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Probability Theory and Related Fields

Furstenberg's Theorem for Nonlinear Stochastic Systems

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Summary. We extend Furstenberg's theorem to the case of an i.i.d. random composition of incompressible diffeomorphisms of a compact manifold M. The original theorem applies to linear maps $\{X_i\}_{i \in \mathbb{N}}$ on $\mathbb{R}^m$ with determinant 1, and says that the highest Lyapunov exponent

$$\beta \equiv \lim_{n \to \infty} \frac{1}{n} \|X_n \circ \ldots \circ X_1\|$$

is strictly positive unless there is a probability measure on the projective $(m-1)$-space which is a.s. invariant under the action of X_i. Our extension refers to a probability measure on the projective bundle over M.

We show that when our diffeomorphism is the flow of a stochastic differential equation, the criterion for $\beta > 0$ is ensured by a Lie algebra condition on the induced system on the principal bundle over M.

1. Aims

In [6], Sect. 8, Furstenberg studies the product $X^n = X_n \circ \ldots \circ X_1$, where $\{X_i\}_{i \in N}$ is a random i.d.d. sequence of elements of $SL(\mathbb{R}^m)$. He shows (proof of Theorem 8.6) that the highest Lyapunov exponent α for this system is strictly positive unless there is a probability measure on the projective $(m-1)$-space P^{m-1}, which is preserved a.s. by the induced action of the system. (The definition and existence of the Lyapunov spectrum is given by the Oseledec (multiplicative ergodic) theorem, see [11, 13]. In fact Furstenberg's paper [6] predates the Oseledec theorem, and he takes α to be the a.s. value of $\lim_{n \to \infty} \frac{1}{n} \log \|X^n\|_{\mathrm{OP}}$ (operator norm). It is easy to see that the largest Lyapunov exponent is given by this expression.)

Our aim in this paper is to extend Furstenberg's result to the case of a stochastic volume preserving diffeomorphism of a C^1 compact Riemannian manifold M. More precisely, we take a random diffeomorphism ξ of M which preserves the (normalised) Riemannian volume ρ a.s. (i.e. $\xi\rho=\rho$ for a.e. ξ) and we work with the product $\xi^n=\xi_n\circ\xi_{n-1}\circ\ldots\circ\xi_1$, where $\{\xi_i\}_{i\in N}$ is a random i.i.d. sequence of diffeomorphisms of M, each distributed like ξ. We require that ξ be C^1 smooth a.s.: let us denote by $(C^1(M,M), \mathscr{F}, P)$ the probability space which gives the distribution of ξ. We also require that the compact open sets for the derivative (i.e., the sets $\{\xi\in C^1(M,M)\colon T\xi(K)\in U\}$, where K is compact and U is open in the tangent bundle TM over M) be P-measurable. This ensures that we can define the Markov process on M with transition probabilities $p(x,B)=P\{\xi\colon \xi(x)\in B\}$ ($x\in M$, B Borel in M), and a corresponding Markov process on TM. From the fact that $\xi\rho=\rho$ a.s. it is easy to deduce that ρ is stationary for this associated Markov process on M, i.e. $\rho(B)=\int\rho(x,B)d\rho(x)$ (B Borel in M). We must assume that ρ is actually ergodic for the Markov process. Finally, we impose the regularity condition

$$\int E[\log^+\|T_x\xi^{\pm1}\|_{\mathrm{OP}}]\,d\rho(x)<\infty.$$

(Here by $T_x\xi$ we mean the derivative $T\xi\colon TM\to TM$, restricted to the tangent space T_xM. Thus, $T_x\xi$ is a linear map $T_xM\to T_{\xi(x)}M$.)

In [2], Theorem 2.1 we give the definition and existence of the Lyapunov spectrum for ξ^n when ρ is a general stationary probability on M for the associated Markov process. (We actually discuss the case when ξ^n arises as the flow of a stochastic dynamical system in the sense of [5, 4], but the proof of Theorem 2.1 works for the system described here.)

The condition $\xi\rho=\rho$ a.s. is equivalent to the linear map $T_x\xi\colon T_xM\to T_{\xi(x)}M$ having determinant ±1 for a.s., and this corresponds to Furstenberg's condition $X\in SL(R^m)$.

