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Robust Prediction and Interpolation for Vector Stationary Processes

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Summary. Robust multivariate prediction and interpolation problems for statistically contaminated vector valued second order stationary processes are considered. The statistical contamination is modeled by requiring that the spectral density matrices of the processes lie within certain nonparametric classes. Both prediction and interpolation are then formalized as games whose saddle point solutions are sought. Finally, such solutions are found and analyzed, for two specific multivariate spectral classes.

1. Introduction

The prediction and interpolation problems for stationary processes have received considerable attention for a number of years. The bulk of the work concentrates around scalar processes and the parametric model. The assumption there is that the measure generating the stochastic process is well known. The initial significant results on prediction and interpolation for the parametric model were given by Wiener (1949) and Kolmogorov (1941).

Strictly speaking, the term prediction refers to the extraction of a datum from the process, when a number of past process data have been observed noiselessly. The term interpolation refers to the same extraction, when past as well as future noiseless process data are available. The two terms are extended sometimes to include noisy observation data. Some results on those extended problems, and for the parametric model, can be found in the papers by Snyders (1973) and Viterbi (1965). We point out here that the majority of studies on the extended problems consider asymptotic and linear prediction and interpolation operations.

The last few years considerable attention has been given to the robust extended prediction problem. Some attention has also been given to the robust

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nonextended interpolation problem. The robust model is nonparametric, and the assumption is that the measure that generates the stationary process is not well known. The existing work on robust extended prediction and interpolation concentrates around scalar stationary processes, linear asymptotic prediction and interpolation operations, and noisy observation data. Representative results here include robust Wiener and Kalman filtering for scalar stationary processes (Masreliez et al. (1977), Kassam et al. (1977), Martin et al. (1976), Cimini et al. (1980), Poor 1980). Related work on time series outliers can be found in Martin et al. (1977). Hosoya (1978) considers the robust nonextended prediction problem, for scalal stationary processes and the additive contamination model. The robust solution is found there within the class of asymptotic linear prediction operations. A more general theoretical treatment of robust filtering problems for scalar processes can be found in the recent papers by Franke and Poor (1984), Franke (1985) and Vastola and Poor (1984).

A survey of most of the up to date existing results in minimax robust methods can be found in Kassam and Poor (1985). A game theoretic formulation on the measures of the stochastic processes is presented by Papantoni-Kazakos (1984), for the robust extended prediction problem. Chen et al. (1981, 1982) consider robust multidimensional matched filtering, for classes with identical eigenvectors. Regarding the robust nonextended interpolation problem, for scalar processes, the interested reader may look into the works by Taniguchi (1981) and Kassam (1982).

The prediction problem for vector processes is considerably more involved than that for scalar processes. The difficulty is mainly due to the cross correlations among the component processes, which have a direct impact on the complexity of the correlation matrix, and the spectral distribution matrix of the vector process. Important questions regarding the structure of a vector process such as rank, regularity, and non-determinacy are treated by Wiener et al. (1957, 1958), Helson et al. (1958), Hannan (1970), and Zasuhin (1941).

In the present paper, we consider the robust nonextended prediction and interpolation problems for vector stationary processes with absolutely continuous spectra whose spectral density matrices lie within certain well defined classes, and we will formulate these problems as games with saddle point solutions. Then, we will find those solutions for two specific classes of spectral density matrices. One of the classes represents additive contamination of a fixed nominal spectral density matrix and includes a power constraint. The other class includes the set of all spectral density matrices with fixed power on prespecified frequency quantiles (p-point class). Vector processes have not been treated in these cases (some limited consideration can be found in Taniguchi 1981), and they present interesting peculiarities both theoretical and practical.

The organization of the paper is as follows. In Sect. 2, we summarize the classical results on multivariate prediction and interpolation. In Sect. 3, we define the spectral classes under consideration, and we formalize the prediction and interpolation games. In Sects. 4 and 5, we find the saddle point solutions for the prediction and interpolation games respectively. Finally, in Sect. 6 we present some conclusions and a brief discussion.

2. Preliminaries

Let $\{\underline{x}_k, k \in Z\}$ be an n-variable, discrete-time, second order stationary process whose spectral distribution matrix is absolutely continuous with respect to the Lebesgue measure in the interval $[-\pi, \pi]$. Let $f(\omega)$, $\omega \varepsilon [-\pi, \pi]$, be the spectral density matrix of the process. Note that $f(\omega)$ is nonnegative definite and Hermitian for all ω.

We consider the space $L_2(f(\omega)\,d\omega)$ of all $n \times n$ matrix valued complex functions $A(\omega)$ on $[-\pi, \pi]$, for which (1) below is true.

$$\operatorname{tr} \int_{-\pi}^{\pi} A(\omega) f(\omega) A^T(\omega)\, d\omega < \infty \tag{1}$$

where, the symbols tr and T stand for trace and conjugate transpose respectively. Considering any two elements $A_1(\omega)$ and $A_2(\omega)$ of $L_2(f(\omega)\,d\omega)$ as equivalent if $\operatorname{tr} \int_{-\pi}^{\pi} (A_1(\omega) - A_2(\omega)) f(\omega)(A_1(\omega) - A_2(\omega))^T d\omega = 0$, then, $L_2(f(\omega)\,d\omega)$ is made into a Hilbert space (Hannan 1970), with inner product and norm defined respectively as follows.

