{1}{2}}\mathbf{E}\bar{S}_h - D^{\frac{1}{2}}h, u)| \leqq c_{12}\|h\|\tau\|D^{\frac{1}{2}}h\|\,\|u\|$$

for all $u \in \mathbb{R}^k$ (it is sufficient to set $u = D^{\frac{1}{2}} t$ and to use (4.3)). Hence

$$\|D^{-\frac{1}{2}} \mathbf{E} \bar{S}_h - D^{\frac{1}{2}} h\| \leqq c_{12} \|h\| \tau \|D^{\frac{1}{2}} h\|. \tag{5.14}$$

Let us suppose $\|D^{\frac{1}{2}} h\| = 6 \|D^{-\frac{1}{2}} x\|$. Then by (5.13), (5.3) (2.5), (4.3), (5.11), (5.12) we obtain

$$\begin{aligned} \varphi(h) &= \tfrac{1}{2} \|D^{\frac{1}{2}} h\|^2 (1 + \theta c_{14} \|h\| \tau) - (D^{\frac{1}{2}} h, D^{-\frac{1}{2}} x) \\ &\geqq 9 \|D^{-\frac{1}{2}} x\|^2 - 6 \|D^{-\frac{1}{2}} x\|^2 = 3 \|D^{-\frac{1}{2}} x\|^2 > 0. \end{aligned}$$

Thus, on the boundary of the ellisoid M the continuously differentiable function $\varphi(h)$ is greater than $\varphi(0)$. Hence there exists a point $\tilde{h}$ in which the smallest value of $\varphi(h)$ on M is achieved. Moreover, $\tilde{h}$ is strictly inside M. Taking (4.2) into account it is easy to check that

$$\operatorname{grad} \varphi(\tilde{h}) = \mathbf{E} \bar{S}_{\tilde{h}} - x = 0.$$

Therefore (5.6) holds. The inequalities (5.7), (5.8) follow directly from (5.11), (5.12) and (5.6), (5.8), (5.14) imply (5.9).

From (5.8), (5.9) we get

$$(x, \tilde{h}) = (D^{-\frac{1}{2}} x, D^{\frac{1}{2}} \tilde{h}) = \|D^{-\frac{1}{2}} x\|^2 + \theta c \frac{\tau}{\sigma} \|D^{-\frac{1}{2}} x\|^3, \tag{5.15}$$

$$(D\tilde{h}, \tilde{h}) = \|D^{\frac{1}{2}} \tilde{h}\|^2 = \|D^{-\frac{1}{2}} x\|^2 + \theta c \frac{\tau}{\sigma} \|D^{-\frac{1}{2}} x\|^3. \tag{5.16}$$

Now (5.10) may be easily deduced from (5.3), (5.8), (2.5), (5.15), (5.16).

6. An Uniform Estimate for the Closeness of Densities

Beginning with this section the smoothing distribution G is supposed to be non-degenerate, $b^2 > 0$, so that the distributions FG and ΦG have continuous densities $p(x)$ and $q(x) = (2\pi)^{-k/2} (\det D)^{-\frac{1}{2}} \exp(-\frac{1}{2} \|D^{-\frac{1}{2}} x\|^2)$ respectively. Corresponding densities $p_h(x)$ and $q_h(x)$ of conjugate distributions also exist and are related to $p(x)$ and $q(x)$ by the equalities

$$p_h(x) = (\mathbf{E} e^{(S,h)})^{-1} e^{(x,h)} p(x), \; q_h(x) = (\mathbf{E} e^{(R,h)})^{-1} e^{(x,h)} q(x) \tag{6.1}$$

(see (4.1)). The notation $r_h(x)$ will be used for the density of Gaussian distribution with the same mean and the same covariance operator as the distribution $\bar{F}(h) \bar{G}(h)$.

Here we get an upper bound for the uniform distance between $p(x)$ and $q(x)$ assuming the eigenvalues of the convariance operator B of the distribution G large enough.

Lemma 6.1. a) *There exist c_{24}, c_{25}, c_{26} such that if $\tau \leqq c_{24}/k$, $b^2 \geqq c_{25} k^3 \tau^2 |\ln \tau|$ then the inequality*

$$\sup_{x \in \mathbb{R}^k} |p(x) - q(x)| \leqq \frac{c_{26}}{(2\pi)^{k/2}} \left(\frac{k^{\frac{3}{2}} \tau}{\sigma (\det D)^{\frac{1}{2}}} + \frac{\tau}{(\det D_0)^{\frac{1}{2}}} \right)$$

holds.

b) *The statement of item* a) *remains true if instead of the condition $F_i \in \mathscr{B}_1(\tau)$ we require $F_i \in \mathscr{B}_1(c_{15} \tau)$ $(i = 1, \ldots, n)$.*

Proof. It is sufficient to prove a). Without loss of generality we suppose the matrix B to be diagonal. Denote

$$K_u=\{t\in\mathbb{R}^k\colon |t-u|\leqq c_{27}(k^{\frac{3}{2}}\tau)^{-1}\}$$

for $u\in\mathbb{R}^k$. The choice of c_{27} will be corrected throughout the proof. By the inversion formula for densities we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^k}|p(x)-q(x)|&\leqq(2\pi)^{-k}\int_{\mathbb{R}^k}|\hat F(t)-\hat\Phi(t)|\,\hat G(t)\,dt\\&=\sum_{\alpha\in\Xi}(2\pi)^{-k}\int_{K_\alpha}|\hat F(t)-\hat\Phi(t)|\,\hat G(t)\,dt.\end{aligned}\tag{6.2}$$

Here α runs over all points of the k-dimensional lattice $\Xi=2c_{27}(k^{\frac{3}{2}}\tau)^{-1}\mathbb{Z}^k$ i.e. all points of the form $\alpha=2c_{27}(k^{\frac{3}{2}}\tau)^{-1}l$ where $l\in\mathbb{R}^k$ is a vector with integer coordinates.

At first we estimate the integral over the cube K_0. Since $\mathscr{L}(\xi_i)\in\mathscr{B}_1(\tau)$ taking (2.1) into account we obtain by choosing c_{24}, c_{27} small enough it is possible to show that

$$\mathbf{E}(\xi_i,t)^2\leqq c\tau^2\|t\|^2\leqq 1,\qquad |\hat F_i(t)-\hat G_i(t)|\leqq c\tau\|t\|\,\mathbf{E}(\xi_i,t)^2,$$

$$\begin{aligned}|\hat F_i(t)|&=1-\tfrac12\mathbf{E}(\xi_i,t)^2+\frac{\theta}{6}\mathbf{E}|(\xi_i,t)|^3\\&\leqq\exp\{-\tfrac12\mathbf{E}(\xi_i,t)^2(1-c\tau\|t\|)\}\leqq\exp\left\{-\tfrac12\mathbf{E}(\xi_i,t)^2\left(1-\frac{1}{4k}\right)\right\},\end{aligned}$$

$$\begin{aligned}|\hat F(t)-\hat\Phi(t)|\,\hat G(t)&\leqq c\tau\|t\|(Dt,t)\exp\left\{-\tfrac12(Dt,t)\left(1-\frac{1}{4k}\right)\right\}\\&\leqq c\frac{\tau}{\sigma}(Dt,t)^{\frac32}\exp\left\{-\tfrac12(Dt,t)\left(1-\frac{1}{4k}\right)\right\}\\&\leqq ck^{\frac32}\tau\sigma^{-1}\exp\left\{-\tfrac12(Dt,t)\left(1-\frac{1}{2k}\right)\right\}\end{aligned}$$

for any $t\in K_0$, $i=1,\dots,n$. Therefore

$$\begin{aligned}\frac{1}{(2\pi)^k}\int_{K_0}|\hat F(t)-\hat\Phi(t)|\,\hat G(t)\,dt&\leqq\frac{ck^{\frac32}\tau}{(2\pi)^k\sigma}\int_{\mathbb{R}^k}\exp\left\{-\tfrac12(Dt,t)\left(1-\frac{1}{2k}\right)\right\}dt\\&=\frac{ck^{\frac32}\tau}{(2\pi)^k\sigma}\,\frac{(2\pi)^{k/2}}{(1-1/2k)^{k/2}(\det D)^{\frac12}}\leqq\frac{ck^{\frac32}\tau}{(2\pi)^{k/2}\sigma(\det D)^{\frac12}}.\end{aligned}\tag{6.3}$$