Our main result is the following:

Theorem 1. *In the situation above, denote by β the largest Lyapunov exponent.*

Then $\beta>0$ inless there is a probability measure v on the projective bundle PM such that $\pi(v)=\rho$ ($\pi\colon PM\to M-$ bundle projection) and $T\xi v=v$ a.s., *where the induced action of $T\xi$ on PM is denoted again by $T\xi$.*

We prove this theorem in Sect. 2.

In Sect. 3 below we discuss the case when ξ_n arises as the flow of a stochastic dynamical system in the sense of [5, 4]. In this case we give a Lie algebra condition related to the induced system on the (special) principal bundle, which ensures the criterion for $\beta>0$ of Theorem 1.

2. Proof of the Main Result

We base this on the following:

Proposition 2. *Let $\{Y_n\colon n\geqq 0\}$ be a Markov chain on the measurable space $(X,\mathscr{X})$ with transition probabilities $Q(x,dy)$, and admitting a stationary ergodic probability m on X. Take a measurable map $A\colon X\to SL(\mathbb{R}^m)$ satisfying $\int\log\|A\|\,dm<\infty$, and put $\beta=\lim_{n\to\infty}1/n\log\|A(Y_{n-1})\ldots A(Y_0)\|$.*

Suppose $\beta=0$. Then there exists an X-measurable family of measures $\{\pi_x\}_x$ on P^{m-1} such that for m – a.e. *x, and given x for $Q(x, -)$* a.e. *y, we have*

$$\pi_y = A(x)\pi_x.$$

Proof. This is a straighforward adaptation of Theorem 2.4 of Royer [12], the difference being that Royer deals with a Markov process on $SL(\mathbb{R}^m)$ itself rather than on X. (n.b.: Virtser [14] has similar results to Royer [12].) □

Proof of Theorem 1. For this we apply Proposition 2 to the situation of Sect. 1, taking $X = C^1(M, M) \times M$ and $m = P \otimes \rho$. To define A identify the tangent spaces $\{T_x M\}_x$ with $\mathbb{R}^m$ in a Borel measurable way, and take $A(\xi, x)$ to correspond to the map $T_x\xi\colon T_x M \to T_{\xi(x)}M$.

Now suppose $\beta=0$. Then we have a measurable collection $\{\pi_{(\xi,x)}\}_{(\xi,x)}$ of measures on P^{m-1} such that for $P\otimes\rho$ - a.e. (ξ, x), and given (ξ, x) for P - a.e. η, we have

$$\pi_{(\eta, \xi(x))} = T_x\xi \cdot \pi_{(\xi, x)}. \tag{$*$}$$

For such (ξ, x) we see from $(*)$ that $\pi_{(\eta,\xi(x))}$ is P - a.s. independent of η, and hence for ρ - a.e. y, $\pi_{(\eta, y)}$ is P - a.s. independent of η. (n.b. the map $(\xi, x) \to \xi(x)$ yields $P \otimes \rho \to \rho$, because ρ is stationary.) For x such that $\pi_{(\xi,x)}$ is P - a.s. independent of ξ, denote its a.s. value by v_x, i.e. $v_x = \int \pi_{(\xi,x)} dP(\xi)$. Then $(*)$ tells us that for such x and P - a.e. ξ, that $\pi_{(\eta,\xi(x))}$ is also a.s. independent of η, and that

$$v_{\xi(x)} = T_x\xi \cdot v_x.$$

Theorem 1 follows from this, taking $v = \rho \otimes \{v_x\}_x$, i.e. taking v to have v_x as its marginals on $P_x M$. □

The author is grateful to an anonymous referee for pointing out that Theorem 1 follows from the work of [12]. An alternative approach, based more directly on [6] is possible. The idea is to take pairs x, y in M and to apply the techniques of [6] to the linear maps $T_x\xi\colon T_x M \to T_y M$, using the probability distribution of $T_x\xi$ conditioned on the event $\xi(x) = y$, so that it is supported on the space of linear maps from $T_x M$ to $T_y M$. Other approaches can be found in [1] and [9].