$$(A_1(\omega), A_2(\omega))_{f(\omega)d\omega} = \operatorname{tr} \int_{-\pi}^{\pi} A_1(\omega) f(\omega) A_2^T(\omega)\, d\omega$$

$$\|A_1(\omega)\|_{f(\omega)d\omega} = (A_1(\omega), A_1(\omega))^{1/2}. \tag{2}$$

Let S_p be the convex set of all matrix trigonometric polynomials, $g^0(\omega)$, of the form.

$$g^0(\omega) = I + \sum_{i=1}^{N} A_i e^{j\omega i} \tag{3}$$

where N runs over all positive integers, $A_1, A_2, \ldots$ are any $n \times n$ complex matrices, and I is the $n \times n$ identity matrix. Then, it is obvious that $S_p \subset L_2(f(\omega)\,d\omega)$. Let $\bar{S}_p$ be the closure of S_p in $L_2(f(\omega)\,d\omega)$. The one step linear prediction problem for the process $\{\underline{x}_k, k \in Z\}$ is defined as the problem of minimizing the functional

$$e(f(\omega), g(\omega)) = (2\pi)^{-1} \operatorname{tr} \int_{-\pi}^{\pi} g(\omega) f(\omega) g^T(\omega)\, d\omega = (2\pi)^{-1} \|g(\omega)\|^2_{f(\omega)d\omega} \tag{4}$$

over all $g(\omega) \in \bar{S}_p$. Since $\bar{S}_p$ is a closed and convex subset of the Hilbert space $L_2(f(\omega)\,d\omega)$, it contains a unique element of minimal norm (unique in the equivalence sense defined above), and that element is the optimal predictor.

For reasons that will be explained below, we are going to consider a more general prediction problem, by enlarging the set S_p, to contain all matrix trigonometric polynomials $g^0(\omega)$ of the form,

$$g^0(\omega) = A_0 + \sum_{i=1}^{N} A_i e^{j\omega i}. \tag{5}$$

As before, N runs over all positive integers, and $A_1, A_2, \ldots$ are any complex $n \times n$ matrices. A_0, however, can now be any $n \times n$ complex matrix, whose determinant is constrained to be equal to one. Let S_p^0 be the set of all polynomials of the form (5). The convex hull S_p^{0c} of S_p^0 contains all polynomials of the form $B_0 + \sum_{i=1}^{N} B_i e^{j\omega i}$, with $\det(B_0) \geqq 1$. This follows from the inequality, $\det(\lambda A + (1-\lambda) B) \geqq (\det(A))^{\lambda} (\det(B))^{1-\lambda}$; $0 \leqq \lambda \leqq 1$ (Bellman 1970). If we take the closure $\bar{S}_p^{0c}(f(\omega)\,d\omega)$ of S_p^{0c} in $L_2(f(\omega)\,d\omega)$, we can define the new prediction problem as follows.

$$\min_{g(\omega) \in \bar{S}_p^{0c}(f(\omega)d\omega)} e(f(\omega), g(\omega)) = (2\pi)^{-1} \min_{g(\omega) \in \bar{S}_p^{0c}(f(\omega)d\omega)} \|g(\omega)\|^2_{f(\omega)d\omega}. \tag{6}$$

As a result of the derivation of the optimal predictor in Helson et al. (1958), the minimum in (6) has a closed form expression. In particular,

$$\min_{g(\omega) \in \bar{S}_p^{0c}(f(\omega)d\omega)} e(f(\omega), g(\omega)) = n \exp\left[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} \log f(\omega)\, d\omega\right]. \tag{7}$$

The right hand side of (7) is interpreted as zero if the scalar function $\operatorname{tr}\log f(\omega)$ is not integrable $\left(\text{since } \int_{-\pi}^{\pi} \operatorname{tr}\log f(\omega)\,d\omega \text{ is bounded from above, as can be verified by using Jensen's inequality, this can only happen if } \int_{-\pi}^{\pi} \operatorname{tr}\log f(\omega)\,d\omega = -\infty\right)$. In the sequel, we will not be concerned with the latter case and we will assume that $\int_{-\pi}^{\pi} \operatorname{tr}\log f(\omega)\,d\omega > -\infty$. An element $g'(\omega)$ in $\bar{S}_p^{0c}(f(\omega)\,d\omega)$, that attains the minimum in (6) is such that,

$$g'(\omega)(g'(\omega))^T = \exp\left[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\log f(\omega)\,d\omega\right] f^{-1}(\omega). \tag{8}$$

It has been proven by Helson et al. (1958) that if $g'(\omega)$ exists, then $g'(\omega) \in \bar{S}_p^0(f(\omega)\,d\omega)$; i.e., the determinant of the leading Fourier coefficient of $g'(\omega)$ is equal to one, or, equivalently, the minimum in (6) over $g(\omega) \in \bar{S}_p^0(f(\omega))\,d\omega)$ exists, although $\bar{S}_p^0(f(\omega)\,d\omega)$ is not convex, and it is attained at $g(\omega) = g'(\omega)$.

Consideration of the prediction problem in $\bar{S}_p^{0c}(f(\omega)\,d\omega)$, or equivalently in $\bar{S}_p^0(f(\omega)\,d\omega)$, has the remarkable advantage of a simple closed form expression for the minimum error, given by (7), which is a direct generalization of Szegö's formula (Grenander et al. 1958), for scalar processes.