Now we pass on the estimating integrals over the rest cubes. We have

$$\begin{aligned}&\int_{\mathbb{R}^k\setminus K_0}|\hat F(t)-\hat\Phi(t)|\,\hat G(t)\,dt\\&\quad\leqq\int_{\mathbb{R}^k\setminus K_0}|\hat F(t)|\,\hat G(t)\,dt+\int_{\mathbb{R}^k\setminus K_0}\hat\Phi(t)\hat G(t)\,dt=I_1+I_2.\end{aligned}\tag{6.4}$$

It is clear that

$$I_1=\sum_{\alpha\in\Xi,\,\alpha\neq0}\int_{K_\alpha}|\hat F(t)|\,\hat G(t)\,dt.\tag{6.5}$$

Let us apply Corollary 3.1 to estimate the characteristic functions $|\hat{F}_i(t)|^2$ of probability distributions $\mathscr{L}(\tilde{\xi}_i - \tilde{\tilde{\xi}}_i)$ where $\tilde{\xi}_i, \tilde{\tilde{\xi}}_i$ are independent random vectors such that $\mathscr{L}(\tilde{\xi}_i) = \mathscr{L}(\tilde{\tilde{\xi}}_i) = F_i$. Let $t_1, t_2 \in K_\alpha$, $\alpha \in \mathbb{R}^k$, $v = t_1 - t_2$. By choosing c_{27} to be small enough we get

$$\begin{aligned}\mathbf{E}|(\tilde{\xi}_i - \tilde{\tilde{\xi}}_i, v)|^3 &\leqq 8\,\mathbf{E}|(\xi_i, v)|^3 \leqq c\tau\,\|v\|\,\mathbf{E}(\xi_i, v)^2\\ &\leqq c k^{\frac{1}{2}} \tau |v|\,\mathbf{E}(\xi_i, v)^2 \leqq c c_{27} k^{-1}\,\mathbf{E}(\xi_i, v)^2\\ &\leqq (2k)^{-1}\,\mathbf{E}(\xi_i, v)^2 = (4k)^{-1}\,\mathbf{E}(\tilde{\xi}_i - \tilde{\tilde{\xi}}_i, v)^2.\end{aligned}$$

Hence, for each $i = 1, \ldots, n$ the conditions of Corollary 3.1 are satisfied for $\hat{H}(t) = |\hat{F}_i(t)|^2$, $\gamma = (4k)^{-1}$. Therefore by (3.6) we get that there exist the points $v_i \in K_\alpha$ such that

$$|\hat{F}_i(t)|^2 \leqq \exp\left\{-\left(\frac{1}{2} - \frac{1}{4k}\right)\mathbf{E}(\tilde{\xi}_i - \tilde{\tilde{\xi}}_i, t - v_i)^2\right\}$$

for all $t \in K_\alpha$ and consequently,

$$|\hat{F}_i(t)| \leqq \exp\left\{-\left(\frac{1}{2} - \frac{1}{4k}\right)\mathbf{E}(\xi_i, t - v_i)^2\right\}. \tag{6.6}$$

From (6.6) it follows that

$$\int_{K_\alpha} |\hat{F}(t)|\,dt \leqq \int_{\mathbb{R}^k} \exp\left\{-\frac{1}{2}\left(1 - \frac{1}{2k}\right)(D_0 t, t)\right\} dt \leqq \frac{c(2\pi)^{k/2}}{(\det D_0)^{\frac{1}{2}}}. \tag{6.7}$$

Put $c_{25} = 4c_{27}^{-2}$ and introduce the function $f(\cdot)$ on the set of all integers by setting $f(m) = 2|m| - 1$ for $m \neq 0$ and $f(0) = 0$. Let $\alpha = 2c_{27}(k^{\frac{3}{2}}\tau)^{-1} l \in \Xi$. For any integer m we have $|m| \leqq f(m) \leqq f^2(m)$. So the inequalities

$$\begin{aligned}\hat{G}(t) &\leqq \exp(-\tfrac{1}{2} b^2 \|t\|^2) \leqq \exp\left\{-\tfrac{1}{2} c_{25} k^3 \tau^2 |\ln \tau| \sum_{j=1}^{k} \left(f(l_j)\frac{c_{27}}{k^{\frac{3}{2}}\tau}\right)^2\right\}\\ &\leqq \prod_{j=1}^{k} \tau^{2(f(l_j))^2} \leqq \prod_{j=1}^{k} \tau^{2|l_j|}\end{aligned} \tag{6.8}$$

are valid for $t \in K_\alpha$, $\tau \leqq e^{-1}$. Therefore if $\tau \leqq (2\mathrm{k})^{-1}$ we have

$$\begin{aligned}\sum_{\alpha \in \Xi, \alpha \neq 0} \max_{t \in K_\alpha} \hat{G}(t) &\leqq \left(1 + 2\sum_{m=1}^{\infty} \tau^{2m}\right)^k - 1 = (1 + 2\tau^2(1-\tau^2)^{-1})^k - 1\\ &\leqq \exp\left(\frac{2\tau^2 k}{1-\tau^2}\right) - 1 \leqq \exp\left(\frac{\tau}{1-\tau^2}\right) - 1 \leqq c\tau.\end{aligned} \tag{6.9}$$

It follows from (6.5), (6.7), (6.9) that

$$I_1 \leqq \sum_{\alpha \in \Xi, \alpha \neq 0} \max_{t \in K_\alpha} \hat{G}(t) \int_{K_\alpha} |\hat{F}(t)|\,dt \leqq \frac{c(2\pi)^{k/2}\tau}{(\det D_0)^{\frac{1}{2}}}. \tag{6.10}$$

Let us estimate I_2. By (6.8), $\max_{t\in\mathbb{R}^k\setminus K_0} \hat{G}(t)\leqq\tau^2\leqq\tau$. Hence

$$I_2\leqq\tau\int_{\mathbb{R}^k}\hat{\Phi}(t)\,dt=\frac{(2\pi)^{k/2}\tau}{(\det D_0)^{\frac{1}{2}}}. \tag{6.11}$$

Now the statement of Lemma 6.1 may be deduced from (6.2)–(6.4), (6.10), (6.11).

7. The Non-Uniform Bound for the Proximity of Densities

Lemma 7.1. a) *Let $B=b^2 I$ where I is the identity operator. There exist $c_{28},\ldots,c_{32}$ such that for $\tau\leqq c_{28}/k$, $b^2\geqq c_{29}k^3\tau^2|\ln\tau|$, $\sigma^2\geqq 2kb^2$, $\tau\sigma^{-1}\|D^{-\frac{1}{2}}x\|\leqq c_{30}/k$ it is possible to find a parameter $\tilde{h}$ for which the relations* (5.6)–(5.10) *are valid, and the density $p(x)$ may be represented in the form*

$$p(x)=(2\pi)^{-k/2}(\det D)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\|D^{-\frac{1}{2}}x\|^2 +\theta(c_{31}\tau+c_{32}\tau\sigma^{-1}(k^{\frac{3}{2}}+\|D^{-\frac{1}{2}}x\|^3))\}. \tag{7.1}$$

Moreover, for any c_{33} we can choose $c_{30}=c_{30}(c_{33})$ so small that the inequality $\|\tilde{h}\|\tau\leqq c_{33}/k$ will be satisfied.

b) *The statement of item* a) *remains true if we change the condition $F_i\in\mathscr{B}_1(\tau)$ by the condition $F_i\in\mathscr{B}_1(c_{15}\tau)$, $i=1,\ldots,n$.*

Proof. It is clear that it suffices to prove a). The statement of item b) follows from a) (may be after a change of numerical values of constants).

It is easy to see that choosing c_{30} sufficiently small we may achieve that $x\in\Pi$ for $\tau\sigma^{-1}\|D^{-\frac{1}{2}}x\|\leqq c_{30}/k$ where Π is the set from Lemma 5.2. In view of this lemma there exists for such x a parameter $\tilde{h}=\tilde{h}(x)$ satisfying (5.6)–(5.10). By choosing c_{30} to be small we may obtain the inequality

$$\|\tilde{h}\|\tau\leqq c_{33}/k\leqq c_{33} \tag{7.2}$$

for any c_{33}. If c_{33} is small enough then for $h=\tilde{h}$ the conditions of Lemmas 4.1 and 5.1 are satisfied. Therefore $\mathscr{L}(\xi^*_{i,\tilde{h}})\in\mathscr{B}_1(c_{15}\tau)$ and (5.1)–(5.5) hold.