3. Flows of Stochastic Dynamical Systems

The flow $\xi_t(\omega)$ of a smooth stochastic dynamical system (SDS) (X, z) on a smooth compact Riemannian manifold M of dimension m, is studied in [5, 4], and as we have said, the Lyapunov exponents for this flow are studied in [2, 3]. We take

$$X(x) \circ dz_t = A(x)dt + \sum_{i=1}^{n} Y^i(x) \circ dB_t^i,$$

where each B_t^i is a Brownian motion on $\mathbb{R}$, and $A, Y^1, \ldots, Y^2$ are smooth vector fields on M. The flow $\xi_t(\omega)$ is a diffeomorphism of M for all t a.s., such that for any $x \in M$, $\xi_t(\omega)x$ is the solution to (X, z) starting from x, i.e. such that

$$d(\xi_t(\omega)x) = A(\xi_t(\omega)x)dt + \sum_{i=1}^{n} Y^i(\xi_t(\omega)x) \circ dB_t.$$

(n.b.: In this section the variable ω is an element of the underlying probability space for the SDS, i.e. the Brownian motions.) Intuitively one can think of the flow as that of the vector field $A + \{$Random choice from vector space spanned by $Y^1, \ldots, Y^n\}$, where the choice is governed by the Brownian motion $(B_t^1, \ldots, B_t^n)$.

We will assume that $\xi_t(\omega)$ a.s. preserves the (normalised) Riemannian volume ρ on M: this is equivalent to $\operatorname{div} A \equiv 0$, $\operatorname{div} Y^i \equiv 0$, and to the condition that $T_x \xi_t(\omega)$: $T_x M \to T_{\xi_t(\omega)x} M$ has determinant ± 1 for all t, x a.s.

From (X, z) we will induce an SDS (SLX, z) on the special principal bundle SLM over M. For $x \in M$ the fibre $SL_x M$ of this bundle consists of the linear maps $\underline{u}$: $\mathbb{R}^m \to T_x M$ with determinant ± 1. If we choose an orthonormal (ordered) basis $e_1, \ldots, e_m$ in $\mathbb{R}^m$ then we can identify each $\underline{u} \in SLM$ with the (ordered) basis $(\underline{u}(e_1), \ldots, \underline{u}(e_m))$ in $T_x M$. We define SLX over a chart U for M by

$$\begin{aligned} SLX\colon U \times SL(\mathbb{R}^m) &\to U \times SL(\mathbb{R}^m) \times \mathbb{R}^m \times GL(\mathbb{R}^m) \\ (x, \underline{u}) &\to (x, \underline{u}, X(x), DX(x) \circ \underline{u}). \end{aligned}$$

Using [4], Remark 4.2(b) we see from this definition that the flow $SL\xi_t(\omega)$ of (SLX, z) is given by

$$[SL\xi_t(\omega)\underline{u}]e = T\xi_t(\omega)(\underline{u}(e)).$$

Thus for $\underline{u} \in SL_x M$, $SL\xi_t(\omega)\underline{u}$ incorporates the derivative $T_x \xi_t(\omega)$ in the sense that

$$[SL\xi_t(\omega)\underline{u}]\underline{u}^{-1} = T_x \xi_t(\omega)\colon T_x M \to T_{\xi_t(\omega)_x} M.$$

In Theorem 3.2 we use the following conditions:

E: Span $\{Y^1, \ldots, Y^n$, multiple Lie brackets involving $Y^1, \ldots, Y^n\}$ $(x) = T_x M$ for all $x \in M$.

H: Span $\{SLY^1, \ldots, SLY^n$, multiple Lie brackets involving $SLA, SLY^1, \ldots, SLY^n\}$ $(\underline{u}) = T_{\underline{u}} SLM$ for some $\underline{u} \in SLM$.

The advantage of these conditions is that they can be verified for (X, z) without calculating the flow. Condition E ensures that the measure ρ on M is ergodic for the Markov process associated with $\xi_t(\omega)$, and that the transition probabilities have smooth nonvanishing densities. See [8], Proposition 6.1 and Theorem 3, also [10]. We use condition H to ensure that the transition probabilities associated with $SL\xi_t(\omega)\underline{u}$ have smooth densities.