The linear interpolation problem for the process $\{\underline{x}_k, k \in Z\}$ is less difficult, due to the lack of a causality requirement of the associated minimization problem. We will denote by S_i the convex set of all matrix trigonometric polynomials of the form,

$$h^0(\omega) = I + \sum_{\substack{i=-N \\ i \neq 0}}^{N} A_i e^{j\omega i} \tag{9}$$

where, N runs over all positive integers and $\{A_i, i \neq 0\}$ run over all complex $n \times n$ matrices. Let $\bar{S}_i(f(\omega)d\omega)$ be the closure of S_i in $L_2(f(\omega)d\omega)$. Then, the interpolation problem is defined as follows.

$$\min_{h(\omega)\in\bar{S}_i(f(\omega)d\omega)} e(f(\omega), h(\omega)) = (2\pi)^{-1} \min_{h(\omega)\in\bar{S}_i(f(\omega)d\omega)} \|h(\omega)\|^2. \tag{10}$$

As derived in Hannan (1970), the minimum in (10) is given by

$$(2\pi)^{-1} \min_{h(\omega)\in\bar{S}_i(f(\omega)d\omega)} \|h(\omega)\|^2_{f(\omega)d\omega} = 2\pi \operatorname{tr}\left[\left(\int_{-\pi}^{\pi} f^{-1}(\omega)\,d\omega\right)^{-1}\right] \tag{11}$$

and it is attained at some $h'(\omega)\in\bar{S}_i(f(\omega)d\omega)$, such that

$$h'(\omega) = 2\pi\left(\int_{-\pi}^{\pi} f^{-1}(\omega)\,d\omega\right)^{-1} f^{-1}(\omega). \tag{12}$$

In (12), $f^{-1}(\omega)$ is the Penrose-Moore generalized inverse of $f(\omega)$, and it is integrable for full rank processes.

3. The Robust Formalization

We now look at the above problems from a different point of view. We assume that the spectral structure of the observed process is only vaguely or incompletely specified. This corresponds to a more realistic situation, since the procedures for obtaining the spectrum of a process always involve errors. This applies even more to vector processes, where the increased complexity results in larger errors. With the above in mind, it is clear that a new formalization of the problems considered in Sect. 2 is needed. Such a formalization is given below, where the spectral density matrix of the process is assumed to be a member of a whole class of spectral density matrices. For the purpose of this work we are going to consider two different types of spectral classes, denoted by F_L and F_Q, which are defined as follows.

(a) $F_L = \Big\{ f(\omega): f(\omega) = (1-\varepsilon) f_0(\omega) + \varepsilon p(\omega),\ \omega \varepsilon [-\pi, \pi],\ \varepsilon$ fixed, $0 < \varepsilon < 1$

$f_0(\omega)$: known nominal spectral density matrix,
$p(\omega)$: arbitrary spectral density matrix satisfying

$$(2\pi)^{-1} \operatorname{tr} \int_{-\pi}^{\pi} p(\omega)\,d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} f_0(\omega)\,d\omega = w > 0,\ w \text{ fixed}\Big\}.$$

(b) $F_Q = \Big\{ f(\omega): (2\pi)^{-1} \operatorname{tr} \int_{D_i} f(\omega)\,d\omega = c_i > 0, \quad i = 1, \ldots, k, \quad c_1, \ldots, c_k$ fixed $D_1, \ldots, D_k$ fixed Lebesque measurable subsets of $[-\pi, \pi]$ with positive measure each, and $D_i \cap D_j = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{k} D_i = [-\pi, \pi] \Big\}$.

F_L is called the additive contamination class, or ε-contamination class, and it corresponds to the case where the observed process $\{\underline{x}_k\}$ is of the form

$$\underline{x}_k=\sqrt{1-\varepsilon}\,\underline{x}_k^0+\sqrt{\varepsilon}\,\underline{v}_k$$

where, $\{\underline{x}_k^0\}$ is a process with spectral density matrix $f_0(\omega)$, and $\{\underline{v}_k\}$ is a noise process uncorrelated to $\{\underline{x}_k^0\}$, whose spectral density matrix is known only to satisfy a power constraint, being arbitrary otherwise. F_Q is called the p-point class, and it contains all the spectra, whose power is specified by a positive number, on a finite collection of mutually exclusive and exhaustive measurable subsets of $[-\pi, \pi]$, with positive Lebesgue measure.

In pursuing a robust formalization for prediction and interpolation, it will be necessary to restrict the classes $\bar{S}_p^0(f(\omega)\,d\omega)$, $\bar{S}_i(f(\omega)\,d\omega)$, for the simple reason that instead of a single $f(\omega)$ a whole class of those is considered. In particular, we will consider the following classes of predictors and interpolators:

$$\begin{aligned} S_{pL}&=\bigcap_{f(\omega)\in F_L}\bar{S}_p^0(f(\omega)\,d\omega) \qquad S_{iL}=\bigcap_{f(\omega)\in F_L}\bar{S}_i(f(\omega)\,d\omega)\\ S_{pQ}&=\bigcap_{f(\omega)\in F_Q}\bar{S}_p^0(f(\omega)\,d\omega) \qquad S_{iQ}=\bigcap_{f(\omega)\in F_Q}\bar{S}_i(f(\omega)\,d\omega). \end{aligned} \tag{13}$$

A predictor (or interpolator), $g^e(\omega)$, is called robust, for the corresponding class of spectral density matrices F_L or F_Q if the following inequality holds.

$$\sup_{f(\omega)\in S_{RT}} e(f(\omega), g^e(\omega)) \leqq \sup_{f(\omega)\in S_{RT}} e(f(\omega), g(\omega));\ \forall g(\omega)\in S_{RT}$$
$$(T=L, Q.\ R=p, i). \tag{14}$$