For all $t\in\mathbb{R}^k$ we have $(Bt,t)=b^2\|t\|^2\leqq\sigma^2\|t\|^2/2k$ and

$$(D_0t,t)=(Dt,t)-(Bt,t)\geqq\left(1-\frac{1}{2k}\right)(Dt,t) \tag{7.3}$$

so, by (2.8), we obtain

$$\det D\geqq\det D_0\geqq\left(1-\frac{1}{2k}\right)^k\det D\geqq c\det D. \tag{7.4}$$

According to Lemma 5.1,

$$\sigma(\tilde{h})=\sigma\exp(\theta c_{21}\tau\|\tilde{h}\|), \tag{7.5}$$

$$(\det D(\tilde{h}))^{\frac{1}{2}}=(\det D)^{\frac{1}{2}}\exp(\theta c_{20}k\tau\|\tilde{h}\|), \tag{7.6}$$

$$(\det D_0(\tilde{h}))^{\frac{1}{2}}=(\det D_0)^{\frac{1}{2}}\exp(\theta c_{20}k\tau\|\tilde{h}\|). \tag{7.7}$$

It follows from (5.6), (7.2), (7.6) that

$$\begin{aligned} r_{\tilde h}(x) &= (2\pi)^{-k/2}(\det D(\tilde h))^{-\frac{1}{2}} \\ &= (2\pi)^{-k/2}(\det D)^{-\frac{1}{2}}\exp(\theta c_{20} k\tau\|\tilde h\|) \\ &\geqq c(2\pi)^{-k/2}(\det D)^{-\frac{1}{2}}. \end{aligned} \tag{7.8}$$

By Lemma 6.1, (7.2), (7.4)–(7.7) we get

$$\begin{aligned} |p_{\tilde h}(x) - r_{\tilde h}(x)| &\leqq \frac{c_{26}}{(2\pi)^{k/2}}\left(\frac{c_{15}k^{\frac{3}{2}}\tau}{\sigma(\tilde h)(\det D(\tilde h))^{\frac{1}{2}}} + \frac{c_{15}\tau}{(\det D_0(\tilde h))^{\frac{1}{2}}}\right) \\ &\leqq c(2\pi)^{-k/2}(\det D)^{-\frac{1}{2}}(k^{\frac{3}{2}}\tau\sigma^{-1} + \tau). \end{aligned} \tag{7.9}$$

Choosing c_{28} small enough and using (7.8), (7.9) we ensure the inequalities

$$|p_{\tilde h}(x) - r_{\tilde h}(x)|/r_{\tilde h}(x) \leqq c(k^{\frac{3}{2}}\tau\sigma^{-1} + \tau) \leqq 1/2 \tag{7.10}$$

to be valid. From (5.8), (7.2), (7.8), (7.10) we obtain

$$\begin{aligned} p_{\tilde h}(x) &= r_{\tilde h}(x)(1 + (p_{\tilde h}(x) - r_{\tilde h}(x))/r_{\tilde h}(x)) \\ &= r_{\tilde h}(x)\exp(\theta c|p_{\tilde h}(x) - r_{\tilde h}(x)|/r_{\tilde h}(x)) \\ &= (2\pi)^{-k/2}(\det D)^{-\frac{1}{2}}\exp\{\theta c(k^{\frac{3}{2}}\tau\sigma^{-1} + k\tau\sigma^{-1}\|D^{-\frac{1}{2}}x\| + \tau)\}. \end{aligned} \tag{7.11}$$

Finally, with the help of (6.1), (5.10), (7.11) we derive (7.1) as follows:

$$\begin{aligned} p(x) &= p_{\tilde h}(x)\mathbf{E}e^{(S,\tilde h) - (x,\tilde h)} \\ &= \frac{1}{(2\pi)^{k/2}(\det D)^{\frac{1}{2}}}\exp\left\{-\tfrac{1}{2}\|D^{-\frac{1}{2}}x\|^2 + \theta c\left(\frac{k^{\frac{3}{2}}\tau}{\sigma} + \frac{k\tau}{\sigma}\|D^{-\frac{1}{2}}x\|\right.\right. \\ &\quad \left.\left. + \frac{\tau}{\sigma}\|D^{-\frac{1}{2}}x\|^3 + \tau\right)\right\} = \frac{1}{(2k)^{k/2}(\det D)^{\frac{1}{2}}}\exp\{-\tfrac{1}{2}\|D^{-\frac{1}{2}}x\|^2 \\ &\quad + \theta(c_{31}\tau + c_{32}\tau\sigma^{-1}(k^{\frac{3}{2}} + \|D^{-\frac{1}{2}}x\|^3))\}. \end{aligned}$$

8. Estimating the Closeness of Conditional Distributions

In addition to the conditions described at the beginning of Sect. 5 we suppose that the covariance matrix D is diagonal and its diagonal elements σ_j^2, $j = 1, \ldots, k$ are non-increasing when j increases:

$$\sigma_1^2 \geqq \sigma_2^2 \geqq \ldots \geqq \sigma_{k-1}^2 = (\sigma')^2 \geqq \sigma_k^2 = \sigma^2. \tag{8.1}$$

Further we suppose that $B = b^2 I$, $b^2 > 0$ so that the probability density $p(x)$ $(x \in \mathbb{R}^k)$ of the distribution $\mathscr{L}(S)$ exists and has good smoothness properties. If $k \geqq 2$, the same may be stated for the density $p'(x')$ $(x' = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1})$ of the vector S' composed from the first $k-1$ coordinates of S. The function $p_k(x_k|x') = p(x)/p'(x')$ of the argument $x_k \in \mathbb{R}^1$ may be considered as the conditional density of the distribution of the k-th coordinate of S, when $S' = x'$ being fixed. The probability measure depending on x' with the density $p_k(x_k|x')$

will be denoted by $U_k = U_{k,x'} \in \mathfrak{F}_1$. In the one-dimensional case it is not necessary to consider conditional distributions and it is convenient to assume that $|x'| = |(D')^{-\frac{1}{2}} x'| = 0$, $p_1(x_1 | x') = p(x)$, $U_k = FG = \mathscr{L}(S)$.

The distribution U_k will be compared with the Gaussian distribution $W \in \mathfrak{F}_1$ having the density $w(x_k) = (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp(-x_k^2/2\sigma^2)$. In what follows we assume that conventions just introduced are valid.

Lemma 8.1. *There exist $c_{34}, \ldots, c_{38}$ such that for $\tau \leqq c_{34}/k$, $b^2 \geqq c_{35} k^3 \tau^2 |\ln \tau|$, $\sigma^2 \geqq 4kb^2$ the following assertions are valid. For $k \geqq 2$, $|(D')^{-\frac{1}{2}} x'| \leqq 2 |\ln \tau|^{\frac{1}{2}}$ there exists a parameter $h' = h'(x') \in \mathbb{R}^{k-1}$ supplying the solution of the equation $\mathbf{E} \bar{S}'_{h'} = x'$. Let $a = (0, \ldots, 0, a_k) \in \mathbb{R}^k$ where $a_k = \mathbf{E} \bar{S}_{h,k}$ and the k-dimensional parameter h is obtained by adding zero as k-th coordinate to the $(k-1)$-dimensional vector h' if $k \geqq 2$ and $h = 0$ if $k = 1$. Denote $v(x_k)$ the density of the distribution $H_k = U_k E_{-a_k}$. Then*

$$v(x_k) = \frac{1}{(2\pi)^{\frac{1}{2}} \sigma} \exp\left\{ -\frac{x_k^2}{2\sigma^2} + \theta \left(c_{36} \tau + c_{37} \frac{k^{\frac{3}{2}} \tau |\ln \tau|^{\frac{1}{2}}}{\sigma} \left(1 + \frac{x_k^2}{\sigma^2} \right) \right) \right\} \tag{8.2}$$

for $|x_k| \leqq 4\sigma |\ln \tau|^{\frac{1}{2}}$ and, moreover,

$$|a_k| \leqq c_{38} k \tau |\ln \tau|. \tag{8.3}$$

Proof. For $k = 1$ the statement of the lemma may be easily deduced from Lemma 7.1.