Note that our condition H is just condition H_1 of [10] applied to (SLX, z), and our condition E implies condition E of [8]. Note also that our condition H can indeed occur: we can find a vector field Z on $\mathbb{R}^m$ to give any $DZ(x)$ we choose, therefore we can find $\{Z^1, \ldots, Z^p\}$ such that $DZ^1(x), \ldots, DZ^p(x)$ span $SL(\mathbb{R}^m)$.

Lemma 3.1. *Assume $m \geqq 2$. Suppose that ν_1, ν_2 are probability measures on P^{m-1}, and μ is a probability measure on $SL(\mathbb{R}^m)$. Suppose $T\nu_1 = \nu_2$ for μ - a.e. $T \in SL(\mathbb{R}^m)$. where the action of T on P^{m-1} is denoted again by T.*

Then μ cannot have C^0 density.

Proof. The set $\{T\colon T\nu_1 = \nu_2\}$ is closed in $SL(\mathbb{R}^m)$, therefore if $T\nu_1 = \nu_2$ μ - a.s., then $\{T\colon T\nu_1 = \nu_2\} \supset \operatorname{Supp}(\mu)$. Suppose μ has a C^0 density. Then $\operatorname{Supp}(\mu)$ has an interior. Take S in this interior, and a neighbourhood U of S with $U \subset \operatorname{Supp}(\mu)$. Then $S^{-1}U$ is an open neighbourhood of the identity in $SL(\mathbb{R}^m)$ and for all $R \in S^{-1}U$, $R\nu_1 = \nu_1$.

But for any orthonormal basis $\{f_1, \ldots, f_m\}$ of $\mathbb{R}^m$ we can find a linear map in $S^{-1}U$ whose limit points in P^{m-1} are exactly the directions $f_1, \ldots, f_m$. Therefore ν_1 must be supported on $\{p(f_1), \ldots, p(f_m)\}$ for any such basis, which is impossible. (Here by $p\colon (\mathbb{R}^m)\setminus\{0\} \to P^{m-1}$ we mean the natural projection.) □

Theorem 3.2. *Assume* $\dim M \geqq 2$. *Suppose the* $\mathrm{SDS}(X, z)$ *described in this section satisfies condition E, and the induced* $\mathrm{SDS}(SLX, z)$ *on SLM satisfies condition H.*

Then the highest Lyapunov exponent β of $\mathrm{SDS}(X, z)$ *is strictly positive.*

Proof. Condition E ensures that ρ is ergodic, and condition H ensures that for $\underline{v}$ in some neighbourhood U of $\underline{u}$ of the statement of that condition, the corresponding condition occurs at $\underline{v}$, and hence the transition probability $SLp_t(\underline{v}, -)$ (any $t > 0$) associated with (SLX, z) has C^0 density. Also, condition E ensures that the transition probability for $p_t(x, -)$ on M (where $\underline{v} \in SL_x M$) has nonvanishing C^0 density, and that for all y, the probability $SLp_t^Y(\underline{v}, -)$ on $SL_y M$, i.e. $SLp_t(\underline{v}, -)$ conditioned on the event $\xi_t(\omega)x = y$, also has C^0 density. Note that $SLp_t^Y(\underline{v}, -)$ is defined for $p_t(x, -)$ - a.e. y and hence for ρ - a.e. y, since the density of $p_t(x, -)$ does not vanish.

Now,

$$SL\xi_t(\omega)\underline{v} = \underline{w} \in SL_y M \leftrightarrow T_x\xi(\omega) = \underline{w} \circ \underline{u}^{-1} \in SL(T_x M, T_y M),$$

so we conclude that via $T_x\xi_t(\omega)$, the conditioning induces a probability on $SL(T_x M, T_y M)$ with C^0 density.

Take any probability ν on PM with $\pi(\nu) = \rho$ and disintegrate ν as $\rho \otimes \{\nu_x\}_x$. Then applying Lemma 3.1, we conclude that for a non ρ-null set of x's and given such x, for $p_t(x, -)$ - a.e. y, we cannot have $T\xi_t(\omega)\nu_x = \nu_y$ a.s. with respect to the conditioned measure, and it follows that we cannot have $\eta_t(\omega)\nu = \nu$ for P - a.e. ω. Now apply Theorem 1. □

Note. If $\dim M = 1$ then M must be essentially the circle and so if the flow preserves volume, it must be a random rotation, with $\beta = 0$.

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