Furthermore, a pair $(f^e(\omega), g^e(\omega)\in F_T\times S_{RT}$ $(T=L, Q.\ R=p, i)$ is called a saddle point solution of the game on $F_T\times S_{RT}$ with payoff functional $e(f(\omega), g(\omega))$, if

$$e(f(\omega), g^e(\omega)) \leqq e(f^e(\omega), g^e(\omega)) \leqq e(f^e(\omega), g(\omega))$$
$$\forall f(\omega)\in S_{RT},\quad \forall g(\omega)\in S_{RT}. \tag{15}$$

If a saddle point, $(f^e(\omega), g^e(\omega))$, satisfying (15) exists, then, $g^e(\omega)$ is a robust predictor or interpolator satisfying (14). The opposite is not in general true, i.e., the existence of a robust predictor or interpolator does not guarantee the existence of a saddle point. However, as we will show, the specific games that we consider here have saddle points. Therefore, the problem of finding robust predictors or interpolators can be reduced to the problem of finding saddle points of the corresponding games.

In Sect. 4, below, we solve the prediction games on $F_L\times S_{pL}$ and $F_Q\times S_{pQ}$. In Sect. 5, we solve the interpolation games on $F_L\times S_{iL}$ and $F_Q\times S_{iQ}$. In all cases we state the solutions and prove them directly by construction.

4. The Solution of the Prediction Games on $F_L \times S_{pL}$ and $F_Q \times S_{pQ}$

Let $\{\lambda_i^0(\omega), \underline{x}_i^0(\omega); i=1, \ldots, n\}$ be the ordered eigenvalues of $f_0(\omega)(\lambda_i^0(\omega) \geqq \lambda_{i+1}^0(\omega),\ i=1, \ldots, n-1, \forall \omega \varepsilon[-\pi, \pi])$, and the corresponding normalized eigenvectors. Since

$$(2\pi)^{-1} \operatorname{tr} \int_{-\pi}^{\pi} (1-\varepsilon) f_0(\omega)\, d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} (1-\varepsilon) \lambda_i^0(\omega)\, d\omega = (1-\varepsilon)\, w < w,$$

there exists a positive number c such that,

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \max((1-\varepsilon) \lambda_i^0(\omega), c)\, d\omega = w. \tag{16}$$

Let us define the set of functions,

$$\lambda_i^e(\omega) = \max((1-\varepsilon) \lambda_i^0(\omega), c), \qquad i=1, \ldots, n \tag{17}$$

and the matrix

$$f_L^e(\omega) = \sum_{i=1}^{n} \lambda_i^e(\omega)\, \underline{x}_i^0(\omega) (\underline{x}_i^0(\omega))^T. \tag{18}$$

$f_L^e(\omega)$ is Hermitian and positive definite, for all $\omega \varepsilon[-\pi, \pi]$, since its smallest eigenvalue is uniformly larger than $c>0$. Furthermore, $f_L^e(\omega) \in F_L$, since $f_L^e(\omega) - (1-\varepsilon) f_0(\omega) = \sum_{i=1}^{n} (\lambda_i^e(\omega) - (1-\varepsilon) \lambda_i^0(\omega))\, \underline{x}_i^0(\omega)(\underline{x}_i^0(\omega))^T$ is nonnegative definite for all ω, and (16) holds. We also note that $(f_L^e(\omega))^{-1}$ exists for all ω and it is integrable, and that the scalar function $\operatorname{tr} \log f_L^e(\omega)$ is integrable as well.

Let $K_{eL} \triangleq \exp\left[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} \log f_L^e(\omega)\, d\omega\right]$. We consider the matrix $K_{eL}(f_L^e(\omega))^{-1}$, which is easily recognized to be equal to the right-hand side of Eq. (8), in Sect. 2, for $f_L^e = f$, and which satisfies the requirements of Theorem 7.13 in Wiener et al. (1957, 1958). From that we conclude that there exists a factorization of $K_{eL}(f_L^e(\omega))^{-1}$ of the following form.

$$g_L^e(\omega)(g_L^e(\omega))^T = K_{eL}(f_L^e(\omega))^{-1} \tag{19}$$

where, if $\{A_n^e, n \in Z\}$ are the Fourier coefficients of $g_L^e(\omega)$, then $A_n^e = 0$ for $n<0$, and $\det A_0^e = 1$. According to the derivations in Helson et al. (1958), $g_L^e(\omega)$ is the element of $\overline{S_p^0}(f_L^e(\omega)\, d\omega)$ that minimizes $e(f_L^e(\omega), g_L(\omega))$, with respect to $g_L(\omega) \in \overline{S_p^0}(f_L^e(\omega)\, d\omega)$. That is,

$$e(f_L^e(\omega), g_L^e(\omega)) \leqq e(f_L^e(\omega), g_L(\omega)); \qquad \forall g_L(\omega) \in \overline{S_p^0}(f_L^e(\omega)\, d\omega). \tag{20}$$

In Lemma 1 below, we prove that $g_L^e(\omega) \in S_{pL}$. Theorem 1 establishes that the pair $(f_L^e(\omega), g_L^e(\omega))$ is the solution of the prediction game on $F_L \times S_{pL}$.

Lemma 1. $g_L^e(\omega) \in S_{pL}$.