Let $k \geqq 2$ and fix $x' \in \mathbb{R}^{k-1}$ such that $|(D')^{-\frac{1}{2}} x'| \leqq 2 |\ln \tau|^{\frac{1}{2}}$. By choosing c_{35} large enough it is possible to ensure that

$$\frac{6\tau}{\sigma'} \|(D')^{-\frac{1}{2}} x'\| \leqq \frac{6(k-1)^{\frac{1}{2}} \tau}{\sigma'} |(D')^{-\frac{1}{2}} x'| \leqq \frac{12 k^{\frac{1}{2}} \tau |\ln \tau|^{\frac{1}{2}}}{\sigma} \leqq c_{22}. \tag{8.4}$$

So, by the $(k-1)$-dimensional version of Lemma 5.2 there exists a parameter $h' \in \mathbb{R}^{k-1}$ supplying the solution of the equation $\mathbf{E} \bar{S}'_{h'} = x'$ and satisfying (5.7)–(5.10) (with a necessary change of notations). Here $\bar{S}'_{h'}$ is a conjugate random vector corresponding to the vector $S' \in \mathbb{R}^{k-1}$. Define now the k-dimensional parameter h by adding zero as a k-th coordinate to the $(k-1)$-dimensional vector h'. The choice of the last coordinate of h to be equal to zero is convenient at once for several reasons.

Firstly, for any $x_k \in \mathbb{R}^1$ we have

$$\mathbf{E} \exp((S, h) - (x + a, h)) = \mathbf{E} \exp((S', h') - (x', h'))$$

where $x \in \mathbb{R}^k$. Therefore by (6.1) we obtain

$$\begin{aligned} v(x_k) &= \frac{p(x+a)}{p'(x')} \\ &= \frac{\mathbf{E} \exp((S, h) - (x + a, h)) p_h(x+a)}{\mathbf{E} \exp((S', h') - (x', h')) p'_{h'}(x')} = \frac{p_h(x+a)}{p'_{h'}(x')} \end{aligned} \tag{8.5}$$

where $p_h(\cdot)$ and $p'_{h'}(\cdot)$ are densities of corresponding conjugate distributions. Secondly, the covariance matrix of $\bar{S}'_{h'}$ coincides with a submatrix $(D(h))'$

composed of $k-1$ initial rows and $k-1$ initial columns of the covariance matrix $D(h)$ of the vector $\bar{S}_h$. This will be used for the calculation of the element $\beta_{k,k}$ of the matrix $(D(h))^{-1}=\{\beta_{ij}\}_{i,j=1}^{k}$. Using the well-known formula for inverse matrices, we obtain

$$\beta_{k,k}=\det(D(h))'/\det D(h). \tag{8.6}$$

Finally, by (4.3) we have (notations are those of Sect. 5)

$$\mathbf{E}\bar{R}_{h,k}=(Dh)_k=0 \tag{8.7}$$

and it can be easily seen that $\mathbf{E}(\bar{S}_h)'=x'$,

$$\|h\|=\|h'\|, \qquad \|(D')^{\frac{1}{2}}h'\|=\|D^{\frac{1}{2}}h\|. \tag{8.8}$$

In view of (5.8), (8.1), (8.4), (8.8) we have

$$\begin{aligned}\|h\| &\leqq (\sigma')^{-1}\|D^{\frac{1}{2}}h\| \leqq 6(\sigma')^{-1}\|(D')^{-\frac{1}{2}}x'\|\\ &\leqq 12(\sigma')^{-1}k^{\frac{1}{2}}|\ln\tau|^{\frac{1}{2}} \leqq 12\sigma^{-1}k^{\frac{1}{2}}|\ln\tau|^{\frac{1}{2}}.\end{aligned} \tag{8.9}$$

So, c_{35} being large enough, it may be ensured that

$$\|h'\|\tau=\|h\|\tau\leqq 12\sigma^{-1}k^{\frac{1}{2}}\tau|\ln\tau|^{\frac{1}{2}}\leqq c_{39}/k \tag{8.10}$$

for arbitrary small c_{39}. In particular, we can obtain the validity of the Lemma 4.1 conditions for h', h and of the Lemma 5.1 conditions for $\mathscr{L}(\bar{S}'_{h'})$, $\mathscr{L}(\bar{S}_h)$.

By Lemma 4.1, $\mathscr{L}(\bar{\xi}'_{i,h'}-\mathbf{E}\bar{\xi}_{i,h'})$ and $\mathscr{L}(\bar{\xi}_{i,h}-\mathbf{E}\bar{\xi}_{i,h})$ belong to $\mathscr{B}_1(c_{15}\tau)$ for $i=1,\ldots,n$. With the help of (5.1), (8.7), (8.9), (8.10) we obtain the inequality (8.3) as follows:

$$\begin{aligned}|a_k| &\leqq c_{12}\|h\|\tau\sigma\|D^{\frac{1}{2}}h\|\\ &\leqq 144kc_{12}\tau|\ln\tau|=c_{38}k\tau|\ln\tau|.\end{aligned}$$

By using (8.8), (8.10) and Lemma 5.1 we find that

$$\sigma(h)=\sigma\exp(\theta c_{21}\tau\|h\|), \tag{8.11}$$

$$\sigma'(h')=\sigma'\exp(\theta c_{21}\tau\|h\|), \tag{8.12}$$

$$\begin{aligned}(\det D(h))^{\frac{1}{2}} &=(\det D)^{\frac{1}{2}}\exp(\theta c_{20}k\tau\|h\|)\\ &=(\det D)^{\frac{1}{2}}\exp(\theta c_{40}k^{\frac{3}{2}}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}}),\end{aligned} \tag{8.13}$$

$$\begin{aligned}(\det(D(h))')^{\frac{1}{2}} &=(\det D')^{\frac{1}{2}}\exp(\theta c_{20}k\tau\|h\|)\\ &=(\det D')^{\frac{1}{2}}\exp(\theta c_{40}k^{\frac{3}{2}}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}}).\end{aligned} \tag{8.14}$$

Here $(\sigma'(h'))^2$ is the minimal eigenvalue of the matrix $(D(h))'$. Since $\det D'=\sigma^{-2}\det D$, it follows from (8.6), (8.13), (8.14) that

$$\beta_{k,k}=\sigma^{-2}\exp(\theta ck\tau\|h\|)=\sigma^{-2}\exp(4\theta c_{40}k^{\frac{3}{2}}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}}). \tag{8.15}$$

Let us return to the formula (8.5). Taking into account that $\mathbf{E}\bar{S}'_{h'}=x'$ and using Lemma 7.1 we obtain

$$p'_{h'}(x')=(2\pi)^{-(k-1)/2}(\det(D(h))')^{-\frac{1}{2}}\exp\left\{\theta\left(c_{31}\tau+\frac{c_{32}\tau k^{\frac{3}{2}}}{\sigma'(h')}\right)\right\}. \tag{8.16}$$

To calculate $p_h(x+a)$ for $|x_k| \leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$ we also apply Lemma 7.1. Let us show that the validity of its conditions can be ensured by choosing c_{34}, c_{39} to be small enough and c_{35} to be large enough.