Proof. From (19) and $(f_L^e(\omega))^{-1} \leqq c^{-1} I$, we conclude that each entry of $g_L^e(\omega)$ is essentially bounded $(d\omega)$. From its Fourier coefficients $A_0^e, A_1^e, \ldots$, we form the sequence $\{G_N^e(e^{j\omega})\}$ of the Fejér-Cesaro partial sums,

$$G_N^e(e^{j\omega}) = \frac{1}{N+1} \sum_{k=0}^{N} S_k^e(e^{j\omega})$$

where

$$S_N^e(e^{j\omega}) = \sum_{i=0}^{N} A_i^e e^{j\omega i}. \qquad \text{Evidently, } G_N^e(e^{j\omega}) \in S_p^0.$$

By the usual theory of this sum, each entry of $G_N^e(e^{j\omega})$ converges a.e. $(d\omega)$ to the corresponding entry of $g_L^e(\omega)$ boundedly, since $g_L^e(\omega)$ is bounded. Put $h_N(\omega) = G_N^e(e^{j\omega}) - g_L^e(\omega)$. Then, for any $f(\omega) \in F_L$ we have:

$$\|h_N(\omega)\|^2_{f(\omega)d\omega} = \int_{-\pi}^{\pi} \operatorname{tr} h_N(\omega) f(\omega) h_N^T(\omega)\, d\omega$$
$$\leqq \int_{-\pi}^{\pi} \lambda_{\max}(h_N^T(\omega) h_N(\omega)) \operatorname{tr} f(\omega)\, d\omega.$$

where $\lambda_{\max}(\cdot)$ denotes maximum eigenvalue. Since $h_N(\omega) \to 0$ a.e. $(d\omega)$ boundedly, it is implied that $\lambda_{\max}(h_N^T(\omega) h_N(\omega)) \to 0$ a.e. $(d\omega!$ boundedly. Now, since $\operatorname{tr} f(\omega)$ contains no singularities, due to the assumed absolute continuity of the members of F_L, it is concluded that $\lambda_{\max}(h_N^T(\omega) h_N(\omega)) \to 0$ a.e. $(\operatorname{tr} f(\omega)\, d\omega)$. Application of the dominated convergence theorem on $\lambda_{\max}(h_N^T(\omega) h_N(\omega))$ yields:

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \lambda_{\max}(h_N^T(\omega) h_N(\omega)) \operatorname{tr} f(\omega)\, d\omega$$
$$= \int_{-\pi}^{\pi} \lim_{N \to \infty} \lambda_{\max}(h_N^T(\omega) h_N(\omega)) \operatorname{tr} f(\omega)\, d\omega = 0, \qquad \text{which implies}$$
$$\|h_N(\omega)\|_{f(\omega)d\omega} \to 0.$$

The preceding arguments show that there always exists a sequence of elements of S_p^0, which tends to $g_L^e(\omega)$, under any norm $\|\cdot\|_{f(\omega)d\omega}$, $f(\omega) \in F_L$. Thus $g_L^e(\omega) \in S_{pL}$.

Remark. The basic constituents for the proof of Lemma 1 are:

1) The fact that the eigenvalues of $f_L^e(\omega)$ are bounded away from zero which implies the a.e. $(d\omega)$ boundedness of $g_L^e(\omega)$. 1) The absolute continuity of the members of the class F_L, which permits the transition from the a.e. $(d\omega)$ to the a.e. $(\operatorname{tr} f(\omega)\, d\omega)$ convergence. Those requirements are satisfied for all the other games we consider in the sequel.

Theorem 1. *The pair* $(f_L^e(\omega), g_L^e(\omega))$ *is a saddle point solution of the prediction game on* $F_L \times S_{pL}$.

Proof. We have to prove:

$$e(f_L, g_L^e) \leqq e(f_L^e, g_L^e) \leqq e(f_L^e, g_L); \ \forall f_L \in F_L; \ \forall g_L \in S_{pL}. \tag{21}$$

The right-hand side inequality in (21) follows from (20) and $\overline{S_p^0}(f_L^e\, d\omega) \supset S_{pL}$. Also

$$e(f_L^e, g_L^e) = n \exp\left[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} \log f_L^e(\omega)\, d\omega\right] = nK_{eL} \tag{22}$$

and

$$e(f_L, g_L^e) = (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\left[g_L^e (g_L^e)^T f_L\right] d\omega. \tag{23}$$

Combining (23) with (19), we get,

$$\begin{aligned} e(f_L, g_L^e) &= (2\pi)^{-1} K_{eL} \int_{-\pi}^{\pi} \operatorname{tr}\left[f_L (f_L^e)^{-1}\right] d\omega \\ &= (2\pi)^{-1} K_{eL} \int_{-\pi}^{\pi} \sum_{i=1}^{n} (\lambda_i^e(\omega))^{-1} (\underline{x}_i^0(\omega))^T f_L(\omega)\, \underline{x}_i^0(\omega)\, d\omega. \end{aligned} \tag{24}$$

Put $\mu_i(\omega) = (\underline{x}_i^0(\omega))^T f_L(\omega)\, \underline{x}_i^0(\omega)$. Since $f_L(\omega) \in F_L$, $f_L(\omega) - (1-\varepsilon) f_0(\omega)$ should be nonnegative definite which implies that

$$\mu_i(\omega) \geqq (1-\varepsilon)\, \lambda_i^0(\omega); \qquad \forall \omega, i = 1, \ldots, n \tag{25}$$

Also

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} f_L(\omega)\, d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \mu_i(\omega)\, d\omega = w. \tag{26}$$

From (24) and (22) we obtain,

$$\begin{aligned} e(f_L, g_L^e) - e(f_L^e, g_L^e) &= K_{eL} \left((2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \frac{\mu_i(\omega)}{\lambda_i^e(\omega)}\, d\omega - n\right) \\ &= \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \left[\int_{(1-\varepsilon)\lambda_i^1(\omega) \geqq c} \frac{\mu_i(\omega) - (1-\varepsilon)\, \lambda_i^0(\omega)}{(1-\varepsilon)\, \lambda_i^0(\omega)}\, d\omega + \int_{(1-\varepsilon)\lambda_i^0(\omega) < c} \frac{\mu_i(\omega) - c}{c}\, d\omega\right] \\ &\leqq \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \left[\int_{(1-\varepsilon)\lambda_i^0(\omega) \geqq c} \frac{\mu_i(\omega) - (1-\varepsilon)\, \lambda_i^0(\omega)}{c}\, d\omega + \int_{(1-\varepsilon)\lambda_i^0(\omega) < c} \frac{\mu_i(\omega) - c}{c}\, d\omega\right] \\ &= \frac{K_{eL}}{2\pi} \sum_{i=1}^{n} \int_{-\pi}^{\pi} \frac{\mu_i(\omega) - \lambda_i^e(\omega)}{c}\, d\omega = \frac{K_{eL}}{2\pi c} \int_{-\pi}^{\pi} \operatorname{tr}(f_L(\omega) - f_L^e(\omega))\, d\omega = 0 \end{aligned}$$

and the left-hand side inequality in (21) follows.

We now proceed to the solution of the prediction game on $F_Q \times S_{pQ}$. We define the spectral density matrix

$$f_Q^e(\omega) = \left(\frac{2\pi}{n} \sum_{i=1}^{k} c_i\, 1_{D_i}(\omega)\, m^{-1}(D_i)\right) \cdot I \tag{27}$$

where $1_{D_i}(\omega)$ is the indicator function of the set D_i, and $m(\cdot)$ is the Lebesque measure in $[-\pi, \pi]$. It can be seen by inspection that $f_Q^e(\omega) \in F_Q$. Since $c_i > 0$, $m(D_i) > 0$, the eigenvalue of $f_Q^e(\omega)$ is bounded away from zero and from infinity, by

$$\frac{2\pi}{n} \min_{i=1,\dots,k} \left\{\frac{c_i}{m(D_i)}\right\}, \frac{2\pi}{n} \max_{i=1,\dots,k} \left\{\frac{c_i}{m(D_i)}\right\}$$

respectively. It follows that $(f_Q^e(\omega))^{-1}$ and $\operatorname{tr}\log(f_Q^e(\omega))^{-1}$ both exist and are integrable. We thus conclude that there exists some $g_Q^e(\omega) \in S_p^0(f_Q^e(\omega)\,d\omega)$, such that

$$g_Q^e (g_Q^e)^T = k_{eQ} (f_Q^e)^{-1}$$

where

$$k_{eQ} = \exp\left[(2\pi n)^{-1} \int_{-\pi}^{\pi} \operatorname{tr}\log f_Q^e \, d\omega\right]. \tag{28}$$

where the Fourier coefficients of g_Q^e, $\{A_i^e, i \in Z\}$, vanish for $i < 0$, and where $\det A_0^e = 1$. Exploiting the assumption of the absolute continuity of all the members of F_Q, we can argue exactly, as in Lemma 1, and establish that $g_Q^e \in S_{pQ} = \bigcap_{f \in F_Q} \overline{S_p^0}(f d\omega)$. Also, g_Q^e is the element of $\overline{S_p^0}(f_Q^e d\omega)$ which minimizes $e(f_Q^e, g_Q)$. We conclude this section with the following theorem.

Theorem 2. *The pair* (f_Q^e, g_Q^e) *given by* (27), (28) *is a saddle point solution of the prediction game on* $F_Q \times S_{pQ}$.

Proof. We have to prove that,

$$e(f_Q, g_Q^e) \leqq e(f_Q^e, g_Q^e) \leqq e(f_Q^e, g_Q); \ \forall f_Q \in F_Q; \ \forall g_Q \in S_{pQ}.$$

The right-hand side inequality follows from the fact that $S_{pQ} \subset \overline{S_p^0}(f_Q^e d\omega)$, and that g_Q^e is the minimizing element of $e(f_Q^e, g_Q)$ for $g_Q \in \overline{S_p^0}(f_Q^e d\omega)$. We thus have:

$$\begin{aligned} e(f_Q, g_Q^e) &= \frac{1}{2\pi} \operatorname{tr} \int_{-\pi}^{\pi} g_Q^e f_Q (g_Q^e)^T d\omega \\ &= \frac{k_{eQ}}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}[f_Q (f_Q^e)^{-1}]\, d\omega = \frac{k_{eQ}}{2\pi} \sum_{i=1}^{n} \frac{n m(D_i)}{2\pi c_i} \int_{D_i} \operatorname{tr} f_Q \, d\omega \\ &= n k_{eQ} = e(f_Q^e, g_Q^e). \end{aligned}$$

5. The Solution of the Interpolation Games on $F_L \times S_{iL}$ and $F_Q \times S_{iQ}$

Let, as in Sect. 4, $\{\lambda_i^0(\omega), \underline{x}_i^0(\omega), i = 1, \dots, n\}$ be the ordered eigenvalues and corresponding eigenvectors of the nominal spectral density matrix $f_0(\omega)$. For each eigenvalue, we define the function,

$$T_i(\gamma) = \gamma \int_{-\pi}^{\pi} [\max(\gamma, (1-\varepsilon)\lambda_i^0(\omega))]^{-1} d\omega, \qquad i = 1, \dots, n.$$