It is clear that the mean value of the probability measure with the density $p_h(x+a)$ is equal to $x - x^{(k)}$ where $x^{(k)} = (0, \ldots, x_k)$, $x \in \mathbb{R}^k$. By a suitable choice of c_{35} and by (8.15) we obtain for $|x_k| \leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$

$$\|(D(h))^{-\frac{1}{2}} x^{(k)}\| = \beta_{k,k}^{\frac{1}{2}} |x_k| \leqq 2|x_k|\sigma^{-1} \leqq 8|\ln\tau|^{\frac{1}{2}}. \tag{8.17}$$

By (8.11) and by the choice of c_{39} it may be ensured that $2\sigma^2(h) \geqq \sigma^2$. Therefore $\sigma^2(h) \geqq \sigma^2/2 \geqq 2kb^2$ and taking sufficiently large c_{35} we find that

$$\tau(\sigma(h))^{-1} \|(D(h))^{-\frac{1}{2}} x^{(k)}\| \leqq 8\sqrt{2}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}} \leqq c_{30}/k.$$

Finally, by the choice of c_{34}, c_{35} we get $\tau \leqq c_{28}/k$, $b^2 \geqq c_{29} k^3 \tau^2 |\ln\tau|$. Now we can apply Lemma 7.1 to conjugate distribution. So, for $|x_k| \leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$ we obtain

$$\begin{aligned} p_h(x+a) = {} & (2\pi)^{-k/2} (\det D(h))^{\frac{1}{2}} \exp\{-\tfrac{1}{2}\|(D(h))^{-\frac{1}{2}} x^{(k)}\|^2 \\ & + \theta(c_{31}\tau + c_{32}\tau(\sigma(h))^{-1}(k^{\frac{3}{2}} + \|(D(h))^{-\frac{1}{2}} x^{(k)}\|^3))\}. \end{aligned} \tag{8.18}$$

It follows from (8.15), (8.17) that

$$\|(D(h))^{-\frac{1}{2}} x^{(k)}\|^2 = \beta_{k,k} x_k^2 = x_k^2 \sigma^{-2}(1 + \theta c k^{\frac{3}{2}} \sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}}) \tag{8.19}$$

(we use again the possibility to choose c_{35} large enough). From (8.5), (8.16), (8.18), (8.11)-(8.14), (8.17), (8.19), (5.7) we get

$$\begin{aligned} v(x_k) &= p_h(x+a)/p'_{h'}(x') = (2\pi)^{-\frac{1}{2}}\sigma^{-1} \exp\Bigg\{2\theta c_{40} k^{\frac{3}{2}}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}} \\ &\quad - \frac{x_k^2}{2\sigma^2}(1 + \theta c k^{\frac{3}{2}}\sigma^{-1}\tau|\ln\tau|^{\frac{1}{2}}) + 2\theta c_{31}\tau \\ &\quad + 2\theta c_{32}\tau\sigma^{-1}\exp(\theta c_{21}\tau\|h\|)\left(k^{\frac{3}{2}} + 32\frac{x_k^2}{\sigma^2}|\ln\tau|^{\frac{1}{2}}\right)\Bigg\} \\ &= \frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\exp\left\{-\frac{x_k^2}{2\sigma^2} + \theta\left(c_{36}\tau + c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\left(1 + \frac{x_k^2}{\sigma^2}\right)\right)\right\} \end{aligned}$$

for $|x_k| \leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$. This completes the proof.

Lemma 8.2. *There exist* $c_{41}, \ldots, c_{44}$ *such that for* $\tau \leqq c_{41}/k$, $b^2 \geqq c_{42} k^3 \tau^2 |\ln\tau|$, $\sigma^2 \geqq 4kb^2$, $\varepsilon = c_{43} k^{\frac{3}{2}}\tau|\ln\tau|$, $z = 2\sigma|\ln\tau|^{\frac{1}{2}}$, $|(D')^{-\frac{1}{2}}x'| \leqq 2|\ln\tau|^{\frac{1}{2}}$ *the inequality* $\varepsilon \leqq 2\sigma|\ln\tau|^{\frac{1}{2}}$ *holds and* $U_k\{X\} \leqq W\{X^{\varepsilon}\} + c_{44}\tau$ *for any closed set* $X \subset [-z, z]$.

Proof. For $c_{41} \leqq c_{34}$, $c_{42} \geqq c_{35}$ the conditions of Lemma 8.1 are satisfied, therefore (8.2) is valid. Put $\varepsilon_1 = 16k^{\frac{3}{2}}c_{37}\tau|\ln\tau|$. Then for $x_k \geqq 0$, $2\sigma \leqq x_k + \varepsilon_1 \leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$ by (8.2) we obtain

$$\begin{aligned} v(x_k + \varepsilon_1) &\leqq w(x_k + \varepsilon_1)\exp\left\{c_{36}\tau + 2c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\frac{(x_k+\varepsilon_1)^2}{\sigma^2}\right\} \\ &\leqq w(x_k)\exp\left\{-\frac{(x_k+\varepsilon_1)}{2\sigma^2} + 2c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\frac{(x_k+\varepsilon_1)^2}{\sigma^2} + c_{36}\tau\right\} \\ &\leqq w(x_k)e^{c_{36}\tau} \end{aligned} \tag{8.20}$$

and, similarly,

$$w(x_k+\varepsilon_1)=w(x_k)\exp\left\{-\frac{2x_k\varepsilon_1+\varepsilon_1^2}{2\sigma^2}\right\}$$
$$\leqq v(x_k)\exp\left\{-\frac{(x_k+\varepsilon_1)\varepsilon_1}{2\sigma^2}+2c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\frac{(x_k+\varepsilon_1)^2}{\sigma^2}\right.$$
$$\left.+c_{36}\tau\right\}\leqq v(x_k)e^{c_{36}\tau}. \tag{8.21}$$

In exactly the same way it may be proved that for $x_k<0$, $-4\sigma|\ln\tau|^{\frac{1}{2}}\leqq x_k-\varepsilon_1\leqq -2\sigma$ the inequalities

$$v(x_k-\varepsilon_1)\leqq w(x_k)e^{c_{36}\tau}, \qquad w(x_k-\varepsilon_1)\leqq v(x_k)e^{c_{36}\tau} \tag{8.22}$$

are true. It is also clear that by choosing small enough c_{41} and sufficiently large c_{42} it is possible to show with the help of (8.2) that

$$c_{36}\tau+c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\left(1+\frac{x_k^2}{\sigma^2}\right)\leqq 1+\frac{x_k^2}{4\sigma^2},$$
$$v(x_k)\leqq\frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\exp\left(-\frac{x_k^2}{2\sigma^2}+1+\frac{x_k^2}{4\sigma^2}\right)$$
$$=\frac{1}{(2\pi)^{\frac{1}{2}}\sigma}\exp\left(1-\frac{x_k^2}{4\sigma^2}\right) \tag{8.23}$$

for $|x_k|\leqq 4\sigma|\ln\tau|^{\frac{1}{2}}$, and

$$v(x_k)\leqq w(x_k)\exp\left(c_{36}\tau+5c_{37}\frac{k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}}{\sigma}\right)\leqq\tfrac{6}{5}w(x_k), \tag{8.24}$$

$$v(x_k)\geqq\tfrac{18}{19}w(x_k) \tag{8.25}$$

for $|x_k|\leqq 2\sigma$.

Let now $\varepsilon_2=2e^2\varepsilon_1$, $\varepsilon_3=c_{38}k^{\frac{3}{2}}\tau|\ln\tau|$, $c_{43}=32e^2c_{37}+c_{38}$, $\varepsilon=\varepsilon_2+\varepsilon_3 =c_{43}k^{\frac{3}{2}}\tau|\ln\tau|$. It can be easily seen that for large c_{42}

$$\varepsilon\leqq 2\sigma|\ln\tau|^{\frac{1}{2}}. \tag{8.26}$$

Let X be an arbitrary closed set contained in the closed interval $[-z,z]$. Let us consider the collection $\{\Pi_\alpha\}_{\alpha\in A}$ of open intervals $\Pi_\alpha\subset\mathbb{R}^1\setminus X$ such that the Lebesgue measure of each Π_α is at least 2ε. Denote $Y=\mathbb{R}^1\setminus\bigcup_\alpha\Pi_\alpha$. Then

$$X\subset Y, \qquad X^\varepsilon=Y^\varepsilon \tag{8.27}$$

and the set Y may be represented as the union of disjoint closed intervals $M_j\subset[-z,z]$ distances between which are $\geqq 2\varepsilon$. Thus $Y=\bigcup_{j=1}^{l}M_j$, $Y^\varepsilon=\bigcup_{j=1}^{l}M_j^\varepsilon$, $M_{j_1}^\varepsilon\cap M_{j_2}^\varepsilon=\varnothing$ if $j_1\neq j_2$ and, consequently,

$$U_k\{Y\}=\sum_{j=1}^{l}U_k\{M_j\}, \qquad W\{Y^\varepsilon\}=\sum_{j=1}^{l}W\{M_j^\varepsilon\}. \tag{8.28}$$