It can be easily verified that $T_i(\gamma)$ is continuous and strictly monotonic for $0 \leqq \gamma \leqq \operatorname{ess\,sup}_{\omega}((1-\varepsilon)\lambda_i^0(\omega))$. Then, for any positive number $c<2\pi$, there exists a unique γ_i, such that, $T_i(\gamma_i)=c$, $i=1,\ldots,n$. We put $\gamma_i(c)=T_i^{-1}(c)$. The inverse mapping $T_i^{-1}(c)$ is also monotonic and continuous. Now, since $(2\pi)^{-1}\int_{-\pi}^{\pi}\sum_{i=1}^{n}(1-\varepsilon)\lambda_i^0(\omega)\,d\omega=(1-\varepsilon)W<W$, there exists a positive number c^*, such that,

$$(2\pi)^{-1}\int_{-\pi}^{\pi}\sum_{i=1}^{n}\max(T_i^{-1}(c^*),(1-\varepsilon)\lambda_i^0(\omega))\,d\omega=w. \tag{29}$$

Before we proceed further, we will make an assumption, concerning the eigenvectors of $f_0(\omega)$, $\{\underline{x}_i^0(\omega), i=1,\ldots,n\}$. For the purpose of obtaining closed form solutions, we will assume that $\{\underline{x}_i^0(\omega)\}$ are constant, independent of ω, for every $i=1,\ldots,n$. We denote them by $\underline{x}_i^0$ omitting their argument. Thus, we consider the class F_L of spectral density matrices, such that the nominal $f_0(\omega)$ has constant eigenvectors. We note that this is different from requiring that all members of F_L have constant eigenvectors.

We define:

$$\lambda_i^e(\omega)=\max(T_i^{-1}(c^*),(1-\varepsilon)\lambda_i^0(\omega)) \tag{30}$$

$$f_L^e(\omega)=\sum_{i=1}^{n}\lambda_i^e(\omega)\,\underline{x}_i^0(\underline{x}_i^0)^T. \tag{31}$$

The eigenvalues of $f_L^e(\omega)$ in (31) are bounded away from zero, since they are all uniformly larger than or equal to $T_i^{-1}(c^*)>0$. Also, $f_L^e(\omega)\in F_L$, due to (31), and to the fact that $f_L^e(\omega)\geqq(1-\varepsilon)f_0(\omega)$; $\forall\omega\in[-\pi,\pi]$. We define

$$h_L^e(\omega)=2\pi\left(\int_{-\pi}^{\pi}(f_L^e(\omega))^{-1}\,d\omega\right)^{-1}(f_L^e(\omega))^{-1} \tag{32}$$

which minimizes $\|h_L(\omega)\|_{f_L^e d\omega}$, $h_L(\omega)\in\overline{S_i}(f_L^e\,d\omega)$. Since $(2\pi)^{-1}\int_{-\pi}^{\pi}h_L^e(\omega)\,d\omega=I$, the Fejér-Cesaro partial sums of the Fourier series of $h_L^e(\omega)$ are trigonometric polynomials belonging to S_i. Furthermore, since the entries of $h_L^e(\omega)$ are bounded by (32), the sequence of the Fejér-Cesaro sums will converge dominatedly a.e. $(d\omega)$ to $h_L^e(\omega)$. The absolute continuity of the members of the class F_L, together with the application of the dominated convergence theorem then implies, in a way similar to that in Lemma 1, that the above sequence will converge to $h_L^e(\omega)$ in the norm $\|\cdot\|_{f(\omega)d\omega}$, for any $f(\omega)\in F_L$. Thus $h_L^e(\omega)\in S_{iL} = \bigcap_{f(\omega)\in F_L}\overline{S_i}(f(\omega)\,d\omega)$. We now state the solution of the game on $F_L\times S_{iL}$.

Theorem 3. *The pair* $(f_L^e(\omega), h_L^e(\omega))$ *defined by* (30), (31), (32) *is a saddle point solution of the interpolation game on* $F_L\times S_{iL}$.

Proof. We restate the theorem, as follows.

$$e(f_L(\omega),h_L^e(\omega))\leqq e(f_L^e(\omega),h_L^e(\omega))\leqq e(f_L^e(\omega),h_L(\omega));$$
$$\forall f_L(\omega)\in F_L,\ \forall h_L(\omega)\in S_{iL}.$$

The second inequality follows immediately from Sect. 2, and the fact that $S_{iL} \subset \overline{S_i}(f_L^e(\omega)\,d\omega)$.

Put $k'_{eL} = \left(\int_{-\pi}^{\pi} (f_L^e(\omega))^{-1} d\omega \right)^{-1}$. The following relationships are valid:

$$\begin{aligned} & e(f_L(\omega), h_L^e(\omega)) - e(f_L^e(\omega), h_L^e(\omega)) \\ & = (2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} h_L^e(\omega)(f_L(\omega) - f_L^e(\omega))(h_L^e(\omega))^T d\omega \\ & = 2\pi \int_{-\pi}^{\pi} \operatorname{tr} [k'_{eL}(f_L^e(\omega))^{-1}(f_L(\omega) - f_L^e(\omega))(k'_{eL}(f_L^e(\omega))^{-1})^T] \, d\omega \\ & = 2\pi \sum_{i=1}^{n} \frac{1}{\left[\int_{-\pi}^{\pi} (\lambda_i^e(\omega))^{-1} d\omega \right]^2} \int_{-\pi}^{\pi} \frac{\underline{x}_i^T f_L(\omega) \underline{x}_i - \lambda_i^e(\omega)}{(\lambda_i^e(\omega))^2} d\omega. \end{aligned} \tag{33}$$

Since $f_L(\omega) \in F_L$, then $f_L(\omega) \geqq (1-\varepsilon) f_0(\omega)$, which implies

$$\underline{x}_i^T f_L(\omega) \underline{x}_i \geqq (1-\varepsilon) \underline{x}_i^T f_0(\omega) \underline{x}_i = (1-\varepsilon) \lambda_i^0(\omega)$$

and

$$(2\pi)^{-1} \int_{-\pi}^{\pi} \operatorname{tr} f_L(\omega) \, d\omega = (2\pi)^{-1} \int_{-\pi}^{\pi} \sum_{i=1}^{n} \underline{x}_i^T f(\omega) \underline{x}_i \, d\omega = w.$$

Put $v_i(\omega) = \underline{x}_i^T f_L(\omega) \underline{x}_i \geqq (1-\varepsilon) \lambda_i^0(\omega)$; $\forall \omega \in [-\pi, \pi]$, $i = 1, \ldots, n$. Then, from (33) and the equality $T_i^{-1}(c^*) \int_{-\pi}^{\pi} (\lambda_i^e(\omega))^{-1} d\omega = c^*$, we get,

$$\begin{aligned} & e(f_L(\omega), h_L^e(\omega)) - e(f_L^e(\omega), h_L^e(\omega)) \\ & = 2\pi \sum_{i=1}^{n} \frac{1}{\left[\int_{-\pi}^{\pi} (\lambda_i^e(\omega))^{-1} d\omega \right]^2} \left[\int_{T_i^{-1}(c^*) > (1-\varepsilon)\lambda_i^0(\omega)} \frac{v_i(\omega) - T_i^{-1}(c^*)}{(T_i^{-1}(c^*))^2} d\omega \right. \\ & \quad \left. + \int_{T_i^{-1}(c^*) \leqq (1-\varepsilon)\lambda_i^0(\omega)} \frac{v_i(\omega) - (1-\varepsilon)\lambda_i^0(\omega)}{((1-\varepsilon)\lambda_i^0(\omega))^2} d\omega \right] \\ & \leqq \frac{2\pi}{(c^*)^2} \sum_{i=1}^{n} \int_{-\pi}^{\pi} (v_i(\omega) - \lambda_i^e(\omega)) \, d\omega = 0. \end{aligned}$$

The proof is now complete.

Finally, we examine the interpolation game on $F_Q \times S_{iQ}$. The result is here summarized in a theorem, whose proof is analogous to that of Theorem 2, and is omitted.

Theorem 4. *The pair* $(f_Q^e(\omega), h_Q^e(\omega))$ *which is defined by the expressions,*

$$f_Q^e(\omega) = \frac{1}{n} \sum_{l=1}^{k} 2\pi c_l 1_{D_l}(\omega) m^{-1}(D_l) \cdot I = \lambda_Q^e(\omega) \cdot I$$

$$h_Q^e(\omega) = 2\pi \left(\int_{-\pi}^{\pi} (f_Q^e(\omega))^{-1} d\omega \right)^{-1} (f_Q^e(\omega))^{-1}$$

is a saddle point solution of the interpolation game on $F_Q \times S_{iQ}$.

6. Conclusions and Discussion

In this paper, we considered the prediction and interpolation problems for vector stationary processes with ill-specified statistical structures. We modeled the uncertainty in the statistical description of the processes, by assuming that their spectral density matrices lie within certain classes. Then, we formalized the problems as games, whose saddle point solutions were found for two specific classes of multivariate spectral densities. The first such class (F_L) represents additive contamination (or ε-contamination) of a nominal spectral matrix, and it includes a power constraint. The second class (F_Q) consists of all spectral matrices whose power is fixed on prespecified number of frequency quantiles.

Both the F_L and the F_Q classes were assumed to consist of absolutely continuous spectra only. If these classes are allowed to include spectra with singularities as well, then the found saddle point solutions cannot be guaranteed to belong to the appropriate product classes. There is an exception for class F_L, where we can allow the nominal spectrum to include singularities, at a certain set of points. Then, each member of F_L has singularities at exactly the same points, and the results we obtained can then be readily extended to include this case. However, when the contaminating spectrum is allowed singularities, then a robust solution does not generally exist. For this latter case, and for scalar processes, an approximate solution is given by Hosoya (1978). Vastola and Poor (1984) consider classes with singular spectra, and they prove existence of robust solutions for classes that satisfy certain compactness requirements.

All the derived solutions for the prediction and interpolation games correspond to the eigenvalues with the "flattest" possible tails, or equivalently to spectral measures with the most evenly spread power. For the F_Q class, we obtained identical spectral density matrices for both the prediction and the interpolation solutions, which are diagonal, with a single eigenvalue that is piece-wise constant. For the F_L class, the spectral density matrices corresponding to the saddle point solutions of the prediction and the interpolation games are not identical (in contrast to the scalar case). In particular, each eigenvalue of the interpolation solution is determined by a different truncation constant ($T_i^{-1}(c^*)$) of the corresponding nominal eigenvalue, while the prediction solution has a single such constant (c), as we can see from Eq. (17) and (30). For dimensionalities higher than one, the solutions obtained for the F_Q class are not unique. We then selected the simplest solutions. The solutions for the F_L class are unique only with respect to the eigenvalues. That is, there may be more than one n-tuples of eigenvectors, which together with the fixed set of eigenvalues, yield solutions for the prediction game. All such solutions attain, however, the same value of the game.

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