Note that in view of (8.26)

$$M_j \subset M_j^{\varepsilon_3} \subset M_j^{\varepsilon} \subset [-4\sigma|\ln\tau|^{\frac{1}{2}}, 4\sigma|\ln\tau|^{\frac{1}{2}}] \tag{8.29}$$

for $j=1,\dots,l$. Since $b^2>0$, the distributions U_k and H_k are absolutely continuous, so we deduce by (8.3) that

$$U_k\{M_j\} \leqq H_k\{M_j^{\varepsilon_3}\}, \qquad j=1,\dots,l. \tag{8.30}$$

We shall compare $H_k\{M_j^{\varepsilon_3}\}$ with $W\{M_j^{\varepsilon}\}$. Let $M_j^{\varepsilon_3}=(a_j,b_j)$. Then $M_j^{\varepsilon} =(M_j^{\varepsilon_3})^{\varepsilon_2}=(a_j-\varepsilon_2, b_j+\varepsilon_2)$. Fix j and consider separately four possible cases:

a) $(a_j,b_j)\cap[-2\sigma,2\sigma]=\varnothing$ and $0\notin(a_j-\varepsilon_2, b_j+\varepsilon_2)$,

b) $0\in(a_j,b_j)$ and $[-2\sigma,2\sigma]\subset(a_j-\varepsilon_2, b_j+\varepsilon_2)$,

c) at least one of the intervals $(a_j-\varepsilon_2, a_j)$ or $(b_j, b_j+\varepsilon_2)$ is contained in the segment $[-2\sigma,2\sigma]$,

d) one of the intervals $(a_j-\varepsilon_2, a_j)$ or $(b_j, b_j+\varepsilon_2)$ contains at least one of the intervals $(0,2\sigma)$ or $(-2\sigma,0)$.

In the case a) let us suppose for example that $0<a_j-\varepsilon_2$, $a_j>2\sigma$. Then by (8.20), (8.29) and since $\varepsilon_2>\varepsilon_1$ we have

$$\begin{aligned} H_k\{M_j^{\varepsilon_3}\} &= \int_{a_j-\varepsilon_1}^{b_j-\varepsilon_1} v(x_k+\varepsilon_1)\,dx_k \\ &\leqq \int_{a_j-\varepsilon_1}^{b_j+\varepsilon_1} w(x_k)e^{c_{36}\tau}dx_k \leqq e^{c_{36}\tau} W\{M_j^{\varepsilon}\}. \end{aligned} \tag{8.31}$$

If $b_j+\varepsilon_2<0$, $b_j<-2\sigma$, then the inequality (8.31) may be obtained in exactly the same way by using (8.22).

Consider the case b). According to Bernstein's inequality

$$W\{(z,\infty)\}=W\{(-\infty,-z)\} \leqq \exp\left(-\frac{z^2}{4\sigma^2}\right)=\tau. \tag{8.32}$$

Thus if $b_j+\varepsilon_2>z$ then $W\{(b_j+\varepsilon_2,\infty)\}\leqq\tau$. Provided that $b_j+\varepsilon_2\leqq z$, since $b_j>0$, $b_j+\varepsilon_2>2\sigma$, $\varepsilon_2>\varepsilon_1$ and by (8.21), (8.32) we get that

$$\begin{aligned} W\{(b_j+\varepsilon_2,\infty)\} &\leqq \tau+\int_{b_j+\varepsilon_2}^{z} w(x_k)\,dx_k = \tau+\int_{b_j+\varepsilon_2-\varepsilon_1}^{z-\varepsilon_1} w(x_k+\varepsilon_1)\,dx_k \\ &\leqq \tau+\int_{b_j}^{z-\varepsilon_1} v(x_k)e^{c_{36}\tau}dx_k \leqq \tau+H_k\{(b_j,\infty)\}e^{c_{36}\tau}. \end{aligned}$$

Hence

$$W\{(b_j+\varepsilon_2,\infty)\} \leqq \tau+e^{c_{36}\tau}H_k\{(b_j,\infty)\}. \tag{8.33}$$

Similarly, with the help of (8.22) one proves that

$$W\{(-\infty,a_j-\varepsilon_2)\} \leqq \tau+e^{c_{36}\tau}H_k\{(-\infty,a_j)\}. \tag{8.34}$$

It follows from (8.33), (8.34) that $1-W\{M_j^{\varepsilon}\}\leqq 2\tau+e^{c_{36}\tau}(1-H_k\{M_j^{\varepsilon_3}\})$ so that

$$H_k\{M_j^{\varepsilon_3}\} \leqq (2+c_{36})\tau+W\{M_j^{\varepsilon}\}. \tag{8.35}$$

To consider the case c) we introduce the following notations:

$$N_j = M_j^{\varepsilon_3} \cap [-2\sigma, 2\sigma],$$

$$K_j = (M_j^{\varepsilon_3} \setminus N_j) \cap ([-2\sigma - \varepsilon_1, -2\sigma] \cup [2\sigma, 2\sigma + \varepsilon_1]),$$

$$T_j = M_j^{\varepsilon_3} \setminus (N_j \cup K_j).$$

Choosing c_{42} large enough we obtain the inequality

$$\exp(5c_{37}k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}\sigma^{-1}) \leqq 1 + 10c_{37}k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}\sigma^{-1}. \tag{8.36}$$

It follows from (8.24), (8.36) that if c_{41} is sufficiently small then

$$\begin{aligned} H_k\{N_j\} &= \int_{N_j} v(x_k)\,dx_k \\ &\leqq \int_{N_j} w(x_k) \exp\{c_{36}\tau + 5c_{37}k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}\sigma^{-1}\}\,dx_k \\ &\leqq W\{N_j\}e^{c_{36}\tau} + 10c_{37}k^{\frac{3}{2}}\tau|\ln\tau|^{\frac{1}{2}}\sigma^{-1}e^{c_{36}\tau} \\ &\leqq e^{c_{36}\tau}W\{N_j\} + \frac{\varepsilon_1}{(2\pi)^{\frac{1}{2}}\sigma}. \end{aligned} \tag{8.37}$$

Further by the condition c), by the definition of K_j and by (8.23), (8.29) we get

$$H_k\{K_j\} \leqq \int_{K_j} \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} \exp\left(1 - \frac{x_k^2}{4\sigma^2}\right) dx_k \leqq \frac{\varepsilon_1}{(2\pi)^{\frac{1}{2}}\sigma}. \tag{8.38}$$

If the set T_j is non-empty then it is entirely lying either on the positive semi-axis or on the negative one. Let $T_j \subset \{x_k\colon x_k \geqq 2\sigma + \varepsilon_1\}$. Then $T_j - \varepsilon_1 \subset T_j \cup K_j$ and by (8.20), (8.29) we obtain

$$\begin{aligned} H_k\{T_j\} &= \int_{T_j - \varepsilon_1} v(x_k + \varepsilon_1)\,dx_k \leqq e^{c_{36}\tau} \int_{T_j - \varepsilon_1} w(x_k)\,dx_k \\ &\leqq e^{c_{36}\tau} W\{T_j \cup K_j\}. \end{aligned} \tag{8.39}$$

When $T_j \subset \{x_k\colon x_k \leqq -2\sigma - \varepsilon_1\}$, (8.39) is established in a similar way. It is also clear that in the case c)

$$W\{M_j^{\varepsilon} \setminus M_j^{\varepsilon_3}\} \geqq \frac{\varepsilon_2 e^{-2}}{(2\pi)^{\frac{1}{2}}\sigma} = \frac{2\varepsilon_1}{(2\pi)^{\frac{1}{2}}\sigma}. \tag{8.40}$$

Now from (8.37)–(8.40) it follows that

$$\begin{aligned} H_k\{M_j^{\varepsilon_3}\} &= H_k\{N_j \cup K_j \cup T_j\} = H_k\{N_j\} + H_k\{K_j\} + H_k\{T_j\} \\ &\leqq e^{c_{36}\tau}W\{N_j\} + \frac{2\varepsilon_1}{(2\pi)^{\frac{1}{2}}\sigma} + e^{c_{36}\tau}W\{K_j \cup T_j\} \\ &\leqq e^{c_{36}\tau}W\{M_j^{\varepsilon_3}\} + W\{M_j^{\varepsilon} \setminus M_j^{\varepsilon_3}\} \leqq e^{c_{36}\tau}W\{M_j^{\varepsilon}\}. \end{aligned} \tag{8.41}$$

Finally, let us consider the case d). In this case we have

$$W\{M_j^{\varepsilon} \setminus M_j^{\varepsilon_3}\} \geqq (2\pi)^{-\frac{1}{2}} \int_0^2 e^{-x^2/2}\,dx > 0.475,$$

$$W\{M_j^{\varepsilon_3} \cap [-2\sigma, 2\sigma]\} < 0.5$$

and, by (8.25),

$$H_k\{(-2\sigma,2\sigma)\}\geqq\tfrac{18}{19}(2\pi)^{-\frac{1}{2}}\int_{-2}^{2}e^{-x^2/2}dx\geqq\tfrac{18}{19}\cdot 0.95=0.9.$$

Therefore, using (8.24) one obtains

$$\begin{aligned}H_k\{M_j^{\varepsilon_3}\}&\leqq H_k\{\mathbb{R}^1\setminus[-2\sigma,2\sigma]\}+H_k\{M_j^{\varepsilon_3}\cap[-2\sigma,2\sigma]\}\\&\leqq 0.1+1.2\,W\{M_j^{\varepsilon_3}\cap[-2\sigma,2\sigma]\}\leqq 0.2+W\{M_j^{\varepsilon_3}\}\\&<W\{M_j^{\varepsilon}\setminus M_j^{\varepsilon_3}\}+W\{M_j^{\varepsilon_3}\}=W\{M_j^{\varepsilon}\}.\end{aligned}\tag{8.42}$$

Thus, according to (8.31), (8.41), (8.42) in the cases a), c) and d) it is proved that

$$H_k\{M_j^{\varepsilon_3}\}\leqq e^{c_{36}\tau}W\{M_j^{\varepsilon}\}.\tag{8.43}$$

In the case b) we have only the inequality (8.35). But only one of the intervals $M_j^{\varepsilon_3}$ may contain zero. Therefore, choosing c_{41} to be small enough we obtain from (8.27), (8.28), (8.30), (8.35), (8.43) that

$$\begin{aligned}U_k\{X\}\leqq U_k\{Y\}&=\sum_{j=1}^{l}U_k\{M_j\}\leqq\sum_{j=1}^{l}H_k\{M_j^{\varepsilon_3}\}\\&\leqq e^{c_{36}\tau}\sum_{j=1}^{l}W\{M_j^{\varepsilon}\}+c\tau=e^{c_{36}\tau}W\{Y^{\varepsilon}\}+c\tau\\&=e^{c_{36}\tau}W\{X^{\varepsilon}\}+c\tau\leqq W\{X^{\varepsilon}\}+c_{44}\tau.\end{aligned}$$

9. Proof of Theorem 1.1

In following Lemmas 9.1–9.3 we suppose the assumptions and the notations introduced in Sects. 5–8 to be valid. Set now $\tilde{F}=FG$, $\tilde{\Phi}=\Phi G$, $\tilde{F}'=\mathscr{L}(S')$, $\tilde{\Phi}'=\mathscr{L}(R')$,

$$P=\{x\in\mathbb{R}^k:\ |D^{-\frac{1}{2}}x|\leqq 2|\ln\tau|^{\frac{1}{2}}\}.$$

Lemma 9.1. *For* $\tau\leqq c_{41}/k$, $b^2\geqq c_{42}k^3\tau^2|\ln\tau|$, $\sigma^2\geqq 4kb^2$, $\varepsilon=c_{43}k^{\frac{3}{2}}\tau|\ln\tau|$ *and for any closed set* $X\subset P$ *the inequality*

$$\tilde{F}\{X\}\leqq\tilde{\Phi}\{X^{(\varepsilon)}\}+c_{44}k\tau$$

is valid.

Proof. It will be carried out by the induction on k (see [19]). For $k=1$ the statement of the lemma coincides with the assertion of Lemma 8.2 since in this case $U_k=\tilde{F}$, $W=\tilde{\Phi}$, $P=[-z,z]$. Let us suppose the assertion of Lemma 9.1 to be valid in $(k-1)$-dimensional case and let us prove it for k-dimensional situation where $k\geqq 2$.

Let X be an arbitrary closed set contained in the parallelepiped P. Let Y be the set $\{x\in\mathbb{R}^k:\ \exists\,y\in X:\ x'=y',\ |x_k-y_k|<\varepsilon\}$, $X_{x'}=\{x_k\in\mathbb{R}^1:\ x=(x',x_k)\in X\}$ be a one-dimensional section of X given by fixing the first $k-1$ coordinates, $Y_{x_k}=\{x'\in\mathbb{R}^{k-1}:\ x=(x',x_k)\in Y\}$ be the $(k-1)$-dimensional section of Y given by fixing the last coordinate.

Using diagonal character of a covariance matrix D and taking (8.1) into account it is not difficult to check that the distribution $\tilde{F}'=\mathscr{L}(S')$ satisfies the same conditions which in k-dimensional situation are satisfied for the distribution $\tilde{F}=\mathscr{L}(S)$. Therefore, by the induction hypothesis, for every set $X'\subset\{x'\in\mathbb{R}^{k-1}: |(D')^{-\frac{1}{2}}x'|\leqq 2|\ln\tau|^{\frac{1}{2}}\}$ the inequality $\tilde{F}'\{X'\}\leqq\tilde{\Phi}'\{(X')^{(\varepsilon)}\}+c_{44}(k-1)\tau$ holds. In particular,

$$\tilde{F}'\{Y_{x_k}\}\leqq\tilde{\Phi}'\{Y_{x_k}^{(\varepsilon)}\}+c_{44}(k-1)\tau$$

for any $x_k\in\mathbb{R}^1$. On the other hand, if $|D^{-\frac{1}{2}}x|\leqq 2|\ln\tau|^{\frac{1}{2}}$ we have $|(D')^{-\frac{1}{2}}x'|\leqq 2|\ln\tau|^{\frac{1}{2}}$, $|x_k|\leqq z=2\sigma|\ln\tau|^{\frac{1}{2}}$, hence by Lemma 8.2 the inequality $U_{k,x'}\{X_{x'}\}\leqq W\{X_{x'}^{\varepsilon}\}+c_{44}\tau$ is valid. Note that

$$\bigcup_{x'}(\{x'\}\otimes X_{x'}^{\varepsilon})=Y,\qquad \bigcup_{x_k}(Y_{x_k}^{(\varepsilon)}\otimes\{x_k\})=X^{(\varepsilon)}$$

(here the sign $\otimes$ is used to denote a direct product of sets). Finally, $q(x)=w(x_k)q'(x')$ where $q'(x')$ is the density of the distribution $\mathscr{L}(R')=\tilde{\Phi}'$.

Taking above mentioned into account and using Fubini's theorem we obtain

$$\begin{aligned}\tilde{F}\{X\}&=\int_{\mathbb{R}^{k-1}}U_{k,x'}\{X_{x'}\}p'(x')dx'\\ &\leqq\int_{\mathbb{R}^{k-1}}W\{X_{x'}^{\varepsilon}\}p'(x')dx'+c_{44}\tau\\ &=\int_{-\infty}^{\infty}\tilde{F}'\{Y_{x_k}\}w(x_k)dx+c_{44}\tau\\ &\leqq\int_{-\infty}^{\infty}\tilde{\Phi}'\{Y_{x_k}^{(\varepsilon)}\}w(x_k)dx_k+c_{44}k\tau\\ &=\int_{X^{(\varepsilon)}}q(x)dx+c_{44}k\tau=\tilde{\Phi}\{X^{(\varepsilon)}\}+c_{44}k\tau.\end{aligned}$$

This completes the proof.

Lemma 9.2. *There exist c_{45}, c_{46}, c_{47} such that for $\tau\leqq c_{41}/k$, $b^2\geqq c_{45}k^3\tau^2|\ln\tau|$, $\sigma^2\geqq 4kb^2$, $\varepsilon=c_{43}k^{\frac{3}{2}}\tau|\ln\tau|$ for any closed set $X\subset\mathbb{R}^k$ the inequality*

$$\tilde{F}\{X\}\leqq\tilde{\Phi}\{X^{(\varepsilon)}\}+c_{46}k\tau \tag{9.1}$$

holds and, consequently,

$$\pi(\tilde{F},\tilde{\Phi})\leqq c_{47}k^2\tau|\ln\tau|. \tag{9.2}$$

Proof. Choose c_{45} so that $b^2\geqq c_{42}k^3\tau^2|\ln\tau|$ and $\sigma>2\tau\ |\ln\tau|^{\frac{1}{2}}$. Hence Lemma 9.1 conditions are satisfied and, in view of (8.1),

$$2\sigma_j|\ln\tau|^{\frac{1}{2}}\leqq\sigma_j^2/\tau \tag{9.3}$$

for $j=1,\dots,k$.

Let $X\subset\mathbb{R}^k$ be an arbitrary closed set. By Lemma 9.1,

$$\tilde{F}\{X\cap P\}\leqq\tilde{\Phi}\{X^{(\varepsilon)}\}+c_{44}k\tau. \tag{9.4}$$

Set $y_j = 2\sigma_j |\ln \tau|^{\frac{1}{2}}$ for $j=1, \dots, k$. By using (9.3) and Bernstein's inequality we obtain

$$\tilde{F}\{X \setminus P\} \leqq \tilde{F}\{\mathbb{R}^k \setminus P\} \leqq \sum_{j=1}^{k} \mathbf{P}\{|S_j| > y_j\}$$
$$\leqq 2 \sum_{j=1}^{k} \exp(-y_j^2/4\sigma_j^2) = 2k\tau. \tag{9.5}$$

Now (9.4), (9.5) imply (9.1) with $c_{46} = c_{44} + 2$.

Obviously, $X^{(\varepsilon)} \subset X^{\varepsilon k^{\frac{1}{2}}}$. Therefore, it follows from (9.1) that $\tilde{F}\{X\} \leqq \tilde{\Phi}\{X^{\varepsilon k^{\frac{1}{2}}}\} + c_{46} k\tau$. In view of (1.2) this means that

$$\pi(\tilde{F}, \tilde{\Phi}) \leqq \max\{\varepsilon k^{\frac{1}{2}}, c_{46} k\tau\} \leqq c_{47} k^2 \tau |\ln \tau|.$$

Proof of Theorem 1.1. It is clear that throughout the proof of (1.4) we can assume $\tau \leqq c_{41}/k$, $\tau \leqq e^{-1}$. Since the Lévy-Prohorov distance is invariant with respect to unitary transformations of $\mathbb{R}^k$, we can suppose, without loss of generality, that the covariance matrix of F is diagonal and its eigenvalues are ordered so that they are non-increasing. We use at once two smoothing distributions: G_0 and G with the covariance matrices $d^2 I$ and $b^2 I$ respectively where $b^2 = c_{45} k^3 \tau^2 |\ln \tau|$, $d^2 = 4kb^2$. Then, by the weak regularity of the Lévy-Prohorov distance, we obtain

$$\pi(F, \Phi) \leqq \pi(F, FG_0) + \pi(FG_0, FG_0 G)$$
$$+ \pi(FG_0 G, \Phi G_0 G) + \pi(\Phi G_0 G, \Phi G_0) + \pi(\Phi G_0, \Phi)$$
$$\leqq \pi(FG_0 G, \Phi G_0 G) + 2\pi(G_0, E) + 2\pi(G, E). \tag{9.6}$$

It can be easily seen that the probability measure FG_0 satisfies all conditions which were imposed on F in Sects. 5–9 (see beginning of Sect. 5). Moreover, the smallest eigenvalue of its covariance operator is at least $4kb^2$. Therefore, for $\tilde{F} = (FG_0)G$, $\tilde{\Phi} = (\Phi G_0)G$ all conditions of Lemma 9.2 are satisfied. Hence

$$\pi((FG_0)G, (\Phi G_0)G) \leqq c_{47} k^2 \tau |\ln \tau|. \tag{9.7}$$

Further, putting $\delta^2 = 4kd^2 |\ln \tau| = 16k^5 c_{45} \tau^2 |\ln \tau|^2$, we obtain that

$$G_0\{\mathbb{R}^k \setminus \{x : \|x\| \leqq \delta\}\} \leqq k \int_{|y| \leqq \delta k^{-\frac{1}{2}}} (2\pi)^{-\frac{1}{2}} d^{-1} \exp(-y^2/2d^2) dy$$
$$\leqq 2k \exp(-\delta^2/4kd^2) = 2k\tau$$

and, consequently,

$$\pi(G_0, E) \leqq \max\{\delta, G_0\{\mathbb{R}^k \setminus \{x : \|x\| \leqq \delta\}\}\} \leqq c k^{\frac{5}{2}} \tau |\ln \tau|. \tag{9.8}$$

Similarly it can be proved that

$$\pi(G, E) \leqq c k^2 \tau |\ln \tau|. \tag{9.9}$$

The derivation of (1.5) from (1.4) essentially repeats the arguments used in [20, 21] to prove a one-dimensional version of (1.5). We shall show that for every

$X \in \mathfrak{B}_k$ and for all $\lambda > 0$

$$F\{X\} \leqq \Phi\{X^\lambda\} + c_2 k^{\frac{5}{2}} \exp\left(-\frac{\lambda}{c_3 k^{\frac{5}{2}} \tau}\right),$$
$$\Phi\{X\} \leqq F\{X^\lambda\} + c_2 k^{\frac{5}{2}} \exp\left(-\frac{\lambda}{c_3 k^{\frac{5}{2}} \tau}\right). \tag{9.10}$$

Consider the random vectors $\delta\xi_1, \ldots, \delta\xi_n$ where $\delta > 0$. It is clear that $\mathscr{L}(\delta\xi_i) \in \mathscr{B}_1(\delta\tau)$ for $i = 1, \ldots, n$ since $\mathscr{L}(\xi_i) \in \mathscr{B}_1(\tau)$. Denote by Φ_δ a zero mean Gaussian distribution whose covariance operator coincides with that of $\mathscr{L}(\delta S)$. It follows from (1.4) that $\pi(\mathscr{L}(\delta S), \Phi_\delta) \leqq c_1 k^{\frac{5}{2}} \delta\tau(|\ln \delta\tau| + 1)$. Setting $\varepsilon = \varepsilon(k, \tau, \delta) = 2c_1 k^{\frac{5}{2}} \delta\tau(|\ln \delta\tau| + 1)$, we obtain that for any $X \in \mathfrak{B}_k$

$$F\{\delta^{-1} X\} = \mathscr{L}(\delta S)\{X\} \leqq \Phi_\delta\{X^\varepsilon\} + \varepsilon = \Phi\{(\delta^{-1} X)^{\varepsilon/\delta}\} + \varepsilon.$$

When X runs over all Borel sets, the same occurs with $\delta^{-1} X$. Therefore,

$$F\{X\} \leqq \Phi\{X^{\varepsilon/\delta}\} + \varepsilon \tag{9.11}$$

for any $X \in \mathfrak{B}_k$. The function $\beta_{k,\tau}(\delta) = 2c_1 k^{\frac{5}{2}}(|\ln \delta\tau| + 1)$ is continuous and decreasing when $0 < \delta \leqq \tau^{-1}$. Since $\beta_{k,\tau}(\tau^{-1}) = 2c_1 k^{\frac{5}{2}}$, for $y \geqq 2c_1 k^{\frac{5}{2}}$ we can define the inverse function $\beta_{k,\tau}^{-1}(y) = \tau^{-1} \exp(1 - y/2c_1 k^{\frac{5}{2}})$. Proving (1.5) we can assume $\lambda/\tau \geqq 2c_1 k^{\frac{5}{2}}$. Put $\delta = \beta_{k,\tau}^{-1}(\lambda/\tau)$. Then

$$\lambda = \tau\beta_{k,\tau}(\delta) = 2c_1 k^{\frac{5}{2}} \tau(|\ln \delta\tau| + 1) = \delta^{-1} \varepsilon(k, \tau, \delta), \tag{9.12}$$

hence

$$\varepsilon(k, \tau, \delta) = \lambda\delta = \lambda\beta_{k,\tau}^{-1}(\lambda/\tau)$$
$$= \frac{\lambda}{\tau} \exp\left(1 - \frac{\lambda}{2c_1 k^{\frac{5}{2}} \tau}\right) \leqq c_{48} k^{\frac{5}{2}} \exp\left(-\frac{\lambda}{4c_1 k^{\frac{5}{2}} \tau}\right). \tag{9.13}$$

Now (9.11)-(9.13) imply the first of the inequalities (9.10). The second one is proved in a similar way.